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The homogeneous q-difference operator

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Abstract

We introduce a q-differential operator D_{xy} on functions in two variables which turns out to be suitable for dealing with the homogeneous form of the q-binomial theorem as studied by Andrews, Goldman, and Rota, Roman, Ihrig, and Ismail, et al. The homogeneous versions of the q-binomial theorem and the Cauchy identity are often useful for their specializations of the two parameters. Using this operator, we derive an equivalent form of the Goldman–Rota binomial identity and show that it is a homogeneous generalization of the q-Vandermonde identity. Moreover, the inverse identity of Goldman and Rota also follows from our unified identity. We also obtain the q-Leibniz formula for this operator. In the last section, we introduce the homogeneous Rogers–Szegö polynomials and derive their generating function by using the homogeneous q-shift operator.

Keywords: q-binomial theorem; Cauchy polynomials; q-Vandermonde identity; Homogeneous q-difference operator; q-Leibniz formula; Homogeneous Rogers–Szegö polynomials

1. Introduction

We adopt the common conventions and notations on q-series. So we always assume that $|q| < 1$ and use the following notation of the q-shifted factorial:

$$(x;q)_0 = 1, \qquad (x;q)_n = \prod_{j=0}^{n-1}\left(1 - q^j x\right), \quad n = 1, 2, \ldots, \infty.$$

The basic hypergeometric series ${}_r\phi_s$ is defined as follows [6]:

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$$
\begin{aligned}
&{}_r\phi_s(x_1, x_2, \ldots, x_r; y_1, y_2, \ldots, y_s; q, t) \\
&\quad = {}_r\phi_s\left[\begin{matrix} x_1, x_2, \ldots, x_r \\ y_1, y_2, \ldots, y_s \end{matrix}; q, t\right] = \sum_{n=0}^{\infty} \frac{(x_1; q)_n (x_2; q)_n \cdots (x_r; q)_n}{(y_1; q)_n (y_2; q)_n \cdots (y_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} t^n,
\end{aligned}
$$

where $q \neq 0$ when $r > s + 1$.

The q-binomial coefficient is given by:

$$
\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_{n-k}(q; q)_k}.
$$

The following is the homogeneous form of the q-shifted factorial:

$$
P_n(x, y) = (y/x; q)_n x^n = (x - y)(x - qy) \cdots \left(x - q^{n-1} y\right).
$$

We also have the following basic relations:

$$
\begin{aligned}
&\begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} = \frac{(q^{-n}; q)_k q^{nk}}{(q; q)_k}, \\
&P_n(x, y) = (-1)^n q^{\binom{n}{2}} P_n\left(y, q^{1-n} x\right), \\
&P_{n-k}\left(x, q^{1-n} y\right) = (-1)^{n-k} q^{\binom{k}{2} - \binom{n}{2}} P_{n-k}\left(y, q^k x\right).
\end{aligned}
$$

The polynomials $P_n(x, y)$ are important in the q-umbral calculus as studied by Andrews [1,2], Goldman and Rota [5], Goulden and Jackson [7], Ihrig and Ismail [8], Roman [13], Johnson [11], et al. In the q-umbral calculus, the polynomial sequence $P_n(x, y)$ is a homogeneous Eulerian family. By vector space arguments, Goldman and Rota [3,5] have shown the following q-binomial identity. This identity may be known earlier, but we do not have accurate information on the reference:

$$
P_n(x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, z) P_{n-k}(z, y). \tag{1.1}
$$

Let V_n be an n-dimensional vector space over the finite field of q elements, and X, Y, Z be vector spaces over $GF(q)$ such that $|X| = x$, $|Y| = y$, and $|Z| = z$ where $|X|$ denotes the number of vectors in X. Assuming that $Z \subset Y \subset X$ and $\dim V_n < \dim Z$, Goldman and Rota [5] show that the above identity counts in two ways the set of all one-to-one linear transformations $f : V_n \to X$ such that $f^{-1}(Z) = 0$. Setting $y = 0$ and $z = 1$ in (1.1), one obtains the following identity due to Cauchy:

$$
x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (x - 1)(x - q) \cdots \left(x - q^{k-1}\right). \tag{1.2}
$$

