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Bijections behind the Ramanujan Polynomials

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Dedicated to Professor Dominique Foata, in honor of his 65th Birthday

Abstract. The Ramanujan polynomials were introduced by Ramanujan in his s-tudy of power series inversions. In an approach to the Cayley formula on the number of trees, Shor discovers a refined recurrence relation in terms of the number of improper edges, without realizing the connection to the Ramanujan polynomials. On the other hand, Dumont and Ramamonjisoa independently take the grammatical approach to a sequence associated with the Ramanujan polynomials and have reached the same conclusion as Shor's. It was a coincidence for Zeng to realize that the Shor polynomials turn out to be the Ramanujan polynomials through an explicit substitution of parameters. Shor also discovers a recursion of Ramanujan polynomials which is equivalent to the Berndt-Evans-Wilson recursion under the substitution of Zeng, and asks for a combinatorial interpretation. The objective of this paper is to present a bijection for the Shor recursion, or and Berndt-Evans-Wilson recursion, answering the question of Shor. Such a bijection also leads to a combinatorial interpretation of the recurrence relation originally given by Ramanujan.

1 Introduction

The original Ramanujan polynomials $\psi_k(r,x)$, where r is any nonnegative integer and $1 \le k \le r+1$, are defined by the following generating function equation:

$$\sum_{k=0}^{\infty} \frac{(x+k)^{r+k} e^{-u(x+k)} u^k}{k!} = \sum_{k=1}^{r+1} \frac{\psi_k(r,x)}{(1-u)^{r+k}}. \tag{1.1}$$

Ramanujan gives a recurrence relation of $\psi_k(r,x)$ as follows:

$$\psi_k(r+1,x) = (x-1)\psi_k(r,x-1) + \psi_{k-1}(r+1,x) - \psi_{k-1}(r+1,x-1). \tag{1.2}$$

where $1 \le k \le r+1$, $\psi_1(0,x) = 1$, and $\psi_k(r,x) = 0$ if $k \notin [r+1]$. Note that here we have adopted the standard notation $[n] := \{1, 2, \ldots, n\}$ for a positive integer n.

Berndt et al. [1, 2] find an elegant proof of (1.1) justifying the existence of the polynomials $\psi_k(r,x)$ and obtain the following recurrence relation:

$$\psi_k(r,x) = (x-r-k+1)\psi_k(r-1,x) + (r+k-2)\psi_{k-1}(r-1,x), \tag{1.3}$$

where the initial value of $\psi_k(r,x)$ and the ranges of indices are given as above.

It is worth noting that the Ramanujan polynomials satisfy the following identity:

$$\sum_{k=1}^{r+1} \psi_k(r,x) = x^r. \tag{1.4}$$

Table of $\psi_k(r,x)$.

$k \backslash r$	0	1	2	3	4
1	1	$x-1$	x^2-3x+2	$x^3-6x^2+11x-6$	$x^4-10x^3+35x^2-50x+24$
2		1	$3x-5$	$6x^2-26x+26$	$10x^3-80x^2+200x-154$
3			3	$15x-35$	$45x^2-255x+340$
4				15	$105x-315$
5					105
$\sum_k$	1	x	x^2	x^3	x^4

It turns out that the Ramanujan polynomials coincide with the polynomials $Q_{n,k}(x)$ introduced by Shor [5], where $n \ge 1$, and $0 \le k \le n-1$. Moreover, for $n = 0$ or $k \notin [n-1]$, we define $Q_{n,k}(x)$ to be zero. Shor's recursive definition of $Q_{n,k}(x)$ goes as follows:

$$Q_{n,k}(x) = (x+n-1)Q_{n-1,k}(x) + (n+k-2)Q_{n-1,k-1}(x), \tag{1.5}$$

for $n \geq 1$ and $k \leq n-1$, where $Q_{1,0}(x) = 1$ and $Q_{n,k}(x) = 0$ if $k \geq n$ or $k < 0$. Zeng [6, Proposition 7] establishes the following remarkable connection:

$$Q_{n,k}(x) = \psi_{k+1}(n-1, x+n). \tag{1.6}$$

The tree enumeration flavor of $Q_{n,k}(x)$ is evidenced by the following identity:

$$\sum_{k=0}^{n-1} Q_{n,k}(x) = (x+n)^{n-1}. \tag{1.7}$$

Table of $Q_{n,k}(x)$.

$k \backslash n$	1	2	3	4	5
0	1	$x+1$	x^2+3x+2	$x^3+6x^2+11x+6$	$x^4+10x^3+35x^2+50x+24$
1		1	$3x+4$	$6x^2+22x+18$	$10x^3+70x^2+150x+96$
2			3	$15x+25$	$45x^2+195x+190$
3				15	$105x+210$
4					105
$\sum_k$	1	$x+2$	$(x+3)^2$	$(x+4)^3$	$(x+5)^4$

In his approach to the enumeration of trees, Shor [5] has considered the following recurrence relation:

$$f_{n,k} = (n-1)f_{n-1,k} + (n+k-2)f_{n-1,k-1}, \tag{1.8}$$

where $f_{1,0} = 1$, $n \geq 1$, $k \leq n-1$, and $f_{n,k} = 0$ otherwise. One sees that $f_{n,k}$ is the value of $Q_{n,k}(x)$ evaluated at $x = 0$, and that $f_{n,k}$ satisfies the following identity:

$$\sum_{k=0}^{n-1} f_{n,k} = n^{n-1}. \tag{1.9}$$

Shor shows that $f_{n,k}$ is in fact the number of rooted trees on $[n]$ with k improper edges. However, he did not seem to have noticed the connection of his formula to the work of Ramanujan. On the other hand, Dumont and Ramamonjisoa [4] use the grammatical method introduced by Chen in [3] to obtain the same combinatorial interpretation.

Besides the recurrence relation (1.5) for $Q_{n,k}(x)$, Shor [5] derives the following recurrence relation, and asks for a combinatorial interpretation:

$$Q_{n,k}(x) = (x-k+1)Q_{n-1,k}(x+1) + (n+k-2)Q_{n-1,k-1}(x+1). \tag{1.10}$$

The above recurrence relation turns out to be equivalent to the Berndt-Evans-Wilson recursion (1.3) by the substitution (1.6) of Zeng.

The aim of this paper is to construct a bijection for (1.10), answering the question of Shor. We note that the above relation is indeed the same as the recurrence relation (1.2) under the substitution (1.6) of Zeng. Therefore, we also obtain a combinatorial interpretation of the recurrence relation (1.2) originally presented by Ramanujan.

2 The Zeng Interpretations and the Shor Recursion

We will follow most notation in Zeng [6]. The set of rooted labeled trees on $[n]$ is denoted by $\mathcal{R}_n$. If $T \in \mathcal{R}_n$, and x is a node of T, the subtree rooted at x is denoted by T_x. We let $\beta(x)$, or $\beta_T(x)$ be the smallest node on T_x. For notational simplicity, we also use β_T or $\beta(T)$ to denote the minimum element in T, and we sometimes write $T(x)$ for T_x in the purpose of avoiding multiple subscripts. We say that a node z of T is a descendant of x, (or x is an ancestor of z), if z is a node of T_x. In particular, each node is a descendant of itself. For any edge $e = (x, y)$ of a tree T, if y is a node of T_x, we call x the *father node* of e, y the *child node* of e, x the father of y, and y a child of x. Assume $e = (x, y)$ is an edge of a tree T, and y is the child node of e, we say that e is a *proper* edge, if $x < \beta_T(y)$. Otherwise, we call e an *improper* edge. The degree of a node x in a rooted tree T is the number of children of x, and is denoted by $\deg(x)$, or $\deg_T(x)$. An unrooted labeled tree will be treated as a rooted tree in which the smallest node is chosen as the root. Then the above definitions are still valid for unrooted trees. Denote by $\mathcal{T}_{n,k}$ and $\mathcal{R}_{n,k}$ the sets of labeled trees and rooted labeled trees on $[n]$ with k improper edges, respectively. Moreover, we may impose some conditions on the sets $\mathcal{T}_{n,k}$ and $\mathcal{R}_{n,k}$ to denote the subsets of trees that satisfy these conditions. For example, $\mathcal{T}_{n+1,k}[\deg(n+1) = 0]$ stands for the subset of $\mathcal{T}_{n+1,k}$ subject to the condition $\deg(n+1) = 0$.

