Applicability of the q-Analogue of Zeilberger's Algorithm

William Y.C. Chen[1], Qing-Hu Hou[2] and Yan-Ping Mu[3]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China

E-mail: [1]chen@nankai.edu.cn, [2]qhhou@nankai.edu.cn, [3]myphb@eyou.com

Abstract

The applicability or terminating condition for the ordinary case of Zeilberger's algorithm was recently obtained by Abramov. For the q-analogue, the question of whether a bivariate q-hypergeometric term has a qZ-pair remains open. Le has found a solution to this problem when the given bivariate q-hypergeometric term is a rational function in certain powers of q. We solve the problem for the general case by giving a characterization of bivariate q-hypergeometric terms for which the q-analogue of Zeilberger's algorithm terminates. Moreover, we give an algorithm to determine whether a bivariate q-hypergeometric term has a qZ-pair.

AMS Classification: 33F10, 68W30

Keywords: Zeilberger's algorithm, q-hypergeometric term, Z-pair, qZ-pair, proper hypergeometric term, q-proper hypergeometric term.

1 Introduction

Zeilberger's algorithm [10,16,19], also known as the method of *creative telescoping*, is devised for proving hypergeometric identities of the form

$$\sum_{k=-\infty}^{\infty} F(n,k) = f(n),$$

where $F(n,k)$ is a bivariate hypergeometric term and $f(n)$ is a given function (for most cases a hypergeometric term plus a constant). The algorithm can be easily adapted to the q-case, which is called the q-analogue of Zeilberger's algorithm [7, 12, 15, 18]. Let N and K be the shift operators with respect to n and k respectively, defined by

$$NT(n,k) = T(n+1,k) \qquad \text{and} \qquad KT(n,k) = T(n,k+1).$$

Given a bivariate q-hypergeometric term $T(n,k)$, the q-analogue of Zeilberger's algorithm aims to find a *qZ-pair* (L,G), where L is a linear difference operator with coefficients in the ring of polynomials in q^n

$$L = a_0(q^n)N^0 + a_1(q^n)N^1 + \cdots + a_r(q^n)N^r$$

and G is a bivariate q-hypergeometric term $G(n,k)$ such that

$$LT(n,k) = (K-1)G(n,k).$$

Zeilberger's algorithm has been widely used as a powerful tool to prove hypergeometric identities. It was an open question when the algorithm terminates.

This problem was solved recently by Abramov [1,2]. For the q-analogue of Zeilberger's algorithm, Le [13] found a solution to the termination problem for the case of rational functions. In this paper we provide a complete solution for the general q-case.

We begin with an additive decomposition of univariate q-hypergeometric terms. Using this decomposition, a univariate q-hypergeometric term $T(n)$ can be represented as

$$T(n) = (N-1)T_1(n) + T_2(n),$$

where $T_1(n)$ and $T_2(n)$ are q-hypergeometric terms, and $T_2(n)$ has the following form

$$T_2(n) = \frac{u_1(q^n)}{u_2(q^n)} \prod_{j=n_0}^{n-1} \frac{f_1(q^j)}{f_2(q^j)},$$

where v_1, v_2, f_1, f_2 are polynomials and for any integer m, $u_2(x)$ and $u_2(xq^m)$ have no common factors except for a power of x. Consequently, a bivariate q-hypergeometric term $T(n,k)$ can be decomposed as

$$T(n,k) = (K-1)T_1(n,k) + T_2(n,k) \tag{1.1}$$

such that

$$T_2(n,k) = T(n,k_0)V(q^n,q^k)\prod_{j=k_0}^{k-1} F(q^n,q^j),$$

where V, F are rational functions and the denominator v_2 of V satisfies the conditions that for any integer m, $v_2(x,y)$ and $v_2(x,yq^m)$ have no common factors except for a power of y. The polynomial $v_2(x,y)$ with the above property is called ε_y-free. We should note that the above decomposition does not solve the minimal additive decomposition problem and is not unique. However, for the purpose of constructing a qZ-pair, it turns out that one may choice any decomposition.

Then we consider the structure of bivariate q-hypergeometric terms. The structure of ordinary hypergeometric terms has been studied by Ore [14], Sato-Shintani-Muro [17], Abramov-Petkovšek [6] and Hou [11]. To a large extent, the q-case is analogous to the ordinary case. For each bivariate q-hypergeometric term, we associate it with a normal representation (q-NR) which consists of four polynomials r, s, u, v. Based on the properties of the representation, we may give a definition of q-proper hypergeometric terms and prove that under the condition that v is ε_y-free, a bivariate q-hypergeometric term has a qZ-pair if and only if it is a q-proper term. Applying the decomposition (1.1), we deduce that for any bivariate q-hypergeometric term T, it has a qZ-pair if and only if T_2 is q-proper.

This paper is concluded with some examples.

2 ε-Free Decomposition

Throughout the paper, we let $\mathbb{Z}, \mathbb{Z}^+$ and $\mathbb{N}$ denote the set of integers, positive integers and nonnegative integers, respectively. For integers (or polynomials) a, b, we denote by $\gcd(a,b)$ the (monic) greatest common divisor of a and b. We also write $a \perp b$ to indicate that a and b are relatively prime, i.e., $\gcd(a,b) = 1$.

Let $\mathbb{F}$ be a field of characteristic zero, $q \in \mathbb{F}$ a nonzero element which is not a root of unity, and x transcendental over $\mathbb{F}$. Denote by ε the unique automorphism of $\mathbb{F}(x)$ which fixes $\mathbb{F}$ and satisfies $\varepsilon x = qx$. Then $\mathbb{F}(x)$ together with the *q-shift operator* ε is a difference field [8]. Let r and s be two polynomials. We say that r/s is *ε-reduced* if $r \perp \varepsilon^h s$ for all $h \in \mathbb{Z}$.

