The Bivariate Rogers-Szegö Polynomials

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Abstract. We obtain Mehler’s formula and the Rogers formula for the continuous big $q$-Hermite polynomials $H_n(x; a|q)$. Instead of working with the polynomials $H_n(x; a|q)$ directly, we consider the equivalent forms in terms of the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ recently introduced by Chen, Fu and Zhang. It turns out that Mehler’s formula for $H_n(x; a|q)$ involves a $3\phi_2$ sum, and the Rogers formula involves a $2\phi_1$ sum. The proofs of these results are based on parameter augmentation with respect to the $q$-exponential operator and the homogeneous $q$-shift operator in two variables.

Keywords: The bivariate Rogers-Szegö polynomials, the continuous big $q$-Hermite polynomials, the Cauchy polynomials, the $q$-exponential operator, the homogeneous $q$-shift operator

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1. Introduction

In this paper, we obtain two formulas for the continuous big $q$-Hermite polynomials $H_n(x; a|q)$ which can be considered as extensions of Mehler’s formula and the Rogers formula for the $q$-Hermite polynomials $H_n(x|q)$.

Let us review the common notation and definitions for basic hypergeometric series in [11]. Throughout this paper, we assume that $|q| < 1$. The $q$-shifted factorial is defined by

$$
(a; q)_0 = 1, \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{Z}.
$$

The following notation stands for the multiple $q$-shifted factorials:

$$
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,
(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.
$$
The $q$-binomial coefficients, or the Gauss coefficients, are given by

$$\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$ 

The basic hypergeometric series $\p_{r+1}^\phi$ are defined by

$$\p_{r+1}^\phi \left( \frac{a_1, \ldots, a_{r+1}}{b_1, \ldots, b_r} : q, x \right) = \sum_{n=0}^\infty \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} x^n.$$ 

The continuous big $q$-Hermite polynomials are defined as

$$H_n(x; a|q) = \sum_{k=0}^n \binom{n}{k}_q (ae^{i\theta}; q)_k e^{i(n-2k)\theta}, \quad x = \cos \theta.$$ 

We first observe that the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ introduced by Chen, Fu and Zhang [8] can be used to derive identities for the continuous big $q$-Hermite polynomials owing to the following relation:

$$H_n(x; a|q) = e^{in\theta} h_n(e^{-2i\theta}, ae^{-i\theta}|q), \quad x = \cos \theta, \quad (1.1)$$

where $h_n(x, y|q)$ are defined as follows. Let

$$P_n(x, y) = (x - y)(x - qy) \cdots (x - q^{n-1}y)$$

be Cauchy polynomials with the generating function

$$\sum_{n=0}^\infty P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty}, \quad |xt| < 1. \quad (1.2)$$

Then the bivariate Rogers-Szegö polynomials are given by

$$h_n(x, y|q) = \sum_{k=0}^n \binom{n}{k}_q P_k(x, y).$$

The Cauchy polynomials $P_n(x, y)$ naturally arise in the $q$-umbral calculus as studied by Andrews [2, 3], Goldman and Rota [12], Goulden and Jackson [13], Ihrig and Ismail [14], Johnson [17], Roman [22]. The generating function (1.2) is the homogeneous version of the Cauchy identity, or the $q$-binomial theorem [11]:

$$\sum_{k=0}^\infty \frac{(a; q)_k}{(q; q)_k} z^k = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1. \quad (1.3)$$

The polynomials $h_n(x, y|q)$ have the generating function [8]

$$\sum_{n=0}^\infty h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(t, xt; q)_\infty}, \quad |t|, \ |xt| < 1. \quad (1.4)$$

2
Notice that the classical Rogers-Szegő polynomials

\[ h_n(x|q) = \sum_{k=0}^{n} \left[ \binom{n}{k} x^k, \right. \]

are a special case of \( h_n(x, y|q) \) when \( y \) is set to zero, and (1.4) reduces to

\[ \sum_{n=0}^{\infty} h_n(x|q) \frac{t^n}{(q; q)_n} = \frac{1}{(t, xt; q)_\infty}, \quad |t|, |xt| < 1. \] (1.5)

The Rogers-Szegő polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey-Wilson polynomials, see [1, 4, 6, 7, 15, 16, 19, 23]. They are closely related to the \( q \)-Hermite polynomials

\[ H_n(x|q) = \sum_{k=0}^{n} \left[ \binom{n}{k} e^{i(n-2k)\theta}, \quad x = \cos \theta. \right. \]

