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Matrix identities on weighted partial Motzkin paths

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Abstract. We give a combinatorial interpretation of a matrix identity on Catalan numbers and the sequence $(1, 4, 4^2, 4^3, \ldots)$ which has been derived by Shapiro, Woan and Getu by using Riordan arrays. By giving a bijection between weighted partial Motzkin paths with an elevation line and weighted free Motzkin paths, we find a matrix identity on the number of weighted Motzkin paths and the sequence $(1, k, k^2, k^3, \ldots)$ for $k \geq 2$. By extending this argument to partial Motzkin paths with multiple elevation lines, we give a combinatorial proof of an identity recently obtained by Cameron and Nkwanta. A matrix identity on colored Dyck paths is also given, leading to a matrix identity for the sequence $(1, t^2 + t, (t^2 + t)^2, \ldots)$.

Key words: Catalan number, Schröder number, Dyck path, Motzkin path, partial Motzkin path, free Motzkin path, weighted Motzkin path, Riordan array

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1 Introduction

This paper is motivated by the following matrix identity obtained by Shapiro et al. [12] in their study of the moments of a Catalan triangle [5, 9, 16]:

$$\begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4^2 \\ 4^3 \\ 4^4 \\ \vdots \end{bmatrix}, \tag{1.1}$$

where the entries in the first column of the matrix are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ and $a_{i,j}$ (the entry in the ith row and jth column) is then determined by the following recurrence relation for $j \geq 2$:

$$a_{i,j} = a_{i-1,j-1} + 2a_{i-1,j} + a_{i-1,j+1}. \tag{1.2}$$

Another proof of the above identity is given by Woan et al. [17] while computing the areas of parallelo-polyominos via generating functions.

The first result of this paper is a combinatorial interpretation of the identity (1.1) in terms of Dyck paths.

This work was motivated by the question of finding a matrix identity that extends the sequence $(1, 4, 4^2, 4^3, \ldots)$ to $(1, k, k^2, k^3, \ldots)$ in (1.1). The following matrix identity was proved by Cameron and Nkwanta [4] that arose in a study of elements of order 2 in *Riordan groups* [1, 10, 11, 14]:

$$\begin{bmatrix} 1 & & & & & \\ 3 & 1 & & & & \\ 11 & 6 & 1 & & & \\ 45 & 31 & 9 & 1 & & \\ 197 & 156 & 60 & 12 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 7 \\ 15 \\ 31 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 6^2 \\ 6^3 \\ 6^4 \\ \vdots \end{bmatrix}, \tag{1.3}$$

where the entry $a_{i,j}$ (ith row and jth column) in the above matrix satisfies the recurrence relation

$$a_{i,j} = a_{i-1,j-1} + 3a_{i-1,j} + 2a_{i-1,j+1} \tag{1.4}$$

for $j \geq 2$ and the $a_{i,1}$ is the ith *little Schröder number* s_i (sequence A001003 in [13]), which counts little Schröder paths of length $2i$. A *little Schröder path* is a lattice path starting at (0, 0) and ending at $(2n, 0)$ and using steps $H = (2, 0)$, $U = (1, 1)$ and

$D = (1, -1)$ such that no step is below the x-axis and there are no peaks at level one. Imposing this condition gives us little Schröder numbers while without it we would have the *large Schröder numbers*.

For $k = 3$, we obtain the following matrix identity on Motzkin numbers:

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 4 & 5 & 3 & 1 & & \\ 9 & 12 & 9 & 4 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3^2 \\ 3^3 \\ 3^4 \\ \vdots \end{bmatrix}, \tag{1.5}$$

where the first column is the sequence of Motzkin numbers, and the matrix $A = (a_{i,j})_{i,j\geq 1}$ is generated by the following recurrence relation:

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j} + a_{i-1,j+1}.$$

For $k = 5$, we find the following matrix identity:

$$\begin{bmatrix} 1 & & & & & \\ 3 & 1 & & & & \\ 10 & 6 & 1 & & & \\ 36 & 29 & 9 & 1 & & \\ 137 & 132 & 57 & 12 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5^2 \\ 5^3 \\ 5^4 \\ \vdots \end{bmatrix}, \tag{1.6}$$

where the first column is the sequence A002212 in [13], which has two interpretations, the number of 3-Motzkin paths or the number of ways to assemble benzene rings into a tree [8]. Recall that a 3-Motzkin path is a lattice path from $(0, 0)$ to $(n, 0)$ that does not go below the x-axis and consists of up steps $U = (1, 1)$, down steps $D = (1, -1)$, and three types of horizontal steps $H = (1, 0)$. The above matrix $A = (a_{i,j})_{i,j\geq 1}$ is generated by the first column together with the following recurrence relation:

