A Telescoping Algorithm for Double Summations

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Abstract

We present an algorithm to prove hypergeometric double summation identities. Given a hypergeometric term $F(n,i,j)$, we aim to find a difference operator $L = a_0(n)N^0 + a_1(n)N^1 + \cdots + a_r(n)N^r$ and rational functions $R_1(n,i,j), R_2(n,i,j)$ such that $LF = \Delta_i(R_1F) + \Delta_j(R_2F)$. Based on simple divisibility considerations, we show that the denominators of R_1 and R_2 must possess certain factors which can be computed from $F(n,i,j)$. Using these factors as estimates, we may find the numerators of R_1 and R_2 by guessing the upper bounds of the degrees and solving systems of linear equations. Our algorithm is valid for the Andrews-Paule identity, the Carlitz's identities, the Apéry-Schmidt-Strehl identity, the Graham-Knuth-Patashnik identity, and the Petkovšek-Wilf-Zeilberger identity.

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1. Introduction

This paper is concerned with double summations of hypergeometric terms $F(n,i,j)$. A function $F(n,k_1,\ldots,k_m)$ is called a *hypergeometric term* if the quotients

$$\frac{F(n+1,k_1,\ldots,k_m)}{F(n,k_1,\ldots,k_m)}, \quad \frac{F(n,k_1+1,\ldots,k_m)}{F(n,k_1,\ldots,k_m)}, \quad \ldots, \quad \frac{F(n,k_1,\ldots,k_m+1)}{F(n,k_1,\ldots,k_m)}$$

are rational functions of $n,k_1,\ldots,k_m$. Throughout the paper, N denotes the shift operator with respect to the variable n, defined by

$$NF(n,k_1,\ldots,k_m) = F(n+1,k_1,\ldots,k_m),$$

and Δ_{k_t} denotes the difference operator with respect to the variable k_t, defined by

$$\Delta_{k_t} F(n, k_1, \ldots, k_m) = F(n, k_1, \ldots, k_t + 1, \ldots, k_m) - F(n, k_1, \ldots, k_t, \ldots, k_m).$$

We also use Δ_i and Δ_j to denote the difference operators with respect to the variables i and j, respectively. For polynomials a and b, we denote by $\gcd(a, b)$ their monic greatest common divisor. When we express a rational function as a quotient p/q, we always assume that p and q are relatively prime unless it is explicitly stated otherwise.

Zeilberger's algorithm [7, 9, 13], also known as the method of *creative telescoping*, is devised for proving hypergeometric identities of the form

$$\sum_k F(n, k) = f(n), \tag{1.1}$$

where $F(n, k)$ is a hypergeometric term and $f(n)$ is a given function. This algorithm has been used to deal with multiple sums by Wilf and Zeilberger [12]. Given a hypergeometric term $F(n, k_1, \ldots, k_m)$, the approach of Wilf and Zeilberger is to try to find a linear difference operator L with coefficients being polynomials in n

$$L = a_0(n)N^0 + a_1(n)N^1 + \cdots + a_r(n)N^r$$

and rational functions $R_1, \ldots, R_m$ of $n, k_1, \ldots, k_m$ such that

$$LF = \sum_{l=1}^{m} \Delta_{k_l}(R_l F). \tag{1.2}$$

The problem of constructing the denominators of $R_1, \ldots, R_m$ for the Wilf-Zeilberger approach has not been solved. In an alternative approach, Wegschaider generalized Sister Celine's technique [11] to multiple summations, and proved many double summation identities. In this paper, we provide estimates of the denominators of R_1 and R_2 for double summations. These estimates turn out to be good enough for several double summation identities, including the Andrews-Paule identity which does not seem to be a suitable example for Wegschaider's approach.

To give a sketch of our approach, we first consider Gosper's algorithm for bivariate hypergeometric terms. Suppose that $F(i,j)$ is a hypergeometric term and $p_1/q_1, p_2/q_2$ are rational functions such that

$$F(i,j) = \Delta_i \left(\frac{p_1(i,j)}{q_1(i,j)} F(i,j) \right) + \Delta_j \left(\frac{p_2(i,j)}{q_2(i,j)} F(i,j) \right).$$

We show that under certain hypotheses (Section 2, $(H1)$–$(H3)$), the denominators q_1, q_2 can be written in the form

$$\begin{aligned} q_1(i,j) &= v_1(i)\, v_2(j)\, v_3(i+j)\, v_4(i,j)\, u_1(j)\, u_2(i,j), \\ q_2(i,j) &= v_1(i)\, v_2(j)\, v_3(i+j)\, v_4(i,j)\, w_1(j)\, w_2(i,j), \end{aligned} \tag{1.3}$$

such that v_1, v_2, v_4 and u_2, w_2 are bounded in the sense that they are factors of certain polynomials. Then we apply these estimates to the telescoping algorithm for double summations. Suppose that

$$LF(n,i,j) = \Delta_i\big(R_1(n,i,j)F(n,i,j)\big) + \Delta_j\big(R_2(n,i,j)F(n,i,j)\big),$$

where

$$R_1(n,i,j) = \frac{1}{d(n,i,j)} \cdot \frac{f_1(n,i,j)}{g_1(n,i,j)}, \qquad R_2(n,i,j) = \frac{1}{d(n,i,j)} \cdot \frac{f_2(n,i,j)}{g_2(n,i,j)}$$

and $d(n,i,j)$ is the denominator of $LF(n,i,j)/F(n,i,j)$. We may deduce that g_1, g_2 can be factored in the form of (1.3) such that v_1, v_2, v_4 and u_2, w_2 are bounded. Although we do not have the universal denominators, these bounds can be used to give estimates of the denominators g_1 and g_2. Then by further guessing the bounds of the degrees of the numerators of R_1 and R_2, we get the desired difference operator if we are lucky.