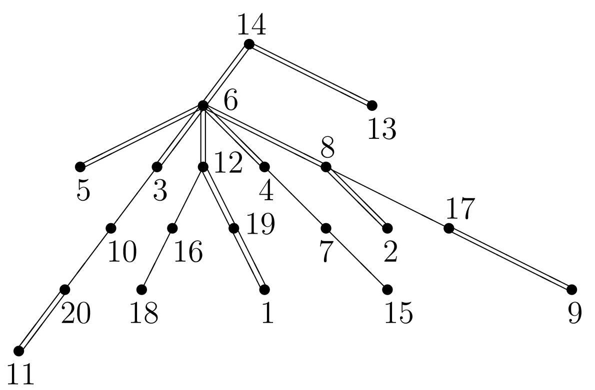

Figure 1: Improper Edges Shown as Double Edges

Theorem 2.1 (Zeng[6, Propositions 1, 2, 7]) *The polynomials $Q_{n,k}(x)$ have the following interpretations:*

$$Q_{n,k}(x) = \sum_{T \in \mathcal{T}_{n+1,k}} x^{\deg_T(1)-1}. \tag{2.1}$$

$$Q_{n,k}(x) = \sum_{T \in \mathcal{R}_{n,k}} (x+1)^{\deg_T(1)}. \tag{2.2}$$

In fact, the above theorem can be reformulated by the following relations:

$$(x+n-1)Q_{n-1,k}(x) = \sum_{T\in\mathcal{T}_{n+1,k}[\deg(n+1)=0]} x^{\deg_T(1)-1}, \tag{2.3}$$

$$(n+k-2)Q_{n-1,k-1}(x) = \sum_{T\in\mathcal{T}_{n+1,k}[\deg(n+1)>0]} x^{\deg_T(1)-1}. \tag{2.4}$$

Zeng [6] proves the two interpretations (2.1) and (2.2) of $Q_{n,k}(x)$ by similar arguments. One naturally expects to make a combinatorial connection bridging these two formulations, and this consideration was mentioned by Zeng. We now provide such an argument for the equivalence between (2.1) and (2.2), that is,

$$\sum_{T\in\mathcal{T}_{n+1,k}} x^{\deg_T(1)-1} = \sum_{T\in\mathcal{R}_{n,k}} (x+1)^{\deg_T(1)}. \tag{2.5}$$

Proof. Let us consider the binomial expansion of the right hand side of (2.5). The binomial expansion can be visualized by coloring the children of the node 1 with black and white colors. Let T be a rooted tree in $\mathcal{R}_{n,k}$, and let T have the children of 1 colored in either black or white. Let B be the set of children of 1 in T which are colored in black. Now we may introduce a new node 0, and move the subtrees of T rooted at the nodes in B as the subtrees of 0, and moreover, move the remaining subtree of T as a subtree of 0. Therefore, we obtain a rooted tree on $\{0,1,2,\ldots,n\}$, say T'. Note that the children of 0 which come from the black nodes can be easily distinguished from the child of 0 which is the original root of T because the node 1 remains in the subtree of original root. Finally, if we relabel the set $\{0,1,2,\ldots,n\}$ by the set $[n+1]$, namely, relabeling i by $i+1$, we get an unrooted tree on $[n+1]$ which preserves the number of improper edges. Furthermore, one sees that the above construction can be reversed. This completes the proof. ∎

Corrolary 2.2 *We have*

$$Q_{n,k}(x-1) = \sum_{T\in\mathcal{T}_{n+1,k}[\deg(2)=0]} x^{\deg_T(1)-1}. \tag{2.6}$$

Proof. It follows from (2.2) that

$$Q_{n,k}(x-1) = \sum_{T\in\mathcal{R}_{n,k}} x^{\deg_T(1)}. \tag{2.7}$$

We now construct a bijection from $\mathcal{R}_{n,k}[\deg(1)=r]$ to $\mathcal{T}_{n+1,k}[\deg(1)=r+1,\deg(2)=0]$. Given $T\in\mathcal{R}_{n,k}[\deg(1)=r]$, we now introduce a new root 0, and put T as a subtree of 0. Then we move all the subtrees of 1 and make them as subtrees of 0. Finally, by relabeling a node i by $i+1$, we obtain a tree $T' \in \mathcal{T}_{n+1,k}[\deg(1) = r+1, \deg(2) = 0]$. It is clear that the construction is reversible. This completes the proof. ∎

Substituting k by $k+1$ in (2.4), we obtain

$$(n+k-1)Q_{n-1,k}(x) = \sum_{T \in \mathcal{T}_{n+1,k+1}[\deg(n+1)>0]} x^{\deg_T(1)-1}. \tag{2.8}$$

We are now ready to give another combinatorial formulation of the Shor recurrence relation (1.10). Rewriting (1.10), by substituting x with $x-1$, we get:

$$Q_{n,k}(x-1) = (x-k)Q_{n-1,k}(x) + (n+k-2)Q_{n-1,k-1}(x). \tag{2.9}$$

If we express the term $(x-k)Q_{n-1,k}(x)$ as

$$(x+n-1)Q_{n-1,k}(x) - [n+(k+1)-2]Q_{n-1,(k+1)-1}(x)\,,$$

then the Shor recurrence relation (1.10) is equivalent to the following combinatorial identity.

Theorem 2.3 *For $n \geq 1$, and $0 \leq k \leq n-1$, we have*

$$\sum_{T \in \mathcal{T}_{n+1,k}[\deg(2)>0]} x^{\deg_T(1)-1} = \sum_{T \in \mathcal{T}_{n+1,k+1}[\deg(n+1)>0]} x^{\deg_T(1)-1}. \tag{2.10}$$

We now present an inductive proof of the above fact, while the next section will be engaged in a purely combinatorial treatment. Clearly, for $n \geq 1$, (2.10) can be restated as follows with the notation $T_{n,k}[\cdots] := |\mathcal{T}_{n,k}[\cdots]|$:

$$T_{n+1,k}[\deg(2) > 0, \deg(1) = r] = T_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r]. \tag{2.11}$$

Proof. For $n \geq 2$, the arguments of Shor [5] or Zeng [6] imply the following identities:

(i) $\quad T_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r] = (n+k-1)T_{n,k}[\deg(1) = r].$

(ii) $\quad T_{n+1,k}[\deg(2) > 0, \deg(1) = r]$

$$\begin{aligned} &= (n-2)T_{n,k}[\deg(2) > 0, \deg(1) = r] + T_{n,k}[\deg(2) > 0, \deg(1) = r-1] \\ &\quad + T_{n,k}[\deg(1) = r] + (n+k-2)T_{n,k-1}[\deg(2) > 0, \deg(1) = r]. \end{aligned}$$

