Linked Partitions and Linked Cycles

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Dedicated to Professor Zhe-Xian Wan on the Occasion of His Eightieth Birthday

Abstract

The notion of noncrossing linked partition arose from the study of certain transforms in free probability theory. It is known that the number of noncrossing linked partitions of $[n+1]$ is equal to the n-th large Schröder number r_n, which counts the number of Schröder paths. In this paper we give a bijective proof of this result. Then we introduce the structures of linked partitions and linked cycles. We present various combinatorial properties of noncrossing linked partitions, linked partitions, and linked cycles, and connect them to other combinatorial structures and results, including increasing trees, partial matchings, k-Stirling numbers of the second kind, and the symmetry between crossings and nestings over certain linear graphs.

Keywords: noncrossing partition, Schröder path, linked partition, linked cycle, increasing trees, generalized k-Stirling number.
MSC Classification: 05A15, 05A18

1   Introduction

One of the most important combinatorial structures is a *partition of a finite set N*, that is, a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of subsets of N such that (i) $B_i \neq \emptyset$ for each i; (ii) $B_i \cap B_j = \emptyset$ if $i \neq j$, and (iii) $B_1 \cup B_2 \cup \cdots \cup B_k = N$. Each element B_i is called a *block* of π.

Let $N = [n]$, the set of integers $\{1, 2, ..., n\}$, and B_i, B_j be two blocks of a partition π of $[n]$. We say that B_i and B_j are *crossing* if there exist a, $c \in B_i$ and b, $d \in B_j$ with $a < b < c < d$.

Otherwise, we say B_i and B_j are *noncrossing*. A *noncrossing partition* σ is a partition of $[n]$ whose blocks are pairwise noncrossing.

Given partitions π and σ of $[n]$ we say that $\pi < \sigma$ if each block of π is contained in a block of σ. This ordering defines a lattice on the set of all partitions of $[n]$, which is called the partition lattice. When restricted to the set of noncrossing partitions on $[n]$, it is called the *noncrossing partition lattice* and denoted NC_n. The noncrossing partition lattice is a combinatorial structure that occurs in a diverse list of mathematical areas, including, for example, combinatorics, noncommutative probability, low-dimensional topology and geometric group theory. A nice expository article on the subject is given in [5].

Recently, in studying the unsymmetrized T-transform in the content of free probability theory, Dykema introduced a new combinatorial structure, the *noncrossing linked partition* [2], which can be viewed as a noncrossing partition with possible some links with restricted nature drawn between certain blocks of the partition. Dykema described two natural partial orderings on the set of noncrossing linked partitions of $[n]$, and compared it with the noncrossing partition lattice NC_n. In particular, he obtained the generating function for the number of noncrossing linked partitions via transforms in free probability theory. It follows that the cardinality of noncrossing linked partitions of $[n+1]$ is equal to the n-th large Schröder number r_n, which counts the number of Schröder paths of length n. A Schröder path of length n is a lattice path from $(0,0)$ to (n,n) consisting of steps East $(1,0)$, North $(0,1)$ and Northeast $(1,1)$, and never lying under the line $y = x$. The first few terms of the large Schröder numbers are $1, 2, 6, 22, 90, 394, 1806 \ldots$ It is the sequence A006318 in the database *On-line Encyclopedia of Integer Sequences* (OEIS) [8].

The restricted link between blocks proposed by Dykema is as follows. Let E and F be two finite subsets of integers. We say that E and F are *nearly disjoint* if for every $i \in E \cap F$, one of the following holds:

    a. $i = \min(E)$, $|E| > 1$ and $i \neq \min(F)$, or

    b. $i = \min(F)$, $|F| > 1$ and $i \neq \min(E)$.

Definition 1.1. A *linked partition* of $[n]$ is a set π of nonempty subsets of $[n]$ whose union is $[n]$ and any two distinct elements of π are nearly disjoint. It is a *noncrossing linked partition* if in addition, any two distinct elements of π are noncrossing.

Denote by $LP(n)$ and $NCL(n))$ the set of all linked partitions and noncrossing linked partitions of $[n]$ respectively. As before, an element of π is called a *block* of π. In the present paper we study the combinatorial properties of linked partitions. Section 2 is devoted to the noncrossing linked partitions. We construct a bijection between the set of noncrossing linked partitions of the set $[n+1]$ and the set of Schröder paths of length n, and derive various generating functions for noncrossing linked partitions. In Section 3 we discuss the set $LP(n)$ of all linked partition of $[n]$. We show that $LP(n)$ is in one-to-one correspondence with the set of increasing trees on $n+1$ labeled vertices, and

derive properties of linked partitions from those of increasing trees. We also define two statistics for a linked partition , 2-crossing and 2-nesting, and show that these two statistics are equally distributed over all linked partitions with the same lefthand and righthand endpoints. Then we propose a notion of *linked cycles*, which is a linked partition equipped with a cycle structure on each of its block. We describe two graphic representations of linked cycles, give the enumeration of the linked cycles on $[n]$, and study certain statistics over linked cycles. In particular, we show that there are two symmetric joint generating functions over all linked cycles on $[n]$: one for 2-crossings and 2-nestings, and the other for the crossing number and the nesting number. This is the content of Section 4.

2 Noncrossing linked partitions

This section studies the combinatorial properties of noncrossing linked partitions. Let $f_n = |NCL(n)|$, the number of noncrossing linked partitions of $[n]$. We establish a recurrence for the sequence f_n, which leads to the generating function. Then we give a bijective proof of the identity $f_{n+1} = r_n$, where r_n is the n-th large Schröder number. Using the bijection, we derive various enumerative results for noncrossing linked partitions.

The following basic properties of noncrossing linked partitions were observed in [2, Remark 5.4].
Property. Let $\pi \in NCL(n)$.

1. Any given element i of $[n]$ belongs to either exactly one or exactly two blocks; we will say i is singly or doubly covered by π, accordingly.

2. The elements 1 and n are singly covered by π.

3. Any two blocks E, F of π have at most one element in common. Moreover, if $|E \cap F| = 1$, then both $|E|$ and $|F|$ have at least two elements.

Noncrossing linked partitions can be represented by graphs. One such graphical representation is described in [2], which is a modification of the usual picture of a noncrossing partition. In this representation, for $\pi \in NCL(n)$, one lists n dots in a horizontal line, and connects the i-th one with the j-th one if and only if i and j are consecutive numbers in a block of π. Here we propose a new graphical representation, called *the linear representation*, which plays an important role in the bijections with other combinatorial objects. Explicitly, for a linked partition π of $[n]$, list n vertices in a horizontal line with labels $1, 2, \ldots, n$. For each block $E = \{i_1, i_2, \ldots, i_k\}$ with $i_1 = \min(E)$ and $k \geq 2$, draw an arc between i_1 and i_j for each $j = 2, \ldots, k$. Denote an arc by (i, j) if $i < j$, and call i the lefthand endpoint, j the righthand endpoint. In drawing the graph we always put the arc (i, j) above (i, k) if $j > k$. Denoted by $\mathcal{G}_\pi$ this linear representation. It is easy to check that a linked partition is noncrossing if and only if there are no two crossing edges in $\mathcal{G}_\pi$.

Example 1. Figure 1 shows the linear representations of all (noncrossing) linked partitions in $NCL(3)$.

π	$\mathcal{G}_\pi$	π	$\mathcal{G}_\pi$
$\{1,2,3\}$		$\{1,2\}\{3\}$	
$\{1,2\}\{2,3\}$		$\{1,3\}\{2\}$	
$\{1\}\{2,3\}$		$\{1\}\{2\}\{3\}$	

Figure 1: The elements of $NCL(3)$ and their linear representations.

Given $\pi \in NCL(n)$, for a singly covered element $i \in [n]$, denote by $B[i]$ the block containing i. If i is the minimal element of a block B of π, we say that it is a minimal element of π. Our first result is a recurrence for the sequence f_n.

Proposition 2.1. *The sequence f_n satisfies the recurrence*

$$f_{n+1} = f_n + f_1 f_n + f_2 f_{n-1} + \cdots + f_n f_1, \tag{1}$$

with the initial condition $f_1 = 1$.

