The srank Conjecture on Schur's Q-Functions

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Abstract. We show that the shifted rank, or srank, of any partition λ with distinct parts equals the lowest degree of the terms appearing in the expansion of Schur's Q_λ function in terms of power sum symmetric functions. This gives an affirmative answer to a conjecture of Clifford. As pointed out by Clifford, the notion of the srank can be naturally extended to a skew partition λ/μ as the minimum number of bars among the corresponding skew bar tableaux. While the srank conjecture is not valid for skew partitions, we give an algorithm to compute the srank.

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1 Introduction

The main objective of this paper is to answer two open problems raised by Clifford [2] on sranks of partitions with distinct parts, skew partitions and Schur's Q-functions. For any partition λ with distinct parts, we give a proof of Clifford's srank conjecture that the lowest degree of the terms in the power sum expansion of Schur's Q-function Q_λ is equal to the number of bars in a minimal bar tableaux of shape λ. Clifford [1, 2] also proposed an open problem of determining the minimum number of bars among bar tableaux of a skew shape λ/μ. As noted by Clifford [1], this minimum number can be naturally regarded as the shifted rank, or srank, of λ/μ, denoted srank(λ/μ). For a skew bar tableau, we present an algorithm to generate a skew bar tableau without increasing the number of bars. This algorithm eventually leads to a bar tableau with the minimum number of bars.

Schur's Q-functions arise in the study of the projective representations of symmetric groups [16], see also, Hoffman and Humphreys [5], Humphreys [6], Józefiak [7], Morris [10, 12] and Nazarov [13]. Shifted tableaux are closely related to Schur's Q-functions analogous to the role of ordinary tableaux to the Schur functions. Sagan [15] and Worley [22] have independently developed a combinatorial theory of shifted tableaux, which includes shifted versions of the Robinson-Schensted-Knuth correspondence, Knuth's equivalence relations, Schützenberger's jeu de taquin, etc. The connections between this combinatorial theory of shifted tableaux and the theory of projective representations of the symmetric groups are further explored by Stembridge [19].

Clifford [2] studied the srank of shifted diagrams for partitions with distinct parts. Recall that the rank of an ordinary partition is defined as the number of boxes on the main diagonal of the corresponding Young diagram. Nazarov and Tarasov [14] found an important generalization of the rank of an ordinary partition to a skew partition in their study of tensor products of Yangian modules. A general theory of border strip decompositions and border strip tableaux of skew partitions is developed by Stanley [17], and it has been shown that the rank of a skew partition is the least number of strips to construct a minimal border strip decomposition of the skew diagram. Motivated by Stanley's theorem, Clifford [2] generalized the rank of a partition to the rank of a shifted partition, called srank, in terms of the minimal bar tableaux.

On the other hand, Clifford has noticed that the srank is closely related to Schur's Q-function, as suggested by the work of Stanley [17] on the rank of a partition. Stanley introduced a degree operator by taking the degree of the power sum symmetric function p_μ as the number of nonzero parts of the indexing partition μ. Furthermore, Clifford and Stanley [3] defined the bottom Schur functions to be the sum of the lowest degree terms in the expansion of the Schur functions in terms of the power sums. In [2] Clifford studied the lowest degree terms in the expansion of Schur's Q-functions in terms of power sum symmetric functions and conjectured that the lowest degree of the Schur's Q-function Q_λ is equal to the srank of λ. Our first result is a proof of this conjecture.

However, in general, the lowest degree of the terms, which appear in the expansion of the skew Schur's Q-function $Q_{\lambda/\mu}$ in terms of the power sums, is not equal to the srank of the shifted skew diagram of λ/μ. This is different from the case for ordinary skew partitions and skew Schur functions. Instead, we will take an algorithmic approach to the computation of the srank of a skew partition. It would be interesting to find an algebraic interpretation in terms of Schur's Q-functions.

2 Shifted diagrams and bar tableaux

Throughout this paper we will adopt the notation and terminology on partitions and symmetric functions in [9]. A *partition* λ is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, denoted $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, and k is called the *length* of λ, denoted $\ell(\lambda)$. For convenience we may add sufficient 0's at the end of λ if necessary. If $\sum_{i=1}^{k} \lambda_i = n$, we say that λ is a partition of the integer n, denoted $\lambda \vdash n$. For each partition λ there exists a geometric representation, known as the Young diagram, which is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in the i-th row. A partition is said to be *odd* (resp. even) if it has an odd (resp. even) number of even parts. Let $\mathcal{P}^o(n)$ denote the set of all partitions of n with only odd parts. We will call a partition *strict* if all its parts are distinct. Let $\mathcal{D}(n)$ denote the set of all strict partitions of n. For each partition $\lambda \in \mathcal{D}(n)$, let $S(\lambda)$ be the shifted diagram of λ, which is obtained from the Young diagram by shifting the i-th row $(i-1)$ squares to the right for each $i > 1$. For instance, Figure 1 illustrates the shifted diagram of shape $(8, 7, 5, 3, 1)$.

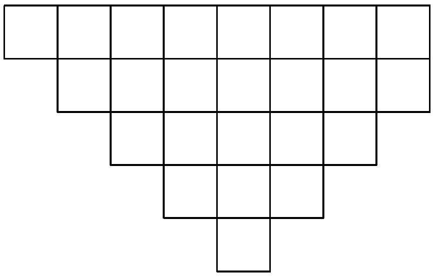

Figure 1: The shifted diagram of shape $(8, 7, 5, 3, 1)$

Given two partitions λ and μ, if for each i we have $\lambda_i \geq \mu_i$, then the skew partition λ/μ is defined to be the diagram obtained from the diagram of λ by removing the diagram of μ at the top-left corner. Similarly, the skew shifted diagram $S(\lambda/\mu)$ is defined as the set-theoretic difference of $S(\lambda)$ and $S(\mu)$.

Now we recall the definitions of bars and bar tableaux as given in Hoffman and Humphreys [5]. Let $\lambda \in \mathcal{D}(n)$ be a partition with length $\ell(\lambda) = k$. Fixing an odd positive integer r, three subsets I_+, I_0, I_- of integers between 1 and k are defined as follows:

$$\begin{aligned}
I_+ &= \{i : \lambda_{j+1} < \lambda_i - r < \lambda_j \text{ for some } j \leq k, \text{ taking } \lambda_{k+1} = 0\},\\
I_0 &= \{i : \lambda_i = r\},\\
I_- &= \{i : r - \lambda_i = \lambda_j \text{ for some } j \text{ with } i < j \leq k\}.
\end{aligned}$$

Let $I(\lambda, r) = I_+ \cup I_0 \cup I_-$. For each $i \in I(\lambda, r)$, we define a new strict partition $\lambda(i, r)$ of $\mathcal{D}(n - r)$ in the following way:

(1) If $i \in I_+$, then $\lambda_i > r$, and let $\lambda(i, r)$ be the partition obtained from λ by removing λ_i and inserting $\lambda_i - r$ between λ_j and λ_{j+1}.

(2) If $i \in I_0$, let $\lambda(i, r)$ be the partition obtained from λ by removing λ_i.

(3) If $i \in I_-$, then let $\lambda(i,r)$ be the partition obtained from λ by removing both λ_i and λ_j.

Meanwhile, for each $i \in I(\lambda, r)$, the associated r-bar is given as follows:

(1') If $i \in I_+$, the r-bar consists of the rightmost r squares in the i-th row of $S(\lambda)$, and we say that the r-bar is of Type 1.

(2') If $i \in I_0$, the r-bar consists of all the squares of the i-th row of $S(\lambda)$, and we say that the r-bar is of Type 2.

(3') If $i \in I_-$, the r-bar consists of all the squares of the i-th and j-th rows, and we say that the r-bar is of Type 3.

For example, as shown in Figure 2, the squares filled with 6 are a 7-bar of Type 1, the squares filled with 4 are a 3-bar of Type 2, and the squares filled with 3 are a 7-bar of Type 3.

1	1	6	6	6	6	6	6	6
	1	2	2	2	5	5	5	
		3	3	3	3	3	3	
			4	4	4			
				3				

Figure 2: A bar tableau of shape $(9, 7, 6, 3, 1)$

A *bar tableau* of shape λ is an array of positive integers of shape $S(\lambda)$ subject to the following conditions:

(1) It is weakly increasing in every row;

(2) The number of parts equal to i is odd for each positive integer i;

(3) Each positive integer i can appear in at most two rows, and if i appears in two rows, then these two rows must begin with i;

(4) The composition obtained by removing all squares filled with integers larger than some i has distinct parts.

