Cauchy Augmentation for Basic Hypergeometric Series

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August 5, 2002

Abstract. We present a technique of deriving basic hypergeometric identities from their specializations with a fewer number of parameters by using the classical Cauchy identity on the expansion of the power of x in terms of the q-binomial coefficients. We call method the Cauchy augmentation. Despite its simple appearance, the Cauchy identity plays a marvelous role for parameter augmentation. For example, from the Euler identity one can reach the q-Gauss summation formula by using the Cauchy augmentation twice. This idea also applies to Jackson's ${}_2\phi_1$ to ${}_3\phi_1$ transformation formula. Moreover, we obtain a transformation formula analogous to Jackson's formula.

Keywords: Cauchy augmentation, Euler identity, Cauchy identity, Gauss identity, basic hypergeometric series, Jackson's transformation formula.

AMS Classification: 05A30, 33D15

1. Introduction

We will follow the standard notation on q-series [1, 5], and we always assume $|q| < 1$. The q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ are defined as

$$\begin{aligned}(a;q)_n &= \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-qa)\cdots(1-q^{n-1}a), & \text{if } n \geq 1, \end{cases} \\ (a;q)_\infty &= (1-a)(1-qa)(1-q^2a)\cdots.\end{aligned}$$

The basic hypergeometric series ${}_r\phi_s$ involved in this paper obeys the general definition:

$${}_r\phi_s\left[\begin{matrix}a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_2\end{matrix}; q; z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

where $q \neq 0$ when $r > s+1$. The q-binomial coefficients, or the *Gauss coefficients*, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

One of the most classical identities in q-series is Euler's identity:

$$\frac{1}{(t;q)_\infty} = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} t^n, \quad |t| < 1,\ |q| < 1. \tag{1.1}$$

The following polynomials

$$P_n(x,y) = (x-y)(x-qy)\cdots(x-q^{n-1}y), \tag{1.2}$$

are called the Cauchy polynomials [2], and a homogeneous difference operator is introduced in [2] for the study of these polynomials and various versions of q-binomial theorems.

Another classical q-series identity is the q-binomial theorem due to Cauchy:

$$\sum_{n=0}^{\infty} \frac{P_n(x,y)}{(q;q)_n} t^n = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \qquad |xt| < 1,\ |q| < 1. \tag{1.3}$$

Note that when setting $y = 0$ and $x = 1$, the Cauchy identity reduces to Euler's identity. Usually, the Cauchy identity is stated for $x = 1$ where $P_n(1,y)$ becomes the q-shifted factorial. On the other hand, the special Cauchy polynomials

$$P_n(x,1) = (x-1)(x-q)\cdots(x-q^{n-1})$$

is of considerable importance because of the following identity often attributed to Cauchy:

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (x-1)(x-q)\cdots(x-q^{k-1}). \tag{1.4}$$

A less straightforward summation formula beyond Euler (1.1) and Cauchy (1.3) is Heine's q-analogue of the Gauss ${}_2F_1$ summation formula:

$${}_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}, \quad |c/ab| < 1. \tag{1.5}$$

As one can see in this paper, the Cauchy identity (1.3) is in fact a specialization of the q-Gauss summation formula (1.5).

The objective of this paper is show how one can obtain a general formula for its specialization. Such an approach by using exponential operators is called parameter augmentation in [3, 4]. It is a little surprising that the Cauchy identity (1.4) plays a marvelous role for parameter augmentation. We will give several classical examples as well as a probably new transformation formula to demonstrate how to implement this idea, and we call this technique *the Cauchy augmentation*.