In fact, the following relation holds

\[ H_n(x|q) = H_n(x; 0|q) = e^{i\theta} h_n(e^{-2i\theta}|q), \quad x = \cos \theta. \] (1.6)

The continuous big \( q \)-Hermite polynomials \( H_n(x; a|q) \) can be expressed explicitly in terms of the \( q \)-Hermite polynomials \( H_n(x|q) \) [5, 10]:

\[ H_n(x; a|q) = \sum_{k=0}^{n} \left[ \binom{n}{k} (-1)^k q^{k(k+1)/2} a^k H_{n-k}(x|q), \right. \] (1.7)

whose inverse expansion takes the form

\[ H_n(x|q) = \sum_{k=0}^{n} \left[ \binom{n}{k} a^k H_{n-k}(x; a|q). \right. \] (1.8)

Based on the recurrence relation for \( H_n(x|q) \), Bressoud [7] gave a proof of Mehler’s formula for the \( q \)-Hermite polynomials:

\[ \sum_{n=0}^{\infty} H_n(x|q)H_n(y|q) \frac{t^n}{(q; q)_n} = \frac{(t^2; q)_\infty}{(te^{i(\theta+\beta)}, te^{-i(\theta-\beta)}, te^{i(\theta-\beta)}, te^{-i(\theta+\beta)}; q)_\infty}, \] (1.9)

where \( x = \cos \theta, \) \( y = \cos \beta. \) Ismail, Stanton and Viennot [16] found a combinatorial proof of (1.9) by using the vector space interpretation of the \( q \)-binomial coefficients. This paper is motivated by the natural question of finding Mehler’s formula for \( H_n(x; a|q) \) which is an extension of the following formula for the Rogers-Szegő polynomials:

\[ \sum_{n=0}^{\infty} h_n(x|q)h_n(y|q) \frac{t^n}{(q; q)_n} = \frac{(xyt^2; q)_\infty}{(t, xt, yt, xyt; q)_\infty}. \] (1.10)

The formula (1.10) has been extensively studied, see [9, 15, 18, 19, 23, 24].
The second result of this paper is the Rogers formula for $H_n(x; a|q)$. The Rogers formula [9, 19, 20] for the Rogers-Szegő polynomials $h_n(x|q)$ reads:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q;q)_n} s^m = (xst; q)_\infty \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n(x|q) h_m(x|q) \frac{t^n}{(q;q)_n} s^m. \quad (1.11)$$

The equivalent form for the $q$-Hermite polynomials $H_n(x|q)$ can be stated as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n+m}(x|q) \frac{t^n}{(q;q)_n} s^m = (st; q)_\infty \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_n(x|q) H_m(x|q) \frac{t^n}{(q;q)_n} s^m, \quad x = \cos \theta. \quad (1.12)$$

It turns out that Mehler’s formula for $H_n(x; a|q)$ (Theorem 2.2) involves a $3\phi_2$ sum, and the Rogers formula for $H_n(x; a|q)$ (Theorem 3.2) involves a $2\phi_1$ sum. Our proofs rely on the $q$-exponential operator $T(bD_q)$ as studied in [9] and the homogeneous $q$-shift operator $\mathfrak{E}(D_{xy})$ introduced by Chen, Fu and Zhang [8].

2. Mehler’s Formula for $h_n(x, y|q)$

The main objective in this section is to derive Mehler’s formula for $H_n(x; a|q)$. To this end, we first obtain Mehler’s formula for the bivariate Rogers-Szegő polynomials $h_n(x, y|q)$ introduced by Chen, Fu and Zhang [8].

**Theorem 2.1** (Mehler’s Formula for $h_n(x, y|q)$). We have

$$\sum_{n=0}^{\infty} h_n(x, y|q) h_n(u, v|q) \frac{t^n}{(q;q)_n} = \frac{(yt, vxt; q)_\infty}{(t, xt, vxt; q)_\infty} 3\phi_2 \left( \begin{array}{c} y, xt, v/u \\ yt, vxt \\ q, ut \end{array} \right), \quad (2.1)$$

provided that $|t|, |xt|, |ut|, |vxt| < 1$.

Before we present the proof, we note that it is not difficult to reformulate the above theorem in terms of $H_n(x; a|q)$.