Proof. Clearly $f_1 = 1$. Let $\pi \in NCL(n+1)$ and $i = \min(B[n+1])$.

If $i = n + 1$, then $n + 1$ is a singleton block of π. There are f_n noncrossing linked partitions satisfying this conditions.

If $1 \leq i \leq n$, then for any two elements $a, b \in [n]$ with $a < i < b$, a and b cannot be in the same block. Hence π can be viewed as a union of two noncrossing linked partitions, one of $\{1, 2, \ldots, i\}$, and the other of $\{i, i+1, \ldots, n+1\}$ where i and $n+1$ belong to the same block. Conversely, given a noncrossing linked partition $\pi_1 = \{B_1, \ldots, B_k\}$ of $\{1, 2, \ldots, i\}$ with $i \in B_k$, and a noncrossing linked partition $\pi_2 = \{C_1, \ldots, C_r\}$ of $\{i, i+1, \ldots, n+1\}$ with i and $n+1 \in C_1$, we can obtain a noncrossing linked partition π of $[n+1]$ by letting

$$\pi = \begin{cases} \pi_1 \cup \pi_2 & \text{if } B_k \neq \{i\}, \\ \pi_1 \cup \pi_2 \setminus \{B_k\} & \text{if } B_k = \{i\}. \end{cases}$$

Also note that a noncrossing linked partition of $\{i, \ldots, n+1\}$ with i and $n+1$ in the same block can be obtained uniquely from a noncrossing linked partition of $\{i, \ldots, n\}$ by adding $n+1$ to the block containing i. Hence we get

$$f_{n+1} = f_n + f_1 f_n + f_2 f_{n-1} + \cdots + f_n f_1,$$

for all $n \geq 0$. ∎

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Prop. 2.1 leads to an equation for the generating function $F(x) = \sum_{n \geq 0} f_{n+1} x^n$.

$$
\begin{aligned}
F(x) = \sum_{n=0}^{\infty} f_{n+1} x^n &= 1 + \sum_{n=1}^{\infty} f_n x^n + \sum_{n=1}^{\infty} (f_1 f_n + f_2 f_{n-1} + \cdots + f_n f_1,) x^n \\
&= 1 + x \cdot F(x) + x \cdot F(x)^2.
\end{aligned}
$$

It follows that

$$
F(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}. \tag{2}
$$

Formula (2) was first obtained by Dykema [2] using transforms in free probability theory. It is the same as the generating function of the large Schröder numbers, where the n-th large Schröder number r_n counts the number of Schröder paths of length n, i.e., lattice paths from $(0,0)$ to (n,n) consisting of steps $(1,0)$, $(0,1)$, and $(1,1)$, and never lying under the line $y = x$. Therefore

Theorem 2.2 (Dykema). *For every $n \geq 0$, the number of elements in $NCL(n+1)$ is equal to the large Schröder number r_n.*

REMARK. The standard definition for a Schröder path is a lattice path starting from $(0,0)$, ending at $(2n, 0)$, consisting only of steps $(1,1)$, $(1,-1)$ and $(2,0)$, and never going below the x-axis. In this paper we rotate the path by $\pi/4$ and rescale it so that it goes from $(0,0)$ to (n,n). This equivalent version makes our construction much easier to describe.

We give a bijection between noncrossing linked partitions of $[n+1]$ and Schröder paths of length n. For convenience, we use E, N and D to denote East, North and Northeast steps, respectively.

A map ϕ from $NCL[n+1]$ to the set of Schröder paths of length n.

Given a noncrossing linked partition π of $[n+1]$ ($n \geq 0$), define a lattice path from the origin $(0,0)$ by the following steps.

Step 1. Initially set $x = 0$. Move $k - 1$ N-steps if the block $B[1]$ contains k elements.

In general, for $x = i > 0$, if $i + 1$ is the minimal element of a block B of π and $|B| = k$, move $(k-1)$ N-steps.

Step 2. Move one D-step if $i + 2$ is a singly covered minimal element of π. Otherwise move one E-step. Increase the value of x by one. (Note that the path reaches the line $x = i + 1$ now).

Iterate Steps 1 and 2 until $x = n$. When the process terminates, the resulting lattice path is $\phi(\pi)$.

Theorem 2.3. *The above defined map ϕ is a bijection from the set of noncrossing linked partitions of $[n+1]$ to the set of Schröder paths of length n.*

Proof. In π each integer $i \in [n+1]$ is of one of the following types:

1. i is a singly covered minimal element;

2. i is singly covered, but $i \neq \min(B[i])$;

3. i is doubly covered. In this case assume i belong to blocks E and F with $i = \min(F)$ and $j = \min(E) < i$;

Each element i of the first type except $i = 1$ contributes one D-step between the lines $x = i - 2$ and $x = i - 1$. Each element i of the second type contributes one N-step at the line $x = \min(B[i])$, and one E-step between the lines $x = i - 2$ and $x = i - 1$. Each element i of the third type contributes one N-step at the line $x = j$, and one E-step between the lines $x = i - 2$ and $x = i - 1$. Hence the path ends at (n, n), and at any middle stage, the number of N-steps is no less than that of E-steps. This proves that the path $\phi(\pi)$ is a Schröder path of length n.

To show that ϕ is a bijection, it is sufficient to give the inverse map of ϕ. Given a Schröder path of length n, for each $i = 2, 3, \ldots, n$, check the segment between the lines $x = i - 2$ and $x = i - 1$. If it is a D-step, then i is a singly covered minimal element. If it is a E-step, draw a line with slope 1 which starts at the middle point of this E-step, and lies between the Schröder path and the line $x = y$. Assume the line meets the given Schröder path for the first time at $x = j < i - 1$. Then i belongs to a block whose minimal element is $j + 1$. We call this diagonal segment between $x = j$ and $x = i - \frac{3}{2}$ a *tunnel* of the Schröder path. See Figure 2 for an illustration.

The resulting collection of subsets of $[n + 1]$ must be pairwise noncrossing. This is because for any tunnel whose endpoints are A and B, where A is an N-step and B is an E-step, there are an equal number of N-step and E-steps between A and B. Therefore for any E-step between A and B, the tunnel starting from it must end at an N-step between A and B as well. Also note that any element $i \in [n + 1] \backslash \{1\}$ can belong to at most two such subsets. If it happens, then both subsets have cardinality at least two, and i is the minimal element of exactly one of them. Hence the collection of subsets obtained forms a noncrossing linked partition. We leave to the reader to check that this gives the inverse of $\phi(\pi)$. ∎

Example 2. Tunnels in a Schröder path of length 5.
The tunnel between $x = 1$ and $x = 3/2$ implies that 3 is in a block B with $\min(B) = 2$.
The tunnel between $x = 2$ and $x = 3/2$ implies that 4 is in a block B with $\min(B) = 3$.
The tunnel between $x = 1$ and $x = 7/2$ implies that 5 is in a block B with $\min(B) = 2$.
The tunnel between $x = 0$ and $x = 9/2$ implies that 6 is in a block B with $\min(B) = 1$.
The corresponding noncrossing linked partition is $\pi = \{\{1, 6\}, \{2, 3, 5\}, \{3, 4\}\}$.

The bijection ϕ can be easily described via the linear representation of π. First in $\mathcal{G}_\pi$, add a mark right before each singly covered minimal element except 1. The bijection ϕ transforms this marked linear representation of π into a lattice path by going through the vertices from left to right, and replacing each left end of an arc with an N-step, each right end of an arc with an E-step, and each mark with a D-step.
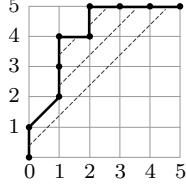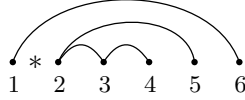
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Figure 2: There are four tunnels in the Schröder path.