Note that the polynomials $P_n(x, 1) = (x - 1)(x - q) \cdots (x - q^{n-1})$ are sometimes called the Gauss polynomials. A direct combinatorial argument for the above identity of Cauchy is also given by Goldman and Rota [5]. For further background on the above q-binomial theorem and its specializations, the reader is referred to the introduction written by Kung [12]. Moreover, by Möbius inversion, Goldman and Rota obtain an identity which leads to a partition identity, generalizing Durfee's identity.

$$P_n(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}P_k(y,1)P_{n-k}\bigl(x,q^k\bigr). \tag{1.3}$$

It was not obvious how to show the equivalence of the above two q-binomial theorems (1.1) and (1.3). Here we give a derivation:

$$\begin{aligned}P_n(x,y)&=(-1)^n q^{\binom{n}{2}}P_n\bigl(y,q^{1-n}x\bigr)\\&=(-1)^n q^{\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(y,1)P_{n-k}\bigl(1,q^{1-n}x\bigr)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}P_k(y,1)P_{n-k}\bigl(x,q^k\bigr).\end{aligned}$$

Goulden and Jackson [7] give a similar derivation of (1.3) from (1.1). Moreover, they give an interpretation of the polynomials $Q_n(x,y)=P_n(x,-y)$ in terms of q-counting of certain permutations (bimodal permutations). The following exchange property of $Q_n(x,y)$ is given by Goulden and Jackson [7]

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}Q_k(x,y)Q_{n-k}(w,z)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}Q_k(w,y)Q_{n-k}(x,z).$$

Note that there is a notation for $Q_n(x,y)$ in the literature following F.H. Jackson [9] as mentioned by Johnson [11]:

$$(x+y)^{[n]}=(x+y)(x+qy)\cdots\bigl(x+q^{n-1}y\bigr).$$

Because the polynomials $P_n(x,y)$ occur so often in q-series that they may deserve a name. We propose to call them the *Cauchy polynomials* for the reason that they are the coefficients in the expansion of the homogeneous version of the Cauchy identity (or the q-binomial theorem):

$$\sum_{n=0}^{\infty}\frac{P_n(x,y)}{(q;q)_n}t^n=\frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{1.4}$$

Setting $y=0$, the Cauchy identity becomes Euler's identity:

$$\frac{1}{(xt;q)_\infty}=\sum_{n=0}^{\infty}\frac{x^n t^n}{(q;q)_n}. \tag{1.5}$$

It seems to be neglected that the q-binomial theorem of Goldman and Rota, and the above exchange property of $Q_n(x, y)$ both are immediate from the above homogeneous form of the Cauchy identity.

The main result of this paper is to introduce the operator D_{xy} on functions in two variables x and y. This operator turns out to be suitable for dealing with the Cauchy polynomials $P_n(x, y)$. We derive a binomial identity which unifies the two identities of Rota and Goldman, as well as the q-Vandermonde identity. Moreover, our identity can be shown to be equivalent to the Goldman–Rota binomial identity, and it can be regarded as a homogeneous generalization of the q-Vandermonde identity.

Based on the q-Leibniz formula for the classical q-difference operator, we obtain the q-Leibniz formula for the homogeneous q-difference operator. It turns out the Cauchy polynomials also appear in the homogeneous q-Leibniz formula. In the last section, we introduce the homogeneous Rogers–Szegö polynomials and the q-shift operator. The generating function of the homogeneous Rogers–Szegö polynomials is derived.