To be more specific, the rational functions involved in the q-hypergeometric terms (see Definition 2.4) are rational functions of q^n. However, for a rational function $R \in \mathbb{F}(x)$, we have

$$N\,R(q^n) = R(q^{n+1}) = \varepsilon R(q^n) \quad \text{and} \quad R(q^n) = 0\ \forall\, n \geq n_0 \Leftrightarrow R(x) = 0.$$

Therefore, there is a natural one-to-one correspondence between the set of rational functions of q^n together with the shift operator N and the field $\mathbb{F}(x)$ together with the q-shift operator ε. In this paper, we adopt the notation of $\mathbb{F}(x)$ as in the work of Abramov-Paule-Petkovšek [4].

The concept of rational normal forms introduced by Abramov and Petkovšek [5] can be extended to the q-case.

Definition 2.1 *Let $R \in \mathbb{F}(x)$ be a rational function. If polynomials $r, s, u, v \in \mathbb{F}[x]$ satisfy*

(i) $R = \frac{r}{s} \cdot \frac{\varepsilon(u/v)}{(u/v)}$ *where $u \perp v$ and u, v have no factor x,*

(ii) *r/s is ε-reduced,*

then (r, s, u, v) is called a q-rational normal form (q-RNF) of R.

Recall that a monic polynomial that has no factor x is called a q-monic polynomials by Abramov, Paule, and Petkovšek [4]. The following factorization theorem was given in [4].

Theorem 2.2 *Let $R \in \mathbb{F}(x) \setminus \{0\}$. Then there exist $z \in \mathbb{F}$ and monic polynomials $a, b, c \in \mathbb{F}[x]$ such that*

$$\begin{aligned} &R(x) = z\frac{a(x)}{b(x)}\frac{c(qx)}{c(x)}, \\ &\gcd(a(x), b(q^n x)) = 1, \quad \textit{for all } n \in \mathbb{N}, \\ &\gcd(a(x), c(x)) = \gcd(b(x), c(qx)) = 1 \quad \textit{and} \quad c(0) \neq 0. \end{aligned} \tag{2.1}$$

We call (az, b, c) a *q-Gosper form* (q-GF) of R.

Theorem 2.3 *Every rational function $R \in \mathbb{F}(x)$ has a q-RNF.*

Proof. It is clear that $(0, 1, 1, 1)$ is a q-RNF of 0. For $R \neq 0$, by Theorem 2.2, there exists a q-GF (az, b, c) of R. Applying Theorem 2.2 again to $b(x)/a(x)$, we get a q-GF (r, s, d). From the construction given in [4], we have $r \mid b$ and $s \mid a$. Hence $s(x) \perp r(xq^n)$ for any $n \in \mathbb{N}$ because (az, b, c) is a q-GF. Since (r, s, d) is also a q-GF, we have $r(x) \perp s(xq^n)$ for any $n \in \mathbb{N}$. Thus s/r is ε-reduced and $(zs, r, c/\gcd(c, d), d/\gcd(c, d))$ is a q-RNF of R. ∎

The above proof provides an algorithm to generate a q-RNF of R.

Algorithm q-RNF

if $R = 0$ then

return $(0, 1, 1, 1)$;
else
compute 'q-GF' of R, we get (a, b, c);
compute 'q-GF' of b/a, we get (r, s, d);
return $(s, r, c/\gcd(c, d), d/\gcd(c, d))$;

We now come to the q-multiplicative representation of a general q-hypergeometric term. This is the starting point of the ε-free decomposition algorithm.

Definition 2.4 *Suppose $T(n)$ is a function from $\mathbb{N}$ to $\mathbb{F}$. If there exist a nonnegative integer n_0 and a nonzero rational function $R(x) \in \mathbb{F}(x)$ such that $T(n+1) = R(q^n)T(n)$ for all $n \geq n_0$, then we call $T(n)$ a (univariate) q-hypergeometric term.*

Suppose (r, s, u, v) is a q-RNF of a rational function R. Then the corresponding q-hypergeometric term $T(n)$ satisfies

$$T(n) = T(n_0) \prod_{j=n_0}^{n-1} R(q^j) = \frac{T(n_0)}{u(q^{n_0})/v(q^{n_0})} \cdot \frac{u(q^n)}{v(q^n)} \prod_{j=n_0}^{n-1} \frac{r(q^j)}{s(q^j)}, \ \forall n \geq n_0.$$

This leads to the following definition.

Definition 2.5 *Let $T(n)$ be a q-hypergeometric term and D, U be two rational functions such that $D(q^n)$ has neither poles nor zeros and $U(q^n)$ has no poles for all $n \geq n_0$. Suppose that*

$$T(n) = U(q^n) \prod_{j=n_0}^{n-1} D(q^j), \qquad \forall n \geq n_0.$$

Then we call (D, U, n_0) a q-multiplicative representation (q-MR) of T.

Let $\Delta = N - 1$ be the difference operator with respect to n. The following lemma can be easily verified.

Lemma 2.6 *Let T and T_1 be two q-hypergeometric terms with q-MRs (D, U, n_0) and (D, U_1, n_0), respectively. Suppose that*

$$T_2 = T - \Delta T_1 \quad \text{and} \quad U_2 = U - D \cdot \varepsilon U_1 + U_1.$$

Then (D, U_2, n_0) is a q-MR of T_2.

For $u, v \in \mathbb{F}[x]$, let $\mathcal{R}$ be the set of all nonnegative integers h such that there exists an irreducible polynomial $p(x) \neq x$ satisfying $p(x) \mid u(x)$ and $p(x) \mid v(q^h x)$. Define $\mathrm{qdis}(u, v)$ to be $\max\{h \in \mathcal{R}\}$ or -1 if $\mathcal{R}$ is empty. Note that $\mathcal{R}$ is a finite set, and "qdis" is well defined. If $\mathrm{qdis}(v, v) = 0$, we say that v is *ε-free*.

Given a q-hypergeometric term T with a q-MR (D, U, n_0). Usually the denominator u of U is not ε-free. However, translating the decomposition algorithm of [5] into the q-case, we have the following ε-free decomposition algorithm "q-decomp", which decomposes $T = \Delta T_1 + T_2$ such that T_2 has a q-MR (F, V, n_0) where the denominator of V is ε-free.