We say that a bar tableau T is of type $\rho = (\rho_1, \rho_2, \ldots)$ if the total number of i's appearing in T is ρ_i. For example, the bar tableau in Figure 3 is of type $(3, 1, 1, 1)$. For a bar tableau T of shape λ, we define its weight $wt(T)$ recursively by the following procedure. If T is empty, let $wt(T) = 1$. Let $\varepsilon(\lambda)$

denote the parity of the partition λ, i.e., $\varepsilon(\lambda) = 0$ if λ has an even number of even parts; otherwise, $\varepsilon(\lambda) = 1$. Suppose that the largest numbers in T form an r-bar, which is associated with an index $i \in I(\lambda, r)$. Let j be the integer that occurrs in the definitions of I_+ and I_-. Let T' be the bar tableau of shape $\lambda(i, r)$ obtained from T by removing this r-bar. Now, let

$$wt(T) = n_i \, wt(T'), \tag{2.1}$$

where

$$n_i = \begin{cases} (-1)^{j-i} 2^{1-\varepsilon(\lambda)}, & \text{if } i \in I_+, \\ (-1)^{\ell(\lambda)-i}, & \text{if } i \in I_0, \\ (-1)^{j-i+\lambda_i} 2^{1-\varepsilon(\lambda)}, & \text{if } i \in I_-. \end{cases} \tag{2.2}$$

For example, the weight of the bar tableau T in Figure 3 equals

$$wt(T) = (-1)^{1-1} 2^{1-0} \cdot (-1)^{1-1} 2^{1-1} \cdot (-1)^{2-2} \cdot (-1)^{1-1} = 2. \tag{2.3}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 4 \\ \hline & 2 & \multicolumn{3}{c}{} \\ \cline{2-2} \end{array}$$

Figure 3: A bar tableau of type $(3, 1, 1, 1)$

The following lemma will be used in Section 3 to determine whether certain terms will vanish in the power sum expansion of Schur's Q-functions indexed by partitions with two distinct parts.

Lemma 2.1 *Let $\lambda = (\lambda_1, \lambda_2)$ be a strict partition with the two parts λ_1 and λ_2 having the same parity. Given an partition $\sigma = (\sigma_1, \sigma_2) \in \mathcal{P}^o(|\lambda|)$, if $\sigma_2 < \lambda_2$, then among all bar tableaux of shape λ there exist only two bar tableaux of type σ, say T_1 and T_2, and furthermore, we have $wt(T_1) + wt(T_2) = 0$.*

Proof. Suppose that both λ_1 and λ_2 are even. The case when λ_1 and λ_2 are odd numbers can be proved similarly. Note that $\sigma_2 < \lambda_2 < \lambda_1$. By putting 2's in the last σ_2 squares of the second row and then filling the remaining squares in the diagram with 1's, we obtain one tableau T_1. By putting 2's in the last σ_2 squares of the first row and then filling the remaining squares with 1's, we obtain another tableau T_2. Clearly, both T_1 and T_2 are bar tableaux of shape λ and type σ, and they are the only two such bar tableaux. We notice that

$$wt(T_1) = (-1)^{2-2} 2^{1-0} \cdot (-1)^{2-1+\lambda_1} 2^{1-1} = -2. \tag{2.4}$$

While, for the weight of T_2, there are two cases to consider. If $\lambda_1 - \sigma_2 > \lambda_2$, then

$$wt(T_2) = (-1)^{1-1}2^{1-0} \cdot (-1)^{2-1+\lambda_1-\sigma_2}2^{1-1} = 2. \tag{2.5}$$

If $\lambda_1 - \sigma_2 < \lambda_2$, then

$$wt(T_2) = (-1)^{2-1}2^{1-0} \cdot (-1)^{2-1+\lambda_2}2^{1-1} = 2. \tag{2.6}$$

Thus we have $wt(T_2) = 2$ in either case, so the relation $wt(T_1) + wt(T_2) = 0$ holds. ∎

For example, taking $\lambda = (8, 6)$ and $\sigma = (11, 3)$, the two bar tableaux T_1 and T_2 in the above lemma are depicted as in Figure 4.

1	1	1	1	1	1	1	1
	1	1	1	2	2	2	

T_1

1	1	1	1	1	2	2	2
	1	1	1	1	1	1	

T_2

Figure 4: Two bar tableaux of shape $(8, 6)$ and type $(11, 3)$

Clifford gave a natural generalization of bar tableaux to skew shapes [2]. Formally, a *skew bar tableau* of shape λ/μ is an assignment of nonnegative integers to the squares of $S(\lambda)$ such that in addition to the above four conditions (1)-(4) we further impose the condition that

(5) the partition obtained by removing all squares filled with positive integers and reordering the remaining rows is μ.

For example, taking the skew partition $(8, 6, 5, 4, 1)/(8, 2, 1)$, Figure 5 is a skew bar tableau of such shape.

A bar tableau of shape λ is said to be *minimal* if there does not exist a bar tableau with fewer bars. Motivated by Stanley's results in [17], Clifford defined the srank of a shifted partition $S(\lambda)$, denoted $\mathrm{srank}(\lambda)$, as the number of bars in a minimal bar tableau of shape λ [2]. Clifford also gave the following formula for $\mathrm{srank}(\lambda)$.

Theorem 2.2 ([2, Theorem 4.1]) *Given a strict partition λ, let o be the number of odd parts of λ, and let e be the number of even parts. Then* $\mathrm{srank}(\lambda) = \max(o, e + (\ell(\lambda) \bmod 2))$.

Next we consider the number of bars in a minimal skew bar tableau of shape λ/μ. Note that the squares filled with 0's in the skew bar tableau give

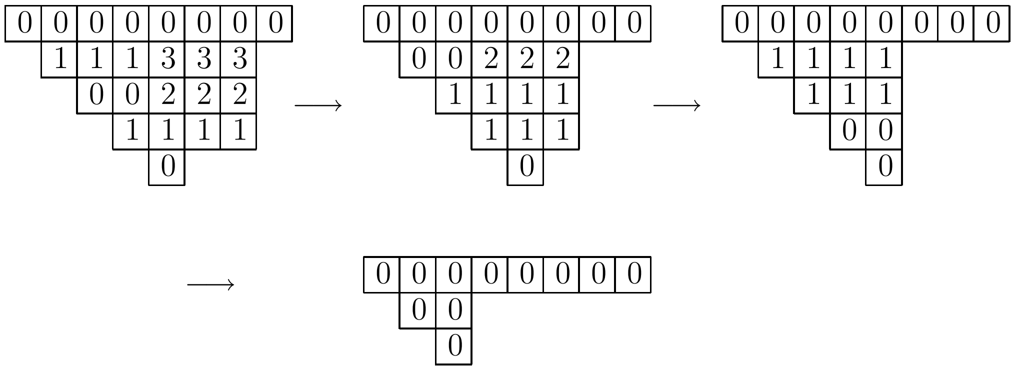

Figure 5: Checking the legality of a skew bar tableau

rise to a shifted diagram of shape μ by reordering the rows. Let o_r (resp. e_r) be the number of nonempty rows of odd (resp. even) length with blank squares, and let o_s (resp. e_s) be the number of rows of λ with some squares filled with 0's and an odd (resp. even) number of blank squares. It is obvious that the number of bars in a minimal skew bar tableau is greater than or equal to

$$o_s + 2e_s + \max(o_r, e_r + ((e_r + o_r) \bmod 2)).$$

In fact the above quantity has been considered by Clifford [1]. Observe that this quantity depends on the positions of the 0's.

It should be remarked that a legal bar tableau of shape λ/μ may not exist once the positions of 0's are fixed. One open problem proposed by Clifford [1] is to find a characterization of $\mathrm{srank}(\lambda/\mu)$. In Section 5 we will give an algorithm to compute the srank of a skew shape.

3 Clifford's conjecture

In this section, we aim to show that the lowest degree of the power sum expansion of a Schur's Q-function Q_λ equals $\mathrm{srank}(\lambda)$. Let us recall relevant terminology on Schur's Q-functions. Let $x = (x_1, x_2, \ldots)$ be an infinite sequence of independent indeterminates. We define the symmetric functions $q_k = q_k(x)$ in $x_1, x_2, \ldots$ for all integers k by the following expansion of the formal power series in t:

$$\prod_{i \geq 1} \frac{1 + x_i t}{1 - x_i t} = \sum_k q_k(x) t^k.$$

In particular, $q_k = 0$ for $k < 0$ and $q_0 = 1$. It immediately follows that

$$\sum_{i+j=n} (-1)^i q_i q_j = 0, \tag{3.7}$$

for all $n \geq 1$. Let $Q_{(a)} = q_a$ and

$$Q_{(a,b)} = q_a q_b + 2 \sum_{m=1}^{b} (-1)^m q_{a+m} q_{b-m}.$$

From (3.7) we see that $Q_{(a,b)} = -Q_{(b,a)}$ and thus $Q_{(a,a)} = 0$ for any a, b. In general, for any strict partition λ, the symmetric function Q_λ is defined by the recurrence relations:

$$Q_{(\lambda_1,\ldots,\lambda_{2k+1})} = \sum_{m=1}^{2k+1} (-1)^{m+1} q_{\lambda_m} Q_{(\lambda_1,\ldots,\hat{\lambda}_m,\ldots,\lambda_{2k+1})}, \tag{3.8}$$

$$Q_{(\lambda_1,\ldots,\lambda_{2k})} = \sum_{m=2}^{2k} (-1)^m Q_{(\lambda_1,\lambda_m)} Q_{(\lambda_2,\ldots,\hat{\lambda}_m,\ldots,\lambda_{2k})}, \tag{3.9}$$

where $\hat{}$ stands for a missing entry.

