Graphs Combin.
21 (2005), no. 1, 51-62

The Flagged Cauchy Determinant

William Y. C. Chen[1], Christian Krattenthaler[2]
and Arthur L. B. Yang[3]

[1,3]Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China

[2]Institut Girard Desargues,
Université Claude Bernard Lyon-I, 21, avenue Claude Bernard, F-69622
Villeurbanne Cedex, France

emails: [1]chen@nankai.edu.cn, [2]kratt@euler.univ-lyon1.fr,
[3]arthuryang@nankai.edu.cn

Abstract. We consider a flagged form of the Cauchy determinant, for which we provide a combinatorial interpretation in terms of nonintersecting lattice paths. In combination with the standard determinant for the enumeration of nonintersecting lattice paths, we are able to give a new proof of the Cauchy identity for Schur functions. Moreover, by choosing different starting and end points for the lattice paths, we are led to a lattice path proof of an identity of Gessel which expresses a Cauchy-like sum of Schur functions in terms of the complete symmetric functions.

Keywords: Divided difference, Cauchy identity, flagged Cauchy determinant, multi-Schur function, nonintersecting lattice paths.

AMS Classification: 05E05, 05A15.

Suggested Running Title: The Flagged Cauchy Determinant

Corresponding Author: William Y. C. Chen, chen@nankai.edu.cn

1. Introduction

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two sets of variables, and let $s_\lambda(X)$ and $s_\lambda(Y)$ be the Schur functions indexed by a partition λ in the sets of variables X and Y, respectively. (We refer the reader to [22] or [26, Ch. 7] for all notation and definitions concerning partitions and symmetric functions.) Then the classical Cauchy identity for Schur functions can be stated as follows:

Theorem 1.1 *For $n \geq 1$, we have*

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(X) s_\lambda(Y), \tag{1.1}$$

where the sum ranges over all partitions of length at most n.

Classical proofs of Theorem 1.1 are by means of the Robinson–Schensted–Knuth correspondence and the Cauchy-Binet formula, respectively [22, 26]. There is also a derivation based on a matrix product involving the elementary symmetric functions as given in Macdonald [22, p. 67].

The aim of this paper is to establish a connection between the Cauchy identity and the lattice path method due to Gessel and Viennot [8, 9].

The key ingredient in our lattice path construction is a flagged form of the Cauchy determinant with respect to the sets of variables. Recall that the Cauchy determinant in the variables X and Y is the determinant

$$\left| \frac{1}{1 - x_i y_j} \right|_{n \times n}.$$

It is well-known that

$$\left| \frac{1}{1 - x_i y_j} \right|_{n \times n} = \Delta(X)\, \Delta(Y) \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}, \tag{1.2}$$

where we have used the common notation for the Vandermonde determinant

$$\Delta(X) := \left| x_i^{n-j} \right|_{n \times n} = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

This paper contains the following results:

1. We define the flagged Cauchy determinant to be

$$F(X,Y) = \left| \sum_k h_{k-n+i}(x_i,\ldots,x_n) h_{k-n+j}(y_j,\ldots,y_n) \right|_{n\times n}, \tag{1.3}$$

where h_i is the complete symmetric function of degree i. As a first result, by simple row and column operations, we prove (see Theorem 2.1):

$$\left| \frac{1}{1-x_i y_j} \right|_{n\times n} = \Delta(X)\cdot\Delta(Y)\cdot F(X,Y).$$

2. We provide an interpretation of the flagged Cauchy determinant $F(X,Y)$ in terms of nonintersecting lattice paths, and we describe as well a relation with semistandard tableaux. This leads us, in particular, to the Cauchy identity (1.1).

3. Choosing different starting and end points for the lattice paths, we obtain the equality of the flagged Cauchy determinant $F(X,Y)$ with a determinant in complete symmetric functions in the *full* sets of variables X and Y:

$$F(X,Y) = \left| \sum_k h_{k-n+i}(X) h_{k-n+j}(Y) \right|_{n\times n}.$$

   It should be noted that in the above formula there appear the same indices as in the flagged formula (1.3). This leads us to a lattice path interpretation of the following identity of Gessel [7, Theorem 16]:

   Theorem 1.2 *We have*

$$\left| \sum_k h_{k-n+i}(X) h_{k-n+j}(Y) \right|_{n\times n} = \sum_\lambda s_\lambda(X) s_\lambda(Y), \tag{1.4}$$

   where the sum ranges over all partitions of length at most n.