Algorithm q-decomp

Input: (D, U, n_0) Output: $U_1, F, V \in \mathbb{F}(x)$

$d_1 :=$ numer(D); $d_2 :=$ denom(D);
$U_1 := 0$; $U_2 := U$; $u_2 :=$ denom(U);
$N :=$ qdis(u_2, u_2);
for $h := N$ down to 1 do
  $v_2 := u_2/\gcd(u_2, d_2)$
  $s(x) := \gcd(v_2(x), v_2(q^{-h}x))$;
  $(\tilde{s}, \tilde{u}_2) :=$ pump(s, u_2);
  write $U_2 = a/\tilde{u}_2 + b/\tilde{s}$ where $a, b \in \mathbb{F}[x]$;
  $U_1' := -b/\tilde{s}$;
  $U_1 := U_1 + U_1'$; $U_2 := U_2 - D \cdot \varepsilon U_1' + U_1'$;
  $u_2 :=$ denom(U_2);
$f_1 := d_1$; $f_2 := d_2$; $v_1 :=$ numer(U_2); $v_2 :=$ denom(U_2);
$w := \gcd(d_2, v_2)$;
$v_2 := v_2/w$; $f_2 := \varepsilon w f_2/w$;
$F := f_1/f_2$; $V := (1/w(q^{n_0})) \cdot v_1/v_2$;
return (U_1, F, V).

The procedure "pump" is the same as in the ordinary case.

Algorithm pump
Input: $f, g \in \mathbb{F}[x]$;   Output: $\tilde{f}, \tilde{g} \in \mathbb{F}[x]$.

$\tilde{f} := f$; $\tilde{g} := g/f$;
repeat
  $d := \gcd(\tilde{f}, \tilde{g})$; $\tilde{f} := \tilde{f}d$; $\tilde{g} := \tilde{g}/d$;
until $\deg d = 0$;
return $(\tilde{f}, \tilde{g})$.

The following theorem shows that the ε-free algorithm generates the desired decomposition.

Theorem 2.7 *Let T be a q-hypergeometric term with a q-MR (D, U, n_0) and U_1, F, V be given by the algorithm q-decomp. Then there exist q-hypergeometric terms T_1 and T_2 such that*

(1) $T = \Delta T_1 + T_2$.

(2) *T_1 has a q-MR (D, U_1, n_0) and T_2 has a q-MR (F, V, n_0).*

(3) *The denominator of V is ε-free.*

Furthermore, if D is ε-reduced, so is F.

Proof. Let u_0 be the denominator of U. We first use induction to show that after iterating the loop of h in the algorithm i times, the denominator u_2 of U_2 satisfies:

(a) qdis$(v_2, v_2) \leq N - i$,

(b) $u_2(q^n)$ has no zeros for all $n \geq n_0$,

where $v_2 = u_2/\gcd(u_2, d_2)$, and d_2 is the denominator of D.

The case for $i = 0$ is trivial. Assume that the assertion holds for $i - 1$. Let u_2 and u_2' be the denominator of U_2 after $i - 1$ and i iterations, respectively.

Set $h = N - (i-1) > 0$ and $w_2 = \gcd(u_2, d_2)$. From the algorithm q-decomp we have

$$v_2 = u_2/w_2 \quad \text{and} \quad s = \gcd(v_2(x), v_2(q^{-h}x)).$$

Suppose the prime decomposition of s is $p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ and $v_2 = p_1^{\beta_1}\cdots p_r^{\beta_r}v', w_2 = p_1^{\gamma_1}\cdots p_r^{\gamma_r}w'$ where $v' \perp s, w' \perp s$. Then the algorithm "pump" enables us to decompose u_2 as $p_1^{\beta_1+\gamma_1}\cdots p_r^{\beta_r+\gamma_r}\cdot(v'w')$. That is, $\tilde{s} = p_1^{\beta_1+\gamma_1}\cdots p_r^{\beta_r+\gamma_r}$ and $\tilde{u}_2 = v'w'$. Since

$$U_2 = \frac{a}{\tilde{u}_2} + \frac{d_1}{d_2}\cdot\varepsilon\left(\frac{b}{\tilde{s}}\right),$$

it follows that u_2' divides the least common multiple of $\tilde{u}_2$ and $d_2\varepsilon\tilde{s}$. Hence we have that u_2' divides $v'd_2\cdot\varepsilon\tilde{s}$. Let $v'' = v'\cdot\varepsilon\tilde{s}$. Assume that there exist an integer $m \geq h$ and an irreducible polynomial $p(x) \neq x$ such that $p \,|\, v''$ and $p \,|\, \varepsilon^m v''$. We may encounter four cases:

- $p \,|\, v'$ and $p \,|\, \varepsilon^m v'$.
  From $v' \,|\, v_2$ and $\operatorname{qdis}(v_2, v_2) \leq h$, it follows that $m = h$. Therefore, $\varepsilon^{-h}p \,|\, \varepsilon^{-h}v_2$ and $\varepsilon^{-h}p \,|\, v_2$. Consequently, we have $\varepsilon^{-h}p \,|\, s$, which contradicts $v' \perp s$.

- $p \,|\, v'$ and $p \,|\, \varepsilon^{m+1}\tilde{s}$.
  Since s and $\tilde{s}$ have the same prime factors, we have $p \,|\, \varepsilon^{m+1}s$, implying that $p \,|\, \varepsilon^{m+1}v_2$. On the other hand, we have $p \,|\, v_2$, which contradicts $\operatorname{qdis}(v_2, v_2) \leq h$.

- $p \,|\, \varepsilon\tilde{s}$ and $p \,|\, \varepsilon^m v'$.
  In this situation, we have $\varepsilon^{-1}p \,|\, \tilde{s}$, which implies that $\varepsilon^{-1}p \,|\, \varepsilon^{-h}v_2$, or equivalently, $\varepsilon^{h-1}p \,|\, v_2$. On the other hand, $\varepsilon^{h-1}p \,|\, \varepsilon^{m+h-1}v_2$. Since $\operatorname{qdis}(v_2, v_2) \leq h$, we get $m + h - 1 \leq h$, and hence $m = 1$. Now we have $p \,|\, \varepsilon s$ and $p \,|\, \varepsilon v'$, which contradicts $v' \perp s$.

