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Factors of the Gaussian Coefficients

William Y. C. Chen[1] and Qing-Hu Hou[2]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China

emails: [1]chen@nankai.edu.cn, [2]hou@nankai.edu.cn

Abstract. We present some simple observations on factors of the q-binomial coefficients, the q-Catalan numbers, and the q-multinomial coefficients. Writing the Gaussian coefficient with numerator n and denominator k in a form such that $2k \leq n$ by the symmetry in k, we show that this coefficient has at least k factors. Some divisibility results of Andrews, Brunetti and Del Lungo are also discussed.

Keywords: q-multinomial coefficient, Gaussian coefficient, q-Catalan number, cyclotomic polynomial.

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Corresponding Author: William Y. C. Chen, chen@nankai.edu.cn

The *q-multinomial coefficients* are defined by

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix} = \frac{(q;q)_n}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_r}},$$

where $n_1 + n_2 + \cdots + n_r = n$ and

$$(q;q)_m = (1-q)(1-q^2)\cdots(1-q^m).$$

For $r = 2$, they are usually called the *q-binomial coefficients* or the *Gaussian coefficients* and are written as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{(1-q^{n-k+1})(1-q^{n-k+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{1}$$

The factorization of q-binomial coefficients plays an important role in the study of divisibility properties of generalized Euler numbers [2, 4, 7, 11]. There are many reasons for the Gaussian coefficients to be polynomials. From the point of view of cyclotomic polynomials, the divisibility for the Gaussian coefficients turns out to be a rather natural fact.

Let $\Phi_n(x)$ be the n-th cyclotomic polynomial defined by

$$\Phi_n(x) = \prod_{\substack{1 \le j \le n \\ \gcd(j,n)=1}} (x - \zeta_n^j),$$

where $\zeta_n = e^{2\pi\sqrt{-1}/n}$ is the n-th root of unity and $\gcd(j,\, n)$ denotes the great common divisor of j and n. It is well-known that $\Phi_n(x) \in \mathbb{Z}[x]$ is the irreducible polynomial for ζ_n (see, for example, [12]). The polynomial $x^n - 1$ has the following factorization into irreducible polynomials over $\mathbb{Z}$:

$$x^n - 1 = \prod_{j \,|\, n} \Phi_j(x). \tag{2}$$

Knuth and Wilf [8] provided the factorization of q-binomial coefficients. In the same manner, one may get the following factorization of q-multinomial coefficients, where the notation $\lfloor x \rfloor$ stands for the largest integer less than or equal to x.

Lemma 1 *The q-multinomial cofficients $\left[\begin{smallmatrix} n \\ n_1, n_2, \ldots, n_r \end{smallmatrix}\right]$ are polynomials in q and can be factored as*

$$\prod_{i=1}^{n} \left(\Phi_i(q)\right)^{\lfloor \frac{n}{i} \rfloor - \lfloor \frac{n_1}{i} \rfloor - \lfloor \frac{n_2}{i} \rfloor - \cdots - \lfloor \frac{n_r}{i} \rfloor}. \tag{3}$$

Proof. By Equation (2), we have

$$(-1)^m (q;q)_m = \prod_{j=1}^{m} \prod_{i \,|\, j} \Phi_i(q) = \prod_{i=1}^{m} \Phi_i^{\lfloor \frac{m}{i} \rfloor}(q) = \prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{m}{i} \rfloor}(q).$$

Therefore,

$$\begin{aligned}
\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix} &= \frac{\prod_{i=1}^{n} \Phi_i^{\lfloor \frac{n}{i} \rfloor}(q)}{\prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{n_1}{i} \rfloor}(q) \cdot \prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{n_2}{i} \rfloor}(q) \cdots \prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{n_r}{i} \rfloor}(q)} \\
&= \prod_{i=1}^{\infty} \left(\Phi_i(q)\right)^{\lfloor \frac{n}{i} \rfloor - \lfloor \frac{n_1}{i} \rfloor - \lfloor \frac{n_2}{i} \rfloor - \cdots - \lfloor \frac{n_r}{i} \rfloor}.
\end{aligned}$$

Since $\sum_{j=1}^{r} n_j = n$ and $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a + b \rfloor$, all the power indices in (3) are nonnegative, which implies that the q-multinomial coefficients are polynomials in q. ∎

From the above lemma we may obtain the following lower bound on the numbers of factors of $\begin{bmatrix} n \\ k \end{bmatrix}$.