(iii) $\quad T_{n+1,k}[\deg(1) = r]$

$$\begin{aligned} &= (n-1)T_{n,k}[\deg(1) = r] + T_{n,k}[\deg(1) = r-1] \\ &\quad + (n+k-2)T_{n,k-1}[\deg(1) = r]. \end{aligned}$$

Because of (i), (2.11) can be deduced from the following relation:

$$T_{n+1,k}[\deg(2) > 0, \deg(1) = r] = (n + k - 1)T_{n,k}[\deg(1) = r], \tag{2.12}$$

for $n \geq 1$. The above claimed identity obviously holds for $n = 1$. Suppose (2.12) holds for $n - 1$. From (i) – (iii) and the inductive hypothesis, it follows that

$$\begin{aligned}
&T_{n+1,k}[\deg(2) > 0, \deg(1) = r]\\
&= (n-2)T_{n,k}[\deg(2) > 0, \deg(1) = r] + T_{n,k}[\deg(2) > 0, \deg(1) = r - 1]\\
&\quad + T_{n,k}[\deg(1) = r] + (n + k - 2)T_{n,k-1}[\deg(2) > 0, \deg(1) = r]\\
&= (n-2)(n+k-2)T_{n-1,k}[\deg(1) = r] + (n+k-2)T_{n-1,k}[\deg(1) = r - 1]\\
&\quad + T_{n,k}[\deg(1) = r] + (n+k-2)(n+k-3)T_{n-1,k-1}[\deg(1) = r]\\
&= (n+k-2)\{(n-2)T_{n-1,k}[\deg(1) = r] + T_{n-1,k}[\deg(1) = r - 1]\\
&\quad +(n+k-3)T_{n-1,k-1}[\deg(1) = r]\} + T_{n,k}[\deg(1) = r]\\
&= (n+k-2)T_{n,k}[\deg(1) = r] + T_{n,k}[\deg(1) = r]\\
&= (n+k-1)T_{n,k}[\deg(1) = r].
\end{aligned}$$

Thus (2.12) holds for n. This completes the proof. ∎

We further remark that the following recurrence relations presented by Zeng [6] also follow from the above combinatorial identity:

$$Q_{n,k}(x) = (x + n - 1)Q_{n-1,k}(x) + Q_{n,k-1}(x) - Q_{n,k-1}(x - 1), \tag{2.13}$$

$$Q_{n,k}(x) = Q_{n,k}(x - 1) + (n + k - 1)Q_{n-1,k}(x). \tag{2.14}$$

Note that the recurrence relation (2.13) is equivalent to the original Ramanujan recursion (1.2). A bijective proof of (2.10) will be the objective of the next section.

3 The Bijections

In order to demonstrate (2.10) combinatorially, it would be ideal to directly construct a bijection from $\mathcal{T}_{n+1,k}[\deg(2) > 0]$ to $\mathcal{T}_{n+1,k+1}[\deg(n+1) > 0]$ which preserves the degree of 1. Although it looks that such a bijection should be easy to construct by moving a child of 2 in a tree in $\mathcal{T}_{n+1,k}[\deg(2) > 0]$ to the node $n+1$, achieving such a task turns out to be quite subtle. To achieve this goal, we first find a stronger bijection on rooted trees subject to certain degree constraints.

Theorem 3.1 *For $n \geq 1$ and $0 \leq k < n$, we have the following bijection:*

$$\mathcal{R}_{n,k}[\deg(1) > 0] \longleftrightarrow \mathcal{R}_{n,k+1}[\deg(n) > 0]. \tag{3.1}$$

Here is an example for $n = 4$, $k = 1$. There are 16 trees for each side of (3.1). The trees in $\mathcal{R}_{4,1}[\deg(1) > 0]$ are listed in Figure 2.

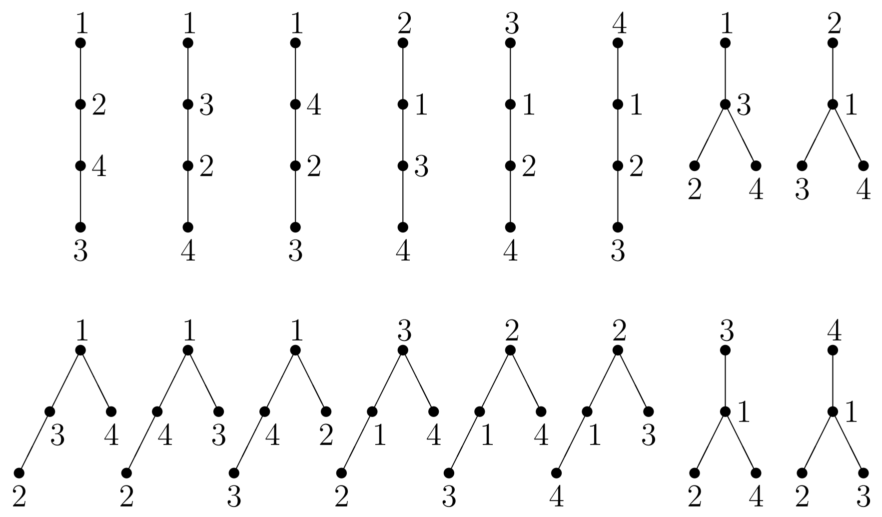

Figure 2: 16 trees in $\mathcal{R}_{4,1}[\deg(1) > 0]$

The trees in $\mathcal{R}_{4,2}[\deg(4) > 0]$ are in Figure 3.

Before we start our journey of constructing the bijection, we present an inductive proof. In principle, it follows from Theorem 2.3 for the case $\deg_T(1) = 1$. For completeness, we include the inductive proof which is slightly simpler than that of (2.11).

Inductive Proof of Theorem 3.1. For $n \geq 2$, the arguments of Shor [5] or Zeng [6] imply the following identities:

(i) $R_{n,k+1}[\deg(n) > 0] = (n + k - 1)R_{n-1,k}$.

(ii) $$\begin{aligned} R_{n,k}[\deg(1) > 0] &= (n-2)R_{n-1,k}[\deg(1) > 0] + R_{n-1,k} \\ &\quad + (n + k - 2)R_{n-1,k-1}[\deg(1) > 0]. \end{aligned}$$

(iii) $R_{n,k} = (n-1)R_{n-1,k} + (n + k - 2)R_{n-1,k-1}$.

