The Bivariate Rogers-Szegö Polynomials

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Abstract. We obtain Mehler's formula and the Rogers formula for the continuous big q-Hermite polynomials $H_n(x;a|q)$. Instead of working with the polynomials $H_n(x;a|q)$ directly, we consider the equivalent forms in terms of the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ recently introduced by Chen, Fu and Zhang. It turns out that Mehler's formula for $H_n(x;a|q)$ involves a ${}_3\phi_2$ sum, and the Rogers formula involves a ${}_2\phi_1$ sum. The proofs of these results are based on parameter augmentation with respect to the q-exponential operator and the homogeneous q-shift operator in two variables.

Keywords: The bivariate Rogers-Szegö polynomials, the continuous big q-Hermite polynomials, the Cauchy polynomials, the q-exponential operator, the homogeneous q-shift operator

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1. Introduction

In this paper, we obtain two formulas for the continuous big q-Hermite polynomials $H_n(x;a|q)$ which can be considered as extensions of Mehler's formula and the Rogers formula for the q-Hermite polynomials $H_n(x|q)$.

Let us review the common notation and definitions for basic hypergeometric series in [11]. Throughout this paper, we assume that $|q|<1$. The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n \in \mathbb{Z}.$$

The following notation stands for the multiple q-shifted factorials:

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The q-binomial coefficients, or the Gauss coefficients, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The basic hypergeometric series ${}_{r+1}\phi_r$ are defined by

$${}_{r+1}\phi_r \left(\begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} x^n.$$

The continuous big q-Hermite polynomials are defined as

$$H_n(x; a|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (ae^{i\theta}; q)_k e^{i(n-2k)\theta}, \quad x = \cos\theta.$$

We first observe that the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ introduced by Chen, Fu and Zhang [8] can be used to derive identities for the continuous big q-Hermite polynomials owing to the following relation:

$$H_n(x; a|q) = e^{in\theta} h_n(e^{-2i\theta}, ae^{-i\theta}|q), \quad x = \cos\theta, \tag{1.1}$$

where $h_n(x, y|q)$ are defined as follows. Let

$$P_n(x, y) = (x - y)(x - qy) \cdots (x - q^{n-1}y)$$

be Cauchy polynomials with the generating function

$$\sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt| < 1. \tag{1.2}$$

Then the bivariate Rogers-Szegö polynomials are given by

$$h_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, y).$$

The Cauchy polynomials $P_n(x, y)$ naturally arise in the q-umbral calculus as studied by Andrews [2, 3], Goldman and Rota [12], Goulden and Jackson [13], Ihrig and Ismail [14], Johnson [17], Roman [22]. The generating function (1.2) is the homogeneous version of the Cauchy identity, or the q-binomial theorem [11]:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{1.3}$$

The polynomials $h_n(x, y|q)$ have the generating function [8]

$$\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t, xt;q)_\infty}, \quad |t|,\ |xt| < 1. \tag{1.4}$$

Notice that the classical Rogers-Szegö polynomials

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k,$$

are a special case of $h_n(x, y|q)$ when y is set to zero, and (1.4) reduces to

$$\sum_{n=0}^{\infty} h_n(x|q) \frac{t^n}{(q;q)_n} = \frac{1}{(t, xt; q)_\infty}, \quad |t|,\ |xt| < 1. \tag{1.5}$$

The Rogers-Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey-Wilson polynomials, see [1, 4, 6, 7, 15, 16, 19, 23]. They are closely related to the q-Hermite polynomials

$$H_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} e^{i(n-2k)\theta}, \quad x = \cos\theta.$$

In fact, the following relation holds

$$H_n(x|q) = H_n(x; 0|q) = e^{in\theta} h_n(e^{-2i\theta}|q), \quad x = \cos\theta. \tag{1.6}$$

The continuous big q-Hermite polynomials $H_n(x; a|q)$ can be expressed explicitly in terms of the q-Hermite polynomials $H_n(x|q)$ [5, 10]:

$$H_n(x; a|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a^k H_{n-k}(x|q), \tag{1.7}$$

whose inverse expansion takes the form

$$H_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k H_{n-k}(x; a|q). \tag{1.8}$$