Indeed, our approach works quite efficiently for many identities such as the Andrews-Paule identity, Carlitz's identities, the Apéry-Schmidt-Strehl identity, the Graham-Knuth-Patashnik identity, and the Petkovšek-Wilf-Zeilberger identity.

2. Denominators in Bivariate Gosper's Algorithm

For a given bivariate hypergeometric term $F(i,j)$, we give estimates of the denominators of the rational functions $R_1(i,j), R_2(i,j)$ satisfying

$$F(i,j) = \Delta_i\big(R_1(i,j)F(i,j)\big) + \Delta_j\big(R_2(i,j)F(i,j)\big). \tag{2.1}$$

Let

$$R_1(i,j) = \frac{f_1(i,j)}{g_1(i,j)}, \qquad R_2(i,j) = \frac{f_2(i,j)}{g_2(i,j)},$$
$$\frac{F(i+1,j)}{F(i,j)} = \frac{r_1(i,j)}{s_1(i,j)}, \quad \frac{F(i,j+1)}{F(i,j)} = \frac{r_2(i,j)}{s_2(i,j)}. \tag{2.2}$$

Dividing $F(i,j)$ on both sides of (2.1) and substituting (2.2) into it, we derive that

$$1 = \frac{r_1(i,j)}{s_1(i,j)} \frac{f_1(i+1,j)}{g_1(i+1,j)} - \frac{f_1(i,j)}{g_1(i,j)} + \frac{r_2(i,j)}{s_2(i,j)} \frac{f_2(i,j+1)}{g_2(i,j+1)} - \frac{f_2(i,j)}{g_2(i,j)}. \tag{2.3}$$

Let

$$u(i,j) = \gcd(s_1(i,j), s_2(i,j)), \qquad v(i,j) = \gcd(g_1(i,j), g_2(i,j)),$$

and

$$s_1'(i,j) = s_1(i,j)/u(i,j), \qquad s_2'(i,j) = s_2(i,j)/u(i,j),$$
$$g_1'(i,j) = g_1(i,j)/v(i,j), \qquad g_2'(i,j) = g_2(i,j)/v(i,j). \tag{2.4}$$

We concern on those $R_1(i,j), R_2(i,j)$ whose denominators g_1, g_2 satisfy the following three hypotheses.

- $(H1)$ For any irreducible factor $p(i,j)$ of $g_1(i,j)$ ($g_2(i,j)$, respectively) and integers h_1, h_2, $p(i+h_1, j+h_2)$ divides $g_1(i,j)$ ($g_2(i,j)$, respectively) implies $p(i+h_1, j+h_2) = p(i,j)$.
- $(H2)$ $\gcd(g_1'(i,j), v(i,j)) = \gcd(g_2'(i,j), v(i,j)) = 1$.
- $(H3)$ For any integers h_1, h_2, $\gcd(g_1'(i+h_1, j+h_2), g_2'(i,j)) = 1$.

Under these hypotheses, we have

Theorem 2.1 *The denominators $g_1(i,j), g_2(i,j)$ can be factored into polynomials:*

$$g_1(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)u_1(j)u_2(i,j),$$
$$g_2(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)w_1(i)w_2(i,j),$$

such that

$$
\begin{aligned}
&v_1(i) \,|\, r_1(i-1,j)s_2'(i-1,j),\\
&v_2(j) \,|\, r_2(i,j-1)s_1'(i,j-1),\\
&v_4(i,j) \,|\, \gcd\big(r_1(i-1,j)s_2'(i-1,j), r_2(i,j-1)s_1'(i,j-1)\big),\\
&u_2(i,j) \,|\, \gcd\big(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j)\big),\\
&w_2(i,j) \,|\, \gcd\big(s_2(i,j)s_1'(i,j), r_2(i,j-1)s_1'(i,j-1)\big).
\end{aligned}
$$

Proof. Substituting (2.4) into (2.3), we get

$$
\begin{aligned}
1 \;=\;& \frac{r_1(i,j)}{s_1'(i,j)u(i,j)}\frac{f_1(i+1,j)}{g_1'(i+1,j)v(i+1,j)} - \frac{f_1(i,j)}{g_1'(i,j)v(i,j)}\\
&+\frac{r_2(i,j)}{s_2'(i,j)u(i,j)}\frac{f_2(i,j+1)}{g_2'(i,j+1)v(i,j+1)} - \frac{f_2(i,j)}{g_2'(i,j)v(i,j)}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
& s_1(i,j)s_2'(i,j)g_1(i,j)g_2'(i,j)g_1(i+1,j)g_2(i,j+1)\\
&\quad= f_1(i+1,j)r_1(i,j)s_2'(i,j)g_1(i,j)g_2'(i,j)g_2(i,j+1)\\
&\qquad -f_1(i,j)s_1(i,j)s_2'(i,j)g_2'(i,j)g_1(i+1,j)g_2(i,j+1)\\
&\qquad +f_2(i,j+1)r_2(i,j)s_1'(i,j)g_1(i,j)g_2'(i,j)g_1(i+1,j)\\
&\qquad -f_2(i,j)s_1(i,j)s_2'(i,j)g_1'(i,j)g_1(i+1,j)g_2(i,j+1).
\end{aligned}
$$