It was known that Q_λ can be also defined as the specialization at $t = -1$ of the Hall-Littlewood functions associated with λ [9]. Originally, these Q_λ symmetric functions were introduced in order to express irreducible projective characters of the symmetric groups [16]. Note that the irreducible projective representations of S_n are in one-to-one correspondence with partitions of n with distinct parts, see [7, 18, 19]. For any $\lambda \in \mathcal{D}(n)$, let $\langle\lambda\rangle$ denote the character of the irreducible projective or spin representation indexed by λ. Morris [11] has found a combinatorial rule for calculating the characters, which is the projective analogue of the Murnaghan-Nakayama rule. In terms of bar tableaux, Morris's theorem reads as follows:

Theorem 3.1 ([11]) *Let $\lambda \in \mathcal{D}(n)$ and $\pi \in \mathcal{P}^o(n)$. Then*

$$\langle\lambda\rangle(\pi) = \sum_T wt(T) \tag{3.10}$$

where the sum ranges over all bar tableaux of shape λ and type π.

The above theorem for projective characters implies the following formula, which will be used later in the proof of Lemma 3.7.

Corollary 3.2 *Let λ be a strict partition of length 2. Suppose that the two parts λ_1, λ_2 are both odd. Then we have*

$$\langle\lambda\rangle(\lambda) = -1. \tag{3.11}$$

Proof. Let T be the bar tableau obtained by filling the last λ_2 squares in the first row of $S(\lambda)$ with 2's and the remaining squares with 1's, and let T' be the bar tableau obtained by filling the first row of $S(\lambda)$ with 1's and the second row with 2's. Clearly, T and T' are of the same type λ. Let us first consider the weight of T. If $\lambda_1 - \lambda_2 < \lambda_2$, then

$$wt(T) = (-1)^{2-1}2^{1-0} \cdot (-1)^{2-1+\lambda_2}2^{1-1} = -2.$$

If $\lambda_1 - \lambda_2 > \lambda_2$, then

$$wt(T) = (-1)^{1-1}2^{1-0} \cdot (-1)^{2-1+\lambda_1-\lambda_2}2^{1-1} = -2.$$

In both cases, the weight of T' equals

$$wt(T') = (-1)^{2-2} \cdot (-1)^{1-1} = 1.$$

Since there are only two bar tableaux, T and T', of type λ, the corollary immediately follows from Theorem 3.1. ∎

Let $p_k(x)$ denote the k-th power sum symmetric functions, i.e., $p_k(x) = \sum_{i\geq 1} x_i^k$. For any partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, let $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$. The fundamental connection between Q_λ symmetric functions and the projective representations of the symmetric group is as follows.

Theorem 3.3 ([16]) *Let $\lambda \in \mathcal{D}(n)$. Then we have*

$$Q_\lambda = \sum_{\pi \in \mathcal{P}^o(n)} 2^{[\ell(\lambda)+\ell(\pi)+\varepsilon(\lambda)]/2}\langle\lambda\rangle(\pi)\frac{p_\pi}{z_\pi}, \tag{3.12}$$

where

$$z_\pi = 1^{m_1}m_1! \cdot 2^{m_2}m_2! \cdot \cdots, \quad \textit{if } \pi = \langle 1^{m_1}2^{m_2}\cdots\rangle.$$

Stanley [17] introduced a degree operator on symmetric functions by defining $\deg(p_i) = 1$, and so $\deg(p_\nu) = \ell(\nu)$. Clifford [2] applied this operator to Schur's Q-functions and obtained the following lower bound from Theorem 3.3.

Corollary 3.4 ([2, Corollary 6.2]) *The terms of the lowest degree in Q_λ have degree at least* $\mathrm{srank}(\lambda)$.

The following conjecture is proposed by Clifford:

Conjecture 3.5 ([2, Conjecture 6.4]) *The terms of the lowest degree in Q_λ have degree* $\mathrm{srank}(\lambda)$.

Our proof of the above conjecture depends on the Pfaffian formula for Schur's Q-functions. Given a skew-symmetric matrix $A = (a_{i,j})$ of even size $2n \times 2n$, the *Pfaffian* of A, denoted Pf(A), is defined by

$$\mathrm{Pf}(A) = \sum_{\pi} (-1)^{\mathrm{cr}(\pi)} a_{i_1 j_1} \cdots a_{i_n j_n},$$

where the sum ranges over all set partitions π of $\{1, 2, \cdots, 2n\}$ into two element blocks $i_k < j_k$ and $cr(\pi)$ is the number of crossings of π, i.e., the number of pairs $h < k$ for which $i_h < i_k < j_h < j_k$.

Theorem 3.6 ([9]) *Given a strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n})$ satisfying $\lambda_1 > \ldots > \lambda_{2n} \geq 0$, let $M_\lambda = (Q_{(\lambda_i, \lambda_j)})$. Then we have*

$$Q_\lambda = \mathrm{Pf}(M_\lambda).$$

We first prove that Clifford's conjecture holds for strict partitions of length less than three. The proof for the general case relies on this special case.

Lemma 3.7 *Let λ be a strict partition of length $\ell(\lambda) < 3$. Then the terms of the lowest degree in Q_λ have degree* srank(λ).

Proof. In view of Theorem 3.1 and Theorem 3.3, if there exists a unique bar tableau of shape λ and type π, then the coefficient of p_π is nonzero in the expansion of Q_λ. There are five cases to consider.

(1) $\ell(\lambda) = 1$ and λ_1 is odd. Clearly, we have srank$(\lambda) = 1$. Note that there exists a unique bar tableau T of shape λ and of type λ with all squares of $S(\lambda)$ filled with 1's. Therefore, the coefficient of p_λ in the power sum expansion of Q_λ is nonzero and the lowest degree of Q_λ is 1.

(2) $\ell(\lambda) = 1$ and λ_1 is even. We see that srank$(\lambda) = 2$. Since the bars are all of odd size, there does not exist any bar tableau of shape λ and of type λ. But there is a unique bar tableau T of shape λ and of type $(\lambda_1 - 1, 1)$, which is obtained by filling the rightmost square of $S(\lambda)$ with 2 and the remaining squares with 1's. So the coefficient of $p_{(\lambda_1-1,1)}$ in the power sum expansion of Q_λ is nonzero and the terms of the lowest degree in Q_λ have degree 2.

(3) $\ell(\lambda) = 2$ and the two parts λ_1, λ_2 have different parity. In this case, we have srank$(\lambda) = 1$. Note that there exists a unique bar tableau T of shape λ and of type $(\lambda_1 + \lambda_2)$, which is obtained by filling all the squares of $S(\lambda)$ with 1's. Thus, the coefficient of $p_{\lambda_1+\lambda_2}$ in the power sum expansion of Q_λ is nonzero and the terms of lowest degree in Q_λ have degree 1.

(4) $\ell(\lambda)=2$ and the two parts λ_1,λ_2 are both even. It is easy to see that $\mathrm{srank}(\lambda)=2$. Since there exists a unique bar tableau T of shape λ and of type $(\lambda_1-1,\lambda_2+1)$, which is obtained by filling the rightmost λ_2+1 squares in the first row of $S(\lambda)$ with 2's and the remaining squares with 1's, the coefficient of $p_{(\lambda_1-1,\lambda_2+1)}$ in the power sum expansion of Q_λ is nonzero; hence the lowest degree of Q_λ is equal to 2.

(5) $\ell(\lambda)=2$ and the two parts λ_1,λ_2 are both odd. In this case, we have $\mathrm{srank}(\lambda)=2$. By Corollary 3.2, the coefficient of p_λ in the power sum expansion of Q_λ is nonzero, and therefore the terms of the lowest degree in Q_λ have degree 2.