   We remark that in Gessel's theorem one can actually replace X and Y by *infinite* sets of variables. As we are going to show, our lattice path proof covers this case as well, as well as a generalization involving *skew* Schur functions (see Theorem 3.5).

To conclude this introduction, we note that the idea of flagged Schur functions and multi-Schur functions has proved to be very efficient in the study of Schubert polynomials in connection with divided difference operators (see Lascoux [19] and Wachs [28]). Flagged determinants with respect to the sets of variables can also be used to give simple character formulas for the symplectic groups and the orthogonal groups, see papers by Chen, Li and Louck [3], Hamel and King [14], and Okada [23].

2. The Flagged Cauchy Determinant

Let $h_k(x_i, x_{i+1}, \ldots, x_n)$ be the complete symmetric function of degree i in $x_i, x_{i+1}, \ldots, x_n$ (cf. [22] or [26, Ch. 7]). The main theorem of this section establishes a relation between the Cauchy determinant and the flagged form (1.3).

Theorem 2.1 *We have*

$$\left|\frac{1}{1 - x_i y_j}\right|_{n\times n} = \Delta(X) \cdot \Delta(Y) \cdot F(X, Y), \tag{2.5}$$

where $F(X, Y)$ denotes the determinant in (1.3)*.*

Proof. First, we express the (i, j)-entry in the Cauchy determinant as

$$\frac{1}{1 - x_i y_j} = \sum_{k\geq 0} (x_i y_j)^k = \sum_{k\geq 0} h_k(x_i) h_k(y_j).$$

Next, we recall the divided difference property of the complete symmetric functions:

$$\frac{h_k(x_i, \ldots, x_j) - h_k(x_{i+1}, \ldots, x_{j+1})}{x_i - x_{j+1}} = h_{k-1}(x_i, \ldots, x_{j+1}).$$

Subtracting the $(i + 1)$-st row from the i-th row and dividing the resulting row by $(x_i - x_{i+1})$ for $i = 1, 2, \ldots, n - 1$, we get the determinant

$$\begin{vmatrix} \sum_k h_{k-1}(x_1, x_2) h_k(y_1) & \sum_k h_{k-1}(x_1, x_2) h_k(y_2) & \cdots & \sum_k h_{k-1}(x_1, x_2) h_k(y_n) \\ \sum_k h_{k-1}(x_2, x_3) h_k(y_1) & \sum_k h_{k-1}(x_2, x_3) h_k(y_2) & \cdots & \sum_k h_{k-1}(x_2, x_3) h_k(y_n) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_k h_k(x_n) h_k(y_1) & \sum_k h_k(x_n) h_k(y_2) & \cdots & \sum_k h_k(x_n) h_k(y_n) \end{vmatrix}.$$

We continue by subtracting the $(i + 1)$-st row from the i-th row and dividing by $(x_i - x_{i+2})$ for $i = 1, 2, \ldots, n - 2$, then subtracting the $(i + 1)$-st row from the i-th row and dividing by $(x_i - x_{i+3})$ for $i = 1, 2, \ldots, n - 3$, etc. Eventually, we obtain the determinant

$$\left|\sum_k h_{k-n+i}(x_i, \ldots, x_n) h_k(y_j)\right|_{n\times n}.$$

Now we apply analogous operations to the columns of the above determinant. That is, we subtract the $(i+1)$-st column from the i-th column and divide the resulting column by $(y_i - y_{i+1})$ for $i = 1, 2, \ldots, n - 1$, then we subtract the $(i + 1)$-st column from the

i-th column and divide by $(y_i - y_{i+2})$ for $i = 1, 2, \dots, n-2$, and so on. We finally get the following flagged determinant of complete symmetric functions in X and Y:

$$\left| \sum_k h_{k-n+i}(x_i, \dots, x_n) h_{k-n+j}(y_j, \dots, y_n) \right|_{n\times n}. \tag{2.6}$$

This is exactly the determinant $F(X, Y)$. On the other hand, the division operations yield the product of the Vandermonde determinants $\Delta(X)$ and $\Delta(Y)$. This completes the proof. ∎

3. Lattice paths and the Cauchy identity

The lattice path method introduced by Gessel and Viennot [8, 9] (but actually dating back to Karlin and McGregor [15, 16] and Lindström [20]) has been widely used as a powerful technique for the study of symmetric functions, plane partitions and many other combinatorial problems (see for example [1, 2, 4, 6, 10, 11, 12, 13, 18, 23, 27, 29]).