$$a_{i,j} = a_{i-1,j-1} + 3a_{i-1,j} + a_{i-1,j+1}.$$

We may prove the above identities (1.5) and (1.6) by using the method of Riordan arrays. So the natural question is how to find a matrix identity for the sequence $(1, k, k^2, k^3, \ldots)$. We need the combinatorial interpretation of the entries in the matrix in terms of weighted partial Motzkin paths, as given by Cameron and Nkwanta [4].

To be precise, a *partial Motzkin path*, also called a Motzkin path from $(0,0)$ to (n,k) in [4], is just a Motzkin path but without the requirement of ending on the x-axis. A *weighted partial Motzkin path* is a partial Motzkin path with the weight assignment that the horizontal steps are endowed with a weight k and the down steps are endowed with a weight t, where k and t are regarded as positive integers. In this sense, weighted Motzkin paths are a generalization of k-Motzkin paths in the sense of 2-Motzkin paths and 3-Motzkin paths [2, 7, 13].

We also introduce the notion of *weighted free Motzkin path* which is a lattice path consisting of Motzkin steps without the restrictions that it has to end with a point on the x-axis and that it cannot go below the x-axis. We then give a bijection between weighted free Motzkin paths and weighted partial Motzkin paths with an elevation line, which leads to a matrix identity involving the number of weighted partial Motzkin paths and the sequence $(1, k, k^2, k^3, \ldots)$. The idea of the elevation operation is also used by Cameron and Nkwanta in their combinatorial proof of the identity (1.1) in a more restricted form. By extending the argument to weighted partial Motzkin paths with multiple elevation lines, we obtain a combinatorial proof of an identity recently derived by Cameron and Nkwanta, in answer to their question.

We also give a generalization of the matrix identity (1.3) and give a combinatorial proof by using colored Dyck paths.

2 Riordan arrays

In this section, we give a brief introduction to the notion of Riordan arrays [10, 11, 14]. Let us use (1.1) and (1.3) as examples. Start with two generating functions $g(x) = 1 + g_1x + g_2x^2 + \cdots$ and $f(x) = f_1x + f_2x^2 + \cdots$ with $f_1 \neq 0$. Let $H = (h_{i,j})_{i,j\geq 0}$ be the infinite lower triangular matrix with nonzero entries on the main diagonal, where $h_{i,j} = [x^i]g(x)(f(x))^j$ for $i \geq j$, namely, $h_{i,j}$ equals the coefficient of x^i in the expansion of the series $g(x)(f(x))^j$. If an infinite lower triangular matrix H can be constructed in this way from two generating functions $g(x)$ and $f(x)$, then it is called a *Riordan array* and is denoted by $H = (g(x), f(x)) = (g, f)$.

Suppose we multiply the matrix $H = (g, f)$ by a column vector $(a_0, a_1, \ldots)^T$ and get a column vector $(b_0, b_1, \ldots)^T$. Let $A(x)$ and $B(x)$ be the generating functions for the sequences $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ respectively. Then it follows quickly that

$$B(x) = g(x)A(f(x)).$$

This allows us to switch easily between the matrix form and generating functions.

For the matrix identity (1.1), let $g(x)$ be the generating function for Catalan numbers $(1,2,5,14,\ldots)$:

$$g(x)=\frac{1-2x-\sqrt{1-4x}}{2x^2}.$$

Let $f(x)=xg(x)$. From the recurrence relation (1.2) one may derive that the generating function for the sequence in the jth ($j\geq 1$) column in the matrix in (1.1) equals $g(xg)^{j-1}$. Let H be the Riordan array (g,xg). Since the generating function of $(1,2,3,4,\ldots)$ equals $A(x)=\frac{1}{(1-x)^2}$, it follows that $B(x)=g(x)A(xg(x))=\frac{1}{1-4x}$ is the generating function for the right hand side of (1.1). Thus we obtain the identity (1.1).