2. The homogeneous q-difference operator

Recall that the classical q-difference operator, or the q-derivative, acting on functions on variable x, D_q is defined by:

$$D_q f(x) = \frac{f(x) - f(qx)}{x}.$$

Note that when the function f is in the context of hypergeometric functions, the variable x is often used as a parameter, but throughout this paper D_q is always acting on x. The operator D_q is also the Euler–Jackson difference operator [10]. It may also be expressed in terms of the q-shift operator on the variable x:

$$\eta_x f(x) = f(qx).$$

Thus, we may write

$$D_q = \frac{1 - \eta_x}{x}.$$

Notice that the inverse of η_x is denoted by $\theta_x = \eta_x^{-1}$.

Andrews [1,2] employs the q-difference operator to study the Cauchy polynomials for the case $y = 1$, and observes the following relation:

$$D_q P_n(x, 1) = (1 - q^n) P_{n-1}(x, 1).$$

The motivation of this paper to introduce a new operator which is suitable for the study of the Cauchy polynomials:

$$D_{xy} f(x, y) = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}, \tag{2.1}$$

where x and y are variables. We now give the first theorem of this paper, which is straightforward to verify.

Theorem 2.1. *We have*

$$D_{xy}\{P_n(x,y)\} = (1-q^n)P_{n-1}(x,y). \tag{2.2}$$

Obviously, for any constant c, one has $D_{xy}c = 0$. Moreover, one may have the following property of the q-difference operator.

Proposition 2.2. *If $f(x,y)$ and $g(x,y)$ are homogeneous polynomials of the same degree n, and $H(x,y) = f(x,y)/g(x,y)$, then we have*

$$D_{xy}H(x,y) = 0.$$

From (2.2), we obtain the following property:

Proposition 2.3. *We have*

$$D_{xy}\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t\frac{(yt;q)_\infty}{(xt;q)_\infty}, \tag{2.3}$$

$$D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t^k\frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{2.4}$$

We use θ_y for the operator acting on the variable y. Clearly,

$$\theta_y\eta_x = \eta_x\theta_y. \tag{2.5}$$

We define $P_n(\theta_y,\eta_x)$ as the following operator:

$$P_n(\theta_y,\eta_x) = (\theta_y-\eta_x)(\theta_y-q\eta_x)\cdots\bigl(\theta_y-q^{n-1}\eta_x\bigr). \tag{2.6}$$

The following theorem gives the expansion of the power of D_{xy} in terms of operations on x and y individually.

Theorem 2.4. *We have*

$$D_{xy}f(x,y) = \frac{(\theta_y-\eta_x)\{f(x,y)\}}{x-q^{-1}y}, \tag{2.7}$$

$$D_{xy}^n f(x,y) = \frac{P_n(\theta_y,q^{1-n}\eta_x)\{f(x,y)\}}{P_n(x,q^{-n}y)}. \tag{2.8}$$

Proof.

$$D_{xy}^{n+1}\{f(x,y)\}\big(x-q^{-1}y\big) = \frac{\theta_y P_n(\theta_y, q^{1-n}\eta_x)\{f(x,y)\}}{P_n(x, q^{-n-1}y)} - \frac{\eta_x P_n(\theta_y, q^{1-n}\eta_x)\{f(x,y)\}}{P_n(qx, q^{-n}y)}$$
$$= \frac{(\theta_y - q^{-n}\eta_x) P_n(\theta_y, q^{1-n}\eta_x)\{f(x,y)\}}{P_n(x, q^{-n-1}y)}$$
$$= \frac{P_{n+1}(\theta_y, q^{-n}\eta_x)\{f(x,y)\}}{P_n(x, q^{-n-1}y)}. \qquad \square$$

From (2.5) and (2.6), we have

Lemma 2.5. *We have*

$$P_n(\theta_y, \eta_x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \eta_x^k \theta_y^{n-k}. \tag{2.9}$$

Theorem 2.4 can be rewritten as:

Theorem 2.6. *The operator D_{xy}^n has the following expansion*:

$$D_{xy}^n\{f(x,y)\} = \frac{1}{\prod_{k=1}^n \theta_y^k\{x-y\}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} \eta_x^k \theta_y^{n-k}\{f(x,y)\}$$
$$= \frac{1}{P_n(x, q^{-n}y)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} f\big(q^k x, q^{k-n} y\big).$$