Because of (i), $R_{n,k}[\deg(1) > 0] = R_{n,k+1}[\deg(n) > 0]$ can be deduced from the following relation:

$$R_{n,k}[\deg(1) > 0] = (n + k - 1)R_{n-1,k}, \tag{3.2}$$

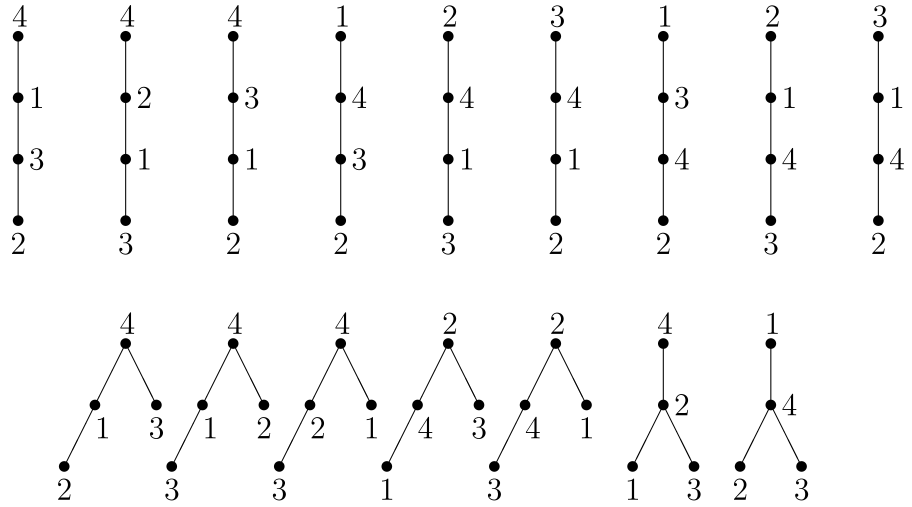

Figure 3: 16 trees in $\mathcal{R}_{4,2}[\deg(4) > 0]$

for $n \geq 1$. The above claimed identity obviously holds for $n = 1$. Suppose (3.2) holds for $n-1$. From (i) – (iii) and the inductive hypothesis, it follows that

$$\begin{aligned}
&R_{n,k}[\deg(1) > 0] \\
&= (n-2)R_{n-1,k}[\deg(1) > 0] + R_{n-1,k} + (n+k-2)R_{n-1,k-1}[\deg(1) > 0] \\
&= (n-2)(n+k-2)R_{n-2,k} + R_{n-1,k} + (n+k-2)(n+k-3)R_{n-2,k-1} \\
&= (n+k-2)\{(n-2)R_{n-2,k} + (n+k-3)R_{n-2,k-1}\} + R_{n-1,k} \\
&= (n+k-2)R_{n-1,k} + R_{n-1,k} \\
&= (n+k-1)R_{n-1,k}.
\end{aligned}$$

This completes the proof. ∎

We note that for $k = n-1$, there does not exist any rooted tree T with n nodes and k improper edges such that $\deg_T(1) > 0$, because any edge with 1 as the father node is proper. Thus, we may assume without loss of generality that $k < n-1$.

It turns out that we need to consider two major cases in the construction of a bijection for (3.1). First, we introduce the notation $\mathcal{R}_{n,k}^{(i)}$ for the set of trees T in $\mathcal{R}_{n,k}$ such that there are i proper edges on the path from the node n to the root. Suppose T is a rooted tree on $[n]$ and x is a node of T such that T_x contains the node n. Then we may define the *lowering operation* L on T_x such that $L(T_x)$ is the rooted tree obtained from T_x by taking n as the new root and letting the ancestor nodes of n fall down to the descendants of n. Under certain circumstances, the lowering operation is reversible, and the reverse will be called the *lifting operation.* The following Theorem 3.2 tells us where we may apply the lowering operations that are reversible. We need to define the *upper critical node* and the *lower critical node* of a rooted tree T.

If T is a rooted tree in $\mathcal{R}_{n,k}^{(i)}$, where $i \geq 1$. Suppose $(n = v_1, v_2, \ldots, v_t)$ is the path from n to the root of T, and v_j is the first node on the path such that (v_{j-1}, v_j) is a proper edge of T. Then we call v_j the *upper critical node* of T. On the other hand, for any rooted tree T on $[n]$ such that $\deg_T(n) > 0$, we define the *lower critical node* of T by the following procedure. First, we note that $\beta(n) < n$. By abuse of the above indices t and j, we assume that $(n = u_1, u_2, \ldots, u_t = \beta(n))$ is the path from n to $\beta(n)$ and that $u_j \neq n$ is the first node on the path such that every node in $T_n - T_{u_j}$, namely, the tree obtained from T_n by removing the subtree T_{u_j}, is greater than u_j, denoted

$$u_j < \beta(T_n - T_{u_j}).$$

Note that such a node u_j must exist because the node $\beta(n)$ is always a candidate to satisfy the above condition. The lower critical node of T will be denoted by $\lambda(T)$, or λ for short, if no confusion arises in the context.

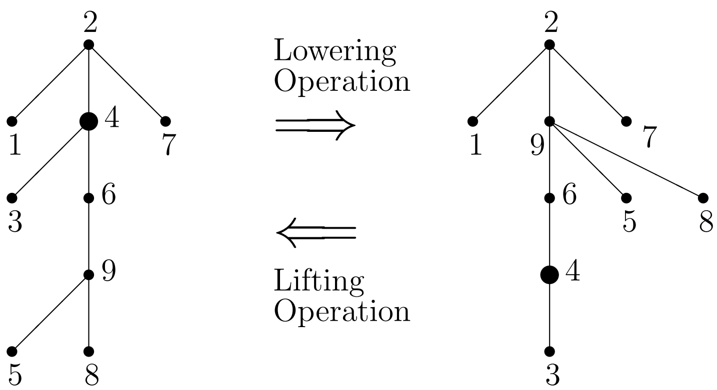

Figure 4: Lowering and Lifting Operations

With the aid of the lifting and lowering operations, we may establish the following bijection which serves as the first case for the bijection (3.1).

Theorem 3.2 *For $i \geq 1$, we have the following bijection:*

$$\mathcal{R}_{n,k}^{(i)}[\deg(1) > 0] \longleftrightarrow \mathcal{R}_{n,k+1}^{(i-1)}[\deg(n) > 0, (\deg(1) > 0 \text{ or } \lambda = 1)]. \tag{3.3}$$

Proof. Suppose T is tree in $\mathcal{R}_{n,k}^{(i)}[\deg(1) > 0]$. We assume that $(n = v_1, v_2, \ldots, v_t)$ is the path from n to the root of T, and v_j is the upper critical node of T. We now apply the lowering operation L on $T(v_j)$. We then obtain a rooted tree T' by substituting the subtree $T(v_j)$ with $L(T(v_j))$. Note that the resulting tree T' has one more improper edge than T because the edge (v_j, v_{j-1}) is proper in T and the edge (v_{j-1}, v_j) is improper in T'. Moreover, we notice that after the lowering operation, the degree of n increases by 1, the degree of the upper critical node decreases by 1, and the degree of any other node remains unchanged. Therefore, if we have $\deg_{T'}(1) = 0$, then 1 must be the upper critical node of T because $\deg_T(1) > 0$.

We now face the task of recovering the original tree T from the tree T' and convincing ourselves of the fact that the upper critical node of T becomes the lower critical node of T'. In order to single out the upper critical node of T in the new environment of T', we first claim that the upper critical node of T, say w, has to be on the path from n to $\beta(n)$ in T'. Assume that v is the child of w that is on the path from w to n. By the definition of w, one sees that $w < \beta(T_v)$. Therefore, after the application of the lowering operation, w has to be on the path from n to $\beta(n)$.

We now assume that $(n, u_1, u_2, \ldots)$ is the path from n to $\beta(n)$ in T'. If $u_1 < \beta(T'_n - T'_{u_1})$, then one sees that (u_1, n) is a proper edge in T and one can lift the edge (u_1, n) up and to restore w as the upper critical node of T. Otherwise, we may consider the next candidate u_2, and so on. Such a process shows that the upper critical node of T can be identified by the lower critical node of T'. This completes the proof. ∎

The next case we should consider is the following theorem.