Let us consider the matrix identity (1.3). Let $g(x)$ be the generating function for the little Schröder numbers which is

$$g(x)=\frac{1-3x-\sqrt{1-6x+x^2}}{4x^2}, \tag{2.1}$$

and let $f(x)=xg(x)$. Note that the generating function for the sequence $(1,3,7,15,\ldots)$ equals $A(x)=\frac{1}{(1-x)(1-2x)}$. From the recurrence relation (1.4) one may verify that the matrix in (1.3) is indeed the Riordan array (g,xg). Therefore, the generating function for the right hand side of (1.3) equals $g(x)A(xg(x))=\frac{1}{1-6x}$, which implies (1.3).

Using the same method, we can verify the matrix identities (1.5) and (1.6). Since we are going to establish a general bijection for weighted Motzkin paths, we omit the proofs here.

3 Dyck path interpretation of (1.1)

In this section, we present a combinatorial interpretation of the matrix identity (1.1) by using Dyck paths. A *Dyck path* of length $2n$ is a path going from the origin $(0,0)$ to $(2n,0)$ using up steps $U=(1,1)$ and down steps $D=(1,-1)$ such that no step is below the x-axis [6, 15]. The number of Dyck paths of length $2n$ equals the Catalan number C_n.

For a Dyck path P, the points on the x-axis except for the initial point are called return points. In this sense, the ending point is always a return point. Formally speaking, a *composition* of a Dyck path P is a sequence of Dyck paths $(P_1,P_2,\ldots,P_j)$ such that $P=P_1P_2\ldots P_j$. For a composition $(P_1,P_2,\ldots,P_j)$ of a Dyck path P, its *length* is meant to be the length of P and j is called the number of segments. We may choose certain return points to cut off a Dyck path into a composition. We use the convention that the ending point is always a cut point. Clearly, a Dyck path with one segment is an ordinary Dyck path.

Lemma 3.1 *For $j \geq 2$, we have the following recurrence relation:*

$$d_{i,j} = d_{i-1,j-1} + 2d_{i-1,j} + d_{i-1,j+1}, \tag{3.1}$$

where $d_{i,j}$ is the number of compositions of Dyck paths of length $2i$ that contain j segments.

Proof. Let $(P_1, P_2, \ldots, P_j)$ be a composition of a Dyck path P of length $2i$. Consider the following cases for P_1. Case 1: $P_1 = UD$. Then we can get a composition of length $2(i-1)$ with $j-1$ segments: $(P_2, \ldots, P_j)$. Case 2: $P_1 = QUD$ and Q is not empty. Then we get a composition $(Q, P_2, \ldots, P_j)$ of length $2(i-1)$ with j segments. Case 3: $P_1 = UQD$ and Q is not empty. We get a composition $(Q, P_2, \ldots, P_j)$ of length $2(i-1)$ with j segments. Case 4: $P_1 = Q_1UQ_2D$, where Q_1 and Q_2 are not empty. Then we get a composition $(Q_1, Q_2, P_2, \ldots, P_j)$ of length $2(i-1)$ with $j+1$ segments. Adding up the terms in the above cases, we obtain the desired recursion (3.1). ∎

From Lemma 3.1 one sees that the entry $a_{i,j}$ in the triangular matrix of the identity (1.1) can be explained as the number of compositions of Dyck paths of length $2i$ that contain j segments. We remark that this combinatorial interpretation can also be derived from the generating function of the entries in the jth column of the matrix in (1.1). The following formula for $a_{i,j}$, essentially a ballot number, has been derived by Cameron and Nkwanta [4]:

$$a_{i,j} = \frac{j}{i}\binom{2i}{i-j}.$$

Let us rewrite the matrix identity (1.1) as follows:

$$\sum_{j=1}^{i} j a_{i,j} = 4^{i-1}. \tag{3.2}$$

A different combinatorial formulation of the above identity is given by Callan [3].

We are now ready to give a combinatorial proof of the above identity. Clearly, 4^n is the number of sequences of length n on four letters, say, $\{1, 2, 3, 4\}$. The term $ja_{i,j}$ suggests that we should specify a segment in a composition as a distinguished segment. We may use a star $*$ to mark the distinguished segment. We call a composition with a distinguished segment a *rooted* composition of a Dyck path. Then $ja_{i,j}$ equals the number of rooted compositions of Dyck paths of length $2i$ that contain j segments.

Theorem 3.2 *There is a bijection ϕ between the set of rooted compositions of Dyck paths of length $2i$ and the set of sequences of length $i-1$ on four letters.*

Proof. We recursively define a map ϕ from rooted compositions of a Dyck path P of length $2i$ to sequences of length $i-1$ on $\{1,2,3,4\}$. For $i=1$, P is unique, and the sequence is set to be the empty sequence. We now assume that $i>1$. Let $(P_1,\ldots,P_t^*,\ldots,P_j)$ be a rooted composition of P with P_t^* being the distinguished segment.