Example 3. Let $\pi = \{\{1,6\}, \{2,3,5\}, \{3,4\}\}$. The marked linear representation is

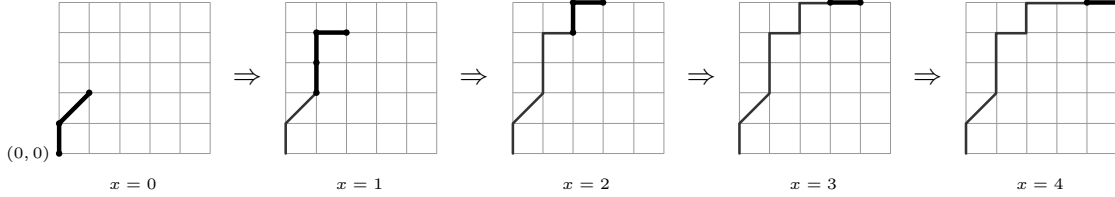The following steps yield the corresponding large Schröder path.

Figure 3: The steps of ϕ that yield the corresponding Schröder path.

Example 4. The elements of $NCL(3)$, their marked linear representations, and the corresponding Schröder paths.

A *peak* of the Schröder path is a pair of consecutive NE steps and a *valley* is a pair of consecutive EN steps. The following results are well-known for Schröder paths.

Proposition 2.4.

1. Let $p(n,k)$ be the number of Schröder paths of length n with k peaks. Then $p(n,k) = C_{n-k}\binom{2n-k}{k} = \binom{n}{k}\binom{2n-k}{n-1}/n$ where the C_n is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It is also the number of Schröder paths with k D-steps. Let $p(0,0) = 1$. The generating function for $p(n,k)$ is

$$\sum_{n,k\geq 0} p(n,k)x^n t^k = \frac{1 - tx - \sqrt{(1-tx)^2 - 4x}}{2}. \qquad (3)$$

2. Let $v(n,k)$ be the number of Schröder paths of length n with k valleys. It also counts the number of Schröder paths of length n with k NN-steps. Let $v(0,0) = 1$. The generating function $V(x,t) = \sum_{n,k\geq 0} v(n,k)x^n t^k$ satisfies

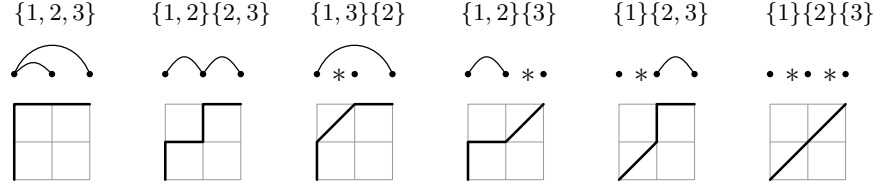$$x(t + x - tx)V(x,t)^2 - (1 - 2x + tx)V(x,t) + 1 = 0.$$

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Figure 4: The elements of $NCL(3)$ and their corresponding Schröder paths.

Explicitly,

$$V(x,t) = \frac{-1 + 2x - tx + \sqrt{1 - 4x - 2tx + t^2x^2}}{2(-tx - x^2 + x^2t)}. \tag{4}$$

3. *Let $d(n,k)$ be the number of Schröder paths of length n, containing k D's not preceded by an E. Let $d(0,0) = 1$. The generating function $D(x,t) = \sum_{n,k \geq 0} D(n,k)x^n t^k$ satisfies*

$$D(x,t) = 1 + txD(x,t) + x(1 + x - tx)D(x,t)^2.$$

Explicitly,

$$D(x,t) = \frac{1 - tx - \sqrt{(1-tx)^2 - 4x(1+x-tx)}}{2x(1+x-tx)}. \tag{5}$$

These results can be found, for example, in OEIS [8], Sequence A060693 for Statement 1, A101282 for Statement 2, and A108916 for Statement 3.

The correspondence $\pi \to \phi(\pi)$ between noncrossing linked partitions of $[n+1]$ and Schröder paths of length n allows us to deduce a number of properties for noncrossing linked partitions. It is easily seen that the properties for an element of $[n+1]$ in a noncrossing linked partition listed on the left correspond to the given steps of Schröder paths, listed on the right.

$NCL(n+1)$	steps in Schröder paths
singly covered minimal element i, $i \neq 1$	D
doubly covered element	EN
singleton block $\{i\}$, $i \neq 1$	D not followed by an N
$i \in B$ where $\min(B) = i - 1$	NE

From Prop. 2.4 and the obvious symmetry between steps ED and DN there follows:

Proposition 2.5. *1. The number of noncrossing linked partitions of $[n+1]$ with k singly covered minimal elements i where $i \neq 1$ is equal to $p(n,k) = C_{n-k}\binom{2n-k}{k} = \binom{n}{k}\binom{2n-k}{n-1}/n$. It is also counts the number of noncrossing linked partitions on $[n+1]$ with k elements x such that $x, x-1$ lie in a block B with $x - 1 = \min(B)$. The generating function of $p(n,k)$ is given by Eqn. (3).*

2. *The number of noncrossing linked partitions of $[n+1]$ with k doubly covered elements is $v(n,k)$, whose generating function is given by Eqn. (4).*

3. *The number of noncrossing linked partitions on $[n+1]$ with k singleton blocks $\{i\}$ where $i \neq 1$ is $d(n,k)$, whose generating function is given by Eqn. (5).*

At the end of this section, we count the noncrossing linked partitions by the number of blocks, using the recurrence (1).

Proposition 2.6. *Let $b(n,k)$ be the number of noncrossing linked partition of $[n]$ with k blocks. Let $B(x,t) = 1 + \sum_{n,k \geq 1} b(n,k)x^n t^k$. Then $B(x,t)$ satisfies the equation*

$$(1 + x - tx)B(x,t)^2 + (2tx - x - 3)B(x,t) + 2 = 0. \tag{6}$$

Explicitly,

$$B(x,t) = \frac{3 + x - tx - \sqrt{(2tx - x - 3)^2 - 8(1 + x - tx)}}{2(1 + x - tx)}. \tag{7}$$

Proof. We derive a recurrence for $b(n,k)$. Given $\pi \in NCL(n)$ with k blocks, again let $i = \min(B[n])$. First, there are $b(n-1, k-1)$ many noncrossing linked partitions in $NCL(n)$ such that $i = n$. Otherwise, assume $1 \leq i \leq n-1$. As in the proof of Prop. 2.1, π is a union of two noncrossing linked partitions, π_1 of $[i]$, and π_2 of $\{i, \ldots, n\}$ with i and n lying in the same block. Assume π_1 has t blocks, and π_2 has r blocks. If i is a singleton of π_1, then π has $t - 1 + r$ many blocks; if i is not a singleton of π_1, then π has $t + r$ many blocks. Finally, note that π_2 can be obtained by taking any noncrossing linked partition on $\{i, i+1, \ldots, n-1\}$, and then adding n to the block containing i. Combining the above, we get the recurrence

$$
\begin{aligned}
b(n,k) &= b(n-1, k-1) \\
&+ \sum_{i=1}^{n-1} \sum_{r+t=k} b(n-i, r)\Big[b(i-1, t) + b(i, t) - b(i-1, t-1)\Big].
\end{aligned} \tag{8}
$$

Note $b(1,1) = b(2,1) = b(2,2) = 1$, and $b(n,0) = 0$ for all $n \geq 1$. If we set $b(0,0) = 1$, then the recurrence (8) holds for all $n \geq 1$ and $k \geq 1$. Now multiply both sides of (8) by $x^n t^k$, and sum over all n, k. Noticing that for any sequence g_i,

$$\sum_{n \geq 1} \sum_{i=1}^{n-1} g_i g_{n-i} x^n = \left(\sum_{n \geq 0} g_n x^n\right)^2 - 2g_0 \sum_{n \geq 0} g_n x^n,$$

we get the equation (6) and hence the formula (7). ∎

3 Linked partitions

In this section we study linked partitions. Recall that a linked partition of $[n]$ is a collection of pairwise nearly disjoint subsets whose union is $[n]$. The set of all the linked partitions on $[n]$ is denoted by $LP(n)$, whose cardinality is lp_n.

