The Terminating Condition of Zeilberger's Algorithm

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Abstract

We obtain a characterization of double hypergeometric terms for which Zeilberger's algorithm terminates. The special case for the rational functions has been solved by Abramov and Le.

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1 Introduction

Zeilberger's algorithm [6, 8, 12], also known as the method of *creative telescoping*, is a useful tool for proving identities of the form

$$\sum_{k=-\infty}^{\infty} F(n,k) = f(n),$$

where $F(n,k)$ is a double hypergeometric term and $f(n)$ is a given function (for most cases a hypergeometric term plus a constant). Given a double hypergeometric term $F(n,k)$, the algorithm tries to construct a *Z-pair* (L,G) which consists of a linear difference operator L with coefficients in the ring of polynomials in n

$$L = a_0(n)N^0 + a_1(n)N^1 + \cdots + a_r(n)N^r$$

and a double hypergeometric term $G(n,k)$ such that

$$LF(n,k) = (E-1)G(n,k),$$

where N is the shift operator with respect to n, defined by $NF(n,k) = F(n+1,k)$ and E is the shift operator with respect to k, defined by $EF(n,k) = F(n,k+1)$. Therefore, the existence of Z-pairs ensures the termination of Zeilberger's algorithm.

While Zeilberger's algorithm has been widely used as a powerful tool to verify and discover hypergeometric identities, it remains an open problem to determine whether the algorithm terminates. An important progress has been made by Abramov and Le for the special case of rational functions $F(n,k)$. In this paper we provide a solution in the general case.

A well known class of double hypergeometric terms for which Zeilberger's algorithm terminates are called the proper hypergeometric terms [6, 8, 10, 11]:

$$F(n,k) = P(n,k)\frac{\prod_{i=1}^{l}(\alpha_i n + \beta_i k + \gamma_i)!}{\prod_{i=1}^{m}(\alpha'_i n + \beta'_i k + \gamma'_i)!}u^n v^k,$$

where $P(n,k) \in \mathbb{C}[n,k], \alpha_i, \beta_i, \alpha_i', \beta_i' \in \mathbb{Z}$, l, m are non-negative integers, and $\gamma_i, \gamma_i', u, v \in \mathbb{C}$. (The factorial of a complex number is considered as the value of the gamma function, i.e., $z! = \Gamma(z+1)$.) However, it is possible to construct a non-proper hypergeometric term for which Zeilberger's algorithm also terminates. For example, $F(n,k) = (E-1)\frac{1}{nk+1}$ [3, Example 6].

Applying any of the algorithms given in [1,2,9], one can represent a rational function $F(n,k)$ in the form

$$F(n,k) = (E-1)S(n,k) + T(n,k),$$

where S, T are rational functions such that the denominator of $T(n,k)$ has the minimal possible degree in k. Based on this decomposition, Abramov and Le [3] prove that for a rational function $F(n,k)$, it has a Z-pair if and only if $T(n,k)$ is a proper hypergeometric term.

In this article, we solve this problem for general double hypergeometric terms by using the properties of the canonical representation. For any hypergeometric term $T(n)$, Abramov and Petkovšek [5] give an algorithm to construct hypergeometric terms $T_1(n), T_2(n)$ such that

$$T(n) = T_1(n+1) - T_1(n) + T_2(n) \tag{1.1}$$

and $T_2(n)$ is minimal in some sense. We show that a double hypergeometric term $F(n,k)$ has a Z-pair if and only if $F_2(n,k)$ is a proper hypergeometric term, where

$$F(n,k) = (E-1)F_1(n,k) + F_2(n,k) \tag{1.2}$$

is the 2-variable analog of the decomposition (1.1). When $F(n,k)$ is a rational function, it reduces to the result of Abramov and Le. Based on this criterion, we present an algorithm to verify the existence of Z-pairs.

We first introduce the canonical representation of double hypergeometric terms and some of its properties in Section 2. Then we obtain a sufficient and necessary condition for the existence of Z-pairs (Theorem 3.6). The corresponding algorithms are presented in Section 4.