We let the underlying (lattice) digraph D be the integer lattice $\mathbb{Z} \times \mathbb{Z}$, where the arcs (or steps) are horizontal, $(i, j) \to (i+1, j)$, or vertical, $(i, j) \to (i, j \pm 1)$, with the following restrictions: if a vertical arc lies strictly to the left of the y-axis, it must be an up step from (i, j) to $(i, j+1)$; if a vertical arc lies strictly to the right of the y-axis, then it must be a down step from (i, j) to $(i, j-1)$; and there are no vertical steps on the y-axis. From now on, when we speak of a (lattice) path then we always mean a path in D.

We define the following weights for the arcs in D:

1. A horizontal arc has weight 1.
2. For $i < 0$, a vertical arc from (i, j) to $(i, j+1)$ has weight x_{n+i+1}.
3. For $i > 0$, a vertical arc from (i, j) to $(i, j-1)$ has weight y_{n-i+1}.

The weight of a path P, denoted by $w(P)$, is defined as the product of the weights of the arcs of the path P. Given an n-tuple $(P_1, P_2, \dots, P_n)$ of lattice paths, its weight is defined to be the product of the weights of the path P_i. We now suppose that $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_n$ are given points in the integer lattice $\mathbb{Z} \times \mathbb{Z}$. Let $\mathcal{P}(A_i, B_j)$ denote the set of lattice paths from A_i to B_j in D. Similarly, we use $\mathcal{P}(A, B)$ to denote the set of all n-tuples $(P_1, P_2, \dots, P_n)$ of lattice paths in D where P_i starts at A_i and ends at B_i. We also adopt the notation $\mathcal{P}_0(A, B)$ for the set of all n-tuples $(P_1, P_2, \dots, P_n)$ of *nonintersecting* lattice paths, where P_i has starting point A_i and end point B_i. Here, as usual, by "nonintersecting" we mean that there are no common points among P_i and P_j for $i \neq j$. By $\mathrm{GF}(\mathcal{P}(A,B))$ and $\mathrm{GF}(\mathcal{P}_0(A,B))$ we denote the generating functions, or the sums of weights, of the n-tuples of lattice paths in $\mathcal{P}(A,B)$ and $\mathcal{P}_0(A,B)$, respectively.

For the purpose of the promised combinatorial interpretation of the flagged Cauchy determinant $F(X,Y)$, we choose

$$A_i = (i-n-1, -i), \quad \text{and} \quad B_i = (n-i+1, -i), \quad i = 1, 2, \ldots, n. \tag{3.7}$$

We refer the reader to Figure 3.1 for an illustration of these points in the case $n = 4$. (The paths should be ignored at this point.)

With the above choice, we arrive at the following lattice path interpretation of the entries in the flagged Cauchy determinant.

Lemma 3.1 *The generating function for the D-paths from A_i to B_j equals*

$$\mathrm{GF}(\mathcal{P}(A_i, B_j)) = \sum_k h_{k-n+i}(x_i, \ldots, x_n) h_{k-n+j}(y_j, \ldots, y_n). \tag{3.8}$$

Proof. We classify the D-paths P from A_i to B_j by their intersection points with the y-axis. To be more specific, assume that P intersects the y-axis in the point Q. Let k be the y-coordinate of Q. Since there are no arcs on the y-axis, the weights of all such paths sum to

$$h_{k+i}(x_i, \ldots, x_n) h_{k+j}(y_j, \ldots, y_n).$$

Summing over k, one gets the right-hand side of (3.8), after having done the replacement $k \to k - n$. This completes the proof of the lemma. ∎

As an immediate corollary of the standard theorem on nonintersecting lattice paths (see [9, Cor. 2] or [27, Theorem 1.2]), we obtain the promised interpretation of the flagged Cauchy determinant in terms of nonintersecting lattice paths. We refer the reader to Figure 3.1 for an example of a set of nonintersecting lattice paths as in the statement of the theorem below (i.e., with the starting and points as in (3.7)), in the case $n = 4$.