1. Suppose $p(i,j)$ is an irreducible factor of $v(i,j)$, and for some non-negative integer l, $p^l \,|\, v$. Since

   $$\gcd(p(i+1,j), f_1(i+1,j)) = \gcd(p(i,j+1), f_2(i,j+1)) = 1,$$

   we have

   $$p^l(i+1,j) \,|\, r_1(i,j)s_2'(i,j)g_1(i,j)g_2'(i,j)g_2(i,j+1)$$

   and

   $$p^l(i,j+1) \,|\, r_2(i,j)s_1'(i,j)g_1(i,j)g_2'(i,j)g_1(i+1,j).$$

   There are three cases:

- $p(i,j)$ is a polynomial depending only on i. Then $\gcd(p(i+1,j), g_1(i,j)) = 1$. Otherwise, by hypothesis $(H1)$ we have that $p(i+1,j) = p(i,j)$ is independent of i, which is a contradiction. Similarly, $\gcd(p(i+1,j), g_2(i,j)) = 1$. Since $p(i,j)$ is a polynomial depending only on i, we have

  $$\gcd(p(i+1,j), g_2(i,j+1)) = \gcd(p(i+1,j+1), g_2(i,j+1)) = 1.$$

  Therefore,

  $$p^l(i+1,j) \mid r_1(i,j)s_2'(i,j).$$

- $p(i,j)$ is a polynomial depending only on j. The same discussion leads to

  $$p^l(i,j+1) \mid r_2(i,j)s_1'(i,j).$$

- $p(i,j)$ is a polynomial depending both on i and on j. Then either

  $$p(i+1,j) = p(i,j+1) \quad \text{or} \quad \gcd(p(i+1,j), p(i,j+1)) = 1.$$

  In the former case, $p(i,j)$ is a polynomial of $i+j$ (see [1, Lemma 3] or [8, Lemma 3.3]). In the later case, by hypothesis $(H1)$, we have

  $$\gcd(p(i+1,j), g_1(i,j)g_2'(i,j)g_2(i,j+1)) = 1$$

  and

  $$\gcd(p(i,j+1), g_1(i,j)g_2'(i,j)g_1(i+1,j)) = 1.$$

  Thus,

  $$p^l(i,j) \mid \gcd\big(r_1(i-1,j)s_2'(i-1,j), r_2(i,j-1)s_1'(i,j-1)\big).$$

2. Suppose p is an irreducible factor of g_1' and $p^l | g_1'$ for some non-negative integer l. If $p(i,j) \mid v(i,j+1)$, then $p(i,j-1) \mid v(i,j)$. By hypothesis $(H1)$, $p(i,j-1) = p(i,j)$, which implies $p(i,j) \mid v(i,j)$, contradicting to hypothesis $(H2)$. Noting further that

   $$\gcd(f_1(i,j), g_1(i,j)) = \gcd(g_1'(i,j), g_2'(i+h_1, j+h_2)) = 1, \quad \forall\, h_1, h_2 \in \mathbb{Z},$$

   we have

   $$p^l(i,j) \mid s_1(i,j)s_2'(i,j)g_1(i+1,j).$$

If $p(i+1,j) \,|\, v(i,j+1)$, then by hypothesis $(H1)$, $p(i+1,j-1) = p(i,j)$, which implies $p(i,j) \,|\, v(i,j)$, contradicting to hypothesis $(H2)$. Therefore,

$$p^l(i+1,j) \,|\, r_1(i,j)s_2'(i,j)g_1(i,j).$$

There are also two cases:

- $p(i,j) = p(i+1,j)$. Then $p(i,j)$ is a polynomial depending only on j.
- $\gcd(p(i,j), p(i+1,j)) = 1$. Then

  $$\gcd(p(i,j), g_1(i+1,j)) = \gcd(p(i+1,j), g_1(i,j)) = 1,$$

  and hence,

  $$p^l(i,j) \,|\, \gcd\big(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j)\big).$$

3. Similarly, suppose p is an irreducible factor of g_2' and $p^l | g_2'$ for some non-negative integer l. Then either $p(i,j)$ is a polynomial depending only on i or

$$p^l(i,j) \,|\, \gcd\big(s_1(i,j)s_2'(i,j), r_2(i,j-1)s_1'(i,j-1)\big).$$

∎

3. Denominators in the Telescoping Algorithm

We are now ready to estimate the denominators of R_1 and R_2 in telescoping algorithm.