Based on the recurrence relation for $H_n(x|q)$, Bressoud [7] gave a proof of Mehler's formula for the q-Hermite polynomials:

$$\sum_{n=0}^{\infty} H_n(x|q) H_n(y|q) \frac{t^n}{(q;q)_n} = \frac{(t^2; q)_\infty}{(te^{i(\theta+\beta)}, te^{-i(\theta-\beta)}, te^{i(\theta-\beta)}, te^{-i(\theta+\beta)}; q)_\infty}, \tag{1.9}$$

where $x = \cos\theta$, $y = \cos\beta$. Ismail, Stanton and Viennot [16] found a combinatorial proof of (1.9) by using the vector space interpretation of the q-binomial coefficients. This paper is motivated by the natural question of finding Mehler's formula for $H_n(x; a|q)$ which is an extension of the following formula for the Rogers-Szegö polynomials:

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(y|q) \frac{t^n}{(q;q)_n} = \frac{(xyt^2; q)_\infty}{(t, xt, yt, xyt; q)_\infty}. \tag{1.10}$$

The formula (1.10) has been extensively studied, see [9, 15, 18, 19, 23, 24].

The second result of this paper is the Rogers formula for $H_n(x;a|q)$. The Rogers formula [9, 19, 20] for the Rogers-Szegö polynomials $h_n(x|q)$ reads:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=(xst;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(x|q)h_m(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}. \tag{1.11}$$

The equivalent form for the q-Hermite polynomials $H_n(x|q)$ can be stated as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_{n+m}(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= (st;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_n(x|q)H_m(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}, \quad x=\cos\theta. \tag{1.12}$$

It turns out that Mehler's formula for $H_n(x;a|q)$ (Theorem 2.2) involves a ${}_3\phi_2$ sum, and the Rogers formula for $H_n(x;a|q)$ (Theorem 3.2) involves a ${}_2\phi_1$ sum. Our proofs rely on the q-exponential operator $T(bD_q)$ as studied in [9] and the homogeneous q-shift operator $\mathbb{E}(D_{xy})$ introduced by Chen, Fu and Zhang [8].

2. Mehler's Formula for $h_n(x,y|q)$

The main objective in this section is to derive Mehler's formula for $H_n(x;a|q)$. To this end, we first obtain Mehler's formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ introduced by Chen, Fu and Zhang [8].

Theorem 2.1 (Mehler's Formula for $h_n(x,y|q)$). *We have*

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} = \frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} y,xt,v/u \\ yt,vxt \end{matrix};q,ut\right), \tag{2.1}$$

provided that $|t|,|xt|,|ut|,|uxt|<1$.

Before we present the proof, we note that it is not difficult to reformulate the above theorem in terms of $H_n(x;a|q)$.

Theorem 2.2. *We have*

$$\sum_{n=0}^{\infty} H_n(x;a|q)H_n(y;b|q)\frac{t^n}{(q;q)_n} = \frac{(ate^{i\beta},bte^{-i\theta};q)_\infty}{(te^{i(\theta+\beta)},te^{-i(\theta-\beta)},te^{-i(\theta+\beta)};q)_\infty}$$
$$\times\ {}_3\phi_2\left(\begin{matrix} ae^{-i\theta},te^{-i(\theta-\beta)},be^{i\beta} \\ ate^{i\beta},bte^{-i\theta} \end{matrix};q,te^{i(\theta-\beta)}\right),$$

provided that $x=\cos\theta,\ y=\cos\beta$ *and* $|te^{i(\theta+\beta)}|,|te^{i(\theta-\beta)}|,|te^{-i(\theta+\beta)}|,|te^{-i(\theta-\beta)}|<1$.

Proof. Substituting (x,y) by $(e^{-2i\theta},ae^{-i\theta})$ and (u,v) by $(e^{-2i\beta},be^{-i\beta})$ in Theorem 2.1, we get

$$\sum_{n=0}^{\infty} h_n(e^{-2i\theta},ae^{-i\theta}|q)h_n(e^{-2i\beta},be^{-i\beta}|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(ate^{-i\theta},bte^{-i(2\theta+\beta)};q)_\infty}{(t,te^{-2i\theta},te^{-2i(\theta+\beta)};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}ae^{-i\theta},te^{-2i\theta},be^{i\beta}\\ate^{-i\theta},bte^{-i(2\theta+\beta)}\end{matrix};q,te^{-2i\beta}\right). \tag{2.2}$$

From (1.1) it follows that

$$h_n(e^{-2i\theta},ae^{-i\theta}|q)=e^{-in\theta}H_n(w;a|q),\;\; w=\cos\theta,$$
$$h_n(e^{-2i\beta},be^{-i\beta}|q)=e^{-in\beta}H_n(z;b|q),\;\; z=\cos\beta.$$