2 The Canonical Representation of Double Hypergeometric Terms

Throughout the paper functions are maps from $\mathbb{N}$ (or $\mathbb{N}^2$) to $\mathbb{C}$ and $\mathbb{C}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{N}$ are the set of complex numbers, integers, positive integers and non-negative integers, respectively. A function $T(k)$ is called a *hypergeometric term* if $T(k+1)/T(k)$ is a rational function of k. A function $F(n,k)$ is called a *double hypergeometric term* if both

$$F(n+1,k)/F(n,k) \quad \text{and} \quad F(n,k+1)/F(n,k)$$

are rational functions of n and k. Two double hypergeometric terms $F(n,k)$ and $G(n,k)$ are said to be *similar* if their ratio is a rational function of n and k.

As in Section 1, N and E denote the shift operators with respect to n and k, respectively. For polynomials $f(n,k), g(n,k) \in \mathbb{C}[n,k]$, we denote by $\gcd(f,g)$ the monic greatest common divisor of f and g. We also write $p \perp q$ to indicate that the polynomials $p(n,k), q(n,k) \in \mathbb{C}[n,k]$ are relatively prime, i.e., $\gcd(p,q) = 1$.

Definition 2.1 *Let $r(n,k), s(n,k) \in \mathbb{C}[n,k]$. If $r \perp E^h s$ for all $h \in \mathbb{Z}$, then the rational function r/s is called k-shift-reduced. If $r \perp E^h r$ for all $h \in \mathbb{Z} \setminus \{0\}$, then the polynomial r is called k-shift-free.*

Definition 2.2 *A non-zero polynomial $f(n,k) \in \mathbb{C}[n,k]$ is called proper if each of its irreducible factor is of the form $g(an+bk)$, where $g(n)$ is a polynomial in one variable and $a, b \in \mathbb{Z}$.*

Now we give the definition of the canonical representation of double hypergeometric terms.

Definition 2.3 *Let $F(n,k)$ be a double hypergeometric term. If there exist polynomials $a(n,k), b(n,k), c(n,k)$ and $d(n,k)$ such that*

$$\frac{F(n,k+1)}{F(n,k)} = \frac{a(n,k)}{b(n,k)} \frac{c(n,k+1)}{c(n,k)} \frac{d(n,k)}{d(n,k+1)}, \tag{2.1}$$

where $c(n,k) \perp d(n,k)$ and $a(n,k)/b(n,k)$ is k-shift-reduced, we call $[a(n,k), b(n,k), c(n,k), d(n,k)]$ a canonical representation of $F(n,k)$.

It is shown that for each double hypergeometric term, there exists a canonical representation [4, 7]. Moreover, we have

Proposition 2.4 ([7]) *Let $F(n,k)$ be a double hypergeometric term with a canonical representation $[a(n,k), b(n,k), c(n,k), d(n,k)]$. Then*

- *$a(n,k)$ and $b(n,k)$ are both proper polynomials;*
- *$F(n,k)$ is a proper hypergeometric term if and only if $d(n,k)$ is a proper polynomial;*
- *There exist polynomials $f_i(x), g_i(x) \in \mathbb{C}[x]$ such that*

$$\prod_{j=0}^{k-1} \left(\frac{a(n+1,j)}{a(n,j)} \cdot \frac{b(n,j)}{b(n+1,j)} \right) = \frac{\prod_{j=1}^{t} f_j(u_j k + u_j' n)}{\prod_{j=1}^{t} g_j(v_j k + v_j' n)}, \tag{2.2}$$

  where u_j, v_j, u_j', v_j' are integers.

3 A Criterion for the Existence of Z-pairs

By Proposition 2.4, we can write the ratio $F(n+i,k)/F(n,k)$ in terms of the canonical representation of $F(n,k)$.

Lemma 3.1 *Let $F(n,k)$ be a double hypergeometric term with a canonical representation $[a(n,k), b(n,k), c(n,k), d(n,k)]$. Then for each $i \geq 0$, there exist proper polynomials $w_1^{(i)}(n,k)$ and $w_2^{(i)}(n,k)$ such that*

$$\frac{F(n+i,k)}{F(n,k)} = \frac{c(n+i,k)}{d(n+i,k)} \cdot \frac{d(n,k)}{c(n,k)} \cdot \frac{w_1^{(i)}(n,k)}{w_2^{(i)}(n,k)}. \tag{3.1}$$