It is not hard to see that $lp_n = n!$. Instead of merely giving a counting argument, we present a one-to-one correspondence between the set of linked partitions of $[n]$ and the set of increasing trees on $n+1$ labeled vertices. The latter is a geometric representation for permutations, originally developed by the French, and outlined in the famous textbook [9, Chap.1.3]. Many properties of linked partitions can be trivially deduced from this correspondence. As a sample, we list the results involving the signless Stirling numbers and the Eulerian numbers. At the end of this section we give the joint distribution for two statistics, 2-crossings and 2-nestings, over linked partitions with given sets of lefthand and righthand endpoints.

Definition 3.1. An *increasing tree* on $n+1$ labeled vertices is a rooted tree on vertices $0, 1, \ldots, n$ such that for any vertex i, $i < j$ if i is a successor of j.

Theorem 3.2. *There is a one-to-one correspondence between the set of linked partitions of $[n]$ and the set of increasing trees on $n+1$ labeled vertices.*

Proof. We use the linear representation $\mathcal{G}_\pi$ for linked partitions. Recall that for $\pi \in LP(n)$, $\mathcal{G}_\pi$ is the graph with n dots listed in a horizontal line with labels $1, 2, \ldots, n$, where i and j are connected by an arc if and only of j lies in a block B with $i = \min(B)$. To get an increasing tree, one simply adds a root 0 to $\mathcal{G}_\pi$ which connects to all the singly covered minimal elements of π. This defines an increasing tree on $[n] \cup \{0\}$, where the children of root 0 are those singly covered minimal elements, and j is a child of i if and only if (i, j) is an arc of $\mathcal{G}_\pi$ and $j > i$. ∎

Example 5. Let $\pi = \{\{126\}, \{248\}, \{3\}, \{57\}\}$. The singly covered minimal elements are $\{1, 3, 5\}$. The corresponding increasing tree is given in Figure 5.
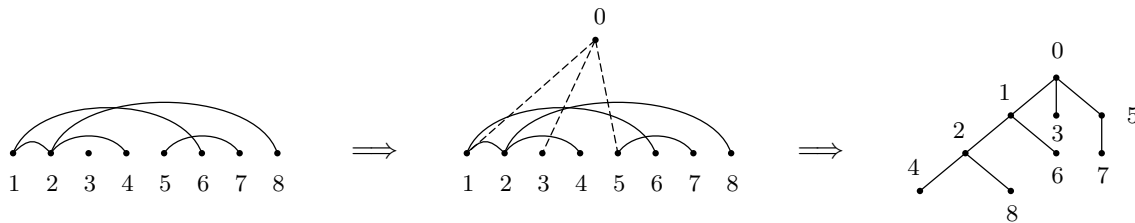

Figure 5: A linked partition and the corresponding increasing tree

The following properties of increasing trees are listed in Proposition 1.3.16 of [9].

Proposition 3.3. *1. The number of increasing trees on $n+1$ labeled vertices is $n!$.*

2. *The number of such trees for which the root has k successors is the signless Stirling number $c(n, k)$ (of the first kind).*

3. *The number of such trees with k endpoints is the Eulerian number $A(n, k)$.*

Let $\beta(\pi) = |\{i \text{ is singly covered and } i \neq \min(B[i])\}| + |\{i : \text{ singleton block of } \pi \}|$.

Corollary 3.4. 1. *The number of linked partitions of $[n]$ is $n!$.*

2. *The number of linked partitions of $[n]$ with k singly covered minimal elements is the signless Stirling number $c(n, k)$ (of the first kind).*

3. *The number of linked partitions of $[n]$ with $\beta(\pi) = k$ is the Eulerian number $A(n, k)$.*

For a linked partition π with linear representation $\mathcal{G}_\pi$, we say that two arcs (i_1, j_1) and (i_2, j_2) form a *2-crossing* if $i_1 < i_2 < j_1 < j_2$; they form a *2-nesting* if $i_1 < i_2 < j_2 < j_1$. Denoted by $cr_2(\pi)$ and $ne_2(\pi)$ the number of 2-crossings and 2-nestings of π, respectively. For example, the linked partition $\pi = \{\{126\}, \{248\}, \{3\}, \{57\}\}$ in Example 5 has $cr_2(\pi) = 2$ and $ne_2(\pi) = 2$.

Given $\pi \in LP(n)$, define two multiple sets

$$\text{left}(\pi) = \{\text{lefthand endpoints of arcs of } \pi\},$$
$$\text{right}(\pi) = \{\text{righthand endpoints of arcs of } \pi\}.$$

For example, for $\pi = \{\{126\}, \{248\}, \{3\}, \{57\}\}$, $\text{left}(\pi) = \{1, 1, 2, 2, 5\}$, and $\text{right}(\pi) = \{2, 6, 4, 7, 8\}$. Clearly, each element of $\text{right}(\pi)$ has multiplicity 1.

Fix S and T where S is a multi-subset of $[n]$, T is a subset of $[n]$, and $|S| = |T|$. Let $LP_n(S, T)$ be the set $\{\pi \in LP(n) : \text{left}(\pi) = S, \text{right}(\pi) = T\}$. We prove that over each set $LP_n(S, T)$, the statistics $cr_2(\pi)$ and $ne_2(\pi)$ have a symmetric joint distribution. Explicitly, let $S = \{a_1^{r_1}, a_2^{r_2}, \ldots, a_m^{r_m}\}$ with $a_1 < a_2 < \cdots < a_m$. For each $1 \leq i \leq m$, let $h(i) = |\{j \in T : j > a_i\}| - |\{j \in S : j > a_i\}|$. We have

Theorem 3.5.

$$\sum_{\pi \in LP_n(S,T)} x^{cr_2(\pi)} y^{ne_2(\pi)} = \sum_{\pi \in LP_n(S,T)} x^{ne_2(\pi)} y^{cr_2(\pi)} = \prod_{i=1}^{m} y^{r_i h(i) - r_i^2} \binom{\mathbf{h(i)}}{\mathbf{r_i}} \bigg|_{q=x/y}, \qquad (9)$$

where $\binom{\mathbf{n}}{\mathbf{m}}$ is the q-binomial coefficient

$$\binom{\mathbf{n}}{\mathbf{m}} = \frac{(\mathbf{n})!}{(\mathbf{m})!(\mathbf{n} - \mathbf{m})!} = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{m-1})}{(q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})}.$$

Proof. For $1 \leq m \leq n$, denote by $\binom{[n]}{m}$ the set of integer sequences $(x_1, x_2, \ldots, x_m)$ such that $1 \leq x_1 < x_2 < \cdots < x_m \leq n$. We give a bijection from the set of linked partitions in $LP_n(S, T)$ to the set $\prod_{i=1}^{m} \binom{[h(i)]}{r_i}$.

Given an element $\mathbf{s} = (\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_m})$ in $\prod_{i=1}^{m} \binom{[h(i)]}{r_i}$ where $\mathbf{s_i} \in \binom{[h(i)]}{r_i}$, we construct a linked partition π by matching each lefthand endpoint in S to a righthand endpoint in T. First, there are r_m lefthand endpoints at node a_m, and on its right there are $h(m)$ many righthand endpoints. Assume $\mathbf{s_m} = (x_1, x_2, \ldots, x_{r_m})$. We connect the r_m lefthand endpoints at node a_m to the x_1-th, x_2-th, ..., x_{r_m}-th righthand endpoints after node a_m.

In general, after matching lefthand endpoints at nodes $a_{i+1}, \ldots, a_m$ to some righthand endpoints, we process the r_i lefthand endpoints at node a_i. At this stage there are exactly $h(i)$ many righthand endpoints available after the node a_i. List them by $1, 2, \ldots, h(i)$, and match the lefthand endpoints at a_i to the y_1-th, y_2-th, ..., y_{r_i}-th of them, if $\mathbf{s_i} = (y_1, y_2, \ldots, y_{r_i})$.

Continue the above procedure until each lefthand endpoint is connected to some righthand endpoint on its right. This gives the desired bijection between $LP_n(S,T)$ and $\prod_{i=1}^{m} \binom{[h(i)]}{r_i}$. In particular, $LP_n(S,T)$ is nonempty if and only if for every $a_i \in S$, $h(i) \geq r_i$.

