Discrete Dynamical Systems on Graphs and Boolean Functions

Chris L. Barrett[1], William Y. C. Chen[2], Michelle J. Zheng[3]

[1]Los Alamos National Laboratory
CCS-5, MS M997, Los Alamos, NM 87545, USA

[2,3]Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China

Email: [1]barrett@lanl.gov, [2]chen@nankai.edu.cn, [3]jzheng@eyou.com

Abstract. Discrete dynamical systems based on dependency graphs have played an important role in the mathematical theory of computer simulations. In this paper, we are concerned with parallel dynamical systems (PDS) and sequential dynamical systems (SDS) with the OR and NOR functions as local functions. It has been recognized by Barrett, Mortveit and Reidys that SDS with the NOR function are closely related to combinatorial properties of the dependency graphs. We present an evaluation scheme for systems with the OR and NOR functions which can be used to clarify some basic properties of the dynamical systems. We show that for forests that does not contain a single edge the number of orientations equals the number of different OR-SDS.

Keywords: Parallel dynamical system (PDS), sequential dynamical system (SDS), Garden of Eden (GOE), state space, fixed point, periodic point.

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Corresponding Author: William Y.C. Chen, chen@nankai.edu.cn

1. Introduction

Computer simulations are used more and more frequently in the modern technological society. This calls for a mathematical theory for simulations. There are two elementary elements associated with a simulation: local rules and dependency relations. In a simulation, there are many (maybe infinitely many) entities and each entity has a state at a given time. The update of states is determined by the local rules and the dependency relations. Moreover, there are two kinds of update schemes: *parallel update* and *sequential update*. Both update schemes have been extensively studied in the literature.

We review the following model of discrete dynamical systems based on a graph, which is called a dependency graph. An entity is represented by a vertex of a graph. Two vertices are joined by an edge if they interfere with each other in the update process. More specifically, an update is implemented by local functions defined for each vertex. For a vertex v, the local function depends on the state of the vertex v itself and the states of the neighbors of v. If the states of the vertices are updated in a parallel manner, the system is called a *parallel dynamical system* (PDS). If the update is carried out in a sequential order, then the system is called a *sequential dynamical system* (SDS). In the sequential case, a permutation on the vertices is used to specify the order of updates, see, for example, [1–5, 7–12].

In this paper, we are particularly concerned with the four kinds of dynamical systems: OR-PDS, OR-SDS, NOR-PDS and NOR-SDS. For each of these systems, we present an evaluation theorem on the update of the global state vectors. In fact, the state vectors are represented by subsets of the vertex set of the dependency graph. Such evaluation schemes turn out to be useful for the study of properties of the dynamical systems. We demonstrate that many properties of the dynamical systems can be characterized by the dependency graphs. For the AND and NAND functions, we may construct the dynamical systems based on the OR and NOR systems, thus, we do not need to consider systems with these two Boolean functions.

We prove that the state spaces of PDS $[OR, G]$ and SDS $[OR, G, \pi]$ have 2^k components if G has k components; The width of PDS $[OR, G]$ equals to the diameter of G, where the width of a dynamical system on a graph is defined to be the maximum distance from a transient state vector to the nearest fixed state vector or periodic state vector; The width of SDS $[OR, G, \pi]$ does not exceed the diameter of G for any $\pi \in S_n$; Any orbit (limit cycle) of PDS $[NOR, G]$ has length 2; There is a bijection between the periodic points of SDS $[NOR, G, \pi]$ and the independent sets of G; The widths of PDS $[NOR, G]$ and SDS $[NOR, G, \pi]$ are both equal to 1. The maximal in-degree in the state space of PDS $[NOR, G]$ is equal to the number of dominant sets of G, which is reached by the state vector $(0, 0, \cdots, 0)$. Analogously, the maximal in-degree in the state space of SDS $[NOR, G, \pi]$ is reached by the state vector $(0, 0, \cdots, 0)$.

We also show that for forests that do not contain a single edge the number of orientations equals the number of different OR-SDS.