Proof. Similar to the proof of Theorem 4.2 in [7], we have

$$\begin{aligned} \frac{F(n+i,k)}{F(n,k)} &= \prod_{j=0}^{k-1}\prod_{m=0}^{i-1}\left\{\frac{a(n+m+1,j)}{a(n+m,j)} \cdot \frac{b(n+m,j)}{b(n+m+1,j)}\right\} \\ &\cdot \prod_{m=0}^{i-1} \frac{F(n+m+1,0)}{F(n+m,0)} \cdot \frac{c(n+i,k)}{c(n,k)} \cdot \frac{c(n,0)}{c(n+i,0)} \\ &\cdot \frac{d(n,k)}{d(n+i,k)} \cdot \frac{d(n+i,0)}{d(n,0)}. \end{aligned}$$

Notice that $\frac{F(n+1,0)}{F(n,0)} = \frac{u(n)}{v(n)}$, where $u(n), v(n)$ are polynomials. By Proposition 2.4, we have

$$\begin{aligned} \frac{F(n+i,k)}{F(n,k)} &= \prod_{m=0}^{i-1} \frac{\prod_{l=1}^{t} f_l(u_l k + u'_l(n+m))}{\prod_{l=1}^{t} g_l(v_l k + v'_l(n+m))} \prod_{m=0}^{i-1} \frac{u(n+m)}{v(n+m)} \\ & \frac{c(n+i,k)}{c(n,k)} \cdot \frac{c(n,0)}{c(n+i,0)} \cdot \frac{d(n,k)}{d(n+i,k)} \cdot \frac{d(n+i,0)}{d(n,0)}. \end{aligned}$$

This completes the proof. □

Proper polynomials can be characterized by the following lemma.

Lemma 3.2 *A polynomial $f(n,k) \in \mathbb{C}[n,k]$ is proper if and only if $f(n,k) \perp p(n,k)$ for any irreducible polynomial $p(n,k)$ which satisfies $p(n,k) \perp p(n+i,k+j)$ for all $(i,j) \in \mathbb{Z}^2 \setminus (0,0)$.*

Proof. Suppose $f(n,k)$ is a proper polynomial and $p(n,k)$ is an irreducible polynomial which satisfies $p(n,k) \perp p(n+i,k+j)$ for all $(i,j) \in \mathbb{Z}^2 \setminus (0,0)$. If $\gcd(f,p) \neq 1$, $p(n,k)$ must be an irreducible factor of $f(n,k)$. By the definition of proper polynomials, $p(n,k) = g(an+bk)$ for some $(a,b) \in \mathbb{Z}^2 \setminus (0,0)$, which implies $p(n+b,k-a) = p(n,k)$. This contradicts the hypothesis of $p(n,k)$. Therefore, $f(n,k) \perp p(n,k)$.

On the other hand, suppose $f(n,k) \perp p(n,k)$ for any irreducible polynomial $p(n,k)$ which satisfies $p(n,k) \perp p(n+i,k+j)$ for all $(i,j) \in \mathbb{Z}^2 \setminus (0,0)$. Let $q(n,k)$ be an irreducible factor of $f(n,k)$, there must be integers a, b which are not both zero such that $\gcd(q(n,k), q(n+a,k+b)) \neq 1$. Since $q(n,k)$ is irreducible, we have $q(n,k) = q(n+a,k+b)$. By Lemma 3.3 in [7], we have $q(n,k) = g(a'n+b'k)$ for some integers a', b'. □

Lemma 3.2 enables us to decompose a non-proper and k-shift-free polynomial.

Lemma 3.3 *Let $d(n,k) \in \mathbb{C}[n,k]$ be a non-proper and k-shift-free polynomial. Then there exists an irreducible polynomial $p(n,k)$ such that*

$$\begin{aligned} d(n,k) &= p^{\alpha}(n,k)\tilde{d}(n,k), \quad \alpha \in \mathbb{Z}^+ \\ p(n,k) &\perp p(n+i,k+j), \quad \forall\,(i,j) \in \mathbb{Z}^2 \setminus (0,0), \\ p(n,k) &\perp \tilde{d}(n+i,k+j), \quad \forall\, i \in \mathbb{N}, j \in \mathbb{Z}. \end{aligned} \tag{3.2}$$

Proof. Since $d(n,k)$ is non-proper, by Lemma 3.2, it has an irreducible factor $q(n,k)$ such that $q(n,k) \perp q(n+i,k+j), \forall\,(i,j) \in \mathbb{Z}^2 \setminus (0,0)$.