Example 6. Let $S = \{1, 1, 2, 2, 2, 3, 3\}$, $T = \{4, 5, 6, 7, 8, 9, 10\}$. Then $h(1) = 2$, $h(2) = 5$, $h(3) = 7$. Figure 6 illustrates how to construct the linked partition for $\mathbf{s} = (\mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3})$ where $\mathbf{s_1} = (1, 2), \mathbf{s_2} = (2, 4, 5)$, and $\mathbf{s_3} = (3, 6)$.
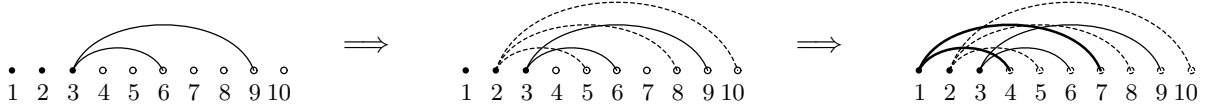

Figure 6: An illustration of the bijection from $LP_n(S,T)$ to $\prod_{i=1}^{m} \binom{[h(i)]}{r_i}$.

The numbers of 2-crossings and 2-nestings are easily expressed in terms of the sequence $\mathbf{s} = (\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_m})$. Assume $\mathbf{s_i} = (x_1, \ldots, x_{r_i})$. By the above construction, the number of 2-crossings formed by arcs (j,k) and (a_i, t) with $j < a_i < k < t$ is $\sum_{t=1}^{r_i}(x_t - t)$, and the number of 2-crossings formed by arcs (j,k) and (a_i, t) with $j < a_i < t < k$ is $\sum_{t=1}^{r_i}(h(i) - x_t - (r_i - t))$. Hence $\mathbf{s_i}$ contributes a factor

$$x^{\left(\sum_{t=1}^{r_i} x_t\right) - \binom{r_i+1}{2}} y^{r_i h(i) - \binom{r_i}{2} - \sum_{t=1}^{r_i} x_t}$$

to the generating function $\sum_{\pi \in LP_n(S,T)} x^{cr_2(\pi)} y^{ne_2(\pi)}$. Since

$$\sum_{(x_1, \ldots, x_{r_i}) \in \binom{[h(i)]}{r_i}} q^{\left(\sum_t x_t\right) - \binom{r_i+1}{2}} = \binom{\mathbf{h(i)}}{\mathbf{r_i}},$$

and $\mathbf{s_i}$ are mutually independent, we have

$$\sum_{\pi \in LP_n(S,T)} x^{cr_2(\pi)} y^{ne_2(\pi)} = \prod_{i=1}^{m} y^{r_i h(i) - r_i^2} \binom{\mathbf{h(i)}}{\mathbf{r_i}} \Bigg|_{q=x/y}.$$

The symmetry between $cr_2(\pi)$ and $ne_2(\pi)$ is obtained by the involution $\tau : (\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_m}) \rightarrow (\mathbf{t_1}, \mathbf{t_2}, \ldots, \mathbf{t_m})$ on $\prod_{i=1}^{m} \binom{[h(i)]}{r_i}$, where $\mathbf{t_i} = (h(i) + 1 - x_{r_i}, \ldots, h(i) + 1 - x_1)$ if $\mathbf{s_i} = (x_1, \ldots, x_{r_i})$. ∎

4   Linked cycles

As with matchings and set partitions, one can define the *crossing number* and the *nesting number* for a given linked partition. Unfortunately, these two statistics do not have the same distribution over all linked partitions of $[n]$. Motivated by the work in [1], we want to find a suitable structure over which the crossing number and nesting number have a symmetric joint distribution. For this purpose we introduce the notion of *linked cycles*, which are linked partitions equipped with a cycle structure on each of its block. It turns out that the set of linked cycles possesses many interesting combinatorial properties.

4.1   Two representations for linked cycles

Definition 4.1. A *linked cycle* $\hat{\pi}$ on $[n]$ is a linked partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$ where for each block B_i the elements are arranged in a cycle.

We call each such block B_i with the cyclic arrangement a *cycle* of $\hat{\pi}$. The set of all linked cycles on $[n]$ is denoted by $LC(n)$.

For a set $B = \{b_1, b_2, \ldots, b_k\}$, we represent by $(b_1, b_2, \ldots, b_k)$ the cycle $b_1 \to b_2 \to \cdots \to b_k \to b_1$. In writing a linked cycle $\hat{\pi}$, we use the convention that: (a) each cycle of $\hat{\pi}$ is written with its minimal element first, (b) the cycles are listed in increasing order of their minimal elements. For example, for the linked partition $\pi = \{\{126\}, \{248\}, \{3\}, \{57\}\}$ with cyclic orders $(1 \to 2 \to 6 \to 1), (2 \to 8 \to 4 \to 2), (3 \to 3)$, and $(5 \to 7 \to 5)$, the linked cycle $\hat{\pi}$ is written as $\hat{\pi} = (126)(284)(3)(57)$.

Again the linked cycles may be represented by certain graphs. Here we introduce two such graphical representations.

1.   Cycle representation $\mathcal{G}_{\hat{\pi}}^c$.   Let $\hat{\pi} \in LC(n)$, the cycle representation $\mathcal{G}_{\hat{\pi}}^c$ of $\hat{\pi}$ is a directed graph on $[n]$ with arcs (i, j) whenever i and j are consecutive elements in a cycle of $\hat{\pi}$.

In drawing the figures, we put elements of a cycle C_i in a circle in clockwise order. If a cycle C_i contains the minimal element of a connected component of $\mathcal{G}_{\hat{\pi}}^c$, then we say that C_i is the *root cycle* of that component.

Example 7. The cycle representation for the linked cycle $\pi = (126)(284)(3)(57)$ is given in Figure 7. There are three connected components and (126) is the root cycle of the connected component $(126)(284)$.
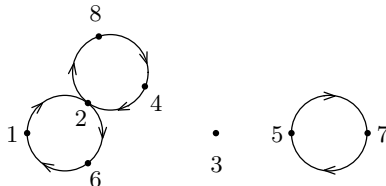

Figure 7: The cycle representation of the linked cycle $\pi = (126)(248)(3)(57)$.

2. Linear representation $\mathcal{G}_{\hat{\pi}}^l$. Let $\hat{\pi} \in LC(n)$, the linear representation $\mathcal{G}_{\hat{\pi}}^l$ of $\hat{\pi}$ is a graph whose vertices lie on a horizontal line, and each vertex is of one of the following kind: (i) a lefthand endpoint, (ii) a righthand endpoint, or (iii) an isolated point.

Start with n vertices in a horizontal line labeled $1, 2, \ldots, n$. We define $\mathcal{G}_{\hat{\pi}}^l$ by first splitting vertices as follows:

1. If i is a singly covered minimal element of a cycle with $k+1$ elements, and $k \geq 0$, split the vertex i into k vertices labeled $i^{(1)}, i^{(2)}, \ldots, i^{(k)}$;

2. If i is singly covered, but not a minimal element, replace the vertex i by a vertex labeled $i^{(0)}$;

3. If i is doubly covered, and is the minimal element of a cycle of size $k+1$, split the vertex i into $k+1$ vertices labeled $i^{(0)}, i^{(1)}, i^{(2)}, \ldots, i^{(k)}$;

For a cycle $C_i = (i_1 i_2 \ldots i_{t_i})$, if $t_i \geq 2$, then we have created vertices with labels $i_1^{(1)}, i_1^{(2)}, \ldots, i_1^{(t_i-1)}$. Add arcs $(i_1^{(1)}, i_{t_i}^{(0)})$, $(i_1^{(2)}, i_{t_i-1}^{(0)})$, $\ldots, (i_1^{(t_i-1)}, i_2^{(0)})$. Do this for each cycle of $\hat{\pi}$, and the resulting graph is $\mathcal{G}_\pi{}^l$.