Theorem 2 *Suppose $2k \leq n$. The Gaussian coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ has at least k irreducible factors.*

Proof. Suppose $n - k + 1 \leq i \leq n$. Since $n \geq 2k$, we have $2i \geq 2n - n + 2 > n$ and $i \geq 2k - k + 1 = k + 1$. Hence,

$$\left\lfloor \frac{n}{i} \right\rfloor = 1 \quad \text{and} \quad \left\lfloor \frac{k}{i} \right\rfloor = \left\lfloor \frac{n-k}{i} \right\rfloor = 0,$$

which implies that $\Phi_i(q)$ is an irreducible factor of $\begin{bmatrix} n \\ k \end{bmatrix}$. Therefore, $\begin{bmatrix} n \\ k \end{bmatrix}$ has at least k irreducible factors: $\Phi_{n-k+1}, \Phi_{n-k+2}, \ldots, \Phi_n$. ∎

Remark 1. Theorem 2 coincides with the observation that applying the command `simplify` in `Maple` to the quotient

$$\frac{(1-q^n)\cdots(1-q^{n-k+1})}{(1-q)\cdots(1-q^k)}$$

gives a product of k factors. In fact, we may factorize $\begin{bmatrix} n \\ k \end{bmatrix}$ into k factors by the following procedure. Let $S_i = \{j \colon j \text{ divides } n - i + 1\}$ and $T_i = \{j \colon j \text{ divides } i\}$, $1 \leq i \leq k$. If an integer r appears in T_i for some i, then there is some S_j containing r. Now we may remove the element r from both T_i and S_j. Repeating this procedure until all T_i's become empty, one gets subsets R_i of S_i for $1 \leq i \leq k$ such that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{k} \left(\prod_{j \in R_i} \Phi_j(q) \right).$$

Note that $n-i+1 \in S_i$, but it does not belong to any T_j. It follows that $n-i+1 \in R_i$ and hence $\prod_{j \in R_i} \Phi_j(q)$ are non-trivial factors. This implies that we obtain k factors if we carry out the computation by first factoring the denominator into irreducible factors and then dividing these factors from the numerator.

Remark 2. Let $A(n,k)$ be the number of irreducible factors of $\left[{n \atop k} \right]$. Let $B(k)$ be the minimum number $A(n,k)$ for $n \geq 2k$. As pointed by one of the referees, it seems that $\lim_{k\to\infty} B(k)/k$ exists and equals approximately 1.3.

The irreducible factors of $\left[{n \atop k} \right]$ can be characterized as follows. The proof is straightforward. Let $\{x\}$ denote the fractional part of x, namely, $\{x\} = x - \lfloor x \rfloor$.

Proposition 3 *$\Phi_i(q)$ is a factor of $\left[{n \atop k} \right]$ if and only if $\left\{ \frac{k}{i} \right\} > \left\{ \frac{n}{i} \right\}$.*

Let us consider the value of $\Phi_n(q)$ at $q = 1$. It is easy to see that $\Phi_1(1) = 0$. For $n > 1$, we have

$$\Phi_n(1) = \begin{cases} p, & \text{if } n = p^m \text{ for some prime number } p, \\ 1, & \text{otherwise,} \end{cases}$$

see [10]. Based on this evaluation and Proposition 3, we obtain an equivalent form of Kummer's theorem [6, §1].

Corollary 4 *The power of prime p dividing $\binom{n}{m}$ is given by the number of integers $j > 0$ for which $\{m/p^j\} > \{n/p^j\}$.*

As a q-generalization of the Catalan numbers, the q-Catalan numbers have been extensively investigated (see [5, 9]). From Theorem 2, we obtain the following divisibility property of the q-Catalan numbers.