Theorem 3.2 *We have the following relation:*

$$F(X,Y) = \mathrm{GF}(\mathcal{P}_0(A,B)). \tag{3.9}$$

Proof. The aforementioned theorem on nonintersecting lattice paths says that

$$\mathrm{GF}(\mathcal{P}_0(A,B)) = \det(\mathrm{GF}(\mathcal{P}(A_i, B_j)))_{n \times n},$$

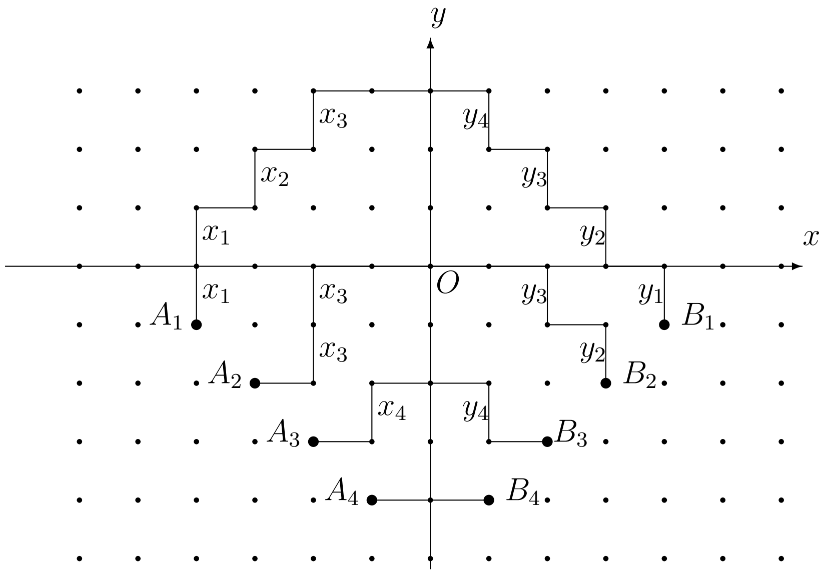

Figure 3.1 Nonintersecting paths from A_i to B_i

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\,, \qquad T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

Figure 3.2 A pair of tableaux of the same shape

under the condition that the starting and end points are "D-compatible" (in the terminology of [27]), i.e., for $i < j$ and $k < l$ any path from A_i to B_l and any path from A_j to B_k have a point in common. However, the latter is entirely obvious from the topology of the digraph D. Hence, in view of Lemma 3.1, the theorem follows. ∎

Given the sets A and B of starting and end points, respectively, we may translate an n-tuple $(P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths into a pair of semistandard tableaux of the same shape with entries from $\{1, 2, \dots, n\}$.

Theorem 3.3 *There is a one-to-one correspondence between n-tuples $(P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths, where P_i runs from A_i to B_i, and pairs of semistandard tableaux of the same shape with entries from $\{1, 2, \dots, n\}$. In particular, we have*

$$\mathrm{GF}(\mathcal{P}_0(A,B)) = \sum_{\lambda, \ell(\lambda) \le n} s_\lambda(X) s_\lambda(Y). \tag{3.10}$$

Proof. While reading this proof, it is advisable to consult in parallel the example in

Figures 3.1 and 3.2. (There, we have chosen $n = 4$.) Given any n-tuple $(P_1, P_2, \dots, P_n)$ of nonintersecting paths such that P_i runs from A_i to B_i, let Q_i be the intersection point of P_i with the y-axis. We now cut each P_i into two segments U_i and V_i, where U_i runs from A_i to Q_i and V_i runs from Q_i to B_i. To the n-tuple $(U_1, U_2, \dots, U_n)$ we can associate a semistandard tableau S with entries from $\{1, 2, \dots, n\}$. This is done as follows: the i-th row of S is obtained from the path U_i by reading the indices of the weights of the vertical steps. (That is, we read i for a vertical step with weight x_i. Compare Figures 3.1 and 3.2.) The column-strictness of S is guaranteed by the fact that the paths $U_1, U_2, \dots, U_n$ are nonintersecting. Similarly, the n-tuple $(V_1, V_2, \dots, V_n)$ corresponds to a semistandard tableau T with entries from $\{1, 2, \dots, n\}$. Thus, the n-tuple $(P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths corresponds to a pair of tableaux (S, T) of the same shape. Clearly, the above procedure is reversible. Hence we obtain the claimed bijection.