This completes the proof. ∎

Given a strict partition λ, we consider the Pfaffian expansion of Q_λ as shown in Theorem 3.6. To prove Clifford's conjecture, we need to determine which terms may appear in the expansion of Q_λ in terms of power sum symmetric functions. Suppose that the Pfaffian expansion of Q_λ is as follows:

$$\mathrm{Pf}(M_\lambda)=\sum_{\pi}(-1)^{\mathrm{cr}(\pi)}Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}, \tag{3.13}$$

where the sum ranges over all set partitions π of $\{1,2,\cdots,2m\}$ into two element blocks $\{(\pi_1,\pi_2),\ldots,(\pi_{2m-1},\pi_{2m})\}$ with $\pi_1<\pi_3<\cdots<\pi_{2m-1}$ and $\pi_{2k-1}<\pi_{2k}$ for any k. For the above expansion of Q_λ, the following two lemmas will be used to choose certain lowest degree terms in the power sum expansion of $Q_{(\lambda_i,\lambda_j)}$ in the matrix M_λ.

Lemma 3.8 *Suppose that λ has both odd parts and even parts. Let λ_{i_1} (resp. λ_{j_1}) be the largest odd (resp. even) part of λ. If the power sum symmetric function $p_{\lambda_{i_1}+\lambda_{j_1}}$ appears in the terms of lowest degree originated from the product $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$ as in the expansion* (3.13)*, then we have $(\pi_1,\pi_2)=(i_1,j_1)$.*

Proof. Without loss of generality, we may assume that $\lambda_{i_1}>\lambda_{j_1}$. By Lemma 3.7, the term $p_{\lambda_{i_1}+\lambda_{j_1}}$ appears in $Q_{(\lambda_{i_1},\lambda_{j_1})}$ with nonzero coefficients. Since $\lambda_{i_1},\lambda_{j_1}$ are the largest odd and even parts, $p_{\lambda_{i_1}+\lambda_{j_1}}$ does not appear as a factor of any term of the lowest degree in the expansion of $Q_{(\lambda_{i_k},\lambda_{j_k})}$, where λ_{i_k} and λ_{j_k} have different parity. Meanwhile, if λ_{i_k} and λ_{j_k} have the same parity, then we consider the bar tableaux of shape $(\lambda_{i_k},\lambda_{j_k})$ and of type $(\lambda_{i_1}+\lambda_{j_1},\lambda_{i_k}+\lambda_{j_k}-\lambda_{i_1}-\lambda_{j_1})$. Observe that $\lambda_{i_k}+\lambda_{j_k}-\lambda_{i_1}-\lambda_{j_1}<\lambda_{j_k}$. Since the lowest degree of $Q_{(\lambda_{i_k},\lambda_{j_k})}$ is 2, from Lemma 2.1 it follows that $p_{\lambda_{i_1}+\lambda_{j_1}}$ can not be a factor of any term of lowest degree in the power sum expansion of $Q_{(\lambda_{i_k},\lambda_{j_k})}$. This completes the proof. ∎

Lemma 3.9 *Suppose that λ only has even parts. Let λ_1, λ_2 be the two largest parts of λ (allowing $\lambda_2 = 0$). If the power sums $p_{\lambda_1-1}p_{\lambda_2+1}$ appears in the terms of the lowest degree given by the product $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$ as in* (3.13)*, then we have $(\pi_1, \pi_2) = (1, 2)$.*

Proof. From Case (4) of the proof of Lemma 3.7 it follows that $p_{\lambda_1-1}p_{\lambda_2+1}$ appears as a term of the lowest degree in the power sum expansion of $Q_{(\lambda_1,\lambda_2)}$. We next consider the power sum expansion of any other $Q_{(\lambda_i,\lambda_j)}$. First, we consider the case when $\lambda_i+\lambda_j > \lambda_2+1$ and $\lambda_i \le \lambda_2$. Since $\lambda_i+\lambda_j-(\lambda_2+1) < \lambda_j$, by Lemma 2.1, the term p_{λ_2+1} is not a factor of any term of the lowest degree in the power sum expansion of $Q_{(\lambda_i,\lambda_j)}$. Now we are left with the case when $\lambda_i + \lambda_j > \lambda_1 - 1$ and $\lambda_i \le \lambda_1 - 2$. Since $\lambda_i + \lambda_j - (\lambda_1 - 1) < \lambda_j$, by Lemma 2.1 the term p_{λ_1-1} does not appear as a factor in the terms of the lowest degree of $Q_{(\lambda_i,\lambda_j)}$. So we have shown that if either p_{λ_2+1} or p_{λ_1-1} appears as a factor of some lowest degree term for $Q_{(\lambda_i,\lambda_j)}$, then we deduce that $\lambda_i = \lambda_1$. Moreover, if both p_{λ_1-1} and p_{λ_2+1} are factors of the lowest degree terms in the power sum expansion of $Q_{(\lambda_1,\lambda_j)}$, then we have $\lambda_j = \lambda_2$. The proof is complete. ∎

We now present the main result of this paper.

Theorem 3.10 *For any $\lambda \in \mathcal{D}(n)$, the terms of the lowest degree in Q_λ have degree* $\mathrm{srank}(\lambda)$.

Proof. We write the strict partition λ in the form $(\lambda_1, \lambda_2, \ldots, \lambda_{2m})$, where $\lambda_1 > \ldots > \lambda_{2m} \ge 0$. Suppose that the partition λ has o odd parts and e even parts (including 0 as a part). For the sake of presentation, let $(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_o})$ denote the sequence of odd parts in decreasing order, and let $(\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_e})$ denote the sequence of even parts in decreasing order.

We first consider the case $o \ge e$. In this case, it will be shown that $\mathrm{srank}(\lambda) = o$. By Theorem 2.2, if $\lambda_{2m} > 0$, i.e., $\ell(\lambda) = 2m$, then we have

$$\mathrm{srank}(\lambda) = \max(o, e + 0) = o.$$

If $\lambda_{2m} = 0$, i.e., $\ell(\lambda) = 2m - 1$, then we still have

$$\mathrm{srank}(\lambda) = \max(o, (e - 1) + 1) = o.$$

Let

$$A = p_{\lambda_{i_1}+\lambda_{j_1}} \cdots p_{\lambda_{i_e}+\lambda_{j_e}} p_{\lambda_{i_{e+1}}} p_{\lambda_{i_{e+2}}} \cdots p_{\lambda_{i_o}}.$$

We claim that A appears as a term of the lowest degree in the power sum expansion of Q_λ. For this purpose, we need to determine those matchings π of $\{1, 2, \ldots, 2m\}$ in (3.13), for which the power sum expansion of the product $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$ contains A as a term of the lowest degree.

By Lemma 3.8, if the $p_{\lambda_{i_1}+\lambda_{j_1}}$ appears as a factor in the lowest degree terms of the power sum expansion of $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$, then we have $\{\pi_1,\pi_2\}=\{i_1,j_1\}$. Iterating this argument, we see that if $p_{\lambda_{i_1}+\lambda_{j_1}}\cdots p_{\lambda_{i_e}+\lambda_{j_e}}$ appears as a factor in the lowest degree terms of $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$, then we have

$$\{\pi_1,\pi_2\}=\{i_1,j_1\},\ldots,\{\pi_{2e-1},\pi_{2e}\}=\{i_e,j_e\}.$$

It remains to determine the ordered pairs

$$\{(\pi_{2e+1},\pi_{2e+2}),\ldots,(\pi_{2m-1},\pi_{2m})\}.$$

By the same argument as in Case (5) of the proof of Lemma 3.7, for any $e+1\leq k<l\leq o$, the term $p_{\lambda_{i_k}}p_{\lambda_{i_l}}$ appears as a term of the lowest degree in the power sum expansion of $Q_{(\lambda_{i_k},\lambda_{i_l})}$. Moreover, if the power sum symmetric function $p_{\lambda_{i_{e+1}}}p_{\lambda_{i_{e+2}}}\cdots p_{\lambda_{i_o}}$ appears as a term of the lowest degree in the power sum expansion of the product $Q_{(\lambda_{\pi_{2e+1}},\lambda_{\pi_{2e+2}})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$, then the composition of the pairs $\{(\pi_{2e+1},\pi_{2e+2}),\ldots,(\pi_{2m-1},\pi_{2m})\}$ could be any matching of $\{1,2,\ldots,2m\}/\{i_1,j_1,\ldots,i_e,j_e\}$.