Example 8. The linear representation for the linked cycle $\hat{\pi} = (126)(284)(3)(57)$.
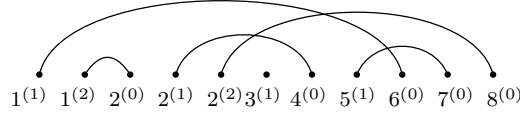
As for the linked partitions, sometimes it is useful to distinguish the set of singly covered minimal elements of $\hat{\pi}$. A vertex i is a singly covered minimal element of $\hat{\pi}$ if and only if in the linear representation $\mathcal{G}_{\hat{\pi}}^l$, there is no vertex $i^{(0)}$. The *marked linear representation* of $\hat{\pi}$ is obtained from $\mathcal{G}_{\hat{\pi}}^l$ by adding a mark before $i^{(1)}$ for each singly covered minimal i, except for $i = 1$. For example, for $\hat{\pi}$ in Example 8 we should add marks before vertices $3^{(1)}$ and $5^{(1)}$.
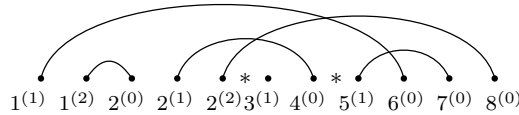
For a linked partition π of $[n]$, the marked linear representation of π has k marks if $\mathcal{G}_\pi$ has $k+1$ connected components (as there is no mark before the vertex 1). In every connected component of $\mathcal{G}_\pi$, the number of vertices is one greater than the number of arcs, so in $\mathcal{G}_\pi$,

$$
\begin{aligned}
n &= \#vertices \\
&= \#arcs + \#components \\
&= \#arcs + \#marks + 1.
\end{aligned}
$$

For a linked cycle $\hat{\pi}$ whose underlying linked partition is π, the marked linear representation of $\hat{\pi}$ has the same number of arcs and marks as that of π. Hence

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Proposition 4.2. *Any marked linear representation of a linked cycle of $[n]$ satisfies*

$$\#arcs + \#marks = n - 1.$$

REMARK. It is clear that both the cycle representation $\mathcal{G}^c_{\hat{\pi}}$ and the linear representation $\mathcal{G}^l_{\hat{\pi}}$ uniquely determine $\hat{\pi}$. In the marked linear representation of $\hat{\pi}$, we may remove the labeling on the vertices, as it can be recovered by the arcs and marks. More precisely, suppose G is a graph of a (partial) matching whose vertices are listed on a horizontal line, and where some vertices $2, 3, \ldots, n$ have marks before them. If the total number of arcs and marks is $n - 1$, then one can partition the vertices of G into n intervals in the following way. Start from the left-most vertex, end an interval before each righthand endpoint or mark. Then for the ith interval, label the righthand endpoint by $i^{(0)}$, the lefthand endpoints consecutively by $i^{(1)}, i^{(2)}, \ldots$, and isolated point by $i^{(1)}$.

4.2 Enumeration of linked cycles

Let lc_n be the cardinality of $LC(n)$, the set of linked cycles on $[n]$. It is easy to get $lc_1 = 1, lc_2 = 2, lc_3 = 7, lc_4 = 37$. For any $\hat{\pi} \in LC(n)$, let $s(\hat{\pi})$ be the number of singly covered minimal elements in $\hat{\pi}$. $s(\hat{\pi})$ is also the number of connected components in the cycle representation $\mathcal{G}^c_{\hat{\pi}}$. We denote by $f(n, m)$ the number of linked cycles in $LC(n)$ with $s(\hat{\pi}) = m$. Let $size(\hat{\pi}) = \sum_{C_i} |C_i|$ where the sum is over all cycles of $\hat{\pi}$, and $double(\hat{\pi})$ be the number of doubly covered elements of $\hat{\pi}$.

Proposition 4.3. *The numbers $f(n, m)$ satisfy the recurrence*

$$f(n, m) = (2(n - 1) - m)f(n - 1, m) + f(n - 1, m - 1)$$

with initial values $f(1, 1) = 1$, $f(n, 0) = 0$ and $f(n, m) = 0$ if $n < m$.

Proof. Clearly $f(1, 1) = 1$ and $f(n, m) = 0$ if $n < m$. For any linked cycle $\hat{\pi} \in LC(n)$, 1 is always a singly covered minimal element. Hence $f(n, 0) = 0$.

Given $\hat{\pi} \in LC(n)$, after removing the vertex n and all edges incident to n in the cycle representation $\mathcal{G}^c_{\hat{\pi}}$, we get a linked cycle on $[n - 1]$.

Conversely, given $\hat{\pi}' = \{B_1, B_2, ..., B_k\} \in LC(n - 1)$, we can obtain a linked cycle $\hat{\pi}$ of $[n]$ by joining the element n in one of the following mutually exclusive ways.

1. $\hat{\pi} = \hat{\pi}' \cup \{n\}$, that is, $\hat{\pi}$ is obtained from $\hat{\pi}'$ by adding a singleton block $\{n\}$. In this case $s(\hat{\pi}) = s(\hat{\pi}') + 1$.

2. $\hat{\pi}$ is obtained from $\hat{\pi}'$ by inserting n into an existing cycle $(a_1 a_2 ... a_k)$ of $\hat{\pi}'$. Since n can be inserted after any element a_i, there are $size(\hat{\pi}')$ many such formed $\hat{\pi}$. For each of them, $s(\hat{\pi}) = s(\hat{\pi}')$.

3. $\hat{\pi}$ is obtained from $\hat{\pi}'$ by adding a cycle of the form (i, n). Such a constructed $\hat{\pi}$ is a linked cycle if and only if i is singly covered and not a minimal element of $\hat{\pi}'$. There are $n - 1 - s(\hat{\pi}') - double(\hat{\pi}')$ many choices for i. For each $\hat{\pi}$ in this case, $s(\hat{\pi}) = s(\hat{\pi}')$.

Combining these three cases, and noting that $size(\hat{\pi}') = n - 1 + double(\hat{\pi}')$, we have $size(\hat{\pi}') + n - 1 - s(\hat{\pi}') - double(\hat{\pi}') = 2(n-1) - s(\hat{\pi}')$, which leads to the desired recurrence

$$f(n, m) = (2(n-1) - m)f(n-1, m) + f(n-1, m-1).$$

∎

The initial values of $f(n, m)$ are

$n \setminus k$	1	2	3	4
1	1			
2	1	1		
3	3	3	1	
4	15	15	6	1

The set of numbers $\{f(n, m) : n \geq m > 0\}$ are the coefficients of Bessel polynomials $y_n(x)$ (with exponents in decreasing order), and has been studied by Riordan [7], and by W. Lang as the signless k-Stirling numbers of the second kind with $k = -1$, [4]. From [7, pp77], we deduce that

$$f(n, m) = \frac{(2n - m - 1)!}{(m-1)!(n-m)!2^{n-m}} = \binom{2n - m - 1}{m - 1} \frac{(2n - 2m)!}{(n - m)!2^{n-m}}. \tag{10}$$

A combinatorial proof of (10) is given at the end of this subsection, using a bijection between linked cycles and certain set partitions. Another definition for $f(n, m)$ is given by the coefficients in the expansion of the operator $(x^{-1}d_x)^n$, i.e.,

$$(x^{-1}d_x)^n = \sum_{m=1}^{n} (-1)^{n-m} f(n, m) x^{m-2n} d_x^m, \qquad n \in N.$$

An extensive algebraic treatment based on this equation was given in [4]. The numbers $\{f(n, m)\}$ can be recorded in an infinite-dimensional lower triangular matrix. In particular, lc_n, the cardinality of $LC(n)$, is the n-th row sum of the matrix. The exponential generating function of lc_n is given in Formula (54) of [4] as

$$\sum_{n=1}^{\infty} lc_n \frac{x^n}{n!} = \exp(1 - \sqrt{1 - 2x}) - 1.$$

In the OEIS [8] $\{lc_n\}$ is the sequence A001515, where the following recurrence is given.

$$a_n = (2n - 3)a_{n-1} + a_{n-2} \tag{11}$$

Here we present a combinatorial proof based on the structure of linked cycles.

Proposition 4.4. *The numbers $f(n, m)$ satisfy the recurrence*

$$f(n, m) = mf(n, m+1) + f(n-1, m-1)$$

with initial values $f(1, 1) = 1$, $f(n, 0) = 0$ and $f(n, m) = 0$ if $n < m$.