The identity (3.10) follows now by using the fact that the generating function $\sum_S w(S)$, where the sum is over all semistandard tableaux S of shape λ with entries from $\{1, 2, \dots, n\}$, and where

$$w(S) = \prod_{i=1}^{n} x_i^{\#(i\text{'s in S})},$$

is the Schur function $s_\lambda(X)$. ∎

A combination of Theorems 2.1, 3.2, and 3.3, and the evaluation (1.2) of the Cauchy determinant yields Cauchy's identity (Theorem 1.1).

We remark that, in fact, the flagged Cauchy determinant can be written as a determinant of complete symmetric functions in the full sets of variables X and Y. Let $A'_i = (-n, -i)$ and $B'_i = (n, -i)$. (See Figure 3.3 for the location of these points in the case that $n = 4$. At this point, the paths should be ignored.) It is easy to see that there is a one-to-one correspondence between n-tuples $(P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths with P_i running from A_i to B_i and n-tuples $(P'_1, P'_2, \dots, P'_n)$ of nonintersecting lattice paths with P'_i running from A'_i to B'_i, for, restricted by the property that paths must be nonintersecting, the path P'_i must pass the points A_i and B_i, $i = 1, 2, \dots, n$; moreover, there is a unique way to extend the path P_i to the points A'_i and B'_i. Figure 3.3 shows the set $(P'_1, P'_2, \dots, P'_n)$ of modified paths for our example in Figure 3.1. Thus, we have the identity

$$\mathrm{GF}(\mathcal{P}_0(A, B)) = \mathrm{GF}(\mathcal{P}_0(A', B')). \tag{3.11}$$

On the other hand, for the generating function of paths from A'_i to B'_j we have the following result.

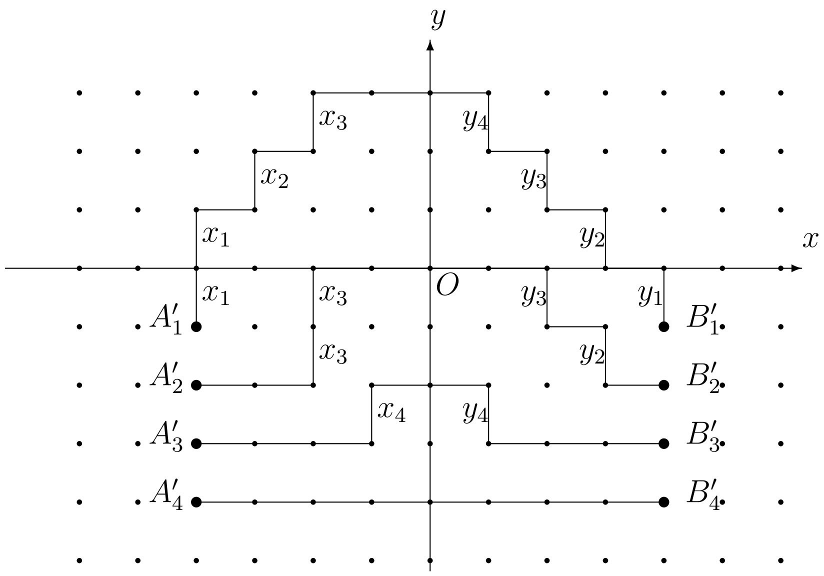

Figure 3.3 Nonintersecting paths from A'_i to B'_i

Lemma 3.4 *Let $A'_i = (-n, -i)$ and $B'_i = (n, -i)$. The generating function for paths from A'_i to B'_j equals*

$$\mathrm{GF}(\mathcal{P}(A'_i, B'_j)) = \sum_k h_{k-n+i}(X) h_{k-n+j}(Y). \tag{3.12}$$

Thus, using the standard theorem on nonintersecting paths (see [9, Cor. 2] or [27, Theorem 1.2]) another time, we obtain

$$\mathrm{GF}(\mathcal{P}_0(A', B')) = \left| \sum_k h_{k-n+i}(X) h_{k-n+j}(Y) \right|_{n\times n}. \tag{3.13}$$