Theorem 3.3 *For $n \geq 1$ and $m \geq 1$, we have the following bijection:*

$$\mathcal{R}_{n,k}^{(0)}[\deg(1) = m] \longleftrightarrow \mathcal{R}_{n,k+1}^{(m-1)}[\deg(n) > 0, (\deg(1) = 0 \text{ and } \lambda > 1)]. \tag{3.4}$$

Note that Theorems 3.2 and 3.3 together lead to a refined version of Theorem 3.1. We now focus on the proof of (3.4). The proof of (3.3) actually implies the following assertion:

Lemma 3.4 *For $i \geq 1, m \geq 1$, we have the following bijection:*

$$\mathcal{R}_{n,k}^{(i)}[\deg(n) = m, \deg(1) = 0, \lambda > 1] \longleftrightarrow$$
$$\mathcal{R}_{n,k+1}^{(i-1)}[\deg(n) = m+1, \deg(1) = 0, \lambda > 1]. \tag{3.5}$$

By iteration, for any $m \geq 1$ it follows that

$$\mathcal{R}_{n,k+1}^{(m-1)}[\deg(1) = 0, \deg(n) \geq 1, \lambda > 1]$$

$$\longleftrightarrow \mathcal{R}^{(0)}_{n,k+m}[\deg(1)=0, \deg(n)\geq m, \lambda>1]. \qquad (3.6)$$

Because of the above bijection, one sees that Theorem 3.3 is equivalent to the following statement.

Theorem 3.5 *For $n \geq 1$ and $m \geq 1$, we have the following bijection:*

$$\mathcal{R}^{(0)}_{n,k}[\deg(1)=m] \longleftrightarrow \mathcal{R}^{(0)}_{n,k+m}[\deg(1)=0, \deg(n)\geq m, \lambda>1]. \qquad (3.7)$$

We now run short of notation and terminology for our unaccomplished mission, and here are more in need.

- $\alpha\ = \alpha_T := \max\{\beta_T(b) :\ b$ is a child of the node $n\}$, for $T \in \mathcal{R}_{n,k}[\deg(n)>0]$. such that $\deg_T(n) > 0$.
- $\beta^* = \beta^*_T := \min\{\beta_T(a) :\ a$ is a child of the node $1\}$, for $T \in \mathcal{R}_{n,k}[\deg(1)>0]$.
- $x \prec y$ denotes that x is a descendant of y, while $x \not\prec y$ means the opposite.
- If we cut off a subtree from a node u and join it to another node v as a subtree, we will simply say that the subtree is moved to another node, or we move the subtree to another node.

Note that, for any $T \in \mathcal{R}^{(0)}_{n,k}$, the node 1 cannot be on the path from n to the root, namely, $n \not\prec 1$. Also, if $T \in \mathcal{R}^{(0)}_{n,k}$, then we have $\deg(n) > 0$; Otherwise, the first edge on the path from n to the root would be proper. Therefore, α is always well-defined for a tree $T \in \mathcal{R}^{(0)}_{n,k}$, and if $1 \not\prec n$, we have $\lambda_T > 1$.

The following lemma is crucial.

Lemma 3.6 *We have the following bijection:*

$$\mathcal{R}_{n,k}[\deg(1)=1, \beta^*=w] \longleftrightarrow \mathcal{R}_{n,k+1}[\deg(1)=0, \mu=w], \qquad (3.8)$$

where $\mu(T)$ is defined for any rooted tree in which 1 is not the root of T. We suppose that $(u_1 = 1, u_2, \ldots, u_t = v)$ is the path from 1 to the root of T. Then $\mu(T) = u_j$ denotes the first node on the path with $u_j > 1$ such that

$$u_j < \beta(T_v - T_{u_j}). \qquad (3.9)$$

Moreover, we always assume that v satisfies the above condition (3.9).

Proof. Suppose $T \in \mathcal{R}_{n,k}[\deg(1) = 1, \beta^* = w]$, where $w \geq 2$. Let v be the unique child of 1, and P: $(v_1, v_2, \ldots, v_t = 1)$ the path from the root of T to 1. The scheme of the construction consists of the following steps:

- Cut off the edge $(1, v)$ and get a tree $S = T - T_v$.
- Cut off some edges on the path from v_1 to 1 to get a forest, say $R_1, R_2, \ldots, R_s$, subject to some conditions to be spelled out later.
- Obtain a tree T' from T_v by joining the each R_i as a subtree of the node $\beta_T(v)$ in T_v.

The tree T' constructed above will be the goal of our bijection. We now make it precise.

First, if $v_1 < w$ then set $j_1 = 1$. Otherwise, we choose j_1 as the minimum index such that

$$v_{j_1} < \beta(T(v_1) - T(v_{j_1})), \quad \text{and} \quad v_{j_1} < w. \tag{3.10}$$

Because $v_t = 1$ is on the path P, j_1 can be determined. Second, we find all indices $j > j_1$ according to the following condition

$$v_j < \beta(T(v_1) - T(v_j)), \tag{3.11}$$

and denote by $j_2, j_3, \ldots, j_s = t$, where $j_2 < j_3 < \cdots < j_s$, the solutions to the above inequality (3.11). Third, set $j_0 = 0$, $v_{t+1} = v$ and

$$R_i = T(v_{j_{i-1}+1}) - T(v_{j_i+1}), \quad 1 \leq i \leq s,$$

namely,

$$R_1 = T(v_1) - T(v_{j_1+1}), \quad R_2 = T(v_{j_1+1}) - T(v_{j_2+1}), \quad \ldots, \quad R_s = T(v_{j_{s-1}+1}) - T(v).$$

Fourth, we construct a tree T' from the above decomposition of $T - T_v$ into $R_1, R_2, \ldots, R_s$ as claimed before.

Let us now take a close look at T and T'. Obviously, all the edges on the path from v_1 to $v_t = 1$ in T are improper. When we cut $T - T_v$ into s pieces $R_1, R_2, \ldots, R_s$, we lose $s - 1$ improper edges. By the definition of $j_1, j_2, \ldots, j_s$, namely, the conditions (3.10) and (3.11), one sees that $v_{j_1} < w$. Since $j_2 > j_1$, the node v_{j_1} must be a node in $T(v_1) - T(v_{j_2})$. It follows that

$$v_{j_2} < \beta(T(v_1) - T(v_{j_2})) \leq v_{j_1}.$$

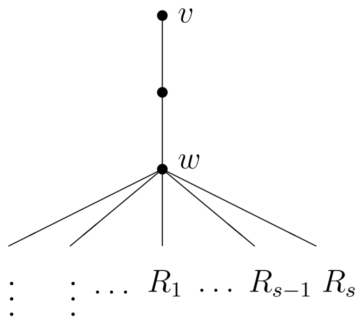

Figure 5: T'

The same reasoning leads to order relation:

$$w > v_{j_1} > v_{j_2} > \cdots > v_{j_s} = 1.$$

Therefore, when joining $R_1, R_2, \ldots, R_s$ as the subtrees of $w = \beta(T_v)$ in T_v, we gain s improper edges. Taking the previously lost improper edges into consideration, one sees that T' has one more improper edge than T.