Corollary 5 *The q-Catalan numbers $\frac{1-q}{1-q^{n+1}} \left[{2n \atop n} \right]$ are polynomials in q and have at least $n-1$ irreducible factors.*

Proof. Since $\Phi_{n+2}, \Phi_{n+3}, \ldots, \Phi_{2n}$ are irreducible factors of $\left[{2n \atop n} \right]$ and are coprime with $1 - q^{n+1}$, they are also irreducible factors of the q-Catalan number. For each factor Φ_i of $1 - q^{n+1}$ other than Φ_1, we have $i \mid n+1$ and

$$\left\{ \frac{n}{i} \right\} = \frac{i-1}{i} > \left\{ \frac{2n}{i} \right\} = \frac{i-2}{i}.$$

From Theorem 2, it follows that Φ_i is a factor of $\left[{2n \atop n} \right]$. ∎

An interesting factor of the q-multinomial coefficient $\begin{bmatrix} n \\ n_1,\dots,n_r \end{bmatrix}$ is

$$(q^n-1)/(q^d-1) = 1+q^d+q^{2d}+\cdots+q^{n-d}.$$

where $d=\gcd(n,n_1,\dots,n_r)$. Andrews [1] proved the existence of this factor for the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ with n and k being relatively prime. Brunetti and Del Lungo [3] extended this result to the q-multinomial coefficients. We note that this divisibility property easily follows from Lemma 1.

Theorem 6 (Brunetti and Del Lungo) *Let $n_1, n_2, \dots, n_r$ be nonnegative integers such that $n = n_1+\cdots+n_r$. If $d=\gcd(n,n_1,\dots,n_r)$, then*

$$f(q) = \begin{bmatrix} n \\ n_1,\dots,n_r \end{bmatrix} \frac{q^d-1}{q^n-1} = \begin{bmatrix} n \\ n_1,\dots,n_r \end{bmatrix} \Big/ \, (1+q^d+q^{2d}+\cdots+q^{n-d})$$

is a polynomial in q with nonnegative coefficients. Moreover, $f(q)$ can be written as a product of $n-M-1$ nonconstant polynomials, where $M=\max\{n_1,\dots,n_r\}$.

Proof. First we show that $f(q)$ is a polynomial in q. Since $q^n-1=\prod_{j|n}\Phi_j(q)$ has no multiple roots, it suffices to show that for any $j\,|\,n$, $\Phi_j(q)$ is a factor of $\begin{bmatrix} n \\ n_1,\dots,n_r \end{bmatrix}(q^d-1)$. In fact, if $j\nmid n_t$ for some $1\le t\le r$, then we have $\lfloor n_t/j\rfloor < n_t/j$ and

$$\left\lfloor \frac{n}{j}\right\rfloor - \left\lfloor \frac{n_1}{j}\right\rfloor - \cdots - \left\lfloor \frac{n_r}{i}\right\rfloor > \frac{n}{j} - \frac{n_1+n_2+\cdots+n_r}{j} = 0.$$

By Lemma 1, $\Phi_j(q)$ is a factor of $\begin{bmatrix} n \\ n_1,\dots,n_r \end{bmatrix}$. Otherwise, we have $j\,|\,n_t$ for any $1\le t\le r$. Thus $j\,|\,\gcd(n,n_1,\dots,n_r)$ and $\Phi_j(q)$ is a factor of q^d-1.

Since

$$f(q) = \begin{bmatrix} n \\ n_1,\dots,n_r \end{bmatrix} \frac{1+q+\cdots+q^{d-1}}{1+q+\cdots+q^{n-1}},$$

the nonnegativity of the coefficients of $f(q)$ follows from the the unimodal property of

$$\begin{bmatrix} n \\ n_1,\dots,n_r \end{bmatrix}(1+q+\cdots+q^{d-1}).$$

Without loss of generality, we may assume that $n_1=\max\{n_1,\dots,n_r\}$. Then

$$f(q) = \frac{(1-q^{n_1+1})(1-q^{n_1+2})\cdots(1-q^{n-1})}{\left(\prod_{k=2}^r\prod_{j=1}^{n_k}(1-q^j)\right)/(1-q^d)}. \tag{4}$$

Since n_1 is maximal, for $n_1+1 \le j \le n-1$, $\Phi_j(q)$ is a factor of $1-q^j$ but not a factor of the polynomial

$$\left(\prod_{k=2}^{r}\prod_{j=1}^{n_k}(1-q^j)\right)/(1-q^d).$$

Thus, after cancelling the common factors of the numerator and denominator of (4), $1-q^{n_1+1},\ldots,1-q^{n-1}$ become $n-n_1-1$ nontrivial factors of $f(q)$. ∎

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