A combination of Theorem 3.3 and Eqs. (3.11) and (3.13) then yields Theorem 1.2 of Gessel. We remark that our lattice path interpretation can also prove the general form of Gessel's theorem, i.e., Theorem 1.2 where X and Y are *infinite* sets of variables. To accomplish the proof, one just has to "move" the starting points "to the left" and the end points "to the right," more precisely, we would choose $A'_i = (-\infty, -i)$ and $B'_i = (+\infty, -i)$, $i = 1, 2, \ldots, n$, and modify the labelling and the weights of the vertical steps accordingly.

More generally, let α and β be two partitions of length at most n. If we choose as starting points the points $A''_i = (-\infty, \alpha_i - i)$ and as end points the points $B''_i = (+\infty, \beta_i - i)$, then, in the same way as above, we obtain the following generalization of Theorem 1.2, also due to Gessel [7, Theorem 16; cf. the paragraph just before Theorem 16].

Theorem 3.5 *We have*

$$\left|\sum_k h_{k-\alpha_i+i}(X)h_{k-\beta_j+j}(Y)\right|_{n\times n} = \sum_\lambda s_{\lambda/\alpha}(X)s_{\lambda/\beta}(Y), \tag{3.14}$$

where the sum ranges over all partitions of length at most n, and where $s_{\lambda/\alpha}(X)$ denotes the skew Schur function of shape λ/α in the variables X (see [22] or [26, Ch. 7]), and similarly for $s_{\lambda/\beta}(Y)$.

Gessel's proof of this theorem (as well as the proof of Theorem 1.2) is *algebraic*, as it makes use of the Cauchy–Binet theorem. For a different *bijective* proof, which makes use of the combinatorics of two-rowed arrays and the skew Robinson–Schensted–Knuth correspondence due to Sagan and Stanley [25], in the formulation of Fomin [5] and Roby [24, Section 4.1], see [17, proof of Theorem 3].

As we observed earlier, Theorem 1.2 and Theorem 3.2 present two different determinants for $F(X,Y)$, which implies that these two determinants are equivalent. To conclude, we present an algebraic proof of this fact by using knowledge about multi-Schur functions. In particular, we need the following property of these functions [19, 21].

Lemma 3.6 *For any families $L_0, L_1, \ldots, L_{n-1}$ of variables such that $|L_i| \le i$, we have*

$$s_\lambda(H_1, H_2, \ldots, H_n) = |h_{\lambda_j+j-i}(H_j)|_{n\times n} = |h_{\lambda_j+j-i}(H_j - L_{n-i})|_{n\times n}, \tag{3.15}$$

where $H_1, H_2, \ldots, H_n$ are sets of variables, and the complete super symmetric function $h_k(X-Y)$ is defined by the generating function

$$\sum_{k\ge 0} h_k(X-Y)\, t^k = \frac{\prod_{y\in Y}(1-yt)}{\prod_{x\in X}(1-xt)}.$$

We note that the matrix in Equation (1.4) can be expressed as the product of two matrices:

$$\left(\sum_k h_{k-n+i}(X)h_{k-n+j}(Y)\right)_{n\times n} = \left(h_{j-i}(X)\right)_{n\times\infty} \cdot \left(h_{i-j}(Y)\right)_{\infty\times n}. \tag{3.16}$$

Let $X_i = \{x_1, x_2, \ldots, x_i\}$ and $Y_i = \{y_1, y_2, \ldots, y_i\}$. On the left-hand side of (3.16) we may replace the pair of sets of variables (X,Y) in the (i,j)-entry by $(X - X_{n-i}, Y - Y_{n-j})$. In accordance with this substitution on the left-hand side, we must make the corresponding substitutions on the right-hand side of (3.16), that is, we must replace

X in the i-th row by $X - X_{n-i}$ in the first matrix and we must replace Y in the j-th column by $Y - Y_{n-j}$ in the second matrix. After these substitutions, Equation (3.16) becomes

$$\begin{aligned}
&\left(\sum_k h_{k-n+i}(X - X_{n-i})h_{k-n+j}(Y - Y_{n-j})\right)_{n\times n} \\
&\quad = \left(h_{j-i}(X - X_{n-i})\right)_{n\times\infty} \cdot \left(h_{i-j}(Y - Y_{n-j})\right)_{\infty\times n}. \qquad (3.17)
\end{aligned}$$