We now come to the justification of the fact that the node w in T can be recovered as the node $\mu(T')$. From the above construction, one sees that w is a node on the path from the node 1 to the root. Moreover, w satisfies condition (3.9). We now need to show that except for 1, there is no other node v_j on the path from 1 to w satisfying the same condition (3.9). Suppose that v_j is such a node, namely, $v_j < \beta(T'(v) - T'(v_j))$. Since w is the minimum node in T_v, we have

$$\beta(T'(v) - T'(v_j)) = \beta(T'(w) - T'(v_j))$$

Thus, we obtain

$$v_j < \beta(T(v_1) - T(v_j)), \quad \text{and} \quad v_j < w. \tag{3.12}$$

Let us consider what happens to the nodes on the path from 1 to the root in the above subtree $R_s = T(v_{j_{s-1}+1}) - T(v)$. Suppose v_j is such a node where $j_{s-1}+1 \le j < j_s = t$. If $s = 1$, (3.12) is contradictory to the definition of j_1. If $s \ge 2$, since j is not a solution to (3.11), we have

$$v_j > \beta(T(v_1) - T(v_j)). \tag{3.13}$$

Note that two relations (3.12) and (3.13) are contradictory to each other. Therefore, we reach the conclusion that $T' \in \mathcal{R}_{n,k+1}[\deg(1) = 0, \mu = w]$.

Before we give the reverse procedure to reconstruct T from T'. We need the following three claims about the above procedure.

Claim 1. The node v_{j_i} must be on the path from v_1 to $\beta(R_i)$ in R_i.

The condition (3.10) says that every node in $R_1' = T(v_1) - T(v_{j_1})$ is greater than v_{j_1}. The subtree R_1 consists of R_1' joined by a subtree rooted at v_{j_1}. Thus, v_{j_1} must be on the path from v_1 to $\beta(R_1)$.

For $R_2, R_3, \ldots, R_s$, the same argument applies. Since $j_{i-1}+1 \le j_i$, $j_i + 1 > j_i$. It follows that $v_{j_i} \in T(v_{j_{i-1}+1})$ and $v_{j_i} \notin T(v_{j_i+1})$, that is $v_{j_i} \in R_i = T(v_{j_{i-1}+1}) - T(v_{j_i+1})$. By the definition of $v_{j_{i+1}}$, we have

$$v_{j_i} < \beta(T(v_1) - T(v_{j_i})) \le \beta(T(v_{j_{i-1}+1}) - T(v_{j_i})) = \beta(R_i - R_i(v_{j_i})).$$

Thus, $\beta(R_i)$ is in $R_i(v_{j_i})$. Note that we have assumed that the above equalities is true for $v_{j_{i-1}+1} = v_{j_i}$.

Claim 2. $w > \beta(R_1) > \beta(R_2) > \cdots > \beta(R_s)$.

From the following relations

$$v_{j_i} < \beta(T(v_1) - T(v_{j_i})) \le \beta(T(v_{j_{i-2}+1}) - T(v_{j_{i-1}+1})) = \beta(R_{i-1}),$$

we obtain $\beta(R_i) \le v_{j_i} < \beta(R_{i-1})$. We have already shown that $\beta(R_1) \le v_{j_1} < w$, so Claim 2 follows.

Claim 3. The node v_{j_i} can be determined as the first node z_j on the path $z_1, z_2, \ldots, z_r = \beta(R_i)$ from the root of R_i to $\beta(R_i)$ such that

$$z_j < \beta(R_i(z_1) - R_i(z_j)) \quad \text{and} \quad z_j < \beta(R_{i-1}). \tag{3.14}$$

Here, we set $\beta(R_0) = w$ and assume the first inequality of (3.14) is always true for $z_j = z_1$.

It is easy to see that Claim 3 holds for $i = 1$. We now assume that $i \ge 2$. In the proofs of Claim 1 and Claim 2, we have also shown that

$$v_{j_i} < \beta(R_i - R_i(v_{j_i})) \quad \text{and} \quad v_{j_i} < \beta(R_{i-1}).$$

Suppose there is another $v_j \ne v_{j_i}$ on the path from the root of R_i to v_{j_i} that satisfies the condition

$$v_j < \beta(R_i - R_i(v_j)) \quad \text{and} \quad v_j < \beta(R_{i-1}).$$

By Claim 2 and the above condition, v_j satisfies the condition

$$v_j < \beta(T(v_1) - T(v_j)).$$

However, we must have $j_{i-1} + 1 \le j < j_i$, because $v_{j_{i-1}+1}$ is the root of R_i. This is a contradiction to the fact that $j_2 < \cdots < j_s$ are the only solutions greater than j_1 to the above inequality.

We now come to the turning point of the bijection. For a tree $T' \in \mathcal{R}_{n,k+1}[\deg(1) = 0, \mu = w]$, we are going to reconstruct the tree T. The first step is easy: the subtrees $R_1, R_2, \ldots, R_s$ can be separated from T' as the subtrees R_i of w such that $\beta(R_i) < w$. By Claim 1, $R_1, R_2, \ldots, R_s$ can be restored by the following order:

$$w > \beta(R_1) > \beta(R_2) > \cdots > \beta(R_s).$$

Let $R = T' - R_1 - R_2 - \cdots - R_s$. By the construction of T', we need to merge the subtrees $R_1, R_2, \ldots, R_s$ into a rooted tree S. In so doing, we need to identify which node on R_i is the last node on the path from v_1 to the node 1 in T. In other words, we need to have the nodes v_{j_1}, v_{j_2}, …, v_{j_s} restored. This job can be left to Claim 2.

The last step would be to put R_1, R_2, …, R_s together with R. For $i = 1, 2, \ldots, s-1$, we join R_{i+1} as a subtree of v_{j_i} of R_i, then join R as a subtree of 1 in R_s. At last, we obtain the tree $T \in \mathcal{R}_{n,k}[\deg(1) = 1, \beta^* = w]$. ∎

Here is an example for $n = 20$, $w = 11$.

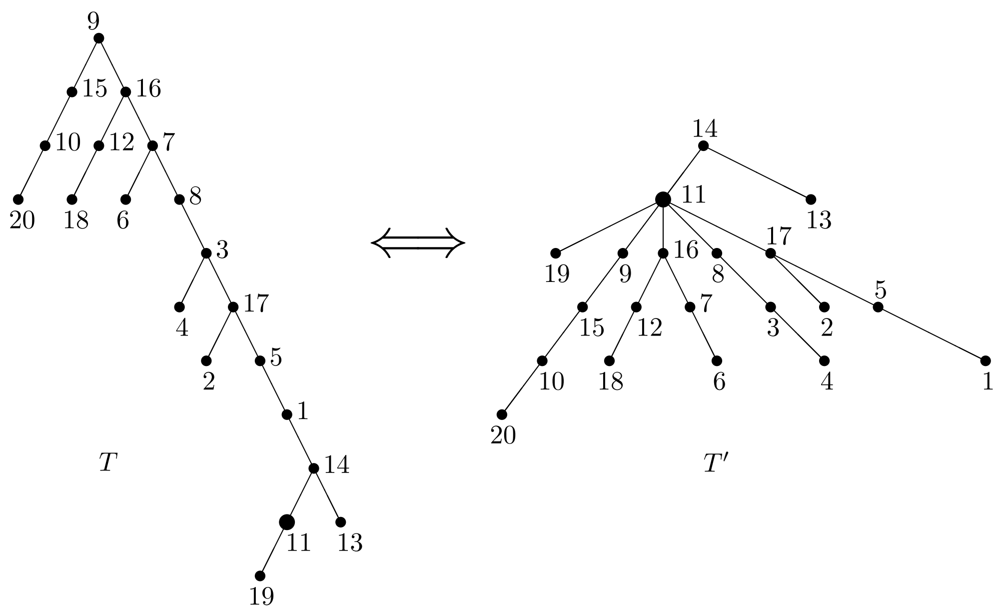

Figure 6: Example for $n = 20$ and $w = 11$

We are now ready to present the proof of Theorem 3.5 in the following refined version.