To summarize, there are $(2(m-e)-1)!!$ matchings π such that A appears as a term of the lowest degree in the power sum expansion of the product $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$. Combining Corollary 3.2 and Theorem 3.3, we find that the coefficient of $p_{\lambda_{i_k}}p_{\lambda_{i_l}}$ $(e+1\leq k<l\leq o)$ in the power sum expansion of $Q_{(\lambda_{i_k},\lambda_{i_l})}$ is $-\frac{4}{\lambda_{i_k}\lambda_{i_l}}$. It follows that the coefficient of A in the expansion of the product $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$ is independent of the choice of π. Since $(2(m-e)-1)!!$ is an odd number, the term A will not vanish in the expansion of Q_λ. Note that the degree of A is $e+(o-e)=o$, which is equal to $\mathrm{srank}(\lambda)$, as desired.

Similarly, we consider the case $e>o$. In this case, we aim to show that $\mathrm{srank}(\lambda)=e$. By Theorem 2.2, if $\lambda_{2m}>0$, i.e., $\ell(\lambda)=2m$, then we have

$$\mathrm{srank}(\lambda)=\max(o,e+0)=e.$$

If $\lambda_{2m}=0$, i.e., $\ell(\lambda)=2m-1$, then we still have

$$\mathrm{srank}(\lambda)=\max(o,(e-1)+1)=e.$$

Let

$$B=p_{\lambda_{i_1}+\lambda_{j_1}}\cdots p_{\lambda_{i_o}+\lambda_{j_o}}p_{\lambda_{j_{o+1}}-1}p_{\lambda_{j_{o+2}}+1}\cdots p_{\lambda_{j_{e-1}}-1}p_{\lambda_{j_e}+1}.$$

We proceed to prove that B appears as a term of the lowest degree in the power sum expansion of Q_λ. Applying Lemma 3.8 repeatedly, we deduce that if $p_{\lambda_{i_1}+\lambda_{j_1}}\cdots p_{\lambda_{i_o}+\lambda_{j_o}}$ appears as a factor in the lowest degree terms of the product $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$, then

$$\{\pi_1,\pi_2\}=\{i_1,j_1\},\ldots,\{\pi_{2o-1},\pi_{2o}\}=\{i_o,j_o\}. \tag{3.14}$$

On the other hand, iteration of Lemma 3.9 reveals that if the power sum symmetric function $p_{\lambda_{j_{o+1}}-1}p_{\lambda_{j_{o+2}}+1}\cdots p_{\lambda_{j_{e-1}}-1}p_{\lambda_{j_e}+1}$ appears as a term of the lowest degree in the power sum expansion of $Q_{(\lambda_{\pi_{2o+1}},\lambda_{\pi_{2o+2}})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$, then

$$\{\pi_{2o+1},\pi_{2o+2}\}=\{j_{o+1},j_{o+2}\},\ldots,\{\pi_{2m-1},\pi_{2m}\}=\{j_{e-1},j_e\}. \tag{3.15}$$

Therefore, if B appears as a term of the lowest degree in the power sum expansion of $Q_{(\lambda_{\pi_1},\lambda_{\pi_2})}\cdots Q_{(\lambda_{\pi_{2m-1}},\lambda_{\pi_{2m}})}$, then the matching π is uniquely determined by (3.14) and (3.15). Note that the degree of B is e, which coincides with $\mathrm{srank}(\lambda)$.

Since there is always a term of degree $\mathrm{srank}(\lambda)$ in the power sum expansion of Q_λ, the theorem follows. ∎

4 Skew Schur's Q-functions

In this section, we show that the srank $\mathrm{srank}(\lambda/\mu)$ is a lower bound of the lowest degree of the terms in the power sum expansion of the skew Schur's Q-function $Q_{\lambda/\mu}$. Note that Clifford's conjecture does not hold for skew shapes.

We first recall a definition of the skew Schur's Q-function in terms of strip tableaux. The concept of strip tableaux were introuduced by Stembridge [18] to describe the Morris rule for the evaluation of irreducible spin characters. Given a skew partition λ/μ, the *j-th diagonal* of the skew shifted diagram $S(\lambda/\mu)$ is defined as the set of squares $(1,j),(2,j+1),(3,j+2),\ldots$ in $S(\lambda/\mu)$. A skew diagram $S(\lambda/\mu)$ is called a *strip* if it is rookwise connected and each diagonal contains at most one box. The *height* h of a strip is defined to be the number of rows it occupies. A *double strip* is a skew diagram formed by the union of two strips which both start on the diagonal consisting of squares (j,j). The *depth* of a double strip is defined to be $\alpha+\beta$ if it has α diagonals of length two and its diagonals of length one occupy β rows. A *strip tableau* of shape λ/μ and type $\pi=(\pi_1,\ldots,\pi_k)$ is defined to be a sequence of shifted diagrams

$$S(\mu)=S(\lambda^0)\subseteq S(\lambda^1)\subseteq\cdots\subseteq S(\lambda^k)=S(\lambda)$$

with $|\lambda^i/\lambda^{i-1}|=\pi_i$ $(1\le i\le k)$ such that each skew shifted diagram $S(\lambda^i/\lambda^{i-1})$ is either a strip or a double strip.

The skew Schur's Q-function can be defined as the weight generating function of strip tableaux in the following way. For a strip of height h we assign the weight $(-1)^{h-1}$, and for a double strip of depth d we assign the weight $2(-1)^{d-1}$. The weight of a strip tableau T, denoted $wt(T)$, is the product of the weights of strips and double strips of which T is composed.

Then the skew Schur's Q-function $Q_{\lambda/\mu}$ is given by

$$Q_{\lambda/\mu} = \sum_{\pi \in \mathcal{P}^o(|\lambda/\mu|)} \sum_{T} 2^{\ell(\pi)} wt(T) \frac{p_\pi}{z_\pi}, \tag{4.16}$$

where T ranges over all strip tableaux T of shape λ/μ and type π, see [18, Theorem 5.1].

Józefiak and Pragacz [8] obtained the following Pfaffian formula for the skew Schur's Q-function.

Theorem 4.1 ([8]) *Let λ, μ be strict partitions with $m = \ell(\lambda)$, $n = \ell(\mu)$, $\mu \subset \lambda$, and let $M(\lambda, \mu)$ denote the skew-symmetric matrix*

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix},$$

where $A = (Q_{(\lambda_i, \lambda_j)})$ and $B = (Q_{(\lambda_i - \mu_{n+1-j})})$.

Then

(1) if $m + n$ is even, we have $Q_{\lambda/\mu} = \mathrm{Pf}(M(\lambda, \mu))$;

(2) if $m+n$ is odd, we have $Q_{\lambda/\mu} = \mathrm{Pf}(M(\lambda, \mu'))$, where $\mu' = (\mu_1, \cdots, \mu_n, 0)$.

A combinatorial proof of the above theorem was given by Stembridge [20] in terms of lattice paths, and later, Hamel [4] gave an interesting generalization by using the border strip decompositions of the shifted diagram.

Given a skew partition λ/μ, Clifford [1] constructed a bijection between skew bar tableaux of shape λ/μ and skew strip tableaux of the same shape, which preserves the type of the tableau. Using this bijection, it is straightforward to derive the following result.

Proposition 4.2 *The terms of the lowest degree in $Q_{\lambda/\mu}$ have degree at least $\mathrm{srank}(\lambda/\mu)$.*

Different from the case of non-skew shapes, in general, the lowest degree terms in $Q_{\lambda/\mu}$ do not have the degree $\mathrm{srank}(\lambda/\mu)$. For example, take the skew partition $(4,3)/3$. It is easy to see that $\mathrm{srank}((4,3)/3) = 2$. While, using Theorem 4.1 and Stembridge's SF Package for Maple [21], we obtain that

$$Q_{(4,3)/3} = \mathrm{Pf} \begin{pmatrix} 0 & Q_{(4,3)} & Q_{(4)} & Q_{(1)} \\ Q_{(3,4)} & 0 & Q_{(3)} & Q_{(0)} \\ -Q_{(4)} & -Q_{(3)} & 0 & 0 \\ -Q_{(1)} & -Q_{(0)} & 0 & 0 \end{pmatrix} = 2p_1^4. \tag{4.17}$$

This shows that the lowest degree of $Q_{(4,3)/3}$ equals 4, which is strictly greater than $\mathrm{srank}((4,3)/3)$.

5 The srank of skew partitions

In this section, we present an algorithm to determine the srank for the skew partition λ/μ. In fact, the algorithm leads to a configuration of 0's. To obtain the srank of a skew partition, we need to minimize the number of bars by adjusting the positions of 0's. Given a configuration $\mathcal{C}$ of 0's in the shifted diagram $S(\lambda)$, let

$$\kappa(\mathcal{C}) = o_s + 2e_s + \max(o_r, e_r + ((e_r + o_r) \bmod 2)),$$

where o_r (resp. e_r) counts the number of nonempty rows in which there are an odd (resp. even) number of squares and no squares are filled with 0, and o_s (resp. e_s) records the number of rows in which at least one square is filled with 0 but there are an odd (resp. nonzero even) number of blank squares.