- $p \,|\, \varepsilon\tilde{s}$ and $p \,|\, \varepsilon^{m+1}\tilde{s}$.
  Similarly, we have $\varepsilon^{-1}p \,|\, s$ and hence $\varepsilon^{-1}p \,|\, \varepsilon^{-h}v_2$, i.e., $\varepsilon^{h-1}p \,|\, v_2$. However, we have $\varepsilon^{h-1}p \,|\, \varepsilon^{m+h}v_2$. Thus, we obtain $m + h \leq h$, which is also a contradiction.

In summary, we may conclude that $\operatorname{qdis}(v'', v'') \leq h - 1$. Because u_2' divides $v''\cdot d_2$, there exist $\bar{v} \,|\, v''$ and $\bar{w} \,|\, d_2$ such that $u_2' = \bar{v}\bar{w}$. Let $v_2' = u_2'/\gcd(u_2', d_2)$. From $\bar{w} \mid \gcd(u_2', d_2)$, it follows that $v_2' \,|\, \bar{v}$. So we get $\operatorname{qdis}(v_2', v_2') \leq h-1 = N-i$. Thus, we have proved (a). Since $u_2' | u_2\cdot\varepsilon u_2\cdot d_2$, (b) immediately follows from the induction hypothesis.

On the other hand, since $\tilde{s} \,|\, u_2$, (b) implies that $U_1(q^n)$ has no poles for all $n \geq n_0$. Let

$$T_1(n) = U_1(q^n)\prod_{j=n_0}^{n-1} D(q^j) \quad \text{and} \quad T_2(n) = U_2(q^n)\prod_{j=n_0}^{n-1} D(q^j). \tag{2.2}$$

Noting that $U_2 = U - D\varepsilon U_1 + U_1$, by Lemma 2.6, we obtain $T = \Delta T_1 + T_2$.

Because $w \,|\, d_2$ and $d_2(q^n) \neq 0$ for all $n \geq n_0$, we can write $T_2(n)$ as

$$T_2(n) = \frac{1}{w(q^{n_0})}U_2(q^n)w(q^n)\prod_{j=n_0}^{n-1} D(q^j)\frac{w(q^j)}{w(q^{j+1})} = V(q^n)\prod_{j=n_0}^{n-1} F(q^j).$$

Let v be the denominator of V. Then (a) implies $\mathrm{qdis}(v, v) = 0$, that is, v is ε-free.

Finally, notice that $f_1 = d_1$ and $f_2 = \varepsilon w \cdot (d_2/w)$ where $w \mid d_2$. Therefore, F is ε-reduced provided that D is ε-reduced. This completes the proof. ∎

3 Bivariate q-Hypergeometric Terms

We begin this section with the definition of bivariate q-hypergeometric terms.

Definition 3.1 *Suppose $T(n,k)$ is a function from $\mathbb{N}^2$ to $\mathbb{F}$. If there exist rational functions $R_1(x,y), R_2(x,y) \in \mathbb{F}(x,y)$ and $n_0 \in \mathbb{N}$ such that*

$$T(n+1,k) = R_1(q^n,q^k)T(n,k) \quad \textit{and} \quad T(n,k+1) = R_2(q^n,q^k)T(n,k),$$

for all $n, k \geq n_0$, then we call $T(n,k)$ a bivariate q-hypergeometric term.

Without loss of generality, from now on we may assume that $n_0 = 0$ and that $R_1(q^n,q^k), R_2(q^n,q^k)$ have neither zeros nor poles for all $n, k \geq 0$.

Denote by ε_x and ε_y the shift operators on $\mathbb{F}(x,y)$ defined by $\varepsilon_x x = qx$, $\varepsilon_x|_{\mathbb{F}(y)} = \mathrm{id}$ (the identity map) and $\varepsilon_y y = qy, \varepsilon_y|_{\mathbb{F}(x)} = \mathrm{id}$, respectively. The idea of q-RNF can be easily adopted to the bivariate case by taking $\mathbb{F}(y)$ as the ground field. Let $R(x,y)$ be a rational function of x and y, its q-rational normal form (q-RNF with respect to ε_x) is represented by (r,s,u,v) as in the univariate case. By using the ground field $\mathbb{F}(x)$, we may find a q-RNF of $R(x,y)$ with respect to ε_y.

Let $T(n,k)$ be a bivariate q-hypergeometric term. By definition, there exists a rational function R such that

$$T(n+1,k)/T(n,k) = R(q^n,q^k).$$

Suppose (r,s,u,v) is a q-RNF of R with respect to ε_x. We call (r,s,u,v) a q-normal representation (q-NR) of $T(n,k)$ with respect to the shift operator N. Similarly, we can define the q-NR of $T(n,k)$ with respect to the shift operator K.

We next give a characterization of the polynomials involved in the q-NR of q-hypergeometric terms.

Theorem 3.2 *Let $T(n,k)$ be a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to N. Then r and s are products of polynomials having the form*

$$(x^c y^d) \cdot \prod_{l=1}^{a} p(q^{w_l} x^a y^b),$$

where p is a Laurent polynomial of one variable, $a \in \mathbb{Z}^+, b, c, d, w_l \in \mathbb{Z}$, $a \perp b$, and $w_i \not\equiv w_j \pmod{a}$, $\forall\, i \neq j$.

Similarly, suppose (r,s,u,v) is a q-NR of T with respect to K. Then r and s are products of polynomials having the form

$$(x^c y^d) \cdot \prod_{l=1}^{a} p(q^{w_l} x^b y^a)$$

under the same conditions.