2. Definitions and Notations

All the dynamical systems are assumed to be built on an undirected graph $G = (V, E)$, which is called the dependency graph. The graph G is supposed to have vertex set $V = \{1, 2, \cdots, n\}$. For each vertex $1 \leq i \leq n$, there is a state $x_i \in \mathbb{F}_2 = \{0, 1\}$. For $1 \leq i \leq n$ and a subset $W \subseteq V$, we define

$$\begin{aligned} N_G(i) &= \{j \in V | (i, j) \in E\}, \\ d_i &= |N_G(i)|, \\ N_G(W) &= \bigcup_{i \in W} N_G(i), \\ \overline{N_G(i)} &= \{i\} \cup N_G(i). \end{aligned}$$

For a dynamical system on G, there is a local function associated with each vertex i. This is a function to update the state of the vertex i based on the state of i itself and the states of the neighbors of i. We will be concerned with the local functions that are symmetric on the input states. Under this assumption, the update of the state of vertex i is determined by the states of these vertices that are related to i, regardless of the order of the related vertices. In fact, the local functions considered in this paper will be the Boolean functions OR and NOR. As to the state vector $X = (x_1, x_2, \ldots, x_n)$ of the vertices $V = \{1, 2, \ldots, n\}$, we can use a subset W of V to represent the state vector X by taking the elements i for which $x_i = 1$ into the subset W.

Definition 2.1 *Let G be a graph on $V = [n]$. The following mapping*

$$\begin{aligned} [F, G] : \mathbb{F}_2^n &\mapsto \mathbb{F}_2^n, \\ [F, G](x_1, x_2, \cdots, x_i, \cdots, x_n) &= (y_1, y_2, \cdots, y_i, \cdots, y_n), \end{aligned}$$

is called a parallel dynamical system (PDS) over G, where y_i is the updated state of vertex i by applying the local function of vertex i with respect to the dependency graph G.

Let $f_{i,G}$ be a local update function of the vertex i with respect to the dependency graph G, and let $F_{i,G}$ be the update function on the global state vector by applying the local function to update the state of vertex i, while keeping other states unchanged. If we compose the functions $F_{i,G}(1 \leq i \leq n)$ according to a given order $\pi \in S_n$, where S_n is the set of permutations on V, then we can get an update function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$.

Definition 2.2 *Let G be a graph on $V = [n]$ and $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$. The mapping*

$$[F, G, \pi] = F_{\pi_1,G}F_{\pi_2,G} \cdots F_{\pi_n,G} : \mathbb{F}_2^n \mapsto \mathbb{F}_2^n$$

is called a sequential dynamical system (SDS) over G.

We note that in the above notation of composition of functions, we assume that the function $F_{\pi_1,G}$ is applied first, $F_{\pi_2,G}$ is applied next, and so on, namely,

$$F_{\pi_1,G}F_{\pi_2,G} \cdots F_{\pi_n,G}(x_1, \ldots, x_n) = F_{\pi_n,G}(\cdots F_{\pi_1,G}(x_1, \ldots, x_n)).$$

We will usually use X to denote a state vector $(x_1, x_2, \ldots, x_n)$, and use g to denote the global update function of a dynamical system which acts on state vectors. The set of all state vectors is called the state space. Given a dynamical system with global update function g, we may describe the following terminology.

1. If $g(X) = X$, then X is called a *fixed point*. The notation $FIX[g]$ represents the set of all fixed points of the dynamical system with global update function g.

2. If there exists an integer $m > 1$ such that $g^m(X) = X$ and for any integer $0 < l < m$, $g^l(X) \neq X$, then X is called a *periodic point* or *periodic state vector* of g and m is called the period of X. We adopt the notation $PER[g]$ to denote the set of all periodic points of g.

3. If $X \notin FIX[g] \cup PER[g]$, then X is called a *transient point*, or a *transient state vector* of g.

4. If there does not exist any state vector Z such that $g(Z) = X$, then X is called a *Garden-of-Eden (GOE)* of g. The set of GOEs of g is denoted by $GOE[g]$.

Definition 2.3 *Let G be a dependency graph on $V = [n]$ and let g be the global update function of a dynamical system on G. The digraph $\Gamma[g]$, called the functional digraph of g, is defined as the digraph with vertex set as the state space of g, and arc set*

$$A[\Gamma[g]] = \{(X, g(X)) | X \in \mathbb{F}_2^n\}.$$

An orbit of g is a cycle or a loop of the functional digraph of g.

We next define the width of a dynamical system in terms of its functional digraph.