We have the following cases.

1. $P_1 = UD$ and $t=1$. Then we set $\phi(P) = 1\,\phi(P_2^*, P_3, \ldots, P_j)$.
2. $P_1 = UD$ and $t \neq 1$. Then we set $\phi(P) = 2\,\phi(P_2, \ldots, P_t^*, \ldots, P_j)$.
3. $P_1 = QUD$ and Q is a nonempty Dyck path. Set $\phi(P) = 3\,\phi(Q^*, P_2, \ldots, P_j)$ if $t=1$ and set $\phi(P_1,\ldots,P_j) = 3\,\phi(Q, P_2, \ldots, P_t^*, \ldots, P_j)$ if $t>1$.
4. $P_1 = Q_1UQ_2D$, where Q_1 and Q_2 are nonempty Dyck paths. Then we set
$$\phi(P_1,\ldots,P_j) = 1\,\phi(Q_1, Q_2, P_2, \ldots, P_t^*, \ldots, P_j) \text{ if } t>1,$$
$$\phi(P) = 1\,\phi(Q_1, Q_2^*, P_2, \ldots, P_j) \text{ if } t=1.$$
5. $P_1 = UQD$ and Q is a nonempty Dyck path. Then we set
$$\phi(P) = 4\,\phi(Q, P_2, \ldots, P_t^*, \ldots, P_j) \text{ if } t>1,$$
$$\phi(P) = 4\,\phi(Q^*, P_2, \ldots, P_j) \text{ if } t=1.$$

In order to show that ϕ is a bijection, we construct the reverse map of ϕ. Let $w = w_1w_2\ldots w_{i-1}$ be a sequence of length $i-1$ on $\{1,2,3,4\}$. If $i=1$, then it corresponds to UD. We now assume that $i>1$. Suppose that $w_2w_3\ldots w_{i-1}$ corresponds to a rooted composition $(R_1, R_2, \ldots, R_m)$ of a Dyck path P of length $2(i-1)$ with R_k being the distinguished segment. We proceed to find a rooted composition $(P_1, P_2, \ldots, P_j)$ with P_t being the distinguished segment such that $\phi(P_1, P_2, \ldots, P_j) = w_1\phi(R_1, R_2, \ldots, R_m)$.

If $w_1 = 2$, we have $P_1 = UD$ and $(P_2, P_3, \ldots, P_j) = (R_1, R_2, \ldots, R_m)$. It follows that $t = k+1$ and $j = m+1$. Thus we can recover $(P_1, P_2, \ldots, P_j)$. For the case $w_1 = 3$, we have $P_1 = R_1UD$ and $t=k$. Also, we can recover $(P_2, \ldots, P_j)$ from $(R_2, \ldots, R_m)$. For the case $w_1 = 4$, we have $t=k$, $P_1 = UR_1D$, and $(P_2, \ldots, P_j) = (R_2, \ldots, R_m)$.

It remains to deal with the situation $w_1 = 1$, which involves Cases 1 and 4 of the bijection. If $k=1$, then we have $t=1$, $P_1 = UD$ and $(P_2, \ldots, P_j) = (R_1, \ldots, R_m)$. If $k=2$, then we have $t=1$, $P_1 = R_1UR_2D$ and $(P_2, \ldots, P_j) = (R_3, \ldots, R_m)$. If $k>2$, then we have $t = k-1$, $P_1 = R_1UR_2D$ and $(P_2, \ldots, P_j) = (R_3, \ldots, R_m)$. Thus, we have shown that ϕ is a bijection. ∎

An example of the bijection ϕ is given in Fig. 1, where normal vertices are drawn with solid dots, return points that cut the Dyck paths into segments are drawn with open circles, and the distinguished segment is marked with a $*$ on its last return step.

$*$ $\longleftrightarrow$ 2 $*$ $\longleftrightarrow$

23 $*$ $\longleftrightarrow$ 234 $*$ $\longleftrightarrow$ 2341 $*$ $\longleftrightarrow$ 2341

Figure 1: The bijection ϕ.