Sketch of the proof. The proof of the ordinary case [11, Theorem 3.4] can be carried over to the q-case except that we need to consider the characterization of polynomials $f(x,y)$ such that $f(q^a x, q^b y) = Cf(x,y)$ for certain integers a, b and $C \in \mathbb{F}$. ∎

Consequently, we have

Corollary 3.3 *Let $T(n,k)$ be a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to N (or K respectively). Then we have*

$$T(n,k) = C \cdot \frac{u(q^n,q^k)}{v(q^n,q^k)} \cdot \frac{\prod_{l=1}^{uu} \prod_{j=0}^{a_l n + b_l k + c_l} f_l(q^j)}{\prod_{l=1}^{vv} \prod_{j=0}^{a'_l n + b'_l k + c'_l} g_l(q^j)},$$

where $C \in \mathbb{F}$, $uu, vv \in \mathbb{N}$, $a_l, b_l, c_l, a'_l, b'_l, c'_l \in \mathbb{Z}$ and f_l, g_l are polynomials.

Corollary 3.3 enables us to give the following definition of q-proper hypergeometric terms.

Definition 3.4 *A polynomial $f \in \mathbb{F}[x,y]$ is said to be q-proper if for each of its irreducible factor $p(x,y) \in \mathbb{F}[x,y]$, there exist $a, b \in \mathbb{Z}$, not both zeros, such that $p(x,y)|p(q^a x, q^b y)$. A bivariate q-hypergeometric term T is said to be q-proper if v is a q-proper polynomial where (r,s,u,v) is a q-NR of T with respect to N or K.*

Suppose that T is a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to N (or K). Theorem 3.2 guarantees that r and s are both q-proper polynomials.

As in the case of ordinary bivariate hypergeometric terms ([11, Theorems 4.2]), we have an analogous "fundamental theorem" for the q-case.

Theorem 3.5 *Let $T(n,k)$ be a bivariate q-hypergeometric term. Then T is q-proper if and only if there exist polynomials $a_{ij}(x) \in \mathbb{F}[x]$, not all zero, such that*

$$\sum_{0 \le i \le I,\ 0 \le j \le J} a_{ij}(q^n) T(n+i, k+j) = 0 \quad \forall\, n, k \ge 0.$$

Based on an analogous argument for the ordinary case as in [16, Theorem 6.2.1], we get

Corollary 3.6 *Any q-proper hypergeometric term has a qZ-pair.*

4 The Existence of qZ-Pairs

In this section, we obtain a necessary and sufficient condition for the existence of qZ-pairs for any bivariate q-hypergeometric term based on its q-NR with respect to K.

From Theorem 3.2, we have

Corollary 4.1 *Let $T(n,k)$ be a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to K. There exist polynomials $f_i(x), g_i(x) \in \mathbb{F}[x]$ and $a_i, a_i', b_i, b_i' \in \mathbb{Z}$ such that*

$$\prod_{j=0}^{k-1} \left(\frac{r(q^{n+1}, q^j)}{r(q^n, q^j)} \cdot \frac{s(q^n, q^j)}{s(q^{n+1}, q^j)} \right) = \prod_{i=1}^{\ell} \frac{f_i(q^{a_i k + b_i n})}{g_i(q^{a_i' k + b_i' n})}.$$

We need to consider the following ratio

$$\frac{T(n+i,k)}{T(n,k)} = \frac{T(n+i,0)}{T(n,0)} \prod_{j=0}^{k-1} \left\{ \frac{T(n+i,j+1)}{T(n+i,j)} \frac{T(n,j)}{T(n,j+1)} \right\},$$

which can be rewritten as

$$\begin{aligned} \frac{T(n+i,k)}{T(n,k)} = & \prod_{l=0}^{i-1} \prod_{j=0}^{k-1} \left\{ \frac{r(q^{n+l+1}, q^j)}{r(q^{n+l}, q^j)} \frac{s(q^{n+l}, q^j)}{s(q^{n+l+1}, q^j)} \right\} \prod_{l=0}^{i-1} \frac{T(n+l+1,0)}{T(n+l,0)} \\ & \cdot \frac{u(q^{n+i}, q^k)}{u(q^{n+i}, q^0)} \frac{u(q^n, q^0)}{u(q^n, q^k)} \frac{v(q^{n+i}, q^0)}{v(q^{n+i}, q^k)} \frac{v(q^n, q^k)}{v(q^n, q^0)}. \end{aligned} \tag{4.1}$$

From Corollary 4.1 we get the following expression.

Lemma 4.2 *Let $T(n,k)$ be a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to K. Then for each $i \geq 0$, there exist q-proper polynomials $w_1^{(i)}(x,y)$ and $w_2^{(i)}(x,y)$ such that*

$$\frac{T(n+i,k)}{T(n,k)} = \frac{u(q^{n+i}, q^k)}{v(q^{n+i}, q^k)} \cdot \frac{v(q^n, q^k)}{u(q^n, q^k)} \cdot \frac{w_1^{(i)}(q^n, q^k)}{w_2^{(i)}(q^n, q^k)}, \quad \forall\, n, k \geq 0. \tag{4.2}$$

An ε_y-free polynomial that is not q-proper has a special factor.

Lemma 4.3 *Let $f \in \mathbb{F}[x,y]$ be a non-q-proper and ε_y-free polynomial. Then there exists an irreducible factor p of f such that*

$$\begin{aligned} p(x,y) \perp p(q^i x, q^j y), \quad & \forall\, (i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\ p(x,y) \perp f(q^i x, q^j y), \quad & \forall\, (i,j) \in (\mathbb{N} \times \mathbb{Z}) \setminus \{(0,0)\}. \end{aligned} \tag{4.3}$$

Proof. Since $f(x,y)$ is non-q-proper, by definition it has an irreducible factor $p_1(x,y)$ such that $p_1(x,y) \perp p_1(q^i x, q^j y), \forall\, (i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$.