As in the case of single summations, the telescoping algorithm for double summations tries to find an operator

$$L = a_0(n) + a_1(n)N + \cdots a_r(n)N^r$$

and rational functions $R_1(n,i,j), R_2(n,i,j)$ such that

$$LF(n,i,j) = \Delta_i(R_1(n,i,j)F(n,i,j)) + \Delta_j(R_2(n,i,j)F(n,i,j)). \qquad (3.1)$$

Let

$$\frac{F(n,i+1,j)}{F(n,i,j)} = \frac{r_1(n,i,j)}{s_1(n,i,j)}, \qquad \frac{F(n,i,j+1)}{F(n,i,j)} = \frac{r_2(n,i,j)}{s_2(n,i,j)}, \tag{3.2}$$

and $d(n,i,j)$ be the common denominator of

$$\frac{F(n+1,i,j)}{F(n,i,j)}, \quad \ldots, \quad \frac{F(n+r,i,j)}{F(n,i,j)}.$$

Then there exists a polynomial $c(n,i,j)$, not necessarily being coprime to d, such that

$$\frac{LF(n,i,j)}{F(n,i,j)} = \sum_{l=0}^{r} a_l(n) \frac{F(n+l,i,j)}{F(n,i,j)} = \frac{c(n,i,j)}{d(n,i,j)}. \tag{3.3}$$

Note that c is related to the polynomials $a_0, a_1, \ldots, a_r$ but d is independent of them.

Now, (3.1) can be written in the form of (2.1):

$$LF(n,i,j) = \Delta_i(R_1'(n,i,j)LF(n,i,j)) + \Delta_j(R_2'(n,i,j)LF(n,i,j)),$$

where

$$R_1'(n,i,j) = R_1(n,i,j)\frac{d(n,i,j)}{c(n,i,j)} \quad \text{and} \quad R_2'(n,i,j) = R_2(n,i,j)\frac{d(n,i,j)}{c(n,i,j)}.$$

This suggests us to assume

$$R_1(n,i,j) = \frac{1}{d(n,i,j)} \frac{f_1(n,i,j)}{g_1(n,i,j)} \quad \text{and} \quad R_2(n,i,j) = \frac{1}{d(n,i,j)} \frac{f_2(n,i,j)}{g_2(n,i,j)}, \tag{3.4}$$

where f_1, g_1 (f_2, g_2, respectively) are relatively prime polynomials.

Since the following discussion is independent of n, we omit the variable n for convenience. For example, we write $R_1(i,j)$ instead of $R_1(n,i,j)$. Using these notations, we have

Theorem 3.1 *Suppose the polynomials g_1, g_2 in (3.4) satisfy the hypotheses $(H1)$–$(H3)$. Suppose further that*

$$\gcd(g_1(i,j), d(i+h_1, j+h_2)) = \gcd(g_2(i,j), d(i+h_1, j+h_2)) = 1, \quad \forall\, h_1, h_1 \in \mathbb{Z}. \tag{3.5}$$

Then $g_1(i,j), g_2(i,j)$ *can be factored into polynomials:*

$$g_1(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)u_1(j)u_2(i,j),$$
$$g_2(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)w_1(i)w_2(i,j),$$

such that

$$\begin{aligned}
&v_1(i) \mid r_1(i-1,j)s_2'(i-1,j),\\
&v_2(j) \mid r_2(i,j-1)s_1'(i,j-1),\\
&v_4(i,j) \mid \gcd\big(r_1(i-1,j)s_2'(i-1,j), r_2(i,j-1)s_1'(i,j-1)\big),\\
&u_2(i,j) \mid \gcd\big(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j)\big),\\
&w_2(i,j) \mid \gcd\big(s_2(i,j)s_1'(i,j), r_2(i,j-1)s_1'(i,j-1)\big),
\end{aligned}$$

where

$$\begin{aligned}
s_1'(i,j) &= s_1(i,j)/\gcd(s_1(i,j), s_2(i,j)),\\
s_2'(i,j) &= s_2(i,j)/\gcd(s_1(i,j), s_2(i,j)).
\end{aligned} \tag{3.6}$$

Proof. Substituting (3.4) into (3.1) and dividing $F(i,j)$ on both sides, we obtain

$$\begin{aligned}
\frac{c(i,j)}{d(i,j)} &= \frac{r_1(i,j)}{s_1(i,j)}\frac{f_1(i+1,j)}{d(i+1,j)g_1(i+1,j)} - \frac{f_1(i,j)}{d(i,j)g_1(i,j)}\\
&\quad + \frac{r_2(i,j)}{s_2(i,j)}\frac{f_2(i,j+1)}{d(i,j+1)g_2(i,j+1)} - \frac{f_2(i,j)}{d(i,j)g_2(i,j)},
\end{aligned} \tag{3.7}$$

i.e.,

$$\begin{aligned}
c(i,j) &= \frac{r_1(i,j)d(i,j)}{s_1(i,j)d(i+1,j)}\frac{f_1(i+1,j)}{g_1(i+1,j)} - \frac{f_1(i,j)}{g_1(i,j)}\\
&\quad + \frac{r_2(i,j)d(i,j)}{s_2(i,j)d(i,j+1)}\frac{f_2(i,j+1)}{g_2(i,j+1)} - \frac{f_2(i,j)}{g_2(i,j)}.
\end{aligned}$$