Definition 2.4 *Let g be the global update function of a dynamical system on the dependency graph G. For a state vector $X \in \mathbb{F}_2^n$, let $h(X)$ be the minimum nonnegative integer such that $g^{h(X)}(X)$ is a fixed point or a periodic point in the functional digraph $\Gamma[g]$, where $g^0(X)$ is defined to be X. The number $max\{h(X)|X \in \mathbb{F}_2^n\}$ is called the width of the dynamical system.*

This paper will be concerned with the properties of the dynamical systems with OR and NOR functions that are related to the above terminology.

3. OR-PDS and OR-SDS

If we take the OR function as the local update functions, then the corresponding dynamical systems are called OR-PDS and OR-SDS, and denoted by $[OR, G]$ and $[OR, G, \pi]$, respectively.

Recall that a system is said to be a fixed point system if all its orbits are loops, see [7]. We will show that PDS $[OR, G]$ and SDS $[OR, G, \pi]$ are fixed point systems, that is, all orbits in their state space are fixed points. To describe the global update functions for the OR systems, we find it more convenient to work with subsets as a representation of state vectors. The following evaluation theorem can be easily verified.

Theorem 3.1 *Let PDS $[OR, G]$ be the OR-PDS on G. Let X be a state vector and W be the subset of V corresponding to X, namely, W is the subset of vertices having state 1 with respect to X. Let $\phi_{[OR,G]}$ be the function acting on subsets of V in accordance with the OR-PDS. In other words, $\phi_{[OR,G]}(W)$ is the subset of vertices having state 1 with respect to the state vector $[OR, G](X)$. Then we have*

$$\phi_{[OR,G]}(W) = W \bigcup N_G(W). \tag{3.1}$$

Theorem 3.2 *Let SDS $[OR, G, \pi]$ be the OR-SDS on G. Let X be a state vector and W be the subset of V corresponding to X. Let $\phi_{[OR,G,\pi]}$ be the function acting on subsets of V in accordance with the OR-SDS. Then we have*

$$\phi_{[OR,G,\pi]}(W) = \vartheta_{\pi_n}(\vartheta_{\pi_{n-1}} \cdots (\vartheta_{\pi_1}(W))), \tag{3.2}$$

where

$$\vartheta_i(W) = \begin{cases} W, & \text{if } N_G(i) \cap W = \emptyset, \\ W \cup \{i\}, & \text{otherwise.} \end{cases}$$

Using the above evaluation theorems, we show that there is no cycle of length bigger than 1 in $\Gamma[OR, G]$ and $\Gamma[OR, G, \pi]$ (for any $\pi \in S_n$), or equivalently, $PER[OR, G] = PER[OR, G, \pi] = \emptyset$. This leads to the following conclusion.

Theorem 3.3 *The systems PDS $[OR, G]$ and SDS $[OR, G, \pi]$ are fixed point systems.*

We next show that the set of fixed points of the system $[OR, G]$ is related to the connected components of G.

Theorem 3.4 *Suppose there are k connected components in the dependency graph G. Then functional digraph $\Gamma[OR, G]$ has 2^k components, and the system PDS $[OR, G]$ has 2^k fixed points.*

Proof. Using the evaluation of $\phi_{[OR,G]}$, one sees that a subset W is a fixed point of $\phi_{[OR,G]}$ if and only if W satisfies the following condition: if $i \in W$, then for any j in the same component as i in the dependency graph G, we have $j \in W$. There are altogether 2^k such subsets W when G has k components. By virtue of Theorem 3.1, PDS $[OR, G]$ has 2^k fixed points. It follows from the fact $PER[OR, G] = \emptyset$ that the functional digraph $\Gamma[OR, G]$ has 2^k components. This completes the proof. ∎

The width of PDS $[OR, G]$ is determined by the diameter of G.

Theorem 3.5 *Assume that G is a connected graph. Then the width of PDS $[OR, G]$ equals to the diameter of G.*

Proof. Suppose that G is a connected graph with vertex set V and d is the diameter of G. Then we may assume that there is a shortest path of length d in G from a vertex i to a vertex j. By Theorem 3.4, there are only two fixed points in the system $[OR, G]$ corresponding to the subsets V and $\emptyset$, namely, the state vectors $(1, 1, \ldots, 1)$ and $(0, 0, \ldots, 0)$. By Theorem 3.1, for a subset $W = \{i\}$ with one element, we have $j \notin \phi^{d-1}_{[OR,G]}(W)$. It follows that

$$\phi^{d-1}_{[OR,G]}(W) \neq V.$$

On the other hand, for any $W\prime \subset V$, we have $\phi^{d}_{[OR,G]}(W') = V$ or $\phi^{d}_{[OR,G]}(W') = \emptyset$; otherwise the diameter of G would exceed d. By Definition 2.4, we obtain that d is the width of PDS $[OR, G]$. ∎