We may factor $f(x,y)$ as

$$f(x,y) = p_1^{\alpha_1}(q^{a_1} x, q^{b_1} y) \cdots p_1^{\alpha_r}(q^{a_r} x, q^{b_r} y) f_1(x,y),$$

where $(a_i, b_i) \in \mathbb{Z}^2$ are distinct pairs, $\alpha_i \in \mathbb{Z}^+$, and $p_1(q^i x, q^j y) \perp f_1(x,y)$ for all $i, j \in \mathbb{Z}$. Since $f(x,y)$ is ε_y-free, it follows that $a_i \neq a_j$ as long as $i \neq j$. Without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_r$. Thus, $p(x,y) = p_1(q^{a_1} x, q^{b_1} y)$ satisfies the condition (4.3). ∎

We are now ready to give a criterion for the existence of qZ-pairs.

Theorem 4.4 *Let $T(n,k)$ be a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to K such that v is ε_y-free. Then $T(n,k)$ has a qZ-pair if and only if v is a q-proper polynomial.*

Proof. Because of Corollary 3.6, it suffices to show that if $T(n,k)$ has a qZ-pair, then it is q-proper. To this end, we assume that $T(n,k)$ is a bivariate q-hypergeometric term. Moreover, we assume that $T(n,k)$ is not q-proper, but it has a qZ-pair. We proceed to find a contradiction.

Clearly, for a difference operator $L \in \mathbb{F}[q^n, N]$, we have

$$(N \cdot L)T(n,k) = (K-1)G(n,k) \Longleftrightarrow LT(n,k) = (K-1)G(n-1,k).$$

Therefore, we may assume that $T(n,k)$ has a qZ-pair (L,G) of the form

$$L = \sum_{i=0}^{I} a_i(q^n)N^i,$$

where $a_i(q^n)$ are polynomials in q^n and $a_0 \neq 0$. Since LT/T and $(K-1)G/G$ are both rational functions of q^n and q^k, we may assume that

$$G(n,k) = \frac{f(q^n,q^k)}{g(q^n,q^k)}T(n,k),$$

where $f, g \in \mathbb{F}[x,y]$ are two relatively prime polynomials.

By the definition of qZ-pairs, we have

$$\sum_{i=0}^{I} a_i(q^n)\frac{T(n+i,k)}{T(n,k)} = \frac{f(q^n,q^{k+1})}{g(q^n,q^{k+1})}\frac{T(n,k+1)}{T(n,k)} - \frac{f(q^n,q^k)}{g(q^n,q^k)}. \tag{4.4}$$

Substituting (4.2) into (4.4), we obtain

$$\sum_{i=0}^{I} a_i(x)\frac{u(q^ix,y)}{v(q^ix,y)}\frac{w_1^{(i)}(x,y)}{w_2^{(i)}(x,y)} = \frac{f(x,qy)}{g(x,qy)}\frac{r(x,y)}{s(x,y)}\frac{u(x,qy)}{v(x,qy)} - \frac{f(x,y)}{g(x,y)}\frac{u(x,y)}{v(x,y)}. \tag{4.5}$$

Let $u_1 = u/\gcd(u,g)$, $g_1 = g/\gcd(u,g)$. Multiplying

$$g_1(x,qy)g_1(x,y)v(x,qy)s(x,y)\prod_{j=0}^{I} v(q^jx,y)w_2^{(j)}(x,y)$$

to both sides of (4.5), we arrive at

$$\begin{aligned}
&g_1(x,qy)g_1(x,y)v(x,qy)s(x,y)\\
&\qquad\cdot\sum_{i=0}^{I} a_i(x)u(q^ix,y)w_1^{(i)}(x,y)\prod_{j\neq i} v(q^jx,y)w_2^{(j)}(x,y)\\
=&f(x,qy)r(x,y)u_1(x,qy)g_1(x,y)\prod_{j=0}^{I} v(q^jx,y)w_2^{(j)}(x,y)\\
&\quad - f(x,y)u_1(x,y)g_1(x,qy)v(x,qy)s(x,y)w_2^{(0)}(x,y)\cdot\prod_{j=1}^{I} v(q^jx,y)w_2^{(j)}(x,y).
\end{aligned} \tag{4.6}$$

Since $T(n,k)$ is not q-proper, from Lemma 4.3 it follows that there exists an irreducible factor p of v satisfying the condition (4.3). Noting that $p(x,y)$

divides each term of the left-hand side of (4.6) except for the first term, we obtain that $p(x,y)$ divides

$$g_1(x,qy)v(x,qy)s(x,y)\prod_{j=1}^{I}v(q^jx,y)w_2^{(j)}(x,y)$$
$$\times\big(g_1(x,y)a_0(x)u(x,y)w_1^{(0)}(x,y)+f(x,y)u_1(x,y)w_2^{(0)}(x,y)\big).$$

From (4.3) it follows that

$$p(x,y)\perp v(x,qy)\prod_{j=1}^{I}v(q^jx,y).$$

Since s and $w_2^{(j)}$ are q-proper, they are also relatively prime to p. This implies that $p(x,y)$ divides

$$g_1(x,qy)\big(g_1(x,y)a_0(x)u(x,y)w_1^{(0)}(x,y)+f(x,y)u_1(x,y)w_2^{(0)}(x,y)\big). \tag{4.7}$$

Similarly, since $p(x,qy)$ divides both sides of (4.6) and $u\perp v$, we have

$$p(x,qy)\,|\,f(x,qy)g_1(x,y). \tag{4.8}$$

Case 1. Suppose $p(x,qy)\,|\,f(x,qy)$. Since $p(x,y)$ divides (4.7), it follows that

$$p(x,y)\,|\,g_1(x,qy)g_1(x,y)a_0(x)u(x,y)w_1^{(0)}(x,y).$$

Since $f\perp g, u\perp v$, a_0 and $w_1^{(0)}$ are q-proper polynomials, we may deduce that $p(x,y)\,|\,g_1(x,qy)$, i.e., $p(x,q^{-1}y)\,|\,g_1(x,y)$. Let $m(>0)$ be the greatest integer such that $p(x,q^{-m}y)\,|\,g_1(x,y)$. By virtue of (4.6), we have that $p(x,q^{-m}y)$ divides

$$f(x,y)u_1(x,y)g_1(x,qy)v(x,qy)s(x,y)w_2^{(0)}(x,y)\prod_{j=1}^{I}v(q^jx,y)w_2^{(j)}(x,y).$$

However, $f\perp g$ and $g_1\perp u_1$ imply that $p(x,q^{-m}y)\,|\,g_1(x,qy)$, which contradicts the choice of m.