Let

$$\tilde{r}_1(i,j) = r_1(i,j)d(i,j), \quad \tilde{s}_1(i,j) = s_1(i,j)d(i+1,j),$$
$$\tilde{r}_2(i,j) = r_2(i,j)d(i,j), \quad \tilde{s}_2(i,j) = s_2(i,j)d(i,j+1).$$

All discussion in the proof of Theorem 2.1 still holds. Thus, we have

$$
\begin{aligned}
&v_1(i) \,|\, \tilde{r}_1(i-1,j)\tilde{s}_2'(i-1,j),\\
&v_2(j) \,|\, \tilde{r}_2(i,j-1)\tilde{s}_1'(i,j-1),\\
&v_4(i,j) \,|\, \gcd\left(\tilde{r}_1(i-1,j)\tilde{s}_2'(i-1,j), \tilde{r}_2(i,j-1)\tilde{s}_1'(i,j-1)\right),\\
&u_2(i,j) \,|\, \gcd\left(\tilde{s}_1(i,j)\tilde{s}_2'(i,j), \tilde{r}_1(i-1,j)\tilde{s}_2'(i-1,j)\right),\\
&w_2(i,j) \,|\, \gcd\left(\tilde{s}_2(i,j)\tilde{s}_1'(i,j), \tilde{r}_2(i,j-1)\tilde{s}_1'(i,j-1)\right),
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
\tilde{s}_1'(i,j) &= \tilde{s}_1(i,j)/\gcd(\tilde{s}_1(i,j), \tilde{s}_2(i,j)),\\
\tilde{s}_2'(i,j) &= \tilde{s}_2(i,j)/\gcd(\tilde{s}_1(i,j), \tilde{s}_2(i,j)).
\end{aligned}
$$

Since we have (3.5), we may replace $\tilde{r}_1, \tilde{s}_1, \tilde{r}_2, \tilde{s}_2$ by r_1, s_1, r_2, s_2 in (3.8), respectively. ∎

4. A Telescoping Algorithm for Bivariate Hypergeometric Terms

Theorem 3.1 provides us a way to choose the denominators in the telescoping algorithm. Given a hypergeometric term $F(n,i,j)$, we have the following algorithm:

Algorithm EstDen

1. Calculate $r_1, r_2, s_1, s_2, s_1', s_2'$ defined by (3.2) and (3.6);

2. Set
   $v_1(i) :=$ the maximal factor of $r_1(i,j)s_2'(i,j)$ depending only on i;
   $v_2(j) :=$ the maximal factor of $r_2(i,j)s_1'(i,j)$ depending only on j;
   and
   $$v(i) := \gcd(v_1(i-1), v_2(i-1));$$

3. Set
   $u_1(j) :=$ the maximal factor of $s_1(i,j)s_2'(i,j)$ depending only on j;
   $w_1(i) :=$ the maximal factor of $s_1(i,j)s_2'(i,j)$ depending only on i;

4. Set $u_2(i,j)$ to be the maximal factor of
   $$\gcd(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j))$$

which depends on i;
Set $w_2(i,j)$ to be the maximal factor of

$$\gcd(s_1(i,j)s_2'(i,j), r_2(i,j-1)s_1'(i,j-1))$$

which depends on j.

5. Return $g_1(i,j) := v(i)u_1(j)u_2(i,j)$ and $g_2(i,j) := v(i)w_1(i)w_2(i,j)$.

Remark. Let $f(i,j)$ be a polynomial in i,j and a be a new variable. Then the maximal factor of $f(i,j)$ depending only on i can be obtained by

$$\gcd(f(i,j), f(i,j+a)),$$

and the maximal factor of $f(i,j)$ depending on i can be obtained by

$$f(i,j)/\gcd(f(i,j), f(i+a,j)).$$

We are now ready to describe the telescoping algorithm for double summations:

Algorithm BiZeil

1. Using algorithm EstDen to obtain g_1 and g_2.
2. Set the order r of the linear difference operator L to be zero.
3. For the order r, calculate the common denominator $d(n,i,j)$ of

$$\frac{F(n+1,i,j)}{F(n,i,j)}, \quad \ldots, \quad \frac{F(n+r,i,j)}{F(n,i,j)}.$$

(If $r=0$, then take $d(n,i,j)=1$.)

4. Set the degrees of f_1 and f_2 to be one more than those of $d \cdot g_1$ and $d \cdot g_2$, respectively.
5. Solve the equation (3.7) by undeterminate coefficients method to obtain $a_0, a_1, \ldots, a_r$ and f_1, f_2.

6. If $a_0 \neq 0$, then return $L, f_1/(d \cdot g_1), f_2/(d \cdot g_2)$ and we are done.

   If $a_0 = 0$ but $\deg f_1 - \deg(d \cdot g_1) \leq 2$, then increase the degrees of f_1 and f_2 by one and repeat Step 5.

   Otherwise, set $r := r + 1$ and repeat the process from Step 3.

Remarks.

1. In most cases, $g_1(i,j)$ and $g_2(i,j)$ can be further reduced by cancelling a factor of degree 1 and 2 from g_1 and g_2, respectively.

2. In all the following examples except Example 4, the degree of the numerator of R_1 (R_2) is one more than that of the denominator. While in Example 4, the difference is two.