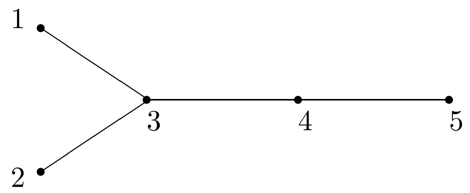
In the spirit of Theorem 3.4 and Theorem 3.5, we obtain the following results for OR-SDS. The proofs are omitted.

Theorem 3.6 *Suppose there are k connected components in the dependency graph G. Then for any permutation $\pi \in S_n$, the functional digraph $\Gamma[OR, G, \pi]$ has 2^k components, and the system SDS $[OR, G, \pi]$ has 2^k fixed points.*

Theorem 3.7 *Let G be a connected graph. Then for any permutation $\pi \in S_n$, the width of SDS $[OR, G, \pi]$ does not exceed the diameter of G. Moreover, there exists a permutation $\sigma \in S_n$ such that the width of the system $[OR, G, \sigma]$ is equal to the diameter of G.*

The following example is an illustration of Theorem 3.7.

Example 3.8 *Let G be the following graph with diameter 3:*



There are 16 different OR-SDS on G (with respect to the global update function) corresponding to the following permutations:

12345	12354	12543	12435	13425	13542	15432	14352
23451	23541	25431	24351	34512	35142	54312	43512.

The widths of these OR-SDS are respectively 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 2, 2, 3, 2.

For a special class of graphs, the number of acyclic orientations equals the number of different OR-SDS based on the these graphs. Let G be an undirected graph on $V = [n]$ and $Acyc(G)$ be the set of acyclic orientations of G. Let $a(G) = |Acyc(G)|$, and let $\mathbb{S}_{(f_k)}(G)$ be the set of different SDS with the local functions $f_k(1 \leq k \leq n)$ and the dependency graph G. We set $a_{(f_k)}(G) = |\mathbb{S}_{(f_k)}(G)|$.

In [10], Reidys shows that the number of different NOR-SDS on G equals to the number of acyclic orientations of G, namely, for any graph G,

$$a_{(nor_k)}(G) = a(G). \tag{3.3}$$

Although the above assertion does not apply to the case of OR functions, we will show that the same result is valid for the forests that do not contain isolated edges.

Theorem 3.9 *Assume that $F = (V, E)$ is a forest on $V = [n]$ which does not contain isolated edges (connected components with two vertices). Then we have*

$$a_{(or_k)}(F) = a(F) = 2^{|E|}.$$

Proof. Let G be a graph on $V = [n]$ and U be the graph with vertex set S_n in which two different vertices $(i_1, \ldots, i_n)$ and $(h_1, \ldots, h_n)$ are adjacent if $(i_1, \ldots, i_n)$ can be obtained from $(h_1, \ldots, h_n)$ by exchanging of two elements h_k, h_{k+1}, where h_k and h_{k+1} are not adjacent in G. We set

$$[\pi]_G = \{\pi' | \pi' \text{ and } \pi \text{ are in the same component of } U\}.$$

Let

$$S_n/_{\sim G} = \{[\pi]_G | \pi \in S_n\}$$

and $h_{(f_k)}$ be the following mapping:

$$\begin{aligned} h_{(f_k)} : S_n/_{\sim G} &\mapsto \mathbb{S}_{(f_k)}(G), \\ [\pi]_G &\mapsto [F, G, \pi]. \end{aligned}$$

Since there is a one-to-one correspondence between $Acyc(G)$ and $S_n/_{\sim G}$ for any graph G ([11], Proposition 1), we only need to show that $h_{(or_k)}$ is a bijection when G is a forest F which does not contain an isolate edge. Clearly, $h_{(or_k)}$ is a surjection. For any $[\pi]_F \neq [\sigma]_F$ in $S_n/_{\sim F}$, there is an edge (i, j) in F such that $i <_\pi j$ and $j <_\sigma i$, where $i <_\pi j$ means that i is on the left of j in the permutation π. Let

$$\begin{aligned} S_i &= \{k \,|\, k \neq i, \ \ i \text{ and } k \text{ are in the same component of } F \setminus (i, j)\}, \\ S_j &= \{k \,|\, k \neq j, \ \ j \text{ and } k \text{ are in the same component of } F \setminus (i, j)\}. \end{aligned}$$

By the assumption that none of the components of F is an isolated edge, it follows that $S_i \neq \emptyset$ or $S_j \neq \emptyset$. Without loss of generality we may assume that $S_i \neq \emptyset$. Since (i, j) is a cut edge, we get that $S_i \cap S_j = \emptyset$.