4 Weighted partial Motzkin paths

A *Motzkin path* of length n is a path going from $(0,0)$ to $(n,0)$ consisting of up steps $U=(1,1)$, down steps $D=(1,-1)$ and horizontal steps $H=(1,0)$, which never goes below the x-axis. A (k,t)-*Motzkin path* is a Motzkin path such that each horizontal step is weighted by k, each down step is weighted by t and each up step is weighted by 1. The case $k=2, t=1$ gives the 2-Motzkin paths which have been introduced by Barcucci et al. [2] and have been studied by Deutsch and Shapiro [7]. The *weight* of a path is the product of the weights of all its steps. Denote by $|P|$ the weight of a path P. The *weight* of a set of paths is the sum of the total weights of all the paths. For any step, we say that it is at level k if the y-coordinate of its end point equals k.

In this section, we aim to establish the following matrix identity on weighted Motzkin paths.

Theorem 4.1 *Let $M=(m_{i,j})_{i,j\geq 1}$ be the lower triangular matrix such that the first column is the sequence of the total weight of $(k-t-1,t)$-Motzkin paths of length $i-1$. Suppose that we have the following recurrence relation for $j\geq 2$:*

$$m_{i,j}=m_{i-1,j-1}+(k-t-1)m_{i-1,j}+tm_{i-1,j+1}. \tag{4.1}$$

Then we have

$$(m_{i,j})_{i,j\geq 1}\times\begin{bmatrix}1\\1+t\\1+t+t^2\\\vdots\end{bmatrix}=\begin{bmatrix}1\\k\\k^2\\\vdots\end{bmatrix}. \tag{4.2}$$

Note that the special case of (4.2) for $t = 1$ has been derived by Aigner in [1] by using sum coefficients. It is well known that the number of 2-Motzkin paths of length n is given by the Catalan number C_{n+1}. It follows that the matrix identity (1.1) is a special case of (4.2) for $k = 4, t = 1$. Denote by $f(x) = \sum_{n\geq 0} f_n x^n$ the generating function for the number of $(3, 2)$-Motzkin paths. Then it is easy to find the functional equation for $f(x)$:

$$f(x) = 1 + 3xf(x) + 2x^2 f^2(x).$$

It follows that

$$f(x) = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x^2}.$$

From the above generating function, one sees that the number of $(3, 2)$-Motzkin paths of length n is the nth little Schröder number. Therefore, the matrix identity (1.3) is a special case of (4.2) for $k = 6, t = 2$.

Let us rewrite the matrix identity (4.2) in the following form:

$$\sum_{j=1}^{i} m_{i,j}(1 + t + \cdots + t^{j-1}) = k^{i-1}. \tag{4.3}$$

The following combinatorial interpretation of the entries in the matrix in (4.2) is due to Cameron and Nkwanta [4]. A *partial (k, t)-Motzkin path* is defined as an initial segment of a (k, t)-Motzkin path. We say that a partial (k, t)-Motzkin path ends at level j if its last step is at level j.

Lemma 4.2 ([4]) *Let $m_{i,j}$ be the entries in the matrix in (4.2). Then $m_{i,j}$ equals the total weight of partial $(k-t-1, t)$-Motzkin paths of length $i-1$ that end at level $j-1$.*

Proof. Regarding the first column of the matrix M, one sees that a partial $(k-t-1, t)$-Motzkin path that ends at level zero is just a $(k-t-1, t)$-Motzkin path. Let $a_{i,j}$ denote the total weight of partial $(k - t - 1, t)$-Motzkin paths of length $i - 1$ ending at level $j - 1$. Let P be a partial $(k - t - 1, t)$-Motzkin path of length $i - 1$ that ends at level $j - 1$ $(j > 1)$. By considering the last step of P and its weight, one sees that $a_{i,j}$ satisfies the recurrence relation (4.1). ∎

Let P be a partial $(k - t - 1, t)$-Motzkin path ending at level $j - 1$. We need the notion of an *elevated* partial Motzkin path introduced by Cameron and Nkwanta [4] in their combinatorial proof of the following identity which is a reformulation of (3.2):

$$4^n = \sum_{k=0}^{n} \frac{(k+1)^2}{n+1} \binom{2n+2}{n-k}.$$

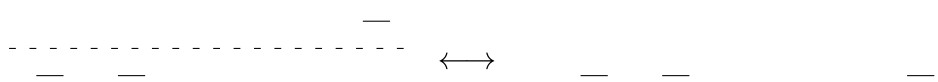

Figure 2: Elevation of a partial Motzkin path.

