BG-ranks and 2-cores

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May 21, 2006

Abstract

We find the number of partitions of n whose BG-rank is j, in terms of $pp(n)$, the number of pairs of partitions whose total number of cells is n, giving both bijective and generating function proofs. Next we find congruences mod 5 for $pp(n)$, and then we use these to give a new proof of a refined system of congruences for $p(n)$ that was found by Berkovich and Garvan.

1 Introduction

If π is a partition of n we define the *BG-rank* $\beta(\pi)$, of π as follows. First draw the Ferrers diagram of π. Then fill the cells with alternating ±1's, chessboard style, beginning with a $+1$ in the $(1,1)$ position. The sum of these entries is $\beta(\pi)$, the BG-rank of π. For example, the BG-rank of the partition $13 = 4+3+3+1+1+1$ is -1.

$+1$	-1	$+1$	-1
-1	$+1$	-1	
$+1$	-1	$+1$	
-1			
$+1$			
-1			

Figure 1: A partition with BG-rank -1

This partition statistic has been encountered by several authors ([1, 2, 3, 6, 7]), but its systematic study was initiated in [1]. Here we wish to study the function

$$p_j(n) = |\{\pi : |\pi| = n \,\text{and}\, \beta(\pi) = j\}|.$$

We will find a fairly explicit formula for it (see (2) below), and a bijective proof for this formula. We will then show that a number of congruences from [1] can all be proved from a single set of congruences for the function $pp(n)$ defined by (1) below.

2 The theorem

We write $p(n)$ for the usual partition function, and $\mathcal{P}(x)$ for its generating function. If π is a partition of n then we will write $|\pi| = n$. $pp(n)$ will be the number of ordered pairs π', π'' of partitions such that $|\pi'| + |\pi''| = n$, i.e., $pp(n)$ is the sequence that is generated by

$$\sum_{n\geq 0} pp(n)x^n = \mathcal{P}(x)^2 = \prod_{i\geq 1} \frac{1}{(1-x^i)^2}. \tag{1}$$

By convention $pp(n)$ vanishes unless its argument is a nonnegative integer. Our main result is as follows.

Theorem 1 *The number of partitions of n whose BG-rank is j is given by*

$$p_j(n) = pp\left(\frac{n - j(2j-1)}{2}\right). \tag{2}$$

A non-bijective proof of this is easy, given the results of [1]. The authors of [1] found the two variable generating function for $\bar{p}_j(m, n)$, the number of partitions of n with BG-rank $= j$ and "2-quotient-rank" $= m$, in the form

$$\sum_{n,m} \bar{p}_j(m,n)x^m q^n = \frac{q^{j(2j-1)}}{(q^2x, q^2/x; q^2)_\infty}.$$

If we simply put $x = 1$ here, and read off the coefficients of like powers of q, we have (2). □

3 Bijective proof

A bijective proof of (2) follows from the theory of 2-cores. The *2-core* of a partition π is obtained as follows. Begin with the Ferrers diagram of π. Then delete a horizontal or a vertical pair of adjacent cells, subject only to the restriction that the result of the deletion must be a valid Ferrers diagram. Repeat this process, making arbitrary choices, until no further such deletions are possible. The remaining diagram is the 2-core of π, $C(\pi)$, say.

The 2-core of a partition is always a staircase partition, i.e., a partition of the form

$$\binom{k+1}{2} = k + (k-1) + \ldots + 1.$$

The following representation theorem is well known, and probably goes back to Littlewood [4] or to Nakayama [5]. For a lucid exposition see Schmidt [6].

Theorem 2 *There is a 1-1 (constructive) correspondence between partitions π of n and triples (S, π', π''), where S is a staircase partition (the 2-core of π), and π', π'' are partitions such that $n = |S| + 2|\pi'| + 2|\pi''|$.*

The proof of Theorem 1 will follow from the following observations:

1. First, the BG-rank of a partition and of its 2-core are equal, since at each stage of the construction of the 2-core we delete a pair of adjacent cells, which does not change the BG-rank.

2. An easy calculation shows that the BG-rank of a staircase partition of height k is $(k+1)/2$, if k is odd, and $-k/2$, if k is even.