We factor $d(n,k)$ to be

$$d(n,k) = q^{\alpha_1}(n+a_1,k+b_1)\cdots q^{\alpha_r}(n+a_r,k+b_r)d_1(n,k),$$

where $(a_i,b_i) \in \mathbb{Z}^2$ are pair-wised distinct, $\alpha_i \in \mathbb{Z}^+$, and $q(n+i,k+j) \perp d_1(n,k)$, $\forall\, i,j \in \mathbb{Z}$. Noting that $d(n,k)$ is k-shift-free, it follows that $a_i \neq a_j$ for $i \neq j$. Without loss of generality, we assume that $a_1 < a_2 < \cdots < a_r$. Then $q^{\alpha_1}(n+a_1,k+b_1)\,|\,d(n,k)$, but

$$\begin{aligned} q(n+a_1,k+b_1) &\perp d(n+i,k+j), \quad \forall\, i > 0, j \in \mathbb{Z}, \\ q(n+a_1,k+b_1) &\perp d(n,k+j), \quad \forall\, j \neq 0. \end{aligned}$$

Taking $\alpha = \alpha_1$ and $p(n,k) = q(n+a_1,k+b_1)$, we get the desired decomposition. $\square$

Now we are ready to give a criterion of the existence of Z-pairs.

Theorem 3.4 *Let $F(n,k)$ be a double hypergeometric term with a canonical representation $[a(n,k),b(n,k),c(n,k),d(n,k)]$ such that $d(n,k)$ is k-shift-free. Then $F(n,k)$ has a Z-pair if and only if $F(n,k)$ is a proper hypergeometric term, or equivalently, if and only if $d(n,k)$ is a proper polynomial.*

Proof. Since proper hypergeometric terms have Z-pairs, we only need the inverse. Suppose $F(n,k)$ has a Z-pair but is not a proper hypergeometric term. Clearly, for a difference operator $L \in \mathbb{C}[n,N]$,

$$(N \cdot L)F(n,k) = (E-1)G(n,k) \Longrightarrow LF(n,k) = (E-1)G(n-1,k).$$

Therefore, we may assume $F(n,k)$ has a Z-pair (L,G) with $L = \sum_{i=0}^{I} a_i(n)N^i$, where $a_i(n)$ are polynomials in n and $a_0(n) \neq 0$. Noting that LF is similar to F and $(E-1)G$ is similar to G, it follows that F and G are similar. Hence we may assume that

$$G(n,k) = \frac{r(n,k)}{s(n,k)}F(n,k),$$

where $r(n,k)$ and $s(n,k)$ are two relatively prime polynomials.

By the definition of Z-pairs, we have

$$\sum_{i=0}^{I} a_i(n)\frac{F(n+i,k)}{F(n,k)} = \frac{r(n,k+1)}{s(n,k+1)}\frac{F(n,k+1)}{F(n,k)} - \frac{r(n,k)}{s(n,k)}. \tag{3.3}$$

Substituting (2.1) and (3.1) into (3.3), we obtain

$$\sum_{i=0}^{I} a_i(n)\frac{c(n+i,k)}{d(n+i,k)}\frac{w_1^{(i)}(n,k)}{w_2^{(i)}(n,k)} = \frac{r(n,k+1)}{s(n,k+1)}\frac{a(n,k)}{b(n,k)}\frac{c(n,k+1)}{d(n,k+1)} - \frac{r(n,k)}{s(n,k)}\frac{c(n,k)}{d(n,k)}. \quad (3.4)$$

Since $F(n,k)$ is not a proper hypergeometric term, $d(n,k)$ is a non-proper polynomial. By Lemma 3.3, there exists an irreducible polynomial $p(n,k)$ such that (3.2) holds. Let $c_1(n,k) = c(n,k)/\gcd(c(n,k),s(n,k))$, $s_1(n,k) = s(n,k)/\gcd(c(n,k),s(n,k))$. Then Equation (3.4) becomes

$$\begin{aligned}
&\sum_{i=0}^{I} a_i(n)\frac{c(n+i,k)w_1^{(i)}(n,k)}{p^\alpha(n+i,k)\tilde{d}(n+i,k)w_2^{(i)}(n,k)} \\
&= \frac{r(n,k+1)c_1(n,k+1)a(n,k)}{s_1(n,k+1)p^\alpha(n,k+1)\tilde{d}(n,k+1)b(n,k)} - \frac{r(n,k)c_1(n,k)}{s_1(n,k)p^\alpha(n,k)\tilde{d}(n,k)}. \quad (3.5)
\end{aligned}$$