Substituting the above relations into (2.2), we see that

$$\sum_{n=0}^{\infty} H_n(w;a|q)H_n(z;b|q)\frac{(te^{-i(\theta+\beta)})^n}{(q;q)_n}$$
$$=\frac{(ate^{-i\theta},b\,te^{-i(2\theta+\beta)};q)_\infty}{(t,te^{-2i\theta},te^{-2i(\theta+\beta)};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}ae^{-i\theta},te^{-2i\theta},be^{i\beta}\\ate^{-i\theta},b\,te^{-i(2\theta+\beta)}\end{matrix};q,te^{-2i\beta}\right).$$

Making the substitutions $t\to te^{i(\theta+\beta)}$, $w\to x$ and $z\to y$, we complete the proof. □

Setting $a=0$ and $b=0$, the above theorem becomes Mehler's formula (1.9) for the q-Hermite polynomials. To prove Theorem 2.1 we need some identities (Lemmas 2.3, 2.4 and 2.5) in connection with the q-exponential operator and the homogeneous q-shift operator.

The q-differential operator, or the q-derivative, acting on the variable a, is defined by

$$D_q f(a)=\frac{f(a)-f(aq)}{a},$$

and the q-exponential operator is given by

$$T(bD_q)=\sum_{n=0}^{\infty}\frac{(bD_q)^n}{(q;q)_n}.$$

Evidently,

$$T(D_q)\{x^n\}=h_n(x|q). \tag{2.3}$$

Lemma 2.3. *We have*

$$T(bD_q)\left\{\frac{(av;q)_\infty}{(as,at;q)_\infty}\right\}=\frac{(bv;q)_\infty}{(as,bs,b\,t;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}v/t,bs\\bv\end{matrix};q,at\right), \tag{2.4}$$

provided that $|bs|,|b\,t|<1$.

Proof. Zhang and Wang [25] have established the following identity:

$$T(bD_q)\left\{\frac{(av;q)_\infty}{(as,at,aw;q)_\infty}\right\}=(av,bv;q)_\infty\frac{(abstw/v;q)_\infty}{(as,at,aw,bs,b\,t,bw;q)_\infty}$$
$$\times\,{}_3\phi_2\left(\begin{matrix}v/s,v/t,v/w\\av,bv\end{matrix};q,abstw/v\right), \tag{2.5}$$

where $|bs|, |b\,t|, |bw|, |abstw/v| < 1$. Setting $w = 0$ in (2.5) and using Jackson's transformation [11, III. 4] and Heine's transformation [11, III. 1], we obtain the claimed identity. □

With the aid of the above lemma, we may reach the following identity.

Lemma 2.4. *We have*

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(z|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xzt,xt,t;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} y,xt \\ yt \end{matrix};q,zt\right), \tag{2.6}$$

provided that $|t|, |xt|, |zt|, |xzt| < 1$.

Proof. Applying (1.4), (2.3) and (2.4), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x,y|q)h_n(z|q)\frac{t^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} h_n(x,y|q)T(D_q)\left\{z^n\right\}\frac{t^n}{(q;q)_n} \\
&= T(D_q)\left\{\sum_{n=0}^{\infty} h_n(x,y|q)\frac{(zt)^n}{(q;q)_n}\right\} \quad (|zt|<1,\ |xzt|<1) \\
&= T(D_q)\left\{\frac{(yzt;q)_\infty}{(xzt,zt;q)_\infty}\right\} \quad (|t|<1,\ |xt|<1) \\
&= \frac{(yt;q)_\infty}{(xzt,xt,t;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} y,xt \\ yt \end{matrix};q,zt\right),
\end{aligned}$$

as desired. □

In [8], Chen, Fu and Zhang defined the homogeneous q-difference operator

$$D_{xy}f(x,y) = \frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}$$

and the homogeneous q-shift operator

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty}\frac{D_{xy}^k}{(q;q)_k}.$$

The following basic facts have been observed in [8]:

$$\begin{aligned}
D_{xy}\{P_n(x,y)\} &= (1-q^n)P_{n-1}(x,y), \\
\mathbb{E}(D_{xy})\{P_n(x,y)\} &= h_n(x,y|q).
\end{aligned} \tag{2.7}$$

Lemma 2.5. *We have*

$$\mathbb{E}(D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_n(x,y)}{(yt;q)_n}\right\} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(y,xt;q)_k}{(yt;q)_k}x^{n-k},$$

provided that $|t|, |xt| < 1$.