Let X_{S_i} be the state vector corresponding to the subset S_i. Notice that the j-th entry of the state vector $[OR, F, \pi](X_{S_i})$ equals 1 and the j-th entry of the state vector $[OR, F, \sigma](X_{S_i})$ equals 0. It follows that $[OR, F, \pi]$ and $[OR, F, \sigma]$ are different systems, that is to say, $h_{(or_k)}$ is injective. Therefore, we have shown that $h_{(or_k)}$ is a bijection. This completes the proof. ∎

4. NOR-PDS

In this section, we are concerned with parallel dynamical systems with the NOR function as local functions. These systems are called NOR-PDS, and denoted by $[NOR, G]$ for a dependency graph G. We first present the following evaluation theorem. The proof is straightforward and hence is omitted.

Theorem 4.1 *Let PDS $[NOR, G]$ be the NOR-PDS on G. Let X be a state vector and W be the subset of V corresponding to X. Let $\phi_{[NOR,G]}$ be the global update function acting on subsets of V in accordance with the NOR-PDS. Then we have*

$$\phi_{[NOR,G]}(W) = V \setminus (W \bigcup N_G(W)). \tag{4.1}$$

As an immediate consequence of Theorem 4.1, the system $[NOR, G]$ does not have any fixed point. Moreover, we have

Theorem 4.2 *All the orbits of PDS $[NOR, G]$ are 2-cycles.*

Proof. Since PDS $[NOR, G]$ has no fixed point, $PER[NOR, G]$ must have some orbits with length exceeding 1 by the structure of the functional digraph of $[NOR, G]$. Let X be a periodic point of PDS $[NOR, G]$ and W be the corresponding subset. Then there exists an integer k such that $\phi^k_{[NOR,G]}(W) = W$. Because $\phi_{[NOR,G]}$ is given by (4.1), k must be an even number. It remains to prove $k = 2$.

Let $W' = \phi_{[NOR,G]}(W)$. We claim that $N_G(W) = N_G(W')$. Assume that there is an element $w \in (N_G(W) \cup N_G(W'))$ and $w \notin (N_G(W) \cap N_G(W'))$. Without loss of generality, we may assume that $w \in N_G(W)$ and $w \notin N_G(W')$. By the definition of $\phi_{[NOR,G]}$, w is contained in $\phi^{2m}_{[NOR,G]}$ for any integer $m > 1$, which is contradictory to the fact that the state vector X corresponding to the subset W is a periodic point. So the claim is justified, and we get

$$\phi_{[NOR,G]}(W) = W' \text{ and } \phi_{[NOR,G]}(W') = W.$$

That is to say, $\phi^2_{[NOR,G]}(W) = W$. This completes the proof. ∎

We next show that the functional digraph of PDS $[NOR, G]$ has quite a simple structure.

Theorem 4.3 *The width of PDS* $[NOR, G]$ *is equal to* 1.

Proof. Let X be a state vector and W be the subset corresponding to X. By Theorem 4.2, it suffices to show that $\phi^3_{[NOR,G]}(W) = \phi_{[NOR,G]}(W)$. Let

$$W_1 = \phi_{[NOR,G]}(W), \quad W_2 = \phi_{[NOR,G]}(W_1), \quad W_3 = \phi_{[NOR,G]}(W_2).$$

It is easy to verify that $N_G(W_1) = N_G(W_2)$. Thus we have

$$\begin{aligned} W_3 &= (V \setminus W_2) \setminus N_G(W_2) \\ &= (V \setminus (V \setminus W_1 \setminus N_G(W_1))) \setminus N_G(W_2) \\ &= W_1 \cup (N_G(W_1) \setminus N_G(W_2)) \\ &= W_1. \end{aligned}$$

This is to say, $\phi^3_{[NOR,G]}(W) = \phi_{[NOR,G]}(W)$. ∎

Note that the above theorem can be restated as

$$GOE[NOR, G] \cup PER[NOR, G] = \mathbb{F}_2^n. \tag{4.2}$$

The following theorem shows that the maximum in-degree of the functional digraph of the system $[NOR, G]$ is related to the dominant sets of the dependency graph G.