If there exists at least one bar tableau of type λ/μ under some configuration $\mathcal{C}$, we say that $\mathcal{C}$ is *admissible*. For a fixed configuration $\mathcal{C}$, each row is one of the following eight possible types:

(1) an even row bounded by an even number of 0's, denoted (e, e),

(2) an odd row bounded by an even number of 0's, denoted (e, o),

(3) an odd row bounded by an odd number of 0's, denoted (o, e),

(4) an even row bounded by an odd number of 0's, denoted (o, o),

(5) an even row without 0's, denoted $(\emptyset, e)$,

(6) an odd row without 0's, denoted $(\emptyset, o)$,

(7) an even row filled with 0's, denoted $(e, \emptyset)$,

(8) an odd row filled with 0's, denoted $(o, \emptyset)$.

Given two rows with respective types s and s' for some configuration $\mathcal{C}$, if we can obtain a new configuration $\mathcal{C}'$ by exchanging the locations of 0's in these two rows such that their new types are t and t' respectively, then denote it by $\mathcal{C}' = \mathcal{C}\left(\left[\begin{smallmatrix} s \\ s' \end{smallmatrix}\right] \to \left[\begin{smallmatrix} t \\ t' \end{smallmatrix}\right]\right)$. Let o_r, e_r, o_s, e_s be defined as above corresponding to configuration $\mathcal{C}$, and let o_r', e_r', o_s', e_s' be those of $\mathcal{C}'$.

In the following we will show that how the quantity $\kappa(\mathcal{C})$ changes when exchanging the locations of 0's in $\mathcal{C}$.

Lemma 5.1 *If $\mathcal{C}' = \mathcal{C}\left(\left[\begin{smallmatrix} s \\ s' \end{smallmatrix}\right] \to \left[\begin{smallmatrix} s \\ s' \end{smallmatrix}\right]\right)$ or $\mathcal{C}' = \mathcal{C}\left(\left[\begin{smallmatrix} s \\ s' \end{smallmatrix}\right] \to \left[\begin{smallmatrix} s' \\ s \end{smallmatrix}\right]\right)$, i.e., the types of the two involved rows are remained or exchanged, where s, s' are any two possible types, then $\kappa(\mathcal{C}') = \kappa(\mathcal{C})$.*

Lemma 5.2 *If* $\mathcal{C}' = \mathcal{C}\left(\begin{bmatrix} {\scriptstyle (e,e)} \\ {\scriptstyle (\emptyset,o)} \end{bmatrix} \to \begin{bmatrix} {\scriptstyle (\emptyset,e)} \\ {\scriptstyle (e,o)} \end{bmatrix}\right)$*, then* $\kappa(\mathcal{C}') \le \kappa(\mathcal{C})$*.*

Proof. In this case we have

$$o'_s = o_s + 1, \quad e'_s = e_s - 1, \quad o'_r = o_r - 1, \quad e'_r = e_r + 1.$$

Note that $o_r + e_r = \ell(\lambda) - \ell(\mu)$. Now there are two cases to consider.
Case I. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 0 \pmod 2$.

(1) If $o_r \le e_r$, then $o'_r \le e'_r$ and

$$\begin{aligned} \kappa(\mathcal{C}) &= o_s + 2e_s + e_r, \\ \kappa(\mathcal{C}') &= o_s + 1 + 2(e_s - 1) + e'_r = o_s + 2e_s + e_r = \kappa(\mathcal{C}). \end{aligned}$$

(2) If $o_r \ge e_r + 2$, then $o'_r = o_r - 1 \ge e_r + 1 = e'_r$ and

$$\begin{aligned} \kappa(\mathcal{C}) &= o_s + 2e_s + o_r, \\ \kappa(\mathcal{C}') &= o_s + 2e_s - 1 + o'_r = o_s + 2e_s + o_r - 2 < \kappa(\mathcal{C}). \end{aligned}$$

Case II. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2$.

(1) If $o_r \le e_r + 1$, then $o'_r < e'_r$ and

$$\begin{aligned} \kappa(\mathcal{C}) &= o_s + 2e_s + e_r + 1, \\ \kappa(\mathcal{C}') &= o_s + 2e_s - 1 + e'_r + 1 = o_s + 2e_s + e_r + 1 = \kappa(\mathcal{C}). \end{aligned}$$

(2) If $o_r \ge e_r + 3$, then $o'_r = o_r - 1 \ge e_r + 2 > e'_r$ and

$$\begin{aligned} \kappa(\mathcal{C}) &= o_s + 2e_s + o_r, \\ \kappa(\mathcal{C}') &= o_s + 2e_s - 1 + o'_r = o_s + 2e_s + o_r - 2 < \kappa(\mathcal{C}). \end{aligned}$$

Therefore, the inequality $\kappa(\mathcal{C}') \le \kappa(\mathcal{C})$ holds under the assumption. ∎

Lemma 5.3 *If* $\mathcal{C}' = \mathcal{C}\left(\begin{bmatrix} {\scriptstyle (o,e)} \\ {\scriptstyle (\emptyset,e)} \end{bmatrix} \to \begin{bmatrix} {\scriptstyle (\emptyset,o)} \\ {\scriptstyle (o,o)} \end{bmatrix}\right)$*, then* $\kappa(\mathcal{C}') \le \kappa(\mathcal{C})$*.*

Proof. In this case we have

$$o'_s = o_s + 1, \quad e'_s = e_s - 1, \quad o'_r = o_r + 1, \quad e'_r = e_r - 1.$$

Now there are two possibilities.
Case I. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 0 \pmod 2$.

(1) If $o_r \le e_r - 2$, then $o'_r \le e'_r$ and

$$\begin{aligned}\kappa(\mathcal{C}) &= o_s + 2e_s + e_r,\\ \kappa(\mathcal{C}') &= o_s + 1 + 2(e_s - 1) + e'_r = o_s + 2e_s + e_r - 2 < \kappa(\mathcal{C}).\end{aligned}$$

(2) If $o_r \ge e_r$, then $o'_r = o_r + 1 > e_r - 1 = e'_r$ and

$$\begin{aligned}\kappa(\mathcal{C}) &= o_s + 2e_s + o_r,\\ \kappa(\mathcal{C}') &= o_s + 2e_s - 1 + o'_r = o_s + 2e_s + o_r = \kappa(\mathcal{C}).\end{aligned}$$

Case II. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2$.

(1) If $o_r \le e_r - 3$, then $o'_r < e'_r$ and

$$\begin{aligned}\kappa(\mathcal{C}) &= o_s + 2e_s + e_r + 1,\\ \kappa(\mathcal{C}') &= o_s + 2e_s - 1 + e'_r + 1 = o_s + 2e_s + e_r - 1 < \kappa(\mathcal{C}).\end{aligned}$$

(2) If $o_r \ge e_r - 1$, then $o'_r = o_r + 1 > e_r - 1 = e'_r$ and

$$\begin{aligned}\kappa(\mathcal{C}) &= o_s + 2e_s + o_r,\\ \kappa(\mathcal{C}') &= o_s + 2e_s - 1 + o'_r = o_s + 2e_s + o_r = \kappa(\mathcal{C}).\end{aligned}$$

In both cases we have $\kappa(\mathcal{C}') \le \kappa(\mathcal{C})$, as required. ∎

Lemma 5.4 *If* $\mathcal{C}' = \mathcal{C}\left(\begin{bmatrix} {\scriptstyle (e,e)} \\ {\scriptstyle (o,e)} \end{bmatrix} \to \begin{bmatrix} {\scriptstyle (o,o)} \\ {\scriptstyle (e,o)} \end{bmatrix}\right)$, *then* $\kappa(\mathcal{C}') < \kappa(\mathcal{C})$.

Proof. In this case, we have

$$o'_s = o_s + 2, \quad e'_s = e_s - 2, \quad o'_r = o_r, \quad e'_r = e_r.$$

Therefore,

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + \max(o'_r, e'_r + ((e'_r + o'_r) \bmod 2)) = \kappa(\mathcal{C}) - 2.$$

The desired inequality immediately follows. ∎

Lemma 5.5 *If* $\mathcal{C}' = \mathcal{C}\left(\begin{bmatrix} {\scriptstyle (e,o)} \\ {\scriptstyle (\emptyset,e)} \end{bmatrix} \to \begin{bmatrix} {\scriptstyle (\emptyset,o)} \\ {\scriptstyle (e,\emptyset)} \end{bmatrix}\right)$, *then* $\kappa(\mathcal{C}') \le \kappa(\mathcal{C})$.