2. From Euler to Gauss

Our first step is to show how to use the Cauchy identity (1.4) for parameter augmentation. Applying the Cauchy identity to Euler's identity, we immediately obtain the Cauchy identity in q-series form (1.3). Substituting t by tx in (1.1), we get

$$\begin{aligned}
\frac{1}{(tx;q)_\infty} &= \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}(tx)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} t^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_n(x,1) \quad \text{(by Cauchy (1.4))} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{P_k(x,1)}{(q;q)_k (q;q)_{n-k}} t^n \\
&= \sum_{k=0}^{\infty} \frac{P_k(x,1)}{(q;q)_k} t^k \sum_{n=k}^{\infty} \frac{1}{(q;q)_{n-k}} t^{n-k} \\
&= \frac{1}{(t;q)_\infty} \sum_{k=0}^{\infty} \frac{P_k(x,1)}{(q;q)_k} t^k. \quad \text{(by Euler (1.1))}
\end{aligned}$$

It follows that

$$\sum_{k=0}^{\infty} \frac{P_k(x,1)}{(q;q)_k} t^k = \frac{(t;q)_\infty}{(tx;q)_\infty}. \tag{2.1}$$

Setting $x \to x/y$ and $t \to ty$ in (2.1), we obtain the Cauchy identity (1.3).

The above idea of Cauchy augmentation, namely, expanding x^n by the Cauchy identity and exchanging the order of summations, can go much further. Using the Cauchy augmentation one more time, we arrive at the q-Gauss summation formula. Notice the following basic relations

$$P_n(x,y) = P_k(x,y)\, P_{n-k}(x, q^k y), \quad 0 \le k \le n. \tag{2.2}$$

$$(a;q)_\infty = (a;q)_k\, (q^k a;q)_\infty, \qquad k \ge 0. \tag{2.3}$$

From the Cauchy identity (1.3), we get

$$\begin{aligned}
\frac{(btx;q)_\infty}{(atx;q)_\infty} &= \sum_{n=0}^{\infty} \frac{P_n(a,b)}{(q;q)_n}(xt)^n \\
&= \sum_{n=0}^{\infty} \frac{P_n(a,b)}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,1)t^n \qquad \text{(by Cauchy (1.4))} \\
&= \sum_{k=0}^{\infty} \frac{P_k(x,1)P_k(a,b)}{(q;q)_k} t^k \sum_{n=k}^{\infty} \frac{P_{n-k}(a,q^k b)}{(q;q)_{n-k}} t^{n-k} \quad \text{(by (2.2))} \\
&= \sum_{k=0}^{\infty} \frac{P_k(x,1)P_k(a,b)}{(q;q)_k} t^k \frac{(q^k bt;q)_\infty}{(at;q)_\infty} \qquad \text{(by Cauchy (1.3))} \\
&= \frac{(bt;q)_\infty}{(at;q)_\infty} \sum_{k=0}^{\infty} \frac{P_k(x,1)P_k(a,b)}{(bt;q)_k(q;q)_k} t^k \qquad \text{(by (2.3))} \\
&= \frac{(bt;q)_\infty}{(at;q)_\infty} {}_2\phi_1(b/a, 1/x; bt; q, atx).
\end{aligned}$$

It follows that

$$ {}_2\phi_1(b/a, 1/x; bt; q, atx) = \frac{(at;q)_\infty}{(bt;q)_\infty} \frac{(btx;q)_\infty}{(atx;q)_\infty}. \tag{2.4}$$

Setting $bt \to c$, $b/a \to a$ and $1/t \to b$ in (2.4), we obtain Heine's q-analogue of the Gauss summation formula (1.5).

3. From Euler to ${}_1\phi_1$

The dual form or the inverse form of Euler's identity (1.1) states

$$(t;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} t^n, \tag{3.1}$$

or one can write it as $(t;q)_\infty = {}_0\phi_0(-;-;q;t)$. Applying the Cauchy augmentation to Euler's identity (3.1), one obtains the following summation formula for ${}_1\phi_1$ [5, p. 21]:

$$ {}_1\phi_1(a;c;q,c/a) = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{3.2}$$