**Theorem 2.2.** We have

$$\sum_{n=0}^{\infty} H_n(x; a|q) H_n(y; b|q) \frac{t^n}{(q;q)_n} = \frac{(ate^{i\beta}, bte^{-i\theta}; q)_\infty}{(te^{i(\theta+\beta)}, te^{-i(\theta-\beta)}, te^{-i(\theta+\beta)}; q)_\infty} \times 3\phi_2 \left( \begin{array}{c} ae^{-i\theta}, te^{-i(\theta-\beta)}, be^{i\beta} \\ ate^{i\beta}, bte^{-i\theta} \\ q, te^{i(\theta-\beta)} \end{array} \right),$$

provided that $x = \cos \theta$, $y = \cos \beta$ and $|te^{i(\theta+\beta)}|, |te^{i(\theta-\beta)}|, |te^{-i(\theta+\beta)}|, |te^{-i(\theta-\beta)}| < 1$. 

4
2.5) in connection with the Proof.

Evidently, and the

Making the substitutions

Substituting the above relations into (2.2), we see that

From (1.1) it follows that

We have

From (2.4) and (2.5) in connection with the q-exponential operator and the homogeneous q-shift operator.

The q-differential operator, or the q-derivative, acting on the variable a, is defined by

and the q-exponential operator is given by

Evidently,

Lemma 2.3. We have

provided that |bs|, |bt| < 1.

Proof. Zhang and Wang [25] have established the following identity:

Proof. Substituting \((x, y) \) by \((e^{-2i\theta}, ae^{-i\theta})\) and \((u, v) \) by \((e^{-2i\beta}, be^{-i\beta})\) in Theorem 2.1, we get

Setting \(a = 0\) and \(b = 0\), the above theorem becomes Mehler’s formula (1.9) for the q-Hermite polynomials. To prove Theorem 2.1 we need some identities (Lemmas 2.3, 2.4 and 2.5) in connection with the q-exponential operator and the homogeneous q-shift operator.
where \( |bs|, |bt|, |bw|, |abstw/v| < 1 \). Setting \( w = 0 \) in (2.5) and using Jackson’s transformation [11, III. 4] and Heine’s transformation [11, III. 1], we obtain the claimed identity.

With the aid of the above lemma, we may reach the following identity.

**Lemma 2.4.** We have

\[
\sum_{n=0}^{\infty} h_n(x,y|q)h_n(z|q) \frac{t^n}{(q;q)_n} = \frac{(yt; q)_\infty}{(xzt, xt, t; q)_\infty} 2\phi_1 \left( y, xt \mid yt, q, zt \right),
\]

provided that \( |t|, |xt|, |zt|, |xzt| < 1 \).

**Proof.** Applying (1.4), (2.3) and (2.4), we have

\[
\sum_{n=0}^{\infty} h_n(x,y|q)h_n(z|q) \frac{t^n}{(q;q)_n} = T(D_q) \left\{ \sum_{n=0}^{\infty} h_n(x,y|q) \frac{(zt)^n}{(q;q)_n} \right\} (|zt| < 1, |xt| < 1)
\]

\[
= T(D_q) \left\{ \frac{(yt; q)_\infty}{(xzt, zt; q)_\infty} \right\} (|t| < 1, |xt| < 1)
\]

\[
= \frac{(yt; q)_\infty}{(xzt, xt, t; q)_\infty} 2\phi_1 \left( y, xt \mid yt, q, zt \right),
\]

as desired. \( \square \)

In [8], Chen, Fu and Zhang defined the homogeneous \( q \)-difference operator

\[
D_{xy}f(x, y) = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}
\]

and the homogeneous \( q \)-shift operator

\[
\mathcal{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D^k_{xy}}{(q; q)_k}.
\]

The following basic facts have been observed in [8]:

\[
D_{xy}\{P_n(x, y)\} = (1 - q^n)P_{n-1}(x, y),
\]

\[
\mathcal{E}(D_{xy})\{P_n(x, y)\} = h_n(x, y|q).
\]

**Lemma 2.5.** We have

\[
\mathcal{E}(D_{xy})\left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} P_n(x, y) \right\} = \frac{(yt; q)_\infty}{(t, xt; q)_\infty} \sum_{k=0}^{n} \frac{[n]}{k} \frac{(y, xt; q)_k}{(yt; q)_k} x^{n-k},
\]

provided that \( |t|, |xt| < 1 \).
Comparing the coefficients of the right hand side of (2.6), we get

\[
\sum_{n=0}^{\infty} \mathbb{E}(D_{xy}) \{ P_n(x, y) \} h_n(z|q) \frac{t^n}{(q; q)_n}
\]

\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) h_n(z|q) \frac{t^n}{(q; q)_n} \right\}
\]

\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \sum_{k=0}^{\infty} \frac{[n]_q}{[k]_q} z^k \frac{t^n}{(q; q)_n} \right\}
\]