Let p be an integer with $0 \leq p \leq j-1$. The *elevation* of P with respect to the horizontal line $y = p$ is defined as follows. For $p = 0$, the elevation of P with respect to $y = 0$ is just P itself. We now assume $0 < p \leq j-1$. Note that there are always up steps of P at levels $j-1$, $j-2$, $\ldots$, 1. Therefore, for each level from 1 to p, one can find a rightmost up step. Note that there are no other steps at the same level to the right of the rightmost up step which is called an *R-visible* up step with respect to the line $y = p$ in the sense that it can be seen far away from the right. The elevation of P with respect to the line $y = p$ is derived from P by changing the R-visible up steps up to level p to down steps by elevating their initial points. The line $y = p$ is called an *elevation line*.

Fig. 2 is an illustration of the elevation of a partial Motzkin path with respect to the line $y = 2$.

We now introduce the notion of *free Motzkin paths* which are lattice paths starting from $(0,0)$ and using up steps $U = (1,1)$, down steps $D = (1,-1)$ and horizontal steps $H = (1,0)$. Note that there is no further restriction so that the paths may go below the x-axis. A free (k,t)-Motzkin path is a free Motzkin path in which the steps are weighted in the same way as for (k,t)-Motzkin paths, namely, an up step has weight one, a horizontal step has weight k and a down step has weight t.

For a free Motzkin path P we may analogously define the *L-visible* down steps as the down steps that are visible from the far left. It is clear that a complete Motzkin path (a partial Motzkin path with ending point on the x-axis) has no R-visible up steps. Similarly, a partial Motzkin path has no L-visible down steps.

We have the following summation formula for weighted free Motzkin paths.

Lemma 4.3 *The sum of weights of free $(k-t-1,t)$-Motzkin paths of length i equals k^i.*

The proof of the above lemma is obvious because of the relation

$$(1 + k - t - 1 + t)^i = k^i.$$

We are now led to establish a bijection for the identity (4.3).

Theorem 4.4 *There is a bijection between the set of partial $(k-t-1,t)$-Motzkin paths of length i with an elevation line and the set of free $(k-t-1,t)$-Motzkin paths of length i.*

Proof. The bijection is just the elevation operation. The reverse map is also easy. For a free Motzkin path, one can identify the L-visible down steps, if any, then change these L-visible down steps to up steps by elevating their end points. ∎

For a partial $(k-t-1,t)$-Motzkin path P with an elevation line $y=p$, suppose that Q is the elevation of P with respect to $y=p$. It is clear that the weight of Q equals $t^p|P|$. If P ends at level j, then the possible elevation lines are $y=0$, $y=1,\ldots,y=j$. Summing over j, we arrive at a combinatorial interpretation of the identity (4.3).

As a consequence of Theorem 4.4, we obtain the matrix identity (4.2).

5 An identity of Cameron and Nkwanta

In their study of involutions in Riordan groups, Cameron and Nkwanta [4] obtained the following identity for $m \geq 0$, and asked for a purely combinatorial proof:

$$\binom{n}{m}4^{n-m} = \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{k+m+1}{k-m}\binom{2n+2}{n-k}.$$

It is clear that identity (3.2) is a special case of the above identity for $m=0$. To be consistent with our notation, we may rewrite the above identity in the following form:

$$\binom{i-1}{m}4^{i-1-m} = \sum_{j=1}^{i} \frac{j}{i}\binom{j+m}{2m+1}\binom{2i}{i-j}. \qquad (5.1)$$

We now give a bijective proof of (5.1).

We recall that the number of partial 2-Motzkin paths of length $i-1$ ending at level $j-1$ is given by $a_{i,j} = \frac{j}{i}\binom{2i}{i-j}$. We now give a combinatorial interpretation of the binomial coefficient $\binom{j+m}{2m+1}$. Let P be a partial 2-Motzkin path of length $i-1$ ending at level $j-1$ with m marked R-visible up steps and $m+1$ elevation lines such that there is exactly one marked up step between two adjacent elevation lines. Such a configuration can be represented as follows:

$$t_1 \,|\, t_2 \,*\, t_3 \,|\, t_4 \,*\, t_5 \,|\, \cdots \,|\, t_{2m} \,*\, t_{2m+1} \,|\, t_{2m+2},$$

where t_i denotes the number of unmarked R-visible up steps. It is clear that we have $t_1 + t_2 + \ldots + t_{2m+2} = j - 1 - m$, and the number of solutions of this equation equals the number of ways to distribute $j - 1 - m$ balls into $2m + 2$ boxes while a box may have more than one ball. So this number equals the binomial coefficient $\binom{j+m}{2m+1}$. This leads to the following Lemma.