3. Therefore, if π is a partition of BG-rank $= j$ then its 2-core is a staircase partition of height $2j-1$, if $j > 0$, and $-2j$, if $j \leq 0$.

4. In either case, if π is a partition whose BG-rank is j, then its 2-core is a diagram of exactly $j(2j-1)$ cells, i.e., a partition of the integer $j(2j-1)$.

Theorem 1 now follows from Theorem 2 and remark 4 above. □

Corollary 1 *There exists a partition of n with BG-rank $= j$ if and only if $j + n$ is even and $j(2j-1) \leq n$.*

4 Congruences

The motivation for introducing the BG-rank lay in the wish to refine some known congruences for $p(n)$. We can give quite elementary proofs of some of their congruences, in particular the following:

$$\begin{aligned}
p_j(5n) &\equiv 0 \pmod 5, \text{ if } j \equiv 1,2 \pmod 5, & (3)\\
p_j(5n+1) &\equiv 0 \pmod 5, \text{ if } j \equiv 0,3,4 \pmod 5, & (4)\\
p_j(5n+2) &\equiv 0 \pmod 5, \text{ if } j \equiv 1,2,4 \pmod 5, & (5)\\
p_j(5n+3) &\equiv 0 \pmod 5, \text{ if } j \equiv 0,3 \pmod 5, & (6)\\
p_j(5n+4) &\equiv 0 \pmod 5, \ \forall j. & (7)
\end{aligned}$$

First, we claim that all of the above congruences would follow if we could prove that

$$pp(n) \equiv 0 \text{ (mod 5) if } n \equiv 2,3,4 \text{ (mod 5)}. \tag{8}$$

This is because of the result

$$p_j(n) = pp\left(\frac{n - j(2j-1)}{2}\right)$$

of Theorem 1 above. There are 15 cases to consider, but fortunately they can all be done at once.

We want to prove that for each of the above pairs (n, j) mod 5, the quantity $(n - j(2j - 1))/2$ is either not an integer or else is 2, 3 or 4 mod 5. For it to be an integer we must have $j \equiv n$ mod 2. Hence we have a pair (n, j) which modulo 5 have given values (n', j'), say, and are such that $j \equiv n$ mod 2. This means that

$$n = 5s + 5j' - 4n' + 10t, \text{ and } j = 5s + j',$$

for some integers s, t. But then

$$\frac{n - j(2j-1)}{2} \equiv 3j' - 2n' - j'^2 \pmod 5. \tag{9}$$

Thus, to prove that (8) imply all of (3)–(7) we need only verify that for each of the 15 pairs (n', j')

$$(0,1), (0,2), (1,0), (1,3), (1,4), (2,1), (2,2), (2,4), (3,0), (3,3), (4, \text{all}),$$

mod 5 it is true that the right side of (9) is 2, 3 or 4 mod 5, which is a trivial exercise. □

It remains to establish (8). We have, modulo 5,

$$\frac{1}{(1-t)^2} \equiv \frac{(1-t)^3}{(1-t^5)},$$

and therefore

$$\prod_{j\geq 1} \frac{1}{(1-x^j)^2} \equiv \frac{\prod_{j\geq 1}(1-x^j)^3}{\prod_{j\geq 1}(1-x^{5j})}.$$

On the other hand it is known that

$$\prod_{j\geq 1}(1-x^j)^3 = \sum_{n\geq 0}(-1)^n(2n+1)x^{\binom{n+1}{2}}.$$

Consequently,

$$\sum_{k\geq 0} pp(k)x^k \equiv \left(\sum_{n\geq 0}(-1)^n(2n+1)x^{\binom{n+1}{2}}\right)\left(\sum_{m\geq 0} p(m)x^{5m}\right).$$

Now all exponents of x on the right are of the form $5m+\binom{n+1}{2}$. Since $\binom{n+1}{2}$ is always 0,1, or 3 mod 5, we have surely that $pp(k)\equiv 0$ if $k\equiv 2,4$ mod 5. Finally, if $\binom{n+1}{2}\equiv 3$ mod 5, then $n\equiv 2$, so $2n+1\equiv 0$, and again the coefficient of x^k vanishes mod 5. $\square$

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