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Proof. Let

$$A = \{\hat{\pi} \in LC(n) \ : \ s(\hat{\pi}) = m \text{ and } n \text{ is not a singleton block}\},$$

and

$$B = \{\hat{\pi} \in LC(n) \ : \ s(\hat{\pi}) = m + 1\},$$

Clearly $|A| = f(n, m) - f(n - 1, m - 1)$. We will construct an m-to-one correspondence between the sets A and B.

Let $\hat{\pi} \in LC(n)$ with cycle representation $\mathcal{G}_{\hat{\pi}}^c$. First we describe an operation τ, which decomposes a non-singleton connected component of $\mathcal{G}_{\hat{\pi}}^c$ into two. Let G_1 be a connected component with at least two vertices. Assume i_1 is the minimal vertex of G_1, which lies in the root cycle $C_1 = (i_1 i_2 \ldots i_t)$. Then i_t must be at least 2. The operation τ removes the arcs (i_1, i_2) and (i_2, i_3), and adds an arc (i_1, i_3).

The inverse operation ρ of τ merges two connected components G_1 and G_2 of a linked cycle as follows. Assume the minimal elements of G_1 and G_2 are i_1 and j_1 respectively and $i_1 < j_1$. Then i_1 and j_1 must be singly covered. Assume the root cycle of G_1 is $C_1 = (i_1 i_2 \ldots i_k)$. The operation ρ inserts j_1 into C_1 to get $(i_1 j_1 i_2 \ldots i_k)$, and keeps all other cycles unchanged.

Now we can define the m-to-1 correspondence between the sets A and B. For any $\hat{\pi} \in A$ the cycle representation $\mathcal{G}_{\hat{\pi}}^c$ has m connected components, where n is not an isolated point. Applying the operation τ to the component containing the vertex n, we get a linked cycle with $m+1$ connected components. (See Figure 8 for an illustration.)

Conversely, given any $\hat{\pi} \in B$, there are $m + 1$ connected components in $\mathcal{G}_{\hat{\pi}}^c$. We may merge the component containing n with any other component to get a linked cycle in A. There are m choices for the other components, hence we get an m-to-one correspondence.
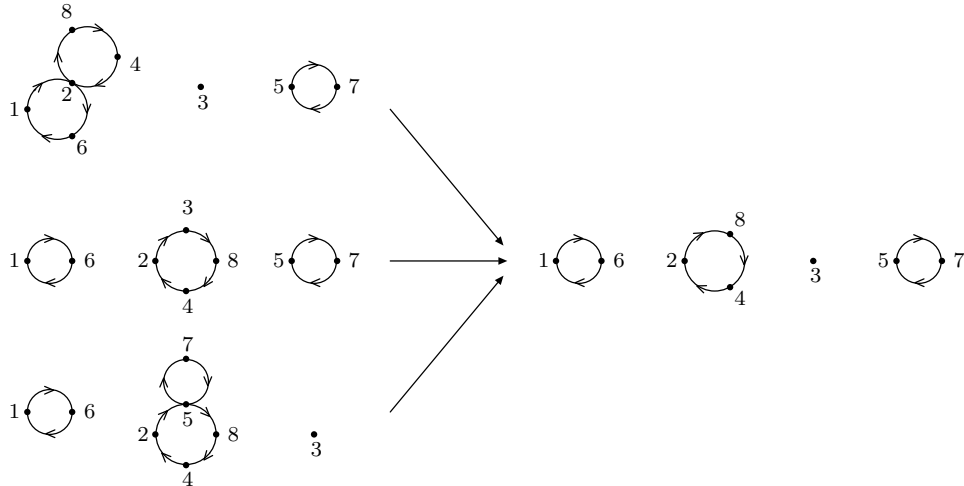

Figure 8: The operation τ on three linked cycles with 3 components all gives the same linked cycle with 4 components.

Theorem 4.5. *The numbers lc_n $(n > 1)$ satisfy the recursive relation*

$$lc_n = (2n - 3)lc_{n-1} + lc_{n-2}$$

with the initial values $lc_1 = 1$ and $lc_2 = 2$.

Proof.

$$
\begin{aligned}
lc_n &= \sum_{m=1}^{n} f(n, m) \\
&= \sum_{m=1}^{n} [(2n - 2 - m)f(n - 1, m) + f(n - 1, m - 1)] \qquad (12) \\
&= \sum_{m=1}^{n} [(2n - 2 - m)f(n - 1, m) + (m - 1)f(n - 1, m) + f(n - 2, m - 2)] \qquad (13) \\
&= (2n - 3)\sum_{m=1}^{n} f(n - 1, m) + \sum_{m=1}^{n} f(n - 2, m - 2) \\
&= (2n - 3)lc_{n-1} + lc_{n-2},
\end{aligned}
$$

where the equation (12) is obtained by applying Prop. 4.3 to $f(n, m)$, and the equation (13) is applying Prop. 4.4 to $f(n - 1, m - 1)$. ∎

In counting various kinds of set-partitions, Proctor found a combinatorial interpretation for the sequence $2, 7, 37, 266, 2431, \ldots$ as the number of partitions of $[k]$ $(0 \leq k \leq 2n)$ into exactly n blocks each having no more than 2 elements, See [6, §7]. For example, for $n = 2$, there are 7 such partitions. They are $p_1 = \{\{1\}, \{2\}\}$, $p_2 = \{\{1, 2\}, \{3\}\}$, $p_3 = \{\{1, 3\}, \{2\}\}$, $p_4 = \{\{1\}, \{2, 3\}\}$, $p_5 = \{\{1, 2\}, \{3, 4\}\}$, $p_6 = \{\{1, 3\}, \{2, 4\}\}$, $p_7 = \{\{1, 4\}, \{2, 3\}\}$. In literature, a partition is called *poor* if each block contains at most two elements [3]. Denote by $P_2(k, n)$ the number of poor partitions of $[k]$ into exactly n blocks. Let $P_2(n)$ be the union $\bigcup_{k=0}^{2n} P_2(k, n)$.

Our linked cycles provide another combinatorial interpretation for the sequence $2, 7, 37, 266, 2431, \ldots$. Using the marked linear representation, we construct a bijection between the linked cycles on $[n+1]$ and the set $P_2(n)$.

A bijection γ between $LC(n + 1)$ and $P_2(n)$.

Given a linked cycle $\hat{\pi} \in LC(n+1)$ with the marked linear representation, removing the vertex labels and all isolated points, then replacing each mark with a vertex, and relabeling the vertices by $1, 2, \ldots, k$ from left to right, we get a graph of a partition of $[k]$ for some $0 \leq k \leq 2n$, where each block has 1 or 2 elements. By Prop. 4.2, there are exactly n blocks in this partition.

Conversely, given a partition of $[k]$ $(0 \leq k \leq 2n)$ in $P_2(n)$ we represent it by a graph whose vertices are listed in a horizontal line, and there is an arc connecting i and j if and only if $\{i, j\}$ is a block. We can define a linked cycle on $[n + 1]$ by the following steps:

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1. Remove the labels of the vertices.

2. Change each singleton block to a mark.

3. If a mark is followed by a righthand endpoint or another mark, add a vertex right after it, and

4. If there is a mark before the first vertex, add a vertex at the very beginning.

The resulting graph is the marked linear representation of a linked cycle of $[n+1]$. By the remark at the end of §4.2, one can recover the labeling of the vertices, and hence the linked cycle.

Example 9. The linked cycle $\pi = (126)(284)(3)(57)$ corresponds to the partition $\{\{1, 10\}, \{2, 3\}, \{4, 7\}, \{5, 12\}, \{6\}, \{8\}, \{9, 11\}\}$, where $n = 7$ and $7 \leq k \leq 12$.
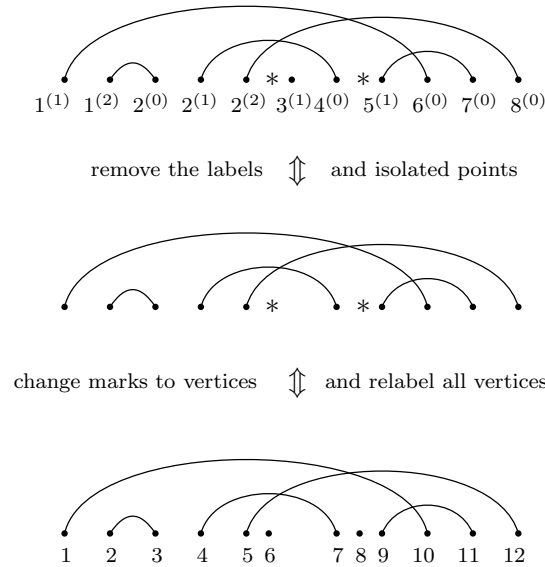

Figure 9: The procedure from a linked cycle to a partition.