Proof. The left hand side of (2.6) equals

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{E}(D_{xy}) \{P_n(x,y)\} h_n(z|q) \frac{t^n}{(q;q)_n} \\
&= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x,y) h_n(z|q) \frac{t^n}{(q;q)_n} \right\} \\
&= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x,y) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k \frac{t^n}{(q;q)_n} \right\} \\
&= \mathbb{E}(D_{xy}) \left\{ \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} P_n(x, q^k y) \frac{t^n}{(q;q)_n} \right) P_k(x,y) \frac{(zt)^k}{(q;q)_k} \right\} \\
&= \sum_{k=0}^{\infty} \frac{(zt)^k}{(q;q)_k} \mathbb{E}(D_{xy}) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{P_k(x,y)}{(yt;q)_k} \right\},
\end{aligned}$$

where $|t|, |xt|, |zt|, |zxt| < 1$. Employing Euler's identity [11, II.1] to expand $1/(zxt;q)_\infty$ on the right hand side of (2.6), we get

$$\sum_{k=0}^{\infty} \frac{(zt)^k}{(q;q)_k} \mathbb{E}(D_{xy}) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{P_k(x,y)}{(yt;q)_k} \right\} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(y,xt;q)_n}{(q,yt;q)_n} \frac{z^{n+k} t^{n+k} x^k}{(q;q)_k}.$$

Comparing the coefficients of z^n, we complete the proof. □

We are now ready to present the proof of Theorem 2.1.

Proof. From (2.7) it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x,y|q) h_n(u,v|q) \frac{t^n}{(q;q)_n} \\
&= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x,y) h_n(u,v|q) \frac{t^n}{(q;q)_n} \right\} \\
&= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(u,v) \right\} \\
&= \mathbb{E}(D_{xy}) \left\{ \sum_{k=0}^{\infty} P_k(u,v) P_k(x,y) \frac{t^k}{(q;q)_k} \left(\sum_{n=0}^{\infty} P_n(x, q^k y) \frac{t^n}{(q;q)_n} \right) \right\} \quad (|xt| < 1) \\
&= \mathbb{E}(D_{xy}) \left\{ \sum_{k=0}^{\infty} P_k(u,v) P_k(x,y) \frac{t^k}{(q;q)_k} \frac{(q^k yt;q)_\infty}{(xt;q)_\infty} \right\} \\
&= \sum_{k=0}^{\infty} P_k(u,v) \frac{t^k}{(q;q)_k} \mathbb{E}(D_{xy}) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{P_k(x,y)}{(yt;q)_k} \right\} \quad (|t|, |xt| < 1).
\end{aligned}$$

By Lemma 2.5, the above summation equals

$$\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{\infty}P_k(u,v)\frac{t^k}{(q;q)_k}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{k-j}.$$

Exchanging the order of summations, we get

$$\begin{aligned}
&\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{j=0}^{\infty}P_j(u,v)\frac{(y,xt;q)_j}{(q,yt;q)_j}t^j\sum_{k=0}^{\infty}\frac{(xt)^kP_k(u,q^jv)}{(q;q)_k}\qquad(|uxt|<1)\\
&=\frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\sum_{j=0}^{\infty}P_j(u,v)\frac{(y,xt;q)_j}{(q,yt,vxt;q)_j}t^j\\
&=\frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}y,xt,v/u\\ yt,vxt\end{matrix};q,ut\right)\qquad(|ut|<1).
\end{aligned}$$

This completes the proof. ∎

Obviously, Mehler's formula (1.10) for the Rogers-Szegö polynomials can be deduced from Theorem 2.1 by setting $y=0, v=0$ and $u=y$.

3. The Rogers Formula for $h_n(x,y|q)$

In this section, we obtain the Rogers formula (Theorem 3.1) for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ using the operator $\mathbb{E}(D_{xy})$ and the technique of parameter augmentation [8, 9]. This formula can be readily restated in terms of the continuous big q-Hermite polynomials $H_n(x;a|q)$ (Theorem 3.2).

Theorem 3.1 (The Rogers Formula for $h_n(x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}=\frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}y,xs\\ ys\end{matrix};q,t\right),\tag{3.1}$$

provided that $|t|,|s|,|xt|,|xs|<1$.