\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} P_n(x, q^k y) \frac{t^n}{(q; q)_n} \right) P_k(x, y) \frac{(z t)^k}{(q; q)_k} \right\}
\]

\[
= \sum_{k=0}^{\infty} \frac{(z t)^k}{(q; q)_k} \mathbb{E}(D_{xy}) \left\{ \frac{(y t; q)_\infty}{(x t; q)_\infty} \frac{P_k(x, y)}{(y t; q)_k} \right\},
\]

where \(|t|, |xt|, |zt|, |zxt| < 1\). Employing Euler's identity [11, II.1] to expand \(1/(zxt; q)_\infty\) on the right hand side of (2.6), we get

\[
\sum_{k=0}^{\infty} \frac{(z t)^k}{(q; q)_k} \mathbb{E}(D_{xy}) \left\{ \frac{(y t; q)_\infty}{(x t; q)_\infty} \frac{P_k(x, y)}{(y t; q)_k} \right\} = \frac{(y t; q)_\infty}{(t, x t; q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(y, x t; q)_n}{(q, y t; q)_n} \frac{z^n + k t^n + k x^k}{(q; q)_k}.
\]

Comparing the coefficients of \(z^n\), we complete the proof.

Proof. From (2.7) it follows that

\[
\sum_{n=0}^{\infty} h_n(x, y|q) h_n(u, v|q) \frac{t^n}{(q; q)_n}
\]

\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) h_n(u, v|q) \frac{t^n}{(q; q)_n} \right\}
\]

\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \sum_{k=0}^{\infty} \frac{[n]_q}{[k]_q} P_k(u, v) \right\}
\]

\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{k=0}^{\infty} P_k(u, v) P_k(x, y) \frac{q^k}{(q; q)_k} \left( \sum_{n=0}^{\infty} P_n(x, q^k y) \frac{t^n}{(q; q)_n} \right) \right\} \quad (|xt| < 1)
\]

\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{k=0}^{\infty} P_k(u, v) P_k(x, y) \frac{q^k}{(q; q)_k} \frac{(q^k y t; q)_\infty}{(x t; q)_\infty} \right\}
\]

\[
= \sum_{k=0}^{\infty} P_k(u, v) \frac{q^k}{(q; q)_k} \mathbb{E}(D_{xy}) \left\{ \frac{(y t; q)_\infty}{(x t; q)_\infty} \frac{P_k(x, y)}{(y t; q)_k} \right\} \quad (|t|, |xt| < 1).
\]
By Lemma 2.5, the above summation equals
\[
\frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}} \sum_{k=0}^{\infty} P_k(u, v) \frac{t^k}{(q; q)_k} \sum_{j=0}^{k} \frac{[k]}{j} (y, xt; q)_j x^{k-j}.
\]
Exchanging the order of summations, we get
\[
\frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}} \sum_{j=0}^{\infty} P_j(u, v) \frac{(y, xt; q)_j t^j}{(q, yt; q)_j} \sum_{k=0}^{\infty} (xt)_k P_k(u, q^j v) \frac{q^k}{(q; q)_k} (|uxt| < 1)
\]
\[
= \frac{(yt, vxt; q)_{\infty}}{(t, xt, uxt; q)_{\infty}} \sum_{j=0}^{\infty} P_j(u, v) \frac{(y, xt; q)_j t^j}{(q, yt, vxt; q)_j} \frac{(yt; vxt; q)_{\infty}}{(t, xt, uxt; q)_{\infty}} \phi_2 \left( \begin{array}{c}
y, xt, v/u \\
y, ut
\end{array} ; q, ut \right) (|ut| < 1).
\]
This completes the proof. □

Obviously, Mehler’s formula (1.10) for the Rogers-Szegö polynomials can be deduced from Theorem 2.1 by setting \( y = 0, v = 0 \) and \( u = y \).

3. The Rogers Formula for \( h_n(x, y|q) \)

In this section, we obtain the Rogers formula (Theorem 3.1) for the bivariate Rogers-Szegö polynomials \( h_n(x, y|q) \) using the operator \( \mathbb{E}(D_{xy}) \) and the technique of parameter augmentation [8, 9]. This formula can be readily restated in terms of the continuous big \( q \)-Hermite polynomials \( H_n(x; a|q) \) (Theorem 3.2).

**Theorem 3.1 (The Rogers Formula for \( h_n(x, y|q) \)).** We have
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n (q; q)_m} s^m = \frac{(ys; q)_{\infty}}{(s, xs, xt; q)_{\infty}} 2\phi_1 \left( \begin{array}{c}
y, xs \\
y, s
\end{array} ; q, t \right),
\]
provided that \(|t|, |s|, |xt|, |xs| < 1\).