   This can be interpreted visually as follows. Let t_1, t_2, t_3, t_4 be the four terms of the right hand side of (3.7) after multiplying the common denominator. In most cases, the leading terms of t_1 and t_2 (t_3 and t_4, respectively) are cancelled and only these terms are cancelled.

3. There is a trick in Step 5 which accelerates the computation. Given g_1 and g_2, we may derive part of the factors of f_1 and f_2 by divisibility. For example, suppose (3.7) becomes

   $$\frac{c(i,j)}{d(i,j)} = \frac{u_1(i,j)}{v_1(i,j)} f_1(i+1,j) - \frac{f_1(i,j)}{w_1(i,j)} + \frac{u_2(i,j)}{v_2(i,j)} f_2(i,j+1) - \frac{f_2(i,j)}{w_2(i,j)},$$

   after substituting and simplification. Suppose further that $D(i,j)$ is the common denominator of the above equation. Then we immediately have that $f_1 \cdot D/w_1$ is divisible by $q_1 = \gcd(cD/d, u_1D/v_1, u_2D/v_2, D/w_2)$ and $f_1(i+1,j) \cdot u_1D/v_1$ is divisible by $q_2 = \gcd(cD/d, D/w_1, u_2D/v_2, D/w_2)$, and hence,

   $$\frac{q_1}{\gcd(D/w_1, q_1)} \quad \text{and} \quad \frac{q_2}{\gcd(u_1D/v_1, q_2)}$$

   are factors of $f_1(i,j)$ and $f_1(i+1,j)$, respectively.

5. Examples

In the following examples, F denotes the summand of the left hand side of the identity.

Example 1. The Andrews-Paule identity:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2. \tag{5.1}$$

It was confirmed by Andrews and Paule [2,3] by proving the more general identity

$$\sum_{i=0}^{\lfloor\frac{m}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{i+j}{i}^2\binom{m+n-2i-2j}{n-2i} = \frac{\lfloor\frac{m+n+1}{2}\rfloor!\lfloor\frac{m+n+2}{2}\rfloor!}{\lfloor\frac{m}{2}\rfloor!\lfloor\frac{m+1}{2}\rfloor!\lfloor\frac{n}{2}\rfloor!\lfloor\frac{n+1}{2}\rfloor!}.$$

Using the algorithm BiZeil, we can deal with (5.1) directly. In fact, we have

$$g_1(i,j) = (2n-2i+1)(n-i+1)(j+1)^2, \quad g_2(i,j) = (2n-2i+1)(n-i+1)(i+1)^2.$$

Cancelling the factors $(n-i+1)$ and $(i+1)^2$ from $g_1(i,j)$ and $g_2(i,j)$, respectively, we obtain

$$\tilde{g}_1(i,j) = (2n-2i+1)(j+1)^2 \quad \text{and} \quad \tilde{g}_2(i,j) = (2n-2i+1)(n-i+1).$$

Finally, we get (in 1 second)

$$(2n+1)F(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j),$$

where

$$R_1 = \frac{i^2(6n^2+5n+1+6jn^2+jn-j-in+2in^2-2i-4j^2n-2j^2-3ij-4ijn)}{(2n-2i+1)(1+j)^2},$$

$$R_2 = \frac{-2n^2+2jn^2+6in^2+9in+3jn-4ijn-4i^2n-n+j-3ij+2i-4i^2}{(2n-2i+1)},$$

which are the same as given in [11, p. 85]. Summing $i,j=0,\ldots,n$, we get

$$\begin{aligned}
&(2n+1)\sum_{i=0}^{n}\sum_{j=0}^{n}F(n,i,j)\\
&= \sum_{i=0}^{n}\big(R_2F(n,i,n+1)-R_2F(n,i,0)\big) + \sum_{j=0}^{n}\big(R_1F(n,n+1,j)-R_1F(n,0,j)\big)\\
&= (G(n+1)-G(0)) + R_1F(n,n+1,n) = (n+1)^2\binom{2n+1}{n}^2,
\end{aligned}$$

where

$$G(i) = \frac{(-2n+i-1)(-4n+2i-1)i}{-1+2i-2n}\binom{4n-2i}{2n-2i}.$$

Example 2. Carlitz's identity [5] (see Also [12, Example 6.1.2]):

$$\sum_i \sum_j \binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{n-i-j} = \sum_{l=0}^{n} \binom{2l}{l}.$$

We have

$$g_1(i,j) = (j+1)^2(-n+j), \quad g_2(i,j) = (i+1)^2(-n+i).$$

Cancelling the factors $(-n+j)$ and $(i+1)(-n+i)$, we obtain

$$\tilde{g}_1(i,j) = (j+1)^2 \quad \text{and} \quad \tilde{g}_2(i,j) = i+1.$$

Noting that for recurrence of order 2, $d(i,j) = (-n+i-1+j)^2(-n+i-2+j)^2$, we finally get (in 2 seconds)