Proof. By (2.7), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&=\mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\right\}\\
&=\mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}\left(\sum_{m=0}^{\infty}P_m(x,q^ny)\frac{s^m}{(q;q)_m}\right)\right\}\qquad(|xs|<1)\\
&=\mathbb{E}(D_{xy})\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}\frac{(q^nys;q)_\infty}{(xs;q)_\infty}\right\}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \mathbb{E}(D_{xy}) \left\{ \frac{(ys;q)_\infty P_n(x,y)}{(xs;q)_\infty (ys;q)_n} \right\} \quad (|s|<1,\ |xs|<1).$$

Applying Lemma 2.5, we get

$$\begin{aligned}
&\frac{(ys;q)_\infty}{(s,xs;q)_\infty} \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(y,xs;q)_k}{(ys;q)_k} x^{n-k} \\
&= \frac{(ys;q)_\infty}{(s,xs;q)_\infty} \sum_{k=0}^{\infty} \frac{(y,xs;q)_k}{(q,ys;q)_k} t^k \sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_n} \quad (|xt|<1) \\
&= \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(y,xs;q)_k}{(q,ys;q)_k} t^k \\
&= \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty} \, {}_2\phi_1 \left(\begin{matrix} y, xs \\ ys \end{matrix} ; q, t \right) \quad (|t|<1),
\end{aligned}$$

as desired. □

Clearly, the Rogers formula (1.11) for $h_n(x|q)$ is the special case of (3.1) when $y = 0$. The following theorem is the Rogers formula for $H_n(x;a|q)$ which contains (1.12) as a special case for $a = 0$.

Theorem 3.2. *We have*

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n+m}(x;a|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} &= \frac{(as;q)_\infty}{(se^{i\theta}, se^{-i\theta}, te^{-i\theta};q)_\infty} \\
&\quad \times \, {}_2\phi_1 \left(\begin{matrix} ae^{-i\theta}, se^{-i\theta} \\ as \end{matrix} ; q, te^{i\theta} \right),
\end{aligned}$$

where $x = \cos\theta$ *and* $|te^{i\theta}|, |se^{i\theta}|, |te^{-i\theta}|, |se^{-i\theta}| < 1$.

The following special case of Theorem 3.1 for $y = 0$ will be useful to verify the relation (3.4) between $h_n(x|q)$ and $h_n(x,y|q)$.

Theorem 3.3. *We have*

$$\begin{aligned}
&\sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k x^k h_{n+m-2k}(x|q) \\
&\quad = \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k \ h_{n-k}(x,y|q) \right) \left(\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} y^j \ h_{m-j}(x,y|q) \right), \qquad (3.2)
\end{aligned}$$

provided that $|x|, |y| < \infty$.

Proof. Setting $y=0$ in Theorem 3.1, by the Cauchy identity (1.3) and (1.4), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&\quad=\frac{1}{(s,xs,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(xs;q)_k}{(q;q)_k}t^k \qquad (|t|<1)\\
&\quad=\frac{(xst;q)_\infty}{(ys,yt;q)_\infty}\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\frac{(ys;q)_\infty}{(s,xs;q)_\infty} \qquad (|t|,|s|,|xt|,|xs|<1)\\
&\quad=\frac{(xst;q)_\infty}{(ys,yt;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(x,y|q)h_m(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{1}{(xst;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q)\ \frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&\qquad=\frac{1}{(yt,ys;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(x,y|q)h_m(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m},
\end{aligned}$$

where $|t|,|s|,|xt|,|xs|<1$.

Assuming that $|xst|,|yt|,|ys|<1$, we can expand $1/(xst;q)_\infty$, $1/(yt;q)_\infty$ and $1/(ys;q)_\infty$ by Euler's identity. By comparing the coefficients of $t^n s^m$, we obtain (3.2). Since $|t|,|s|,|xt|$, $|xs|<1$ and $|xst|,|yt|,|ys|<1$, we see that $|x|$ and $|y|$ must be finite. ∎

When $y=0$, (3.2) reduces to the well-known linearization formula [7, 15, 21] for $h_n(x|q)$:

$$h_n(x|q)h_m(x|q)=\sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k x^k h_{n+m-2k}(x|q), \quad |x|<\infty. \tag{3.3}$$

Setting $m=0$ in (3.2), we are led to the following relation between $h_n(x|q)$ and $h_n(x,y|q)$:

$$h_n(x|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}y^k\ h_{n-k}(x,y|q), \quad |x|,|y|<\infty. \tag{3.4}$$

The inverse expansion for (3.4) is

$$h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}} y^k h_{n-k}(x|q), \quad |x|,|y|<\infty. \tag{3.5}$$

Note that (3.4) and (3.5) are equivalent to the relations (1.8) and (1.7) between $H_n(x|q)$ and $H_n(x;a|q)$.

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