Theorem 4.4 *In the functional digraph* $\Gamma[NOR, G]$, *the maximum in-degree is reached by the state vector* $X = (0, 0, \cdots, 0)$, *and it is equal to the number of dominant sets of* G.

Proof. By the evaluation theorem of PDS $[NOR, G]$, one sees that for any subset W', $\phi_{[NOR,G]}(W) = W'$ if and only if W is a dominant set of the graph $G[V \setminus W']$, the induced subgraph of G restricted to $V \setminus W'$. In particular, $W' = \emptyset$ if and only if W is a dominant set of G. From Theorem 4.1, it follows that the in-degree of $X = (0, 0, \cdots, 0)$ equals to the number of dominant sets of G.

We continue to show that the in-degree of the state vector $X = (0, 0, \ldots, 0)$ in $\Gamma[NOR, G]$ is indeed the maximum in-degree. For any subset W', if $\phi_{[NOR,G]}(W) = W'$, then $\phi_{[NOR,G]}(W \cup W') = \emptyset$. This implies that from a pre-image of W', we can find a pre-image of $\emptyset$. Moreover, for any two distinct pre-images W_1 and W_2 of W', we have $W_1 \cap W' = \emptyset$ and $W_2 \cap W' = \emptyset$. It follows that

$$W_1 \cup W' \neq W_2 \cup W'. \tag{4.3}$$

Furthermore, $W_1 \cup W'$ and $W_2 \cup W'$ are both dominant sets of G, namely,

$$\phi_{[NOR,G]}(W_1 \cup W') = \phi_{[NOR,G]}(W_2 \cup W') = \emptyset. \tag{4.4}$$

Therefore, for any two distinct pre-images of W', there are two distinct pre-images of $\emptyset$. So we obtain that

$$|\phi^{-1}_{[NOR,G]}(W')| \leq |\phi^{-1}_{[NOR,G]}(\emptyset)|.$$

This completes the proof. ∎

5. *NOR*-SDS

In this section, we study sequential dynamical systems with NOR function as the local update functions, denoted by $[NOR, G, \pi]$ for a given dependency graph G. These systems have been studied by Reidys [12]. We give the following evaluation theorem which is useful to give a simpler augment for the results obtained by Reidys.

Theorem 5.1 *Let SDS $[NOR, G, \pi]$ be a NOR-SDS on G. Let X be a state vector and W be the subset of V corresponding to X. Let $\phi_{[NOR,G,\pi]}$ be the global update function acting on subsets of V in accordance with the NOR-SDS with dependency graph G. Then we have*

$$\phi_{[NOR,G,\pi]}(W) = \rho_{\pi_n}(\rho_{\pi_{n-1}} \cdots (\rho_{\pi_1}(W))), \tag{5.1}$$

where

$$\rho_i(W) \;=\; \begin{cases} W \setminus \{i\}, & \textit{if } i \in W, \\ W \cup \{i\}, & \textit{if } \overline{N_G(i)} \cap W = \emptyset, \\ W, & \textit{otherwise.} \end{cases}$$

An instant consequence of the above theorem is that the functional digraph $\Gamma[NOR, G, \pi]$ has no fixed points. Furthermore, we may use the above evaluation scheme to simplify proofs of several results of Reidys. The set of independent sets of G will be denoted by D_G [6].

Lemma 5.2 *For any subset W of V, $\phi_{[NOR,G,\pi]}(W)$ forms an independent set of G.*

Proof. Assume that there exists a subset W such that $\phi_{[NOR,G,\pi]}(W) \notin D_G$. Then there are two elements i, j in $\phi_{[NOR,G,\pi]}(W)$ such that (i, j) is an edge of G. Without loss of generality, we may assume that $i <_\pi j$ and $j = \pi_k$ for some k. Since i is in the subset $\rho_{\pi_{k-1}}(\rho_{\pi_{k-2}} \cdots (\rho_{\pi_1}(W)))$, after the action of ρ_j on the subset $\rho_{\pi_{k-1}}(\rho_{\pi_{k-2}} \cdots (\rho_{\pi_1}(W)))$, j is not contained in $\rho_{\pi_k}(\rho_{\pi_{k-1}} \cdots (\rho_{\pi_1}(W)))$. This leads to a contradiction. ∎

Next we show that the mapping $\phi_{[NOR,G,\pi]}$ induces a bijection on D_G. In other words, $\phi_{[NOR,G,\pi]}$ maps an independent set to an independent, and the mapping is one-to-one.