$$LF(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j),$$

where

$$L = (4n+6) - (8+5n)N + (n+2)N^2,$$

and

$$\begin{aligned} R_1 = \big(&- i^2(-n+i-1)(36 - 10ji^2n - 13j^2ni + 60j^2 + 60ji - 2i^2 - 38j^2i \\ &- 8ji^2 + 10i^3 + 36n^3 - 11in^3 - 14jn^3 - 2i^4 - 92jn^2 + 8i^2n - 80in + 5j^2n^2 \\ &+ 8j^2i^2 + 88jin + 42j^2n - 172jn + 24jin^2 + 5i^2n^2 + 3i^3n - 54in^2 + 88n^2 + 4j^3n \\ &- 90j + 6j^3 - 40i + 5n^4 + 90n)\big) \Big/ \big((-n+i-1+j)^2(-n+i-2+j)^2(j+1)^2\big), \end{aligned}$$

$$\begin{aligned} R_2 = \big(&(64 - 19ji^2n - 6j^2ni + 14j^2 + 74ji + 54i^2 - 10j^2i - 36ji^2 + 2i^3 + 39n^3 \\ &- 16in^3 - 9jn^3 - 4i^4 + 6ji^3 - 53jn^2 + 50i^2n - 176in + 4j^2n^2 + 4j^2i^2 + 5n^4 \\ &+ 83jin + 16j^2n - 100jn + 22jin^2 + 11i^2n^2 + 4i^3n - 93in^2 + 112n^2 - 60j \\ &- 108i + 140n)(-n-1+j)\big) \Big/ \big((-n+i-2+j)^2(-n+i-1+j)^2\big). \end{aligned}$$

Example 3. Carlitz's identity [4] (see Also [12, Example 6.1.3]):

$$\sum_i \sum_j \binom{i+j}{i}\binom{m-i+j}{j}\binom{n-j+i}{i}\binom{m+n-i-j}{m-i}$$
$$= \frac{(m+n+1)!}{m!n!}\sum_k \frac{1}{2k+1}\binom{m}{k}\binom{n}{k}.$$

We have

$$g_1(i,j) = (n-j+i)(1+j)^2, \quad g_2(i,j) = (m-i+j)(i+1)^2.$$

Cancelling the factors $(1+j)$ and $(i+1)^2$, we obtain

$$\tilde{g}_1(i,j) = (n-j+i)(1+j) \quad \text{and} \quad \tilde{g}_2(i,j) = m-i+j.$$

Noting that for recurrence of order 2, $d(i,j) = (-n+j-1)^2(-n+j-2)^2$, we finally get (in 37 seconds)

$$LF(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j),$$

where

$$L = 2(m+3+n)(2+m+n)^2 -$$
$$(3m+2nm+4n^2+14+15n)(n+m+3)N + (2n+5)(n+2)^2N^2,$$

and the denominators of R_1, R_2 are $d(i,j)\tilde{g}_1(i,j)$ and $d(i,j)\tilde{g}_2(i,j)$, respectively. The degrees of denominators and numerators of R_1, R_2 are both less than those given in [12].

Example 4. The Apéry-Schmidt-Strehl identity [10]:

$$\sum_i \sum_j \binom{n}{j}\binom{n+j}{j}\binom{j}{i}^3 = \sum_k \binom{n}{k}^2\binom{n+k}{k}^2.$$

We have

$$g_1(i,j) = (-j-1+i)^3 \quad \text{and} \quad g_2(i,j) = (i+1)^3.$$

Cancelling the factors $(-j-1+i)$ and $(i+1)^2$, we obtain

$$\tilde{g}_1(i,j) = (-j-1+i)^2 \quad \text{and} \quad \tilde{g}_2(i,j) = i+1.$$

Noting that for recurrence of order 2, $d(i,j) = (n+2-j)(n+1-j)$, we finally get (in 1 second)

$$LF(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j),$$

where

$$L = (n+1)^3 - (3+2n)(17n^2+51n+39)N + (n+2)^3 N^2,$$

and

$$R_1 = \big(-2i^2(3+2n)(-10+30j^2-49n^2-j^3-4n^4-24n^3-2n^2i^2+n^2i-6ni^2 + 3ni + 3nji + n^2ji + 3j^2i^2 - 3j^3i + 3ji - 4i^2 - 2j^2i - 2ji^2 + 11n^2j^2 + 6n^2j + 33nj^2+18nj-6j^4+2i+15j-39n)\big)\Big/\big((n+2-j)(n+1-j)(-j-1+i)^2\big),$$

$$R_2 = \big(2(-j+i)(3+2n)(-8n^2i - 4n^2i^2 - 4n^2ji + 4n^2j + 4n^2j^2 + 12nj - 12nji - 24ni + 12nj^2 - 12ni^2 + 12j^2 - 4ji^2 + j^3 + 6j^2i^2 - 3j^4 + 8j + 5j^2i - 8i^2 + 3j^3i - 16i - 16ji)\big)\Big/\big((n+2-j)(n+1-j)(i+1)\big).$$

The rational functions R_1, R_2 are simpler than those given in [10]. The operator L was used by Apéry in his proof of the irrationality of $\zeta(3)$ and Chyzak and Salvy obtained it using Ore algebras [6].