Multiplying

$$s_1(n,k+1)s_1(n,k)p^\alpha(n,k+1)\tilde{d}(n,k+1)b(n,k)\prod_{j=0}^{I} p^\alpha(n+j,k)\tilde{d}(n+j,k)w_2^{(j)}(n,k)$$

on both sides of (3.5), we get

$$\begin{aligned}
&s_1(n,k+1)s_1(n,k)p^\alpha(n,k+1)\tilde{d}(n,k+1)b(n,k) \\
&\quad\cdot\sum_{i=0}^{I} a_i(n)c(n+i,k)w_1^{(i)}(n,k)\prod_{j\neq i} p^\alpha(n+j,k)\tilde{d}(n+j,k)w_2^{(j)}(n,k) \\
&= r(n,k+1)c_1(n,k+1)a(n,k)s_1(n,k)\prod_{j=0}^{I} p^\alpha(n+j,k)\tilde{d}(n+j,k)w_2^{(j)}(n,k) \\
&\quad - r(n,k)c_1(n,k)s_1(n,k+1)p^\alpha(n,k+1)\tilde{d}(n,k+1)b(n,k)w_2^{(0)}(n,k) \\
&\quad\cdot\prod_{j=1}^{I} p^\alpha(n+j,k)\tilde{d}(n+j,k)w_2^{(j)}(n,k). \quad (3.6)
\end{aligned}$$

Notice that $p(n,k)$ divides each term of the left hand side except for the first one. Therefore, $p(n,k)$ divides

$$\begin{aligned}
&s_1(n,k+1)p^\alpha(n,k+1)\tilde{d}(n,k+1)b(n,k)\prod_{j=1}^{I} p^\alpha(n+j,k)\tilde{d}(n+j,k)w_2^{(j)}(n,k) \\
&\quad\cdot\big(s_1(n,k)a_0(n)c(n,k)w_1^{(0)}(n,k) + r(n,k)c_1(n,k)w_2^{(0)}(n,k)\big).
\end{aligned}$$

From (3.2), it follows that

$$p(n,k)\perp p^\alpha(n,k+1)\tilde{d}(n,k+1)\prod_{j=1}^{I} p^\alpha(n+j,k)\tilde{d}(n+j,k).$$

By Proposition 2.4 and Lemma 3.2, $b(n,k)$ and $w_2^{(j)}(n,k)$ are also relatively prime to $p(n,k)$. Hence,

$$p(n,k)|s_1(n,k+1)\big(s_1(n,k)a_0(n)c(n,k)w_1^{(0)}(n,k)+r(n,k)c_1(n,k)w_2^{(0)}(n,k)\big). \quad (3.7)$$

Similarly, since $p(n,k+1)$ divides both sides of Equation (3.6) and $c(n,k)\perp d(n,k)$, we have

$$p(n,k+1)|r(n,k+1)s_1(n,k), \quad (3.8)$$

and hence $p(n,k+1)|r(n,k+1)$ or $p(n,k+1)|s_1(n,k)$.

Case 1. Suppose $p(n,k+1)|r(n,k+1)$. From (3.7), we have

$$p(n,k)|s_1(n,k+1)s_1(n,k)a_0(n)c(n,k)w_1^{(0)}(n,k).$$

Since $r(n,k)\perp s(n,k)$, $c(n,k)\perp d(n,k)$, $a_0(n)$ and $w_1^{(0)}(n,k)$ are proper polynomials, it follows that $p(n,k)|s_1(n,k+1)$, i.e., $p(n,k-1)|s_1(n,k)$. Let $m(>0)$ be the greatest integer such that $p(n,k-m)|s_1(n,k)$. By Equation (3.6), $p(n,k-m)$ divides

$$r(n,k)c_1(n,k)s_1(n,k+1)p^\alpha(n,k+1)\tilde{d}(n,k+1)b(n,k)w_2^{(0)}(n,k) \cdot\prod_{j=1}^{I}p^\alpha(n+j,k)\tilde{d}(n+j,k)w_2^{(j)}(n,k).$$

However, $r(n,k)\perp s(n,k)$ and $c_1(n,k)\perp s_1(n,k)$ imply that $p(n,k-m)|s_1(n,k+1)$, which contradicts the choice of m.