From (2.4) and Theorem 2.6, we have

$$D_{xy}^n\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \frac{1}{P_n(x, q^{-n}y)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} \frac{(q^{k-n}yt;q)_\infty}{(q^k xt;q)_\infty}$$
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{1}{P_n(x, q^{-n}y)}$$
$$\times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} (xt;q)_k \big(q^{k-n} yt; q\big)_{n-k}.$$

We now arrive at the following identity:

$$t^n P_n(x, q^{-n}y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} (xt;q)_k \big(q^{k-n} yt; q\big)_{n-k}. \tag{2.10}$$

Note that the above identity is an equivalent form of the Goldman–Rota q-binomial identity. However, this form has the advantage of specializing to the inverse Goldman–Rota identity (1.3) and it can be viewed as a homogeneous version of the q-Vandermonde identity:

$$ {}_2\phi_1(q^{-n}, x; y; q, q) = \frac{(y/x;q)_n}{(y;q)_n} x^n. \tag{2.11}$$

For given n, we may specialize the values of the parameters in (2.10) to obtain some classical results.

- Setting $t \to 1/z$, $q^{-1}y \to y$, and exchanging x and y, we obtain Goldman–Rota q-binomial identity (1.1). Thus, we may say that the formula (2.10) is equivalent to the Goldman–Rota q-binomial theorem.
- Setting $t \to 1$ and $q^{-n}y \to y$, we obtain the q-Vandermonde identity (2.11). Indeed, setting $1/t \to z$ and $q^{-n}y \to y$ one may rewrite (2.10) in the following form:

$$P_n(x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{(1-n)k} P_k\big(q^{k-1}x, z\big) P_{n-k}\big(z, q^k y\big).$$

- Setting $t \to q^{1-n}$ and $q^{-n}y \to y$, we get the inverse Goldman–Rota identity (1.3). In (1.3), setting $1/y$ by y and $1/x$ by x then setting $n \to \infty$, we obtain the following identity [6]:

$$ {}_1\phi_1(y; x; q, x/y) = \frac{(x/y;q)_\infty}{(x;q)_\infty}.$$

3. A homogeneous q-Leibniz formula

In this section, we give a homogeneous q-Leibniz formula for the operator D_{xy}. In order to present a non-inductive proof, we will use the q-Leibniz formula for the classical q-difference operator D_q [13,14]

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(x)\} D_q^{n-k}\big\{g\big(q^k x\big)\big\}.$$

Theorem 3.1. *For $n \geqslant 0$, we have*

$$D_{xy}^n\{f(x, y)g(x, y)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_{n-k}(q^{-1}y, x)}{P_{n-k}(q^{-1}y, q^k x)} D_{xy}^k\big\{g\big(q^{n-k}x, y\big)\big\} D_{xy}^{n-k}\big\{f\big(x, q^{-k}y\big)\big\}.$$

Proof. Let $y = xzq$, then we have $F(x,z) = f(x,y)$, and $G(x,z) = g(x,y)$. It follows that