**Proof.** By (2.7), we have
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n (q; q)_m} s^m
\]
\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n+m}(x, y) \frac{t^n}{(q; q)_n (q; q)_m} s^m \right\}
\]
\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \left( \sum_{m=0}^{\infty} P_m(x, q^n y) \frac{s^m}{(q; q)_m} \right) \right\} (|xs| < 1)
\]
\[
= \mathbb{E}(D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \frac{(q^n ys; q)_{\infty}}{(xs; q)_{\infty}} \right\}
\]

8
Applying Lemma 2.5, we get

\[ \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} (s; x, y)_n (x, y; q)_n \]

\[ = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} (y; x; q)_n (x; y; q)_n \]

as desired. \( \square \)

Clearly, the Rogers formula (1.11) for \( h_n(x; q) \) is the special case of (3.1) when \( y = 0 \). The following theorem is the Rogers formula for \( H_n(x; a|q) \) which contains (1.12) as a special case for \( a = 0 \).

**Theorem 3.2.** We have

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n+m}(x; a|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(as; q)_\infty}{(se^{i\theta}, se^{-i\theta}, te^{-i\theta}; q)_\infty} \times 2\phi_1 \left( \begin{array}{c} a e^{-i\theta}, se^{-i\theta} \\ as \end{array} ; q, te^{i\theta} \right), \]

where \( x = \cos \theta \) and \( |te^{i\theta}|, |se^{i\theta}|, |te^{-i\theta}|, |se^{-i\theta}| < 1 \).

The following special case of Theorem 3.1 for \( y = 0 \) will be useful to verify the relation (3.4) between \( h_n(x; q) \) and \( h_n(x, y|q) \).

**Theorem 3.3.** We have

\[ \sum_{k=0}^{\min\{n, m\}} \binom{n}{k} \binom{m}{k} (q; q)_k x^k h_{n+m-2k}(x; q) \]

\[ = \left( \sum_{k=0}^{n} \binom{n}{k} y^k h_{n-k}(x, y|q) \right) \left( \sum_{j=0}^{m} \binom{m}{j} y^j h_{m-j}(x, y|q) \right), \]

provided that \( |x|, |y| < \infty \).
Proof. Setting \( y = 0 \) in Theorem 3.1, by the Cauchy identity (1.3) and (1.4), we have

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} = \frac{1}{(s,xs;st;q)_\infty} \sum_{k=0}^{\infty} (xs;q)_k \frac{t^k}{(q;q)_k} (|t| < 1)
\]

\[
= \frac{(xs;t;q)_\infty}{(ys;yt;q)_\infty} \frac{(yt;q)_\infty}{(s,xs;q)_\infty} \frac{(ys;q)_\infty}{(s,xt;q)_\infty} \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m},
\]

which can be rewritten as

\[
\frac{1}{(xst;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} = \frac{1}{(yt,ys;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n(x,y|q) h_m(x,y|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m},
\]

where \(|t|, |s|, |xt|, |xs| < 1\).

Assuming that \(|xst|, |yt|, |ys| < 1\), we can expand \(1/(xst;q)_\infty\), \(1/(yt;q)_\infty\) and \(1/(ys;q)_\infty\) by Euler’s identity. By comparing the coefficients of \(t^n s^m\), we obtain (3.2). Since \(|t|, |s|, |xt|, |xs| < 1\) and \(|xst|, |yt|, |ys| < 1\), we see that \(|x|\) and \(|y|\) must be finite. \(\square\)

When \(y = 0\), (3.2) reduces to the well-known linearization formula [7, 15, 21] for \(h_n(x|q)\):

\[
h_n(x|q) h_m(x|q) = \sum_{k=0}^{\min\{n,m\}} \binom{n}{k} \binom{m}{k} (q;q)_k x^k h_{n+m-2k}(x|q), \quad |x| < \infty. \tag{3.3}
\]

Setting \(m = 0\) in (3.2), we are led to the following relation between \(h_n(x|q)\) and \(h_n(x,y|q)\):

\[
h_n(x|q) = \sum_{k=0}^{n} \binom{n}{k} y^k h_{n-k}(x,y|q), \quad |x|, |y| < \infty. \tag{3.4}
\]

The inverse expansion for (3.4) is

\[
h_n(x,y|q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{(k)} x^k h_{n-k}(x|q), \quad |x|, |y| < \infty. \tag{3.5}
\]

Note that (3.4) and (3.5) are equivalent to the relations (1.8) and (1.7) between \(H_n(x|q)\) and \(H_n(x;a|q)\).

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References