Here is the proof by the Cauchy augmentation:

$$
\begin{aligned}
(cx;q)_\infty &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}(cx)^n \qquad \text{(by Euler (3.1))}\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} c^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,1) \qquad \text{(by Cauchy (1.4))}\\
&= \sum_{k=0}^{\infty} \frac{P_k(x,1)(-1)^k q^{\binom{k}{2}}}{(q;q)_k} c^k \sum_{n=k}^{\infty} \frac{(-1)^{n-k} q^{\binom{n-k}{2}}}{(q;q)_{n-k}} (q^k c)^{n-k}\\
&= \sum_{k=0}^{\infty} \frac{P_k(x,1)(-1)^k q^{\binom{k}{2}}}{(q;q)_k} c^k (q^k c;q)_\infty \qquad \text{(by Euler (3.1))}\\
&= (c;q)_\infty \sum_{k=0}^{\infty} \frac{(1/x;q)_k(-1)^k q^{\binom{k}{2}}}{(c;q)_k(q;q)_k}(cx)^k. \qquad \text{(by 2.3)}
\end{aligned}
$$

Thus, we get

$$
\sum_{k=0}^{\infty} \frac{(1/x;q)_k(-1)^k q^{\binom{k}{2}}}{(c;q)_k(q;q)_k}(cx)^k = \frac{(cx;q)_\infty}{(c;q)_\infty}. \tag{3.3}
$$

Substituting x with $1/a$, one obtains (3.2).

4. From q-Vandermonde-Chu to Jackson

The following transformation formula from ${}_2\phi_1$ to ${}_3\phi_1$ of Jackson [5, p. 23] naturally falls into the framework of the Cauchy augmentation:

$$
{}_2\phi_1(q^{-n}, b; c; q, z) = \frac{(c/b;q)_n}{(c;q)_n} b^n {}_3\phi_1(q^{-n}, b, q/z; bq^{1-n}/c; q, z/c). \tag{4.1}
$$

Recall the q-Vandermonde-Chu formula:

$$
{}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b;q)_n}{(c;q)_n} b^n. \tag{4.2}
$$

We now give the Cauchy augmentation procedure for Jackson's transformation formula starting from the q-Vandermonde-Chu identity.

Proof. Substituting z with qz, the left hand side of (4.1) equals

$$\begin{aligned}
&\sum_{m=0}^{n} \frac{(q^{-n};q)_m (b;q)_m}{(c;q)_m (q;q)_m}\,(zq)^m \\
&= \sum_{m=0}^{n} \frac{(q^{-n};q)_m (b;q)_m}{(c;q)_m (q;q)_m}\,q^m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} P_k(z,1) \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (b;q)_k P_k(z,1)}{(c;q)_k (q;q)_k}\,q^k \sum_{m=k}^{n} \frac{(q^{-n+k};q)_{m-k} (q^k b;q)_{m-k}}{(q^k c;q)_{m-k} (q;q)_{m-k}} q^{m-k} \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (b;q)_k P_k(z,1)}{(c;q)_k (q;q)_k}\,q^k {}_2\phi_1(q^{-(n-k)}, q^k b; q^k c; q, q) \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (b;q)_k P_k(z,1)}{(c;q)_k (q;q)_k}\,q^k \frac{(c/b;q)_{n-k}}{(q^k c;q)_{n-k}} (q^k b)^{n-k} \qquad \text{(by (4.2))} \\
&= \frac{1}{(c;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (b;q)_k (1/z;q)_k (qz)^k}{(q;q)_k}\,(c/b;q)_{n-k} (q^k b)^{n-k}.
\end{aligned}$$

Let us write $(c/b;q)_{n-k}$ as

$$(c/b;q)_{n-k} = \frac{(c/b;q)_n}{(q^{n-k} c/b;q)_k}.$$

Using the inversion formulas [5]:

$$\begin{aligned}
(a;q)_n &= (a^{-1};q^{-1})_n\, a^n\, (-1)^n q^{\binom{n}{2}}, \\
(a;q^{-1})_n &= (q^{1-n} a;q)_n,
\end{aligned}$$

we get

$$\begin{aligned}
(q^{n-k} c/b;q)_k &= (q^{n-k} c/b)^k (-1)^k q^{\binom{k}{2}} (q^{-(n-k)} b/c;q^{-1})_k \\
&= (q^{n-k} c/b)^k (-1)^k q^{\binom{k}{2}} (q^{1-n} b/c;q)_k. \qquad (4.3)
\end{aligned}$$