Case 2. Suppose $p(n,k+1)|s_1(n,k)$. Let $M>0$ be the greatest integer such that $p(n,k+M)|s_1(n,k)$, i.e., $p(n,k+M+1)|s_1(n,k+1)$. Also by (3.6), we have $p(n,k+M+1)$ divides

$$r(n,k+1)c_1(n,k+1)a(n,k)s_1(n,k)\prod_{j=0}^{I}p^\alpha(n+j,k)\tilde{d}(n+j,k)w_2^{(j)}(n,k).$$

Similarly, we get $p(n,k+M+1)|s_1(n,k)$, which is also a contradiction. $\Box$

Similar to Lemma 1 in [3], the existence of Z-pairs is preserved under addition.

Lemma 3.5 *Suppose there exist Z-pairs for two similar double hypergeometric terms $F_1(n,k)$ and $F_2(n,k)$. Then there exists a Z-pair for $F(n,k)=F_1(n,k)+F_2(n,k)$.*

Proof. Let $(L_1,G_1),(L_2,G_2)$ be the Z-pairs of F_1 and F_2 respectively. Let L be the least common left multiple of L_1 and L_2, i.e., $L=L_1'\circ L_1=L_2'\circ L_2\in\mathbb{C}[n,N]$. Therefore,

$$LF=LF_1+LF_2=L_1'(L_1F_1)+L_2'(L_2F_2)=L_1'((E-1)G_1)+L_2'((E-1)G_2).$$

Since $NE = EN$ and $nE = En$,

$$LF = (E-1)(L_1'G_1 + L_2'G_2).$$

Since $L_1'G_1$ is similar to G_1, it is similar to F_1. Also, $L_2'G_2$ is similar to F_2. Since F_1 and F_2 are similar, it follows that $L_1'G_1 + L_2'G_2$ is a double hypergeometric term similar to F. $\square$

Notice that $F(n,k) = (E-1)G(n,k)$ has a Z-pair $(1, G)$, and (1.2) implies that F_1, F_2 are similar to F. Combining Theorem 3.4 and Lemma 3.5, we obtain the main result of this paper.

Theorem 3.6 *Let $F(n,k)$ be a double hypergeometric term. Suppose $F_1(n,k), F_2(n,k)$ be two double hypergeometric terms which satisfies*

$$F(n,k) = (E-1)F_1(n,k) + F_2(n,k)$$

and $F_2(n,k)$ has a canonical representation $[a(n,k), b(n,k), c(n,k), d(n,k)]$ such that $d(n,k)$ is k-shift-free. Then $F(n,k)$ has a Z-pair if and only if $F_2(n,k)$ is a proper hypergeometric term, or equivalently, if and only if $d(n,k)$ is a proper polynomial.

4 The Algorithms

Theorem 3.6 transforms the problem of the existence of Z-pairs into the following two problems.

1. Find an algorithm to decompose $F(n,k)$ into

$$F(n,k) = (E-1)F_1(n,k) + F_2(n,k),$$

   such that $F_2(n,k)$ has a canonical representation $[a(n,k), b(n,k), c(n,k), d(n,k)]$ with $d(n,k)$ being k-shift-free.

2. Find an algorithm to determine whether $d(n,k)$ is a proper polynomial.

These two problems can be solved as follows.