Lemma 5.1 *The summand in (5.1) counts partial* 2*-Motzkin paths of length* $i - 1$ *ending at level* $j - 1$ *with* m *marked R-visible up steps and* $m + 1$ *elevation lines such that there is exactly one marked step between two adjacent elevation lines.*

We are now ready to give a combinatorial proof of the identity of Cameron and Nkwanta. We recall that a 2-Motzkin path has two kinds of horizontal steps, straight steps and wavy steps. We need to introduce the third kind of horizontal steps – dotted steps. Therefore, the left hand side of (5.1) is the number of free 3-Motzkin paths of length $i-1$ with exactly m dotted horizontal steps. We now give the following bijection that leads to a combinatorial interpretation of (5.1).

Theorem 5.2 *There is a bijection between partial* 2*-Motzkin paths of length* i *with* m *marked R-visible up steps and* $m + 1$ *elevation lines such that there is exactly one marked step between two adjacent elevation lines and free* 3*-Motzkin paths of length* i *with exactly* m *dotted horizontal steps.*

Proof. Suppose that $P = P_1 U^* P_2 U^* \ldots P_m U^* P_{m+1}$ is a partial 2-Motzkin path with m marked R-visible up steps and $m + 1$ elevation lines, then we get a free 3-Motzkin path by changing all the marked up steps to dotted horizontal steps and applying the elevation operation for each P_k.

Conversely, given a free 3-Motzkin path $P = P_1 \dashrightarrow P_2 \dashrightarrow \cdots P_m \dashrightarrow P_{m+1}$ with m dotted horizontal steps, where $\dashrightarrow$ denotes a dotted horizontal step, then we can get a partial 2-Motzkin path by changing each dotted horizontal step to a marked up step and the L-visible down steps of each P_k to up steps by elevating their end points. ∎

Fig. 3 is an illustration of the elevation operation with respect to multiple elevation lines. We conclude this section by giving a more general identity. Let $a_{i,j,k}$ be the number of partial k-Motzkin paths of length $i - 1$ ending at level $j - 1$. Then we have

$$\binom{i-1}{m} k^{i-1-m} = \sum_{j=1}^{i} a_{i,j,k-2} \binom{j+m}{2m+1}. \tag{5.2}$$

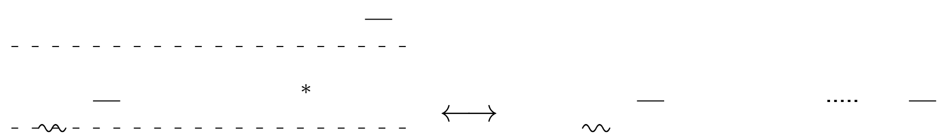

Figure 3: Elevation of a partial 2-Motzkin path.

6 A Dyck path generalization of (1.3)

In this section, we give a Dyck path generalization of the matrix identity (1.3) on the little Schröder numbers. A *k-Dyck path* is a Dyck path in which an up step is colored by one of the k colors in $\{1, 2, \ldots, k\}$ if it is not immediately followed by a down step. In this section, we aim to give the following generalization of (1.3).

Theorem 6.1 *Let $M = (m_{i,j})_{i,j\geq 1}$ be a lower triangular matrix with the first column being the number of (t^2-t)-Dyck paths of length $2i$. The other columns of M are given by the following relation:*

$$m_{i,j} = m_{i-1,j-1} + (t^2 - t + 1)m_{i-1,j} + (t^2 - t)m_{i-1,j+1}. \tag{6.1}$$

Then we have the following matrix identity:

$$(m_{i,j})_{i,j\geq 1} \times \begin{bmatrix} 1 \\ t^2 - (t-1)^2 \\ t^3 - (t-1)^3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ t^2 + t \\ (t^2+t)^2 \\ \vdots \end{bmatrix}. \tag{6.2}$$

The matrix identity (1.3) is a consequence of (6.2) by setting $t = 2$. By using generating functions, one can verify that the number of 2-Dyck paths of length $2n$ equals the number of little Schröder paths of length n.

We now proceed to give a combinatorial proof of (6.2). To this end, we need to give a combinatorial interpretation of the entries in the matrix M in (6.2). We may define a composition of a k-Dyck path P as a sequence of k-Dyck paths $(P_1, P_2, \ldots, P_j)$ such that $P = P_1 P_2 \cdots P_j$, where j is the number of segments.