Example 5. The Strehl identity [10]:

$$\sum_i \sum_j \binom{n}{j}\binom{n+j}{j}\binom{j}{i}^2\binom{2i}{i}^2\binom{2i}{j-i} = \sum_k \binom{n}{k}^3\binom{n+k}{k}^3. \tag{5.2}$$

We have

$$g_1(i,j) = (j+1-i)^3, g_2(i,j) = (-3i-3+j)(-3i-2+j)(-3i-1+j)(i+1)^3.$$

Cancelling the factor $(-3i-3+j)(-3i-2+j)$ from g_2, we obtain

$$\tilde{g}_1(i,j) = (j+1-i)^3 \quad \text{and} \quad \tilde{g}_2(i,j) = (-3i-1+j)(i+1)^3.$$

Noting that for recurrence of order 6,

$$d(i,j) = (n+1-j)(n+2-j)(n+3-j)(n+4-j)(n+5-j)(n+6-j),$$

we finally get (in 2510 seconds)

$$LF(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j),$$

where L is a linear difference operator of order $r = 6$ and the denominators of R_1, R_2 are $d(i,j)\tilde{g}_1(i,j)$ and $d(i,j)\tilde{g}_2(i,j)$, respectively. The operator L is the same as the operator obtained by applying Zeilberger's algorithm to the right hand side of (5.2).

Example 6. The Graham-Knuth-Patashnik identity [7, p. 172]:

$$\sum_j \sum_k (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}. \tag{5.3}$$

We have

$$g_1(j,k) = (k+1)(k+l+1), \quad g_2(j,k) = (j+1)(j+1-l).$$

Cancelling the factor $(j+1)(j+1-l)$ from g_2, we obtain

$$\tilde{g}_1(j,k) = (k+1)(k+l+1) \quad \text{and} \quad \tilde{g}_2(j,k) = 1.$$

Noting that for recurrence of order 1 (with respect to r), $d(j,k) = r - j + 1$, we finally get (in 8 seconds)

$$LF(r,j,k) = \Delta_j R_1 F(r,j,k) + \Delta_k R_2 F(r,j,k),$$

where

$$L = (r+n+1)(n+s+l-m-r) + (r-l+1)(r-s)R$$

is a linear difference operator with respect to the variable r and the denominators of R_1, R_2 are $d(j,k)\tilde{g}_1(j,k)$ and $d(j,k)\tilde{g}_2(j,k)$, respectively. Then (5.3) follows from the evaluation of the initial value ($r = 0$) by Zeilberger's algorithm:

$$\sum_k (-1)^k \binom{k}{k+l} \binom{n}{k} \binom{s+n-k}{m} = (-1)^l \binom{n}{n+l} \binom{s}{m-n-l}.$$

Example 7. The Petkovšek-Wilf-Zeilberger identity [9, p. 33]:

$$\sum_r \sum_s (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+s}{s} \binom{n+r}{r} \binom{2n-r-s}{n} = \sum_k \binom{n}{k}^4. \tag{5.4}$$

We have

$$g_1(r,s) = (n+r)(n+1-r)(s+1)^2, \quad g_2(r,s) = (n+r)(n+1-r)(r+1)^2.$$

Cancelling the factors $s+1$ and $(r+1)^2$, we obtain

$$\tilde{g}_1(r,s) = (n+r)(n+1-r)(s+1) \quad \text{and} \quad \tilde{g}_2(r,s) = (n+r)(n+1-r).$$

Noting that for recurrence of order 2, $d(r,s)$ equals

$$(n+1)(n+2)(n+1-r)(n+2-r)(n+1-s)(n+2-s)(n-r-s+1)(n+2-r-s),$$

we finally get (in 35 seconds)

$$LF(n,r,s) = \Delta_r R_1 F(n,r,s) + \Delta_s R_2 F(n,r,s),$$

where

$$L = 4(4n+5)(4n+3)(n+1) + 2(2n+3)(3n^2+9n+7)N - (n+2)^3 N^2$$

is a linear difference operator and the denominators of R_1, R_2 are $d(r,s)\tilde{g}_1(r,s)$ and $d(r,s)\tilde{g}_2(r,s)$, respectively. The recursion is the same as that obtained by applying Zeilberger's algorithm to the right hand side of (5.4).

Example 8**.** Calculate

$$f(n) = \sum_{i=0}^{n} \sum_{j=0}^{i} (i^2 + j^3) \binom{n}{i} \binom{i}{j}.$$

We have

$$g_1(i,j) = (j+1)(i^2+j^3), \quad g_2(i,j) = (i+1-j)(i^2+j^3).$$

Cancelling the factors $j+1$ and i^2+j^3, we obtain

$$\tilde{g}_1(i,j) = i^2 + j^3 \quad \text{and} \quad \tilde{g}_2(i,j) = i+1-j.$$

Noting that for recurrence of order 1, $d(i,j) = n+1-i$, we finally get (in 1 second)

$$LF(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j),$$

where

$$L = 3(n^2+20n+27)(n+1) - (n^2+18n+8)nN$$

and the denominators of R_1, R_2 are $d(i,j)\tilde{g}_1(i,j)$ and $d(i,j)\tilde{g}_2(i,j)$, respectively. Solving $Lf(n) = 0$, we immediately get

$$f(n) = 3^{n-3}n(n^2 + 18n + 8).$$

Using the package `MultiSum.m` by K. Wegschaider and A. Riese, we only find recurrences of order greater than 2, which are not easy to solve.

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