Proof. Under this transformation we have

$$o'_s = o_s - 1, \quad e'_s = e_s, \quad o'_r = o_r + 1, \quad e'_r = e_r - 1.$$

Since $o_r + e_r = \ell(\lambda) - \ell(\mu)$ is invariant, there are two cases.
Case I. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 0 \pmod 2$.

(1) If $o_r \geq e_r$, then $o'_r \geq e'_r$ and

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + o'_r = o_s - 1 + 2e_s + o_r + 1 = \kappa(\mathcal{C}).$$

(2) If $o_r \leq e_r - 2$, then $o'_r = o_r + 1 \leq e_r - 1 = e'_r$ and

$$\kappa(\mathcal{C}') = o_s - 1 + 2e_s + e'_r = o_s + 2e_s + e_r - 2 < \kappa(\mathcal{C}).$$

Case II. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2$.

(1) If $o_r \geq e_r + 1$, then $o'_r = o_r + 1 \geq e_r + 2 > e'_r + 1$ and

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + o'_r = o_s + 2e_s + o_r = \kappa(\mathcal{C}).$$

(2) If $o_r \leq e_r - 1$, then $o'_r = o_r + 1 \leq e_r = e'_r + 1$ and

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + e'_r + 1 = o_s + 2e_s + e_r - 1 < \kappa(\mathcal{C}).$$

Hence the proof is complete. ∎

Lemma 5.6 *If* $\mathcal{C}' = \mathcal{C}\left(\begin{bmatrix} {\scriptstyle (o,o)} \\ {\scriptstyle (\emptyset,o)} \end{bmatrix} \to \begin{bmatrix} {\scriptstyle (\emptyset,e)} \\ {\scriptstyle (o,\emptyset)} \end{bmatrix}\right)$, *then* $\kappa(\mathcal{C}') \leq \kappa(\mathcal{C})$.

Proof. In this case we have

$$o'_s = o_s - 1, \quad e'_s = e_s, \quad o'_r = o_r - 1, \quad e'_r = e_r + 1.$$

There are two possibilities:
Case I. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 0 \pmod 2$.

(1) If $o_r \geq e_r + 2$, then $o'_r = o_r - 1 \geq e_r + 1 = e'_r$ and

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + o'_r = o_s - 1 + 2e_s + o_r - 1 < \kappa(\mathcal{C}).$$

(2) If $o_r \leq e_r$, then $o'_r = o_r - 1 \leq e_r - 1 < e'_r$ and

$$\kappa(\mathcal{C}') = o_s - 1 + 2e_s + e'_r = o_s + 2e_s + e_r = \kappa(\mathcal{C}).$$

Case II. The skew partition λ/μ satisfies that $\ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2$.

(1) If $o_r \geq e_r + 3$, then $o'_r = o_r - 1 \geq e_r + 2 = e'_r + 1$ and

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + o'_r = o_s + 2e_s + o_r - 2 < \kappa(\mathcal{C}).$$

(2) If $o_r \le e_r + 1$, then $o'_r = o_r - 1 \le e_r < e'_r + 1$ and

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + e'_r + 1 = o_s + 2e_s + e_r + 1 = \kappa(\mathcal{C}).$$

Therefore, in both cases we have $\kappa(\mathcal{C}') \le \kappa(\mathcal{C})$. ∎

Lemma 5.7 *If* $\mathcal{C}' = \mathcal{C}\left(\begin{bmatrix} (e,o) \\ (o,o) \end{bmatrix} \to \begin{bmatrix} (o,e) \\ (e,\emptyset) \end{bmatrix}\right)$ *or* $\mathcal{C}' = \mathcal{C}\left(\begin{bmatrix} (o,o) \\ (e,o) \end{bmatrix} \to \begin{bmatrix} (e,e) \\ (o,\emptyset) \end{bmatrix}\right)$, *then* $\kappa(\mathcal{C}') = \kappa(\mathcal{C})$.

Proof. In each case we have

$$o'_s = o_s - 2, \quad e'_s = e_s + 1, \quad o'_r = o_r, \quad e'_r = e_r.$$

Therefore

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + \max(o'_r, e'_r + ((e'_r + o'_r) \bmod 2)) = \kappa(\mathcal{C}),$$

as desired. ∎

Lemma 5.8 *If* $\mathcal{C}'$ *is one of the following possible cases:*

$$\mathcal{C}\left(\begin{bmatrix} (e,e) \\ (e,e) \end{bmatrix} \to \begin{bmatrix} (e,e) \\ (e,\emptyset) \end{bmatrix}\right), \ \mathcal{C}\left(\begin{bmatrix} (e,e) \\ (o,o) \end{bmatrix} \to \begin{bmatrix} (o,o) \\ (e,\emptyset) \end{bmatrix}\right), \ \mathcal{C}\left(\begin{bmatrix} (e,o) \\ (e,e) \end{bmatrix} \to \begin{bmatrix} (e,o) \\ (e,\emptyset) \end{bmatrix}\right),$$
$$\mathcal{C}\left(\begin{bmatrix} (e,e) \\ (\emptyset,e) \end{bmatrix} \to \begin{bmatrix} (\emptyset,e) \\ (e,\emptyset) \end{bmatrix}\right), \ \mathcal{C}\left(\begin{bmatrix} (o,o) \\ (o,e) \end{bmatrix} \to \begin{bmatrix} (o,o) \\ (o,\emptyset) \end{bmatrix}\right), \ \mathcal{C}\left(\begin{bmatrix} (o,e) \\ (e,o) \end{bmatrix} \to \begin{bmatrix} (e,o) \\ (o,\emptyset) \end{bmatrix}\right),$$
$$\mathcal{C}\left(\begin{bmatrix} (o,e) \\ (o,e) \end{bmatrix} \to \begin{bmatrix} (o,e) \\ (o,\emptyset) \end{bmatrix}\right), \ \mathcal{C}\left(\begin{bmatrix} (o,e) \\ (\emptyset,o) \end{bmatrix} \to \begin{bmatrix} (\emptyset,o) \\ (o,\emptyset) \end{bmatrix}\right),$$

then $\kappa(\mathcal{C}') < \kappa(\mathcal{C})$.

Proof. In each case we have

$$o'_s = o_s, \quad e'_s = e_s - 1, \quad o'_r = o_r, \quad e'_r = e_r.$$

Therefore

$$\kappa(\mathcal{C}') = o'_s + 2e'_s + \max(o'_r, e'_r + ((e'_r + o'_r) \bmod 2)) < \kappa(\mathcal{C}),$$

as required. ∎

Note that Lemmas 5.1-5.8 cover all possible transformations of exchanging the locations of 0's in two involved rows. Lemmas 5.2-5.4 imply that, to minimize the number of bars, we should put 0's in the skew shifted diagram such that there are as more as possible rows for which the first several squares are filled with 0's and then followed by an odd number of blank squares. Meanwhile, from Lemmas 5.5-5.8 we know that the number of rows fully filled with 0's should be as more as possible. Based on these observations, we have the following algorithm to determine the location of 0's for a given skew partition λ/μ, where both λ and μ are strict partitions. Using this algorithm we will obtain a shifted diagram with some squares filled with 0's such that the corresponding quantity $\kappa(\mathcal{C})$ is minimized. This property allows us to determine the srank of λ/μ.

The Algorithm for Determining the Locations of 0's:

(S1) Let $\mathcal{C}_1 = S(\lambda)$ be the initial configuration of λ/μ with blank square. Set $i = 1$ and $J = \{1, \ldots, \ell(\lambda)\}$.

(S2) For $i \leq \ell(\mu)$, iterate the following procedure:

- (A) If $\mu_i = \lambda_j$ for some $j \in J$, then we fill the j-th row of $\mathcal{C}_i$ with 0.
- (B) If $\mu_i \neq \lambda_j$ for any $j \in J$, then there are two possibilities.
  - (B1) $\lambda_j - \mu_i$ is odd for some $j \in J$ and $\lambda_j > \mu_i$. Then we take the largest such j and fill the leftmost μ_i squares with 0 in the j-th row of $\mathcal{C}_i$.
  - (B2) $\lambda_j - \mu_i$ is even for any $j \in J$ and $\lambda_j > \mu_i$. Then we take the largest such j and fill the leftmost μ_i squares by 0 in the j-th row of $\mathcal{C}_i$.

Denote the new configuration by $\mathcal{C}_{i+1}$. Set $J = J \backslash \{j\}$.

(S3) Set $\mathcal{C}^* = \mathcal{C}_i$, and we get the desired configuration.

It should be emphasized that although the above algorithm does not necessarily generate a bar tableau, it is sufficient for the computation of the srank of a skew partition.