We conclude this subsection with a combinatorial proof of Formula (10). Recall that $f(n, m)$ is the number of linked cycles on $[n]$ with m singly covered minimal elements. Under the above bijection γ, it is the number of partitions in $P_2(n-1)$ with $m-1$ isolated points, and hence $n-m$ blocks of size 2. For such a partition, the total number of points is $k = 2(n-m)+m-1 = 2n-m-1$. To obtain such a partition, we can first choose $m-1$ elements from $[k] = [2n - m - 1]$ as isolated points, and then construct a complete matching on the remaining $2(n-m)$ elements. The number of ways to do this is then

$$\binom{2n-m-1}{m-1} \frac{2(n-m)!}{(n-m)! 2^{n-m}}.$$

4.3 Crossings and nestings of linked cycles

In this subsection we present results on the enumeration of crossings and nestings, as well as 2-crossings and 2-nestings, for linked cycles. This was our original motivation to introduce the notion of linked cycles.

Given a linked cycle $\hat{\pi} \in LC(n)$ with linear representation $\mathcal{G}_{\hat{\pi}}^l$. Denote by $\mathcal{L} = \mathcal{L}(\pi)$ the vertex labeling of $\mathcal{G}_{\hat{\pi}}^l$. Two different linked cycles $\hat{\pi}$ and $\hat{\pi}'$ on $[n]$ may have the same vertex labeling. If this happens, then $\hat{\pi}$ and $\hat{\pi}'$ share the following properties:

1. $\hat{\pi}$ and $\hat{\pi}'$ have the same number of cycles.

2. $\hat{\pi}$ and $\hat{\pi}'$ have the same set of singly covered minimal elements;

3. $\hat{\pi}$ and $\hat{\pi}'$ have the same set of doubly covered elements;

4. Each cycle C_i of $\hat{\pi}$ can be paired with a unique cycle C_i' of $\hat{\pi}'$ such that $|C_i| = |C_i'|$, and C_i and C_i' have the same minimal element.

Fix a vertex labeling $\mathcal{L}$, denote by $LC_n(\mathcal{L})$ the set of all linked cycles $\hat{\pi}$ with $\mathcal{L}(\hat{\pi}) = \mathcal{L}$. In $\mathcal{L}$, if a vertex has a label $i^{(0)}$, we say it is a lefthand endpoint; if it has a label $i^{(k)}$ with $k \geq 1$, we say it is a righthand endpoint; if it has a label i, we say it is an isolated point. A $\hat{\pi} \in LC_n(\mathcal{L})$ corresponds to a matching between the set of lefthand endpoints to the set of righthand endpoints.

Let $k \geq 2$ be an integer. A k-crossing of $\hat{\pi}$ is a set of k arcs $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ of $\mathcal{G}_{\hat{\pi}}^l$ such that the vertices appear in the order $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ from left to right. A k-nesting is a set of k arcs $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ such that the vertices appear in the order $x_1, x_2, \ldots, x_k, y_k, \ldots, y_2, y_1$. Denoted by $cr_k(\hat{\pi})$ the number of k-crossings of $\hat{\pi}$, and $ne_k(\hat{\pi})$ the number of k-nestings of $\hat{\pi}$. Finally, let $cr(\hat{\pi})$ be the maximal i such that $\hat{\pi}$ has a i-crossing and $ne(\hat{\pi})$ the maximal j such that $\hat{\pi}$ has a j-nesting.

Our first result is an analog of Theorem 3.5, on the joint generating function of cr_2 and ne_2. For any lefthand endpoint $i^{(0)}$, let

$$h(i) = |\{\text{righthand endpoints on the right of } i^{(0)}\}| - |\{\text{lefthand endpoints on the right of } i^{(0)}\}|.$$

Then

Theorem 4.6.

$$\sum_{\hat{\pi} \in LC_n(\mathcal{L})} x^{cr_2(\hat{\pi})} y^{ne_2(\hat{\pi})} = \sum_{\hat{\pi} \in LC_n(\mathcal{L})} x^{ne_2(\hat{\pi})} y^{cr_2(\hat{\pi})}$$

$$= \prod_{i^{(0)} \in \mathcal{L}} (x^{h(i)-1} + x^{h(i)-2}y + \cdots + x^{h(i)-k}y^{k-1} + \cdots + xy^{h(i)-2} + y^{h(i)}).$$

In particular, the statistics cr_2 and ne_2 have a symmetric joint distribution over each set $LC_n(\mathcal{L})$.

The proof is basically the same as that of Theorem 3.5. It is even simpler since every vertex is the endpoint of at most one arc, that is, all $r_i = 1$ in the proof of Theorem 3.5.

Perhaps more interesting is the joint distribution of $cr(\hat{\pi})$ and $ne(\hat{\pi})$ over $LC_n(\mathcal{L})$. Recall the following result in [1]: Given a partition P of $[n]$, let

$$\min(P) = \{\text{minimal block elements of } P\},$$
$$\max(P) = \{\text{maximal block elements of } P\}.$$

Fix $S, T \subset [n]$ with $|S| = |T|$. Let $P_n(S,T)$ be the set $\{P \in \Pi_n : \min(P) = S, \max(P) = T, \}$. Then

Theorem 4.7 (CDDSY).

$$\sum_{P \in P_n(S,T)} x^{cr(P)} y^{ne(P)} = \sum_{P \in P_n(S,T)} x^{ne(P)} y^{cr(P)}.$$

That is, the statistics $cr(P)$ and $ne(P)$ have a symmetric joint distribution over each set $P_n(S,T)$.

Although the *standard representation* for a partition of $[n]$ given in [1] is different than the linear representation defined in the present paper, they coincide on partial matchings. View the graph $\mathcal{G}_{\hat{\pi}}^l$ as the graph of a partial matching P. Observe that by fixing the vertex labeling, we actually fix the number of vertices in $\mathcal{G}_{\hat{\pi}}^l$, the set of minimal block elements of P (which is the set of lefthand endpoints and isolated points), and the set of maximal block element of P (which is the set of righthand endpoints and isolated points). Taking all linked cycles with the vertex labeling $\mathcal{L}$ is equivalent to taking all possible partial matchings with the given sets of minimal block elements and maximal block elements. Hence Theorem 4.7 applies to the set of linked cycles, and we obtain the following theorem.

Theorem 4.8.
$$\sum_{\hat{\pi} \in LC_n(\mathcal{L})} x^{cr(\hat{\pi})} y^{ne(\hat{\pi})} = \sum_{\hat{\pi} \in LC_n(\mathcal{L})} x^{ne(\hat{\pi})} y^{cr(\hat{\pi})}.$$
That is, the statistics $cr(\hat{\pi})$ and $ne(\hat{\pi})$ have a symmetric joint distribution over each set $LC_n(\mathcal{L})$.

We say that a linked cycle $\hat{\pi}$ is *k-noncrossing* if $cr(\hat{\pi}) < k$. It is *k-nonnesting* if $ne(\hat{\pi}) < k$.

Corollary 4.9. *The number of linked cycles in $LC_n(\mathcal{L})$ that are k-noncrossing and l-nonnesting is equal to the number of linked cycles in in $LC_n(\mathcal{L})$ that are l-noncrossing and k-nonnesting.*

Corollary 4.10. *The number of k-noncrossing linked cycles on $[n]$ is equal to the number of k-nonnesting linked cycles on $[n]$.*

In particular, when $k = 2$, a 2-noncrossing linked cycle is just a noncrossing linked partition. The set of 2-noncrossing linked cycles is enumerated by the larger Schröder numbers.

Acknowledgments

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