On Postnikov’s Hook Length Formula for Binary Trees

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Abstract. We present a combinatorial proof of Postnikov’s hook length formula for binary trees.

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Let \([n]\) = \{1, 2, \ldots, n\}. It is well known that the number of labeled trees on \([n]\) equals \(n^{n-2}\), and the number of rooted trees on \([n]\) equals \(n^{n-1}\) [5, 8]. Recently, Postnikov [6] derived an identity on binary trees and asked for a combinatorial proof [6]. We adopt the terminology of Postnikov [6]. Given a binary tree \(T\) and a vertex \(v\) of \(T\), we use \(h(v)\) to denote the hook-length of \(v\), namely, the number of descendants of \(v\) (including \(v\) itself). Postnikov’s hook length formula for binary trees reads as follows [6].

Theorem 1. For \(n \geq 1\), we have

\[
(n + 1)^{n-1} = \sum_T \frac{n!}{2^m} \prod_{v \in T} \left(1 + \frac{1}{h(v)}\right),
\]

where the sum ranges over all binary trees \(T\) with \(n\) vertices.
Our combinatorial proof is based on the following reformulation of (1) in terms of rooted trees:

\[(n + 1)^n = \sum_T \frac{(n + 1)!}{2^n} \prod_{v \in T} \left(1 + \frac{1}{h(v)}\right). \tag{2}\]

**Proof of (2).** Let \(F_n\) denote the sum on the right hand side of (2). For any unlabeled binary tree \(T\) with \(n\) vertices, the hook length of the root is always \(n\). Let us consider a binary tree \(T\) such that the left subtree \(T_1\) has \(k\) vertices and the right subtree \(T_2\) has \(n - k - 1\) vertices. From the relation

\[\frac{(n + 1)!}{2^n} \left(1 + \frac{1}{n}\right) = \frac{n + 1}{2n} \left(\begin{array}{c} n + 1 \\ k + 1 \end{array}\right) \frac{(k + 1)!}{2^k} \frac{(n - k)!}{2^{n-k-1}},\]

it can be deduced that

\[F_n = \frac{n + 1}{2n} \sum_{k=0}^{n-1} \left(\begin{array}{c} n + 1 \\ k + 1 \end{array}\right) \sum_{T_1} \frac{(k + 1)!}{2^k} \prod_{v \in T_1} \left(1 + \frac{1}{h(v)}\right) \sum_{T_2} \frac{(n - k)!}{2^{n-k-1}} \prod_{v \in T_2} \left(1 + \frac{1}{h(v)}\right),\]

where \(T_1\) and \(T_2\) range over all binary trees on \(k\) and \(n - k - 1\) vertices, respectively. Hence \(F_n\) satisfies the following recurrence relation:

\[F_n = \frac{n + 1}{2n} \sum_{k=0}^{n-1} \left(\begin{array}{c} n + 1 \\ k + 1 \end{array}\right) F_k F_{n-k-1}. \tag{3}\]

It is known that the number \(T_n = n^{n-2}\) of labeled trees with \(n\) vertices satisfies the recurrence relation:

\[2nT_{n+1} = \sum_{k=0}^{n-1} \left(\begin{array}{c} n + 1 \\ k + 1 \end{array}\right) (k + 1)T_{k+1}(n - k)T_{n-k}. \tag{4}\]

Let \(R_n = nT_n\) denote the number of rooted tree on \(n\) vertices. Then the above recurrence (4) can be recast as

\[R_{n+1} = \frac{n + 1}{2n} \sum_{k=0}^{n-1} \left(\begin{array}{c} n + 1 \\ k + 1 \end{array}\right) R_{k+1} R_{n-k}. \tag{5}\]

A combinatorial interpretation of (4) is given by Moon [5]: The left hand side of (4) equals the number of labeled trees on \([n + 1]\) with a distinguished edge and a direction on this distinguished edge. Let \(T\) be such a tree, we may decompose it into an ordered pair of rooted trees by cutting off the distinguished edge.

Combining the recurrence (3) of \(F_n\) with the recurrence (5) of \(R_n\), we arrive at the conclusion that \(F_n = R_{n+1} = (n + 1)^n\). Thus we obtain (2).
We note that Seo [7] also found a combinatorial proof of the identity (1). Further studies related to Postnikov’s hook length formula (1) have been carried out by Du and Liu [1], Gessel and Seo [2], Liu [4], and Hivert, Novelli and Thibon [3].

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References