We now apply the Cauchy-Binet formula to (3.16), to obtain the identity

$$\begin{aligned}
&\left|\sum_k h_{k-n+i}(X)h_{k-n+j}(Y)\right|_{n\times n} \\
&\quad = \sum_{1\le k_1<\cdots<k_n} \left|h_{k_j-1+j-i}(X)\right| \cdot \left|h_{k_i-1+i-j}(Y)\right|. \qquad (3.18)
\end{aligned}$$

On the other hand, the Cauchy-Binet formula applied to (3.17) yields the identity

$$\begin{aligned}
&\left|\sum_k h_{k-n+i}(X - X_{n-i})h_{k-n+j}(Y - Y_{n-j})\right|_{n\times n} \\
&\quad = \sum_{1\le k_1<\cdots<k_n} \left|h_{k_j-1+j-i}(X - X_{n-i})\right| \cdot \left|h_{k_i-1+i-j}(Y - Y_{n-j})\right|.
\end{aligned}$$

From Lemma 3.6 it follows that

$$\left|h_{k_j-1+j-i}(X)\right|_{n\times n} = \left|h_{k_j-1+j-i}(X - X_{n-i})\right|_{n\times n}, \qquad (3.19)$$

$$\left|h_{k_i-1+i-j}(Y)\right|_{n\times n} = \left|h_{k_i-1+i-j}(Y - Y_{n-j})\right|_{n\times n}. \qquad (3.20)$$

Applying (3.19) and (3.20) to (3.18), we have

$$\begin{aligned}
&\left|\sum_k h_{k-n+i}(X)h_{k-n+j}(Y)\right|_{n\times n} \\
&\quad = \sum_{k_1<\cdots<k_n} \left|h_{k_j-1+j-i}(X - X_{n-i})\right| \cdot \left|h_{k_i-1+i-j}(Y - Y_{n-j})\right| \\
&\quad = \left|\sum_k h_{k-n+i}(X - X_{n-i})h_{k-n+j}(Y - Y_{n-j})\right|_{n\times n} \\
&\quad = \left|\sum_k h_{k-n+i}(x_i, \ldots, x_n)h_{k-n+j}(y_j, \ldots, y_n)\right|_{n\times n}.
\end{aligned}$$

The last equality comes from simultaneously reversing the order of rows and columns of the determinant. Therefore, we have accomplished an algebraic proof of the equality of the flagged Cauchy determinant (3.9) and the determinant (1.4) in the full sets of variables. ∎

In the same way, we may derive a more general theorem:

Theorem 3.7 *For any two families $L_0, L_1, \ldots, L_{n-1}$ and $G_0, G_1, \ldots, G_{n-1}$ of variables such that $|L_i| \le i$, $|G_i| \le i$, we have*

$$\begin{aligned} &\left| \sum_k h_{k-n+i}(X) h_{k-n+j}(Y) \right|_{n\times n} \\ &\quad = \left| \sum_k h_{k-n+i}(X - L_{i-1}) h_{k-n+j}(Y - G_{j-1}) \right|_{n\times n} . \end{aligned} \tag{3.21}$$

Acknowledgments. This work was done under the auspices of the 973 Project on Mathematical Mechanization of the Ministry of Science and Technology, and the National Science Foundation of China. We thank Professor Alain Lascoux for his lectures on symmetric functions.

References

[1] Brenti, F.: Determinants of super-Schur functions, lattice paths, and dotted plane partitions. Adv. Math. **98**, 27-64 (1993)

[2] Bressoud, D.M., Wei, S.-Y.: Determinental formulae for complete symmetric functions. J. Combin. Theory, Ser. A **60**, 277-286 (1992)

[3] Chen, W.Y.C., Li, B.-Q., Louck, J.D.: The flagged double Schur function. J. Alg. Combin. **15**, 7-26 (2002)

[4] Ciucu, M., Eisenkölbl, T., Krattenthaler, C., Zare, D.: Enumeration of lozenge tilings of hexagons with a central triangular hole. J. Combin. Theory, Ser. A **95**, 251-334 (2001)