Theorem 3.7 *For $m \geq 1$, we have the following bijections:*

(a) $\mathcal{R}^{(0)}_{n,k}[\deg(1) = m, 1 \not\prec n, \alpha < \beta^*]$

$$\longleftrightarrow \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) \geq m + 1, 1 \not\prec n].$$

(b) $\mathcal{R}^{(0)}_{n,k}[\deg(1) = m, \alpha > \beta^*]$

$$\longleftrightarrow \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) = m, 1 \not\prec n].$$

(c) $\mathcal{R}^{(0)}_{n,k}[\deg(1) = m, 1 \prec n, \deg(n) \geq 2, \alpha < \beta^*]$.

$$\longleftrightarrow \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) \geq m + 1, 1 \prec n, \lambda > 1].$$

(d) $\mathcal{R}^{(0)}_{n,k}[\deg(1) = m, 1 \prec n, \deg(n) = 1]$

$$\longleftrightarrow \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) = m, 1 \prec n, \lambda > 1].$$

Proof. Recall the facts that for any tree $T \in \mathcal{R}^{(0)}_{n,k}$ we have $\deg(n) > 0$ and that $\alpha(T)$ is well defined. Moreover, we do not get into the detailed discussion about the range of m, because when m is out of range the bijection would simply do nothing.

(a) Suppose $T \in \mathcal{R}^{(0)}_{n,k}[\deg(1) = m, 1 \not\prec n, \alpha < \beta^*]$.

Since n is not on the path from 1 to the root and 1 is not on the path from n to the root, 1 and n lie in different branches of their minimum common ancestor, in other words, the common ancestor furthest from the root. Moving all subtrees of 1 to the node n, we are led to a tree

$$T' \in \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) \geq m + 1, 1 \not\prec n].$$

Conversely, given the tree T', we assume that $b_1, b_2, \ldots, b_j (j \geq m + 1)$ are the children of n ordered by $\beta(b_1) > \beta(b_2) > \cdots > \beta(b_j)$. We now move the first m subtrees $T'_{b_i} (1 \leq i \leq m)$ to node 1. Thus, we have recovered the above tree T.

(b) Suppose $T \in \mathcal{R}^{(0)}_{n,k}[\deg(1) = m, \alpha > \beta^*]$. Exchange the node n and the subtree T_α. Thus the degree of n becomes zero, the edges on the path from the root to α are all improper by the definition of $\mathcal{R}^{(0)}_{n,k}$, while the first edge on the path from α to n is proper by the definition of α_T. Then move all subtrees of 1 to the node n. Since $\alpha > \beta^*$, we obtain a tree

$$T' \in \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) = m, 1 \not\prec n].$$

The reverse of the above procedure is strictly the other way around. Starting with the above tree T', move all the subtrees of n back to the node 1. Suppose that we obtain a tree T'' and that the path from the root to n in T'' is $P : (y_1, y_2, \ldots, y_s = n)$. Let (y_i, y_{i+1}) be the first proper edge on the path P. Suppose $R_1, R_2, \ldots, R_s$ are all of the subtrees of y_i such that $\beta(R_j) > y_i$, and $n \notin R_j$, $\forall j = 1, 2, \ldots, s$. Move these subtrees to the node n, and exchange labels of the nodes y_i and n. Therefore, we get the above tree T such that $y_i = \alpha(T)$.

(c) Suppose $T \in \mathcal{R}^{(0)}_{n,k}[\deg(1) = m, 1 \prec n, \deg(n) \geq 2, \alpha < \beta^*]$. Assume that $b_1, b_2, \ldots, b_j (j \geq 2)$ are the children of n ordered by $1 = \beta(b_1) < \beta(b_2) < \cdots < \beta(b_j) < \beta^*$. Suppose $Q: (b_2 = c_1, c_2, \ldots, c_t = \beta(b_2))$ is the path from b_2 to $\beta(b_2)$. We locate the first c_i such that

$$c_i < \beta(T(b_2) - T(c_i)), \quad c_i < \beta^*, \quad \text{and} \quad c_i < \beta(b_3), \quad \text{if } j \geq 3.$$

Moving the subtree T_{b_1} to the node c_i and moving all subtrees of 1 to the node n, we obtain a tree

$$T' \in \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) \geq m+1, 1 \prec n, \lambda > 1],$$

with the property that $c_i = \lambda(T')$.

Conversely, for this tree T', we have $x = \lambda(T') > 1$. We assume that $d_1, d_2, \ldots, d_s (s \geq m+1)$ are the children of n ordered by $\beta(d_1) > \beta(d_2) > \cdots > \beta(d_s) = 1$. Moving $T'_{d_i} (1 \leq i \leq m)$ to the node 1, and moving the subtree of x that contains 1 to the node n, we get the above tree T.

(d) Suppose $T \in \mathcal{R}^{(0)}_{n,k}[\deg(1) = m, 1 \prec n, \deg(n) = 1]$, and let b be the unique child of n. Assume that $a_1, a_2, \ldots, a_m$ are the children of 1 ordered by $\beta(a_1) < \beta(a_2) < \cdots < \beta(a_m)$. Moving T_{a_i} $(2 \leq i \leq m)$ to the node n, and let R be the resulting tree. Substituting $S = R_b$ with S' by applying Lemma 3.6, we obtain a tree

$$T' \in \mathcal{R}^{(0)}_{n,k+m}[\deg(1) = 0, \deg(n) = m, 1 \prec n, \lambda > 1].$$

Conversely, for this tree T', we assume that the subtree of n that contains the node 1 is S'. Applying Lemma 3.6 to S', we may recover S. Moving the other $m-1$ subtrees of n to the node 1, we obtain the above tree T. ∎

After such an exciting and exhausting journey, we finally come to our destination—Theorem 3.1. The essence of the Theorem 3.1 is the duality between the minimum element and the maximum element in a rooted tree. It is easy to imagine that the labels of a rooted tree do not have to be a consecutive segment of integers in order for the bijection to hold. For this reason, we say that a rooted tree T is relabeled by a set V of the same number of nodes if the minimum node of T is relabeled by the minimum node in V, the second minimum node is relabeled by the second minimum node in V, and so forth. By applying Theorem 3.1, we can construct the main bijection of this paper, leading to a combinatorial proof of Theorem 2.3.

Theorem 3.8 *For $1 \leq r \leq n$ and $0 \leq k < n - r$, we have the following bijection:*

$$\mathcal{T}_{n+1,k}[\deg(2) > 0, \deg(1) = r] \longleftrightarrow \mathcal{T}_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r]. \qquad (3.15)$$

Proof. It is obvious that for $r = n$ both sides of (3.8) are empty. So we may assume that $1 \leq r \leq n-1$. First, it is easy to see that the case $r = 1$ reduces to Theorem 3.1, by relabeling the set $\{2, 3, \ldots, n+1\}$ with $\{1, 2, \ldots, n\}$. Thus, we may assume that $r \geq 2$. Suppose $T \in \mathcal{T}_{n+1,k}[\deg(2) > 0, \deg(1) = r]$. Assume x is the child of the root 1 such that 2 is a descendant of x in T, and y is the child of the root 1 such that $n+1$ is a descendant of y in T. Note that it is possible that $x = y$. We now proceed to construct a tree $T' \in \mathcal{T}_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r]$. We have three cases to consider:

Case 1. $x = y$. In this case, the minimum element 2 and the maximum element $n+1$ both appear in the subtree T_x. Applying the Theorem 3.1 on T_x, we are led to a rooted tree T'_x. Substituting T_x by T'_x in T, we obtain a rooted tree $T' \in \mathcal{T}_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r]$.