Using the arguments in the proofs of Lemmas 5.1-5.8, we can derive the following crucial property of the configuration $\mathcal{C}^*$. The proof is omitted since it is tedious and straightforward.

Proposition 5.9 *For any configuration $\mathcal{C}$ of 0's in the skew shifted diagram of λ/μ, we have $\kappa(\mathcal{C}^*) \leq \kappa(\mathcal{C})$.*

Theorem 5.10 *Given a skew partition λ/μ, let $\mathcal{C}^*$ be the configuration of 0's obtained by applying the algorithm described above. Then*

$$\mathrm{srank}(\lambda/\mu) = \kappa(\mathcal{C}^*). \tag{5.18}$$

Proof. Suppose that for the configuration $\mathcal{C}^*$ there are o_r^* rows of odd size with blank squares, and there are o_s^* rows with at least one square filled with

0 and an odd number of squares filled with positive integers. Likewise we let e_r^* and e_s^* denote the number of remaining rows. Therefore,

$$\kappa(\mathcal{C}^*) = o_s^* + 2e_s^* + \max(o_r^*, e_r^* + ((e_r^* + o_r^*) \bmod 2)).$$

Since for each configuration $\mathcal{C}$ the number of bars in a minimal bar tableau is greater than or equal to $\kappa(\mathcal{C})$, by Proposition 5.9, it suffices to confirm the existence of a skew bar tableau, say T, with $\kappa(\mathcal{C}^*)$ bars.

Note that it is possible that the configuration $\mathcal{C}^*$ is not admissible. The key idea of our proof is to move 0's in the diagram such that the resulting configuration $\mathcal{C}'$ is admissible and $\kappa(\mathcal{C}') = \kappa(\mathcal{C}^*)$. To achieve this goal, we will use the numbers $\{1, 2, \ldots, \kappa(\mathcal{C}^*)\}$ to fill up the blank squares of $\mathcal{C}^*$ guided by the rule that the bars of Type 2 or Type 3 will occur before bars of Type 1.

Let us consider the rows without 0's, and there are two possibilities: (A) $o_r^* \geq e_r^*$, (B) $o_r^* < e_r^*$.

In Case (A) we choose a row of even size and a row of odd size, and fill up these two rows with $\kappa(\mathcal{C}^*)$ to generate a bar of Type 3. Then we continue to choose a row of even size and a row of odd size, and fill up these two rows with $\kappa(\mathcal{C}^*) - 1$. Repeat this procedure until all even rows are filled up. Finally, we fill the remaining rows of odd size with $\kappa(\mathcal{C}^*) - e_r^*, \kappa(\mathcal{C}^*) - e_r^* - 1, \ldots, \kappa(\mathcal{C}^*) - o_r^* + 1$ to generate bars of Type 2.

In Case (B) we choose the row with the i-th smallest even size and the row with the i-th smallest odd size and fill their squares with the number $\kappa(\mathcal{C}^*) - i + 1$ for $i = 1, \ldots, o_r^*$. In this way, we obtain o_r^* bars of Type 3. Now consider the remaining rows of even size without 0's. There are two subcases.

(B1) The remaining diagram, obtained by removing the previous o_r^* bars of Type 3, does not contain any row with only one square. Under this assumption, it is possible to fill the squares of a row of even size with the number $\kappa(\mathcal{C}^*) - o_r^*$ except the leftmost square. This operation will result in a bar of Type 1. After removing this bar from the diagram, we may combine this leftmost square of the current row and another row of even size, if it exists, and to generate a bar of Type 3. Repeating this procedure until there are no more rows of even size, we obtain a sequence of bars of Type 1 and Type 3. Evidently, there is a bar of Type 2 with only one square. To summarize, we have $\max(o_r^*, e_r^* + ((e_r^* + o_r^*) \bmod 2))$ bars.

(B2) The remaining diagram contains a row composed of the unique square filled with 0. In this case, we will move this 0 into the leftmost square of a row of even size, see Figure 6. Denote this new configuration by $\mathcal{C}'$, and from Lemma 5.6 we see that $\kappa(\mathcal{C}^*) = \kappa(\mathcal{C}')$. If we start with $\mathcal{C}'$ instead of $\mathcal{C}^*$, by a similar construction, we get $\max(o_r', e_r' + ((e_r' + o_r') \bmod 2))$ bars, occupying the rows without 0's in the diagram.

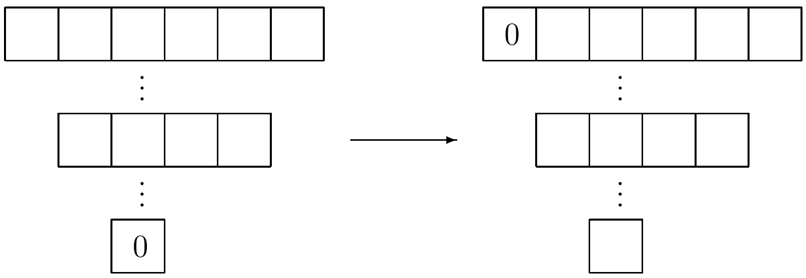

Figure 6: Vacating the unique square at the bottom of the diagram

Without loss of generality, we may assume that for the configuration $\mathcal{C}^*$ the rows without 0's in the diagram have been occupied by the bars with the first $\max(o_r^*, e_r^* + ((e_r^* + o_r^*) \bmod 2))$ positive integers in the decreasing order, namely, $(\kappa(\mathcal{C}^*), \ldots, 2, 1, 0)$. By removing these bars and reordering the remaining rows, we may get a shifted diagram with which we can continue the above procedure to construct a bar tableau.

At this point, it is necessary to show that it is possible to use $o_s^* + 2e_s^*$ bars to fill this diagram. In doing so, we process the rows from bottom to top. If the bottom row has an odd number of blank squares, then we simply assign the symbol $o_s^* + 2e_s^*$ to these squares to produce a bar of Type 1. If the bottom row are completely filled with 0's, then we continue to deal with the row above the bottom row. Otherwise, we fill the rightmost square of the bottom row with $o_s^* + 2e_s^*$ and the remaining squares with $o_s^* + 2e_s^* - 1$. Suppose that we have filled i rows from the bottom and all the involved bars have been removed from the diagram. Then we consider the $(i+1)$-th row from the bottom. Let t denote the largest number not greater than $o_s^* + 2e_s^*$ which has not been used before. If all squares in the $(i+1)$-th row are filled with 0's, then we continue to deal with the $(i+2)$-th row. If the number of blank squares in the $(i+1)$-th row is odd, then we fill these squares with t. If the number of blank squares in the $(i+1)$-th row is even, then we are left with two cases:

(A') The rows of the diagram obtained by removing the rightmost square of the $(i+1)$-th row have distinct lengths. In this case, we fill the rightmost square with t and the remaining blank squares of the $(i+1)$-th row with $t-1$.

(B') The removal of the rightmost square of the $(i+1)$-th row does not result in a bar tableau. Suppose that the $(i+1)$-th row has m squares in total. It can only happen that the row underneath the $(i+1)$-th row has $m-1$ squares and all these squares are filled with 0's. By interchanging the location of 0's in these two rows, we get a new configuration $\mathcal{C}'$, see

Figure 7. From Lemma 5.7 we deduce that $\kappa(\mathcal{C}^*) = \kappa(\mathcal{C}')$. So we can transform $\mathcal{C}^*$ to $\mathcal{C}'$ and continue to fill up the $(i+1)$-th row.

$$\begin{array}{|c|c|c|c|c|c|c|} \hline 0 & 0 & 0 & & & & \\ \hline \multicolumn{1}{c|}{} & 0 & 0 & 0 & 0 & 0 & 0 \\ \cline{2-7} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline \multicolumn{1}{c|}{} & 0 & 0 & 0 & & & \\ \cline{2-7} \end{array}$$

Figure 7: Interchanging the location of 0's in two neighbored rows

Finally, we arrive at a shifted diagram whose rows are all filled up. Clearly, for those rows containing at least one 0 there are $o_s^* + 2e_s^*$ bars that are generated in the construction, and for those rows containing no 0's there are $\max(o_r^*, e_r^* + ((e_r^* + o_r^*) \bmod 2))$ bars that are generated. It has been shown that during the procedure of filling the diagram with nonnegative numbers if the configuration $\mathcal{C}^*$ is transformed to another configuration $\mathcal{C}'$, then $\kappa(\mathcal{C}')$ remains equal to $\kappa(\mathcal{C}^*)$. Hence the above procedure leads to a skew bar tableau of shape λ/μ with $\kappa(\mathcal{C}^*)$ bars. This completes the proof. ∎

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