[5] Fomin, S.: Schensted algorithms for graded graphs. J. Alg. Combin. **4**, 5-45 (1995)

[6] Fulmek, M., Krattenthaler, C.: Lattice path proofs for determinantal formulas for symplectic and orthogonal characters. J. Combin. Theory, Ser. A **77**, 3-50 (1997)

[7] Gessel, I.M.: Symmetric functions and P-recursiveness. J. Combin. Theory, Ser. A **53**, 257-285, (1990)

[8] Gessel, I.M., Viennot, X.: Binomial determinants, paths, and hook length formulae. Adv. Math. **58**, 300-321 (1985)

[9] Gessel, I.M., Viennot, X.: Determinants, paths, and plane partitions. Preprint, 1989; available at `http://www.cs.brandeis.edu/~ira`.

[10] Goulden, I.P., Greene, C.: A new tableau representation for supersymmetric Schur function. J. Alg. **170**, 687-703 (1994)

[11] Hamel, A.M.: Pfaffians and determinants for Schur Q-functions. J. Combin. Theory, Ser. A **75**, 328-340 (1996)

[12] Hamel, A.M.: Determinantal forms for symplectic and orthogonal Schur functions. Canad. J. Math. **49/2**, 263-282 (1997)

[13] Hamel, A.M., Goulden, I.P.: Planar decompositions of tableaux and Schur function determinants. Europ. J. Combin. **16**, 461-477 (1995)

[14] Hamel, A.M., King, R.C.: Symplectic shifted tableaux and deformations of Weyl's denominator formula for $sp(2n)$. J. Alg. Combin. **16**, 269-300 (2002)

[15] Karlin, S., McGregor, J.L.: Coincidence properties of birth-and-death processes. Pacific J. Math. **9**, 1109-1140 (1959)

[16] Karlin, S., McGregor, J.L.: Coincidence probabilities. Pacific J. Math. **9**, 1141-1164 (1959)

[17] Krattenthaler, C.: Non-crossing two-rowed arrays and summations for Schur functions, in: Proc. of the 5th Conference on Formal Power Series and Algebraic Combinatorics, A. Barlotti, M. Delest, R. Pinzani, eds., Florence, 1993, D.S.I., Università di Firenze, pp. 301–314; available at `http://igd.univ-lyon1.fr/~kratt`.

[18] Krattenthaler, C.: Oscillating tableaux and nonintersecting lattice paths. J. Statist. Plann. Inference **54**, 75-85 (1996)

[19] Lascoux, A.: Symmetric Functions and Combinatorial Operators on Polynomials, CBMS Regional Conference Series in Mathematics **99**, Amer. Math. Soc., 2003.

[20] Lindström, B.: On the vector representations of induced matroids. Bull. London Math. Soc. **5**, 85-90 (1973)

[21] Macdonald, I.G.: Notes on Schubert Polynomials, Publication du LACIM, Université du Québec à Montréal **6**, 1991.

[22] Macdonald, I.G.: Symmetric Functions and Hall Polynomials, second ed., Oxford University Press, Oxford, 1995.

[23] Okada, S.: Lattice path method and characters of classical groups. Unpublished manuscript, 1992.

[24] Roby, T.W.: Applications and extensions of Fomin's generalization of the Robinson–Schensted correspondence to differential posets, Ph.D. thesis, M.I.T., Cambridge, Massachusetts, 1991.

[25] Sagan, B.E., Stanley, R.P.: Robinson–Schensted algorithms for skew tableaux. J. Combin. Theory, Ser. A **55**, 161-193 (1990)

[26] Stanley, R.P.: Enumerative Combinatorics, Vol. II, Cambridge University Press, 1999.

[27] Stembridge, J.R.: Nonintersecting paths, pfaffians, and plane partitions. Adv. Math. **83**, 96-131 (1990)

[28] Wachs, M.L.: Flagged Schur functions, Schubert polynomials, and symmetrizing operators. J. Combin. Theory, Ser. A **40**, 276-289 (1985)

[29] Ueno, K.: Lattice path proof of the ribbon determinant formula for Schur functions. Nagoya Math. J. **124**, 55-59 (1991)