Thus we have

$$\begin{aligned}
(c/b;q)_{n-k} (q^k b)^{n-k} &= \frac{(c/b;q)_n (q^k b)^{n-k}}{(q^{n-k} c/b)^k (-1)^k q^{\binom{k}{2}} (q^{1-n} b/c;q)_k} \\
&= \frac{(c/b;q)_n (-1)^k q^{-\binom{k}{2}}}{c^k (q^{1-n} b/c;q)_k}.
\end{aligned}$$

In accordance with the above definition of ${}_r\phi_s$, we obtain

$${}_2\phi_1(q^{-n}, b; c; q, qz) = \frac{(c/b;q)_n}{(c;q)_n} b^n {}_3\phi_1(q^{-n}, b, 1/z; bq^{1-n}/c; q, qz/c).$$

Setting $zq \to z$, we arrive at (4.1). ∎

5. A Transformation Formula

Analogous to Jackson's formula (4.1), we obtain a transformation formula based on the following summation formula [5, p. 21]:

$$ {}_2\phi_1(q^{-n}, q^{1-n}; qb^2; q^2; q^2) = \frac{(b^2;q^2)_n}{(b^2;q)_n} q^{-\binom{n}{2}}. \tag{5.1} $$

Theorem 5.1 *We have*

$$ \begin{aligned} &{}_2\phi_1(q^{-n}, q^{1-n}; qb^2; q^2; z) \\ &\quad = \frac{(b^2;q^2)_n}{(b^2;q)_n} q^{-\binom{n}{2}} {}_3\phi_1(q^{-n}, q^{1-n}, q^2/z; q^{2(1-n)}/b^2; q^2; z/qb^2). \end{aligned} \tag{5.2} $$

It is delightful to see that the Cauchy augmentation can be carried through to reach the above transformation formula. In the following proof, we need the Cauchy identity for parameter q^2:

$$ z^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} P_k(z, 1; q^2), \tag{5.3} $$

where

$$ P_n(x, y; q^2) = (x-y)(x-q^2y)\cdots(x-q^{2(n-1)}y), $$

$\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$ stands for the Gauss coefficients with q replaced by q^2. We also need to recall that the summation (5.1) terminates because $(q^{-n};q^2)_m(q^{1-n};q^2)_m = 0$ for any integer m whenever $2m > n$ regardless of its parity. Because of this vanishing property, we may relax the ranges of summations in the following proof.

Proof. Substituting z with q^2z, the left hand side of (5.2) equals

$$ \begin{aligned} &\sum_{m=0}^{n} \frac{(q^{-n};q^2)_m(q^{1-n};q^2)_m}{(q^2;q^2)_m(qb^2;q^2)_m} (q^2z)^m \\ &\quad = \sum_{m=0}^{n} \frac{(q^{-n};q^2)_m(q^{1-n};q^2)_m}{(q^2;q^2)_m(qb^2;q^2)_m} (q^2)^m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} P_k(z,1;q^2) \\ &\quad = \sum_{k=0}^{n} \frac{(q^{-n};q^2)_k(q^{1-n};q^2)_k}{(q^2;q^2)_k(qb^2;q^2)_k} (q^2)^k P_k(z,1;q^2) \\ &\qquad\quad \cdot \sum_{m=k}^{n} \frac{(q^{-(n-2k)};q^2)_{m-k}(q^{1-(n-2k)};q^2)_{m-k}}{(q^2;q^2)_{m-k}(qq^{2k}b^2;q^2)_{m-k}} (q^2)^{m-k}. \end{aligned} \tag{5.4} $$