Case 2. Suppose $p(x,qy)\,|\,g_1(x,y)$. Let $M>0$ be the greatest integer such that $p(x,q^My)\,|\,g_1(x,y)$. Similarly, from (4.6) it follows that $p(x,q^{M+1}y)$ divides

$$f(x,qy)r(x,y)u_1(x,qy)g_1(x,y)\prod_{j=0}^{I}v(q^jx,y)w_2^{(j)}(x,y).$$

Hence we get $p(x,q^{M+1}y)\,|\,g_1(x,y)$, which is again a contradiction. ∎

To extend the above result to general bivariate q-hypergeometric terms, we need the concept of similar q-hypergeometric terms. Two bivariate q-hypergeometric terms T_1,T_2 are called *similar* if there exists a rational function $R\in\mathbb{F}(x,y)$ such that $T_1(n,k)/T_2(n,k)=R(q^n,q^k)$.

As in the ordinary case, the existence of qZ-pairs is preserved under addition of similar bivariate q-hypergeometric terms.

Lemma 4.5 *Suppose there exist qZ-pairs for two similar bivariate q-hypergeometric terms $T_1(n,k)$ and $T_2(n,k)$. Then there exists a qZ-pair for $T(n,k) = T_1(n,k) + T_2(n,k)$.*

Notice that $T(n,k) = (K-1)G(n,k)$ has a qZ-pair $(1, G)$. Combining Theorem 4.4 and Lemma 4.5, we obtain the main result of this paper.

Theorem 4.6 *Let $T(n,k)$ be a bivariate q-hypergeometric term. Let T_1, T_2 be two similar bivariate q-hypergeometric terms satisfying*

$$T(n,k) = (K-1)T_1(n,k) + T_2(n,k)$$

and $T_2(n,k)$ has a q-NR (r,s,u,v) with respect to K such that v is ε_y-free. Then $T(n,k)$ has a qZ-pair if and only if $T_2(n,k)$ is a q-proper hypergeometric term, or equivalently, if and only if $v(x,y)$ is a q-proper polynomial.

5 Algorithms

Let $T(n,k)$ be a bivariate q-hypergeometric term. By the algorithm "q-RNF", we may find a q-RNF (r,s,u,v) of $T(n,k)$ with respect to K. Let

$$F(k) = \frac{u(x,q^k)}{v(x,q^k)} \prod_{j=0}^{k-1} \frac{r(x,q^j)}{s(x,q^j)}, \quad \forall\, k \in \mathbb{N}.$$

Then $F(k)$ is a univariate q-hypergeometric term over the field $\mathbb{F}(x)$ with a q-MR $(r/s, u/v, 0)$. On the other hand, by Equation (4.1), we have

$$\begin{aligned}\frac{F(k)|_{x=q^{n+1}}}{F(k)|_{x=q^n}} &= \frac{u(q^{n+1},q^k)v(q^n,q^k)}{u(q^n,q^k)v(q^{n+1},q^k)} \prod_{j=0}^{k-1} \frac{r(q^{n+1},q^j)s(q^n,q^j)}{r(q^n,q^j)s(q^{n+1},q^j)} \\ &= \frac{T(n+1,k)}{T(n,k)} \cdot \frac{T(n,0)}{T(n+1,0)} \cdot \frac{u(q^{n+1},q^0)v(q^n,q^0)}{u(q^n,q^0)v(q^{n+1},q^0)},\end{aligned}$$

which is also a rational function on q^n and q^k. Hence $\widetilde{F}(n,k) = F(k)|_{x=q^n}$ is a bivariate q-hypergeometric term.

Using the algorithm "q-decomp" given in Section 2, one may find univariate q-hypergeometric terms $F_1(k), F_2(k)$ such that

$$F(k) = (K-1)F_1(k) + F_2(k)$$

and $F_2(k)$ has a q-MR $(f_1/f_2, v_1/v_2, 0)$ with v_2 being ε_y-free. Since $f_1/f_2, v_1/v_2 \in \mathbb{F}(x)(y)$, we may assume that $f_1, f_2, v_1, v_2 \in \mathbb{F}[x,y]$ and $f_1 \perp f_2, v_1 \perp v_2$. From the fact that r/s is ε_y-reduced, it follows that f_1/f_2 is also ε_y-reduced.

Let

$$\begin{aligned}T_1(n,k) &= T(n,0)\frac{v(q^n,q^0)}{u(q^n,q^0)} \cdot F_1(k)|_{x=q^n}, \\ T_2(n,k) &= T(n,0)\frac{v(q^n,q^0)}{u(q^n,q^0)} \cdot F_2(k)|_{x=q^n}.\end{aligned}$$

Since Equation (2.2) implies that

$$F_1(k) = \frac{U_1}{u/v} \cdot F(k) \quad \text{and} \quad F_2(k) = \frac{v_1/v_2}{u/v} \cdot F(k),$$

it follows that $T_1(n,k)$ and $T_2(n,k)$ are similar bivariate q-hypergeometric terms. It is easily verified that

$$T(n,k) = (K-1)T_1(n,k) + T_2(n,k)$$

and (f_1, f_2, v_1, v_2) is a q-NR of T_2 with respect to K. Therefore, Theorem 4.6 implies that $T(n,k)$ has a qZ-pair if and only if v_2 is a q-proper polynomial.

Finally, we need the algorithm given by Le [13] for determining whether or not a polynomial is q-proper.

We are now ready to describe the algorithm to determine whether a bivariate q-hypergeometric term $T(n,k)$ has a qZ-pair.