Lemma 5.3 *For any permutation π on $[n]$, the mapping $\phi_{[NOR,G,\pi]}$ yields a bijection on the set of independent sets of G.*

Proof. By the finiteness of D_G, we only need to prove that $\phi_{[NOR,G,\pi]}$ over D_G is an injection. Suppose that W_1 and W_2 are two different independent set of G such that

$$\phi_{[NOR,G,\pi]}(W_1) = \phi_{[NOR,G,\pi]}(W_2). \tag{5.2}$$

Let k_0 be the first element in π that appears in $W_1 \cup W_2 \setminus (W_1 \cap W_2)$. Without loss of generality, we assume that $k_0 \in W_1$ and $k_0 \notin W_2$. Then there exists $k_1 >_\pi k_0$ such that $k_1 \in W_2$, $k_1 \notin W_1$ and $(k_0, k_1) \in E$. Similarly, for $k_1 \in W_1 \cup W_2 \setminus (W_1 \cap W_2)$, there exists $k_2 >_\pi k_1$ such that $k_2 \in W_1$, $k_2 \notin W_2$ and $(k_1, k_2) \in E$. By iterating this procedure, we will fail at certain point to find $k_i >_\pi k_{i-1}$ such that $k_i \in W_1$, $k_i \notin W_2$ and $(k_{i-1}, k_i) \in E$, or to find $k_i >_\pi k_{i-1}$ such that $k_i \in W_2$, $k_i \notin W_1$ and $(k_{i-1}, k_i) \in E$. Hence we get $k_{i-1} \in \phi_{[NOR,G,\pi]}(W_1)$ and $k_{i-1} \notin \phi_{[NOR,G,\pi]}(W_2)$, or get $k_{i-1} \in \phi_{[NOR,G,\pi]}(W_2)$ and $k_{i-1} \notin \phi_{[NOR,G,\pi]}(W_1)$. This is contradictory to the assumption (5.2). ∎

The above Theorem 4.4 has the following counterpart for SDS $[NOR, G, \pi]$. The proof is essentially the same.

Lemma 5.4 *For any subset W of V, we have*

$$|\phi^{-1}_{[NOR,G,\pi]}(W)| \leq |\phi^{-1}_{[NOR,G,\pi]}(\emptyset)|.$$

From the above three lemmas, we get the following results due to Reidys [12].

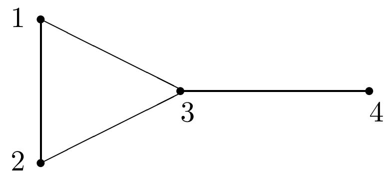
Theorem 5.5 *The width of SDS* $[NOR, G, \pi]$ *equals to* 1.

Theorem 5.6 *There is a bijection between the set of the periodic points of* $[NOR, G, \pi]$ *and the set of the independent sets of* G.

Theorem 5.7 *In the functional digraph* $\Gamma[NOR, G, \pi]$, *the maximal in-degree is reach by the state vector* $X = (0, 0, \cdots, 0)$.

The following example is an illustration of Theorem 5.5, Theorem 5.6 and Theorem 5.7.

Example 5.8 *Let* G *be the following graph with independent sets:* $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 4\}$, $\{2, 4\}$.



By Theorem 5.5 and Theorem 5.6, we obtain that for any $\pi \in S_n$,

$$PER[NOR, G, \pi] = \{0000, 1000, 0100, 0010, 0001, 1001, 0101\},$$

and

$$GOE[NOR, G, \pi] = \{0011, 0110, 0111, 1010, 1011, 1100, 1101, 1110, 1111\}.$$

Moreover, for the permutation $\pi = 3124$, *the in-degrees of the periodic points* 0000, 1000, 0100, 0010, 0001, 1001, 0101 *in the functional digraph* $\Gamma[NOR, G, 3124]$ *are respectively* 4, 2, 2, 1, 4, 1, 2.

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