Now the second summation can be indexed by m ranging from 0 to $n-k$:

$$\sum_{m=0}^{n-k} \frac{(q^{-(n-2k)};q^2)_m (q^{1-(n-2k)};q^2)_m}{(q^2;q^2)_m (qq^{2k}b^2;q^2)_m} (q^2)^m. \tag{5.5}$$

Because of the vanishing property of $(q^{-(n-2k)};q^2)_m(q^{1-(n-2k)};q^2)_m$, the summation (5.5) can be further reduced to the range of m from 0 to $n-2k$. Applying (5.8), (5.5) sums to

$$\frac{(q^{2k}b^2;q^2)_{n-2k}}{(q^{2k}b^2;q)_{n-2k}}\, q^{-\binom{n-2k}{2}}. \tag{5.6}$$

From the relations

$$\begin{aligned}
(b^2;q^2)_n &= (b^2;q^2)_k (q^{2k}b^2;q^2)_{n-2k} (q^{2(n-k)}b^2;q^2)_k,\\
(b^2;q)_n &= (b^2;q)_{2k} (q^{2k}b^2;q)_{n-2k},\\
(b^2;q)_{2k} &= (b^2;q^2)_k\, (qb^2;q^2)_k,
\end{aligned}$$

it follows that

$$\frac{(q^{2k}b^2;q^2)_{n-2k}}{(q^{2k}b^2;q)_{n-2k}} = \frac{(b^2;q^2)_n}{(b^2;q)_n} \frac{(qb^2;q^2)_k}{(q^{2(n-k)}b^2;q^2)_k}.$$

Using the inverse relation (4.3) with q replaced by q^2, we get

$$(q^{2(n-k)}b^2;q^2)_k = (q^{2(n-k)}b^2)^k (-1)^k q^{2\binom{k}{2}} (q^{2(1-n)}/b^2;q^2)_k. \tag{5.7}$$

Hence (5.6) equals

$$\begin{aligned}
&\frac{(b^2;q^2)_n}{(b^2;q)_n} \frac{(qb^2;q^2)_k (-1)^k q^{-2\binom{k}{2}} q^{-2(n-k)k}}{(b^2)^k (q^{2(1-n)}/b^2;q^2)_k}\, q^{-\binom{n-2k}{2}}\\
&= \frac{(b^2;q^2)_n}{(b^2;q)_n}\, q^{-\binom{n}{2}} \frac{(qb^2;q^2)_k (-1)^k q^{-2\binom{k}{2}} q^{-k}}{(b^2)^k (q^{2(1-n)}/b^2;q^2)_k}.
\end{aligned} \tag{5.8}$$

Writing $P_k(z,1;q^2)$ as $(1/z;q^2)_k\, z^k$, from (5.4), (5.6) and (5.8) it follows that

$$\begin{aligned}
&{}_2\phi_1(q^{-n}, q^{1-n}; qb^2; q^2; q^2 z)\\
&= \frac{(b^2;q^2)_n}{(b^2;q)_n} q^{-\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q^2)_k (q^{1-n};q^2)_k (1/z;q^2)_k}{(q^2;q^2)_k (q^{2(1-n)}/b^2;q^2)_k} (-1)^k (q^2)^{-\binom{k}{2}} (qz/b^2)^k\\
&= \frac{(b^2;q^2)_n}{(b^2;q)_n} q^{-\binom{n}{2}} {}_3\phi_1(q^{-n}, q^{1-n}, 1/z; q^{2(1-n)}/b^2; q^2, qz/b^2).
\end{aligned}$$

Setting $q^2 z \to z$, we obtain (5.2). ∎

Acknowledgments. This work was done under the auspices of the Key Laboratory of Pure Mathematics and Combinatorics of the Ministry of Education of China, and the 973 Project on Mathematical Mechanization of the Ministry of Science and Technology of China.

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