Let $F(n,k)$ be a double hypergeometric term. It is shown in [4, 7] that there is an algorithm, denoted by `DGosper`, to find the canonical representation of $F(n,k)$, i.e., to find polynomials $a(n,k), b(n,k), c(n,k)$ and $d(n,k)$ such that

$$\frac{F(n,k+1)}{F(n,k)} = \frac{a(n,k)}{b(n,k)} \frac{c(n,k+1)}{c(n,k)} \frac{d(n,k)}{d(n,k+1)},$$

where $c(n,k) \perp d(n,k)$ and $a(n,k)/b(n,k)$ is k-shift-reduced. Therefore,

$$T(n,k) = \frac{F(n,k)}{F(n,0)} = \frac{d(n,0)}{c(n,0)} \frac{c(n,k)}{d(n,k)} \prod_{i=0}^{k-1} \frac{a(n,i)}{b(n,i)}.$$

Viewing $a(n,k)/b(n,k)$ and $c(n,k)/d(n,k)$ as rational functions of k with coefficients in the field $\mathbb{C}(n)$ (the fractional field of $\mathbb{C}[n]$), $T(n,k)$ becomes a hypergeometric term over the field $\mathbb{C}(n)$. Therefore, by the algorithm `dcert` given in [5], one can find hypergeometric terms $F_1'(k), F_2'(k)$ similar to $T(n,k)$ such that

$$T(n,k) = (E-1)F_1'(k) + F_2'(k)$$

and $F_2'(k)$ has a canonical representation $[\tilde{f}_1(k), \tilde{f}_2(k), \tilde{v}_1(k), \tilde{v}_2(k)]$ having the following property: $\gcd_k(\tilde{v}_2(k), \tilde{v}_2(k+h)) = 1$, $\forall\, h \in \mathbb{Z}\backslash\{0\}$, where $\gcd_k$ denotes the monic greatest common divisor in the field $\mathbb{C}(n)$.

Noting that for any $r(n), s_1(n), s_2(n) \in \mathbb{C}(n)$ which are not zero, we have

$$\left[r(n)\tilde{f}_1(k),\ r(n)\tilde{f}_2(k),\ s_1(n)\tilde{v}_1(k),\ s_2(n)\tilde{v}_2(k)\right]$$

is also a canonical representation of $F_2'(k)$, we may transform $\tilde{f}_1, \tilde{f}_2, \tilde{v}_1, \tilde{v}_2$ to $f_1, f_2, v_1, v_2 \in \mathbb{C}[n,k]$ such that $[f_1, f_2, v_1, v_2]$ is a canonical representation of $F_2'(k)$ and v_2 is k-shift-free. Since $F_1'(k)$ and $F_2'(k)$ are similar to $T(n,k)$, we have $F_1'(k)F(n,0)$ and $F_2'(k)F(n,0)$ are similar to $F(n,k)$, and hence, they are both double hypergeometric terms. Clearly,

$$F(n,k) = (E-1)(F_1'(n,k)F(n,0)) + (F_2'(n,k)F(n,0)).$$

Noting that $[f_1, f_2, v_1, v_2]$ is also a canonical representation of $F_2'(k)F(n,0)$, Theorem 3.6 states that $F(n,k)$ has a Z-pair if and only if $v_2(n,k)$ is a proper polynomial.

Finally, Abramov and Le [3, Section 4] give the algorithm to determine whether or not a polynomial is proper.

We provide two examples.

Example 1. Let

$$F_1(n,k) = \frac{kn^2 + kn + 2n + 1}{(kn+1)(kn+n+1)k!}.$$

Applying the algorithms `DGosper` and `dcert`, we get

$$\frac{\tilde{v}_1(k)}{\tilde{v}_2(k)} = -\frac{n+1}{n}k + \frac{2n+1}{n}.$$

Since $v_2(n,k) = 1$ is naturally a proper polynomial, we see that $F_1(n,k)$ has a Z-pair. In fact, by the Maple package `EKHAD` we find that

$$L = 1 - N, \quad G = -\frac{1}{(kn+k+1)(kn+1)(k-1)!}$$

is a Z-pair for $F_1(n,k)$.

Example 2. Let

$$F_2(n,k) = kF_1(n,k) = \frac{kn^2 + kn + 2n + 1}{(kn+1)(kn+n+1)(k-1)!}.$$

Applying the algorithms `DGosper` and `dcert`, we get

$$\frac{\tilde{v}_1(k)}{\tilde{v}_2(k)} = \left(-nk^3 + \frac{3n^2-2}{n+1}k^2 + \frac{n^3+8n^2+3n-1}{n(n+1)}k - \frac{2n^3+n^2-4n-2}{n(n+1)}\right) \Big/ (kn+n+1).$$

Since $v_2(n,k) = kn+n+1$ is not a proper polynomial, we conclude that $F_2(n,k)$ has no Z-pair.

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