1. `Apply the algorithm in [7] to find a rational function` $R \in \mathbb{F}(x,y)$ `such that`
$$\frac{T(n,k+1)}{T(n,k)} = R(q^n, q^k).$$
2. `Find a` q`-RNF` (r,s,u,v) `of` R.
3. `For` $D = r/s, U = u/v$ `and` $n_0 = 0$, `apply the algorithm '`q`-decomp' to get` $V = v_1/v_2$.
4. `Use the algorithm in [13] to determine whether` v_2 `is` q`-proper. If the answer is yes, then` T `has a` qZ`-pair; otherwise,` T `does not have any` qZ`-pair.`

Here are two examples.

Example 1. Let

$$T(n,k) = \frac{q^k(1+q^{n+1}+q^{k+2})}{(q^n+q^k+1)(q^n+q^{k+1}+1)\prod_{j=1}^{k+1}(1-q^j)}.$$

Then
$$\frac{T(n,k+1)}{T(n,k)} = \frac{q(1+q^{n+1}+q^{k+3})(q^n+q^k+1)}{(q^n+q^{k+2}+1)(1+q^{n+1}+q^{k+2})(1-q^{k+2})},$$

and we have

$$r = q,\ s = 1-q^2y,\ u = 1+qx+q^2y,\ v = (x+y+1)(x+qy+1)$$

is a q-NR of T with respect to K. For $D = r/s, U = u/v$ and $n_0 = 0$, applying the algorithm "q-decomp", we get

$$V = v_1/v_2 = \frac{-q^2}{(-1+q^2)(x+1)}.$$

Clearly, v_2 is q-proper, so $T(n,k)$ has a qZ-pair. Indeed, we can check that

$$L = 1, \quad G = \frac{1}{(q^n+q^k+1)\prod_{j=1}^{k}(1-q^j)}$$

is a qZ-pair for $T(n,k)$.

Example 2. Let

$$T(n,k) = \frac{q^k(1+q^{n+1}+q^{k+2})}{(q^n+q^k+1)(q^n+q^{k+1}+1)\prod_{j=1}^{k}(1-q^j)}.$$

Then

$$\frac{T(n,k+1)}{T(n,k)} = \frac{q(1+q^{n+1}+q^{k+3})(q^n+q^k+1)}{(q^n+q^{k+2}+1)(1+q^{n+1}+q^{k+2})(1-q^{k+1})},$$

and we have

$$r = q,\ s = 1 - qy,\ u = 1 + qx + q^2 y,\ v = (x+y+1)(x+qy+1)$$

is a q-NR of T with respect to K. For $D = r/s, U = u/v$ and $n_0 = 0$, applying the algorithm "q-decomp", we get

$$V = v_1/v_2 = \frac{-(x+y+1)q^2}{(q-1)(x+1)(x+qy+1)}.$$

Since $x + qy + 1$ is not a q-proper polynomial, it follows that $T(n,k)$ has no qZ-pair.

Acknowledgments. The authors are grateful to the referees for many helpful comments and suggestions. This work was done under the auspices of the 973 Project on Mathematical Mechanization and the National Science Foundation of China. We thank Professor Marko Petkovšek for bringing our attention to the work of Ore and Sato.

References

[1] S.A. Abramov, Applicability of Zeilberger's algorithm to hypergeometric terms, In: Proc. Int. Symp. on Symbolic and Algebraic Computation, ACM Press, 2002.

[2] S.A. Abramov, When does Zeilberger's algorithm succeed?, Adv. Appl. Math., **30** (2003) 424–441.

[3] S.A. Abramov and H.Q. Le, A criterion for the applicability of Zeilberger's algorithm to rational functions, Discrete Math., **259** (2002) 1–17.

[4] S.A. Abramov, P. Paule, and M. Petkovšek, q-Hypergeometric solutions of q-difference equations, Discrete Math. **180** (1998) 3–22.

[5] S.A. Abramov and M. Petkovšek, Rational normal forms and minimal decompositions of hypergeometric terms, J. Symbolic Comput. **33** (2002) 521–543.

[6] S.A. Abramov and M. Petkovšek, On the structure of multivariate hypergeometric terms, Adv. Appl. Math. **29** (2002) 386–411.

[7] H. Böing and W. Koepf, Algorithms for q-hypergeometric summation in computer algebra, J. Symbolic Comput. **28** (1999) 777–799.

[8] R.M. Cohn, Difference Algebra, Interscience Publishers, New York, 1965.

[9] I. M. Gel'fand, M. I. Graev and V. S. Retakh, General hypergeometric systems of equations and series of hypergeometric type (Russian), Uspekhi Mat. Nauk **47** (1992) 3–82, 235; translation in Russian Math. Surveys **47** (1992) 1–88.

[10] R. Graham, D. Knuth, and O. Patashnik, Concrete Mathematics, 2nd Edition, Addison-Wesley, Reading, MA, 1994.

[11] Q.H. Hou, k-Free recurrences of double hypergeometric terms, Adv. Appl. Math. **32** (2004) 468-484.

[12] T.H. Koornwinder, On Zeilberger's algorithm and its q-analogue, J. Comput. Appl. Math. **48** (1993) 91–111.

[13] H.Q. Le, On the q-analogue of Zeilberger's algorithm to rational functions, Program. Comput. Software (Programmirovanie) **27** (2001) 49–58.

[14] O. Ore, Sur la forme des fonctions hypergéométriques de plusieurs variables, J. Math. Pures Appl. **9** (1930) 311–326.

[15] P. Paule and A. Riese, A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping, In: Special Functions, q-Series and Related Topics (Toronto, ON, 1995), 179–210, Fields Inst. Commun., **14**, Amer. Math. Soc., Providence, RI, 1997.

[16] M. Petkovšek, H.S. Wilf, and D. Zeilberger, A=B, A.K. Peters, Wellesley, MA, 1996.

[17] M. Sato, T. Shintani and M. Muro, Theory of prehomogeneous vector spaces (algebraic part), Nagoya Math. J. 120 (1990) 1–34.

[18] H. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities, Invent. Math. **108** (1992) 575–633.

[19] D. Zeilberger, The method of creative telescoping, J. Symbolic Comput. **11** (1991) 195–204.