Case 2. $x \neq y$, and $\deg_T(n+1) > 0$. We also apply Theorem 3.1 on T_x, and we may obtain a rooted tree $T' \in \mathcal{T}_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r]$ as in Case 1.

Case 3. $x \neq y$, and $\deg_T(n+1) = 0$. Let us relabel the subtrees T_x and T_y. Suppose T_x has nodes 2 and $u_1 < u_2 < \cdots < u_i$ and T_y has nodes $n+1$ and $v_1 < v_2 < \cdots < v_j$. Let R be the rooted tree obtained from T_x by relabeled by $u_1 < u_2 < \cdots < u_i$ and $n+1$, and S be the rooted tree obtained from T_y relabled by 2 and $v_1 < v_2 < \cdots < v_j$. Applying Theorem 3.1 on R, we obtain a rooted tree R' with $\deg_{R'}(n+1) > 0$. Now substituting T_x by R' and T_y by S, we are led to a rooted tree T', which is clearly in $\mathcal{T}_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r]$.

Since all the above steps are reversible. We now only need to classify the cases for a tree $T' \in \mathcal{T}_{n+1,k+1}[\deg(n+1) > 0, \deg(1) = r]$ so that they can fit into one of the above three cases.

Case A. If 2 and $n+1$ are in the same subtree T'_x where x is a child of the root 1, then we resort to the reverse of Case 1 to recover the tree $T \in \mathcal{T}_{n+1,k}[\deg(2) > 0, \deg(1) = r]$.

Case B. Suppose u and v are the children of 1 in T' such that T'_u contains $n+1$ and T'_v contains 2. If the degree of the maximum element in T'_v is nonzero, then we may resort the reverse of the construction in Case 2 to recover T. Otherwise, the degree of the maximum element in T'_v equals zero. In this case, we may count on the reverse procedure of Case 3 to recover the desired T. This completes the proof. ∎

4 Open Problems

In evaluation of the bijections presented in this paper, the construction of (3.4) seems to be much more technical than it should be, especially when compared with the case

(3.3). We would very much like to propose the following problem.

Problem 4.1 *Find an intrinsic construction for the bijection (3.1). In particular, Lemma 3.6 deserves a better explanation.*

The inductive proofs of Theorem 2.3 and Theorem 3.1 might serve a hint if they can be informatively translated into a bijective scheme.

The next problem is concerned with a refined version of the recurrence relation for the numbers $f_{n,k} = |\mathcal{R}_{n,k}|$. Recall that $\lambda(T)$ denotes the lower critical node of T, as defined in the previous section. Notice that $\lambda(T)$ is defined only for a tree T such that $\deg_T(n) > 0$, where n is the maximum node. For notational simplicity, we leave out the condition $\deg_T(n) > 0$ when the condition $\lambda = i$ is present. We have the following conjecture:

Conjecture 4.2 *For $n \geq 3$ and $1 \leq i \leq n-2$, we have the recurrence relation:*

$$|\mathcal{R}_{n,k}[\lambda = i]| = (n-2)\,|\mathcal{R}_{n-1,k}[\lambda = i]| + (n+k-3)\,|\mathcal{R}_{n-1,k-1}[\lambda = i]|. \tag{4.1}$$

Some numerical evidence in support of the above conjecture is presented below for speculation.

Table of $R_{n,k}[\lambda = 1]$.

$k\backslash n$	2	3	4	5
1	1	1	2	6
2		2	7	29
3			8	59
4				48

Table of $R_{n,k}[\lambda = 2]$.

$k\backslash n$	2	3	4	5
1		1	2	6
2		1	5	23
3			4	37
4				24

Table of $R_{n,k}[\lambda = 3]$.

$k\backslash n$	2	3	4	5
1			2	6
2			4	20
3			3	29
4				18

Here are some very special cases:

$$|\mathcal{R}_{n,1}[\lambda = i]| = |\mathcal{R}_{n-1,0}| = (n-2)!, \quad 1 \leq i \leq n-1, \tag{4.2}$$

$$|\mathcal{R}_{n,k}[\lambda = n-1]| = |\mathcal{R}_{n-1,k-1}|, \quad 1 \leq k \leq n-1. \tag{4.3}$$

If the above conjecture is true, then we can use induction to derive the recurrence relation (1.8) from (4.1), (4.3), and the obvious identity

$$|\mathcal{R}_{n,k}[\deg(n) = 0]| = (n-1)|\mathcal{R}_{n-1,k}|.$$

The following special case is worth mentioning:

$$|\mathcal{R}_{n,n-1}| = (2n-3)!! = 1 \cdot 3 \cdot \cdots \cdot (2n-3). \tag{4.4}$$

We may make a connection to increasing plane trees. A rooted tree on $[n]$ is called *increasing* if any path from the root to another vertex forms an increasing sequence. As an equivalent statement, we may say that an increasing tree is a rooted tree without improper edges. We have the following observation

Proposition 4.3 *There is a bijection between the set of rooted trees on $[n]$ with $n-1$ improper edges and the set of plane trees on $[n]$ without improper edges.*

Note that $(2n-3)!!$ is also the number of increasing plane trees on $[n]$. Here is a combinatorial interpretation. Let T be a tree in $\mathcal{R}_{n,n-1}$. Then 1 has to be a leaf of T, and all edges of T are improper. Suppose that $(1, v_1, v_2, \ldots, v_t)$ is the path from 1 to the root. Then we may recursively construct an increasing plane tree T'. If T has only one node, then T' is the same as T. If $n > 1$, then for each $T(v_i)$ construct the corresponding increasing plane tree, and put them together by joining them to the minimum node in the order of $(v_1, v_2, \ldots, v_t)$. An example is given in Figure 7. The tree on the left is a rooted tree in which every edge is improper, and the tree on the right is an increasing plane tree.

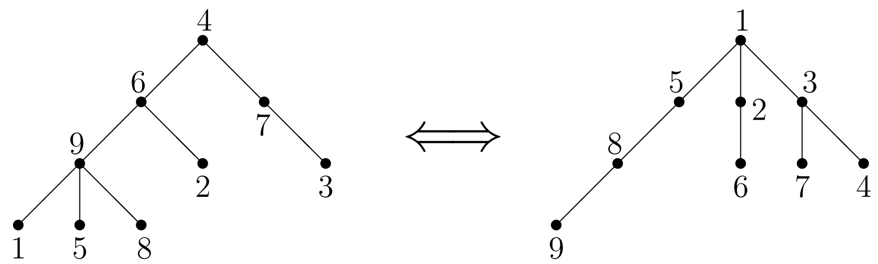

Figure 7: Example for Proposition 4.3

We now state a problem based on the above simple observation, yet to be better understood.

Problem 4.4 *Since $Q_{n,0}(x)$ corresponds to increasing trees on $[n]$ while $Q_{n,n-1}(x)$ corresponds to increasing plane trees on $[n]$, there must be some kind of combinatorial structure like partial increasing plane trees, a notion of interpolation of increasing trees and increasing plane trees. Such a structure should serve the purpose as an alternative combinatorial interpretation of the Ramanujan polynomials.*

It is quite intriguing that there lie rich combinatorial structures behind the Ramanujan polynomials. No doubt that we may expect more episodes of uncovering further mysteries plotted by these polynomials. Hopefully, we have made some room for imagination, and we may (in any case) keep our fingers crossed with respect to further developments.

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