Lemma 6.2 *Let $a_{i,j}$ be the number of compositions of $(t^2 - t)$-Dyck paths of length $2i$ with j segments. Then $a_{i,j}$ satisfies the recurrence relation (6.1).*

The proof of the above lemma is similar to that of Lemma 3.1. Let us rewrite (6.2) as follows:

$$\sum_{j\geq 1} m_{i,j}(t^j-(t-1)^j)=(t^2+t)^{i-1}. \tag{6.3}$$

In order to deal with $m_{i,j}(t^j-(t-1)^j)$ combinatorially, we introduce a coloring scheme on a composition of a (t^2-t)-Dyck path with j segments. Suppose that we have t colors $c_1, c_2, \ldots, c_t$. If we use these t colors to color the j segments such that the first color c_1 must be used, then there are $t^j-(t-1)^j$ ways to accomplish such colorings. We simply call such colorings *t-feasible colorings*.

Here is a bijection leading to a combinatorial proof of (6.3).

Theorem 6.3 *There is a bijection between the set of compositions of (t^2-t)-Dyck paths of length $2i$ with a t-feasible coloring on the segments and the set of sequences of length $i-1$ on t^2+t letters.*

Proof. The desired bijection σ is constructed as follows. Let $(P_1, P_2, \ldots, P_j)$ be a composition of a (t^2-t)-Dyck path P of length $2i$ with a t-feasible coloring on the segments. We will use the following alphabet that contains t^2+t letters:

$$\{\alpha_r,\ \mid 1\leq r\leq t\}\cup\{\beta_s,\ \mid 1\leq s\leq t-1\}\cup\{\gamma_k\mid 1\leq k\leq t^2-t\}\cup\{\delta\}. \tag{6.4}$$

For $i=1$, both the composition and the t-feasible coloring are unique. We set the corresponding sequence to be empty. For $i\geq 2$, we consider the following cases:

1. $P_1=UD$, P_1 is colored by c_r $(1\leq r\leq t)$ and $(P_2,\ldots,P_j)$ still has a t-feasible coloring. Then we set $\sigma(P_1,\ldots,P_j)=\alpha_r\sigma(P_2,\ldots,P_j)$.

2. $P_1=UD$, P_1 is colored by c_1 and $(P_2,\ldots,P_j)$ does not inherit a t-feasible coloring. Assume that P_2 is colored by c_{s+1} $(1\leq s\leq t-1)$. Then we change the color of P_2 to c_1 and set $\sigma(P_1,\ldots,P_j)=\beta_s\sigma(P_2,\ldots,P_j)$.

3. $P_1=UDQ$, where Q is not empty. Then we set $\sigma(P_1,\ldots,P_j)=\delta\sigma(Q,P_2,\ldots,P_j)$.

4. $P_1=UQD$, where Q is not empty and the first up step of P has color k $(1\leq k\leq t^2-t)$. Then we set $\sigma(P_1,\ldots,P_j)=\gamma_k\sigma(Q,P_2,\ldots,P_j)$.

5. $P_1=UQ_1DQ_2$, neither Q_1 nor Q_2 is empty and the first up step of P has color k. Since k ranges from 1 to $t(t-1)$, we may encode a color k by a pair of colors (c_p,c_q) where p ranges from 1 to t and q ranges from 1 to $t-1$. Moreover, we may use (c_r,β_s) to denote a color k. Then we assign color c_r to Q_1, pass the color of P_1 to Q_2, and set $\sigma(P)=\beta_s\sigma(Q_1,Q_2,P_2,\ldots,P_j)$.

For each case, the resulting path is always a sequence of length $i-1$.

In order to show that σ is a bijection, we proceed to construct the inverse map of σ. Let S be a sequence of length $i-1$ on the alphabet (6.4). If $i=1$, then we get the unique Dyck path UD and the unique composition with a t-feasible coloring. Note that the up step in the Dyck path UD is not colored. We now assume that $i>1$. It is easy to check that Cases 1, 3, and 4 are reversible. It remains to show that Cases 2 and 5 are reversible. In fact, we only need to ensure that Case 2 and Case 5 can be distinguished from each other. For Case 2, either $j=2$ or $(P_3,\ldots,P_j)$ does not have a t-feasible coloring. On the other hand, for Case 5, $(Q_2,P_2,\ldots,P_j)$ is always nonempty and it has a t-feasible coloring. This completes the proof. ∎

We also have a combinatorial interpretation of the matrix identity (1.3) based on little Schröder paths. The idea is similar to the proof given above, so the details are omitted.

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