$$D_{xy} = \frac{1}{1-z} D_q \theta_z \tag{3.1}$$

and

$$D_q \theta_z = \theta_z D_q. \tag{3.2}$$

Therefore,

$$D_{xy}^k = \frac{1}{(q^{1-k}z;q)_k} D_q^k \theta_z^k. \tag{3.3}$$

Thus, we have

$$\begin{aligned}
D_{xy}^n\{f(x,y)g(x,y)\} &= \frac{1}{(q^{1-n}z;q)_n} D_q^n \theta_z^n \{F(x,z)G(x,z)\} \\
&= \frac{1}{(q^{1-n}z;q)_n} \theta_z^n D_q^n \{F(x,z)G(x,z)\} \\
&= \frac{1}{(q^{1-n}z;q)_n} \theta_z^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{F(x,z)\} D_q^{n-k}\bigl\{G\bigl(q^k x, z\bigr)\bigr\} \\
&= \frac{1}{(q^{1-n}z;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \theta_z^k \bigl\{F\bigl(x, q^{k-n}z\bigr)\bigr\} \\
&\quad \times D_q^{n-k} \theta_z^{n-k} \bigl\{G\bigl(q^k x, q^{-k}z\bigr)\bigr\} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_k(q^{-1}y,x)}{P_k(q^{-1}y,q^{n-k}x)} D_{xy}^k \bigl\{f\bigl(x, q^{k-n}y\bigr)\bigr\} D_{xy}^{n-k}\bigl\{g\bigl(q^k x, y\bigr)\bigr\} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_{n-k}(q^{-1}y,x)}{P_{n-k}(q^{-1}y,q^{k}x)} D_{xy}^k \bigl\{g\bigl(q^{n-k}x, y\bigr)\bigr\} \\
&\quad \times D_{xy}^{n-k} \bigl\{f\bigl(x, q^{-k}y\bigr)\bigr\}. \qquad \square
\end{aligned}$$

Clearly, setting $z = 0$, namely, $y = 0$, we have:

$$D_{xy}^k = D_q^k.$$

Corollary 3.2. *We have*

$$D_{xy}^n\{f(x,y)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-x)^k q^{\binom{k}{2}}}{P_k(q^{-1}y,q^{n-k}x)} D_q^k \bigl\{g\bigl(q^{n-k}x\bigr)\bigr\} D_{xy}^{n-k}\bigl\{f\bigl(x,q^{-k}y\bigr)\bigr\}.$$

4. The homogeneous q-shift operator

Based on the homogeneous q-difference operator, one can build up the homogeneous q-shift operator as the q-exponential of the homogeneous q-difference operator:

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D_{xy}^k}{(q;q)_k}. \tag{4.1}$$

The following proposition for the homogeneous q-shift operator immediately follows from Proposition 2.3:

Proposition 4.1. *We have*

$$\mathbb{E}(D_{xy}) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \right\} = \frac{(yt;q)_\infty}{(t;q)_\infty (xt;q)_\infty}.$$

The q-shift operator is suitable for the study of the homogeneous Rogers–Szegö polynomials which are defined by

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y).$$

Note that setting $y = 0$ the polynomials $h_n(x,y)$ reduces to the classical Rogers–Szegö polynomials $h_n(x|q)$. Recall that $h_n(x|q)$ can be expressed in terms of the q-shift operator $T(D_q)x^n$, where

$$T(D_q) = \sum_{n=0}^{\infty} \frac{D_q^n}{(q;q)_n}.$$

The operator $T(D_q)$ is called the augmentation operator in [4], which can be used to derive the generating function of $h_n(x|q)$:

$$\sum_{n=0}^{\infty} \frac{h_n(x|q)t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty (xt;q)_\infty}. \tag{4.2}$$

From (2.2), we obtain the following formula:

$$E(D_{xy})\{P_n(x,y)\} = h_n(x,y|q). \tag{4.3}$$

Next we present the generating function for the homogeneous Rogers–Szegö polynomials.

Theorem 4.2. *We have*

$$\sum_{n=0}^{\infty} \frac{h_n(x, y|q)t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t;q)_\infty (xt;q)_\infty}.$$

Proof. By Proposition 4.1, we have

$$\sum_{n=0}^{\infty} \frac{h_n(x, y|q)t^n}{(q;q)_n} = E(D_{xy})\left\{\frac{P_n(x, y)t^n}{(q;q)_n}\right\} = E(D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}$$
$$= \frac{(yt;q)_\infty}{(t;q)_\infty (xt;q)_\infty}.$$

This completes the proof. □

Setting $y = 1$ in the above theorem, by Euler's identity (1.5) we are led to the evaluation $h_n(x, 1|q) = x^n$, which is the Cauchy identity (1.2).

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