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The Limiting Distribution of the Coefficients of the q-Catalan Numbers

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Abstract. We show that the limiting distributions of the coefficients of the q-Catalan numbers and the generalized q-Catalan numbers are normal. Despite the fact that these coefficients are not unimodal for small n, we conjecture that for sufficiently large n, the coefficients are unimodal and even log-concave except for a few terms of the head and tail.

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1 Introduction

The main objective of this paper is to show that the limiting distribution of the coefficients of the q-Catalan numbers is normal. The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

have many combinatorial interpretations, see Stanley [10]. The usual q-analog of the Catalan numbers is given by

$$C_n(q) = \frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}, \tag{1.1}$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$, and

$$\begin{bmatrix}n\\ k\end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

There are also other types of q-analogs of the Catalan numbers, see, for example, Andrews [2], Gessel and Stanton [4], Krattenthaler [5].

We further consider the limiting distribution of the coefficients of the quotient of two products, which includes the result for the q-Catalan numbers as a special case. We conclude this paper with two conjectures on the unimodality and log-concavity for almost all the coefficients of the q-Catalan numbers and the generalized q-Catalan numbers provided that n is sufficiently large.

2 The Limiting Distribution

In this section, we use the moment generating function technique to obtain the limiting distribution of the coefficients of the q-Catalan numbers. We introduce the random variable ξ_n corresponding to the probability generating function

$$\phi_n(q) = C_n(q)/C_n.$$

As far as the computations are concerned, we will not need the following combinatorial interpretation of $C_n(q)$. However, for the sake of completeness, we recall that ξ_n reflects the distribution of the major indices of Catalan words of length $2n$, see, for example, [3]. Moreover, we write

$$C_n(q) = \sum m_n(k)q^k,$$

where $m_n(k)$ stands for the number of Catalan words of length $2n$ with major index k. The following lemma gives the expectation and variance of ξ_n.

Lemma 2.1. *We have*

$$E(\xi_n) = \frac{n(n-1)}{2} \quad \textit{and} \quad \mathrm{Var}(\xi_n) = \frac{n(n-1)(n+1)}{6}. \tag{2.1}$$

Proof. By the definition of $C_n(q)$, it is easy to check the following symmetry property of $m_n(k)$:

$$m_n(k) = m_n(n(n-1) - k).$$

Hence

$$E(\xi_n) = \frac{n(n-1)}{2}.$$

Let

$$F = F(q) = \prod_{i=1}^{n-1}(1 + q + \cdots + q^{n+i}) \quad \text{and} \quad G = G(q) = \prod_{i=1}^{n-1}(1 + q + \cdots + q^{i}).$$

It is easily verified that $C_n(q) = F/G$. Since

$$\begin{aligned} C_n(q)''\big|_{q=1} &= \left(\frac{F''}{G} - \frac{FG''}{G^2} - \frac{2G'F'}{G^2} + \frac{2G'^2F}{G^3}\right)\bigg|_{q=1} \\ &= \frac{1}{12}n(n-1)(3n^2 - n - 4)C_n, \end{aligned}$$

we obtain

$$\mathrm{Var}(\xi_n) = \frac{C_n(q)''\big|_{q=1}}{C_n} + E(\xi_n) - E(\xi_n)^2 = \frac{1}{6}n(n-1)(n+1).$$

This completes the proof. ∎

Lemma 2.2. *When $n \to \infty$, we have*

$$\sum_{k=2}^{\infty} B_{2k} \frac{t^{2k}}{2k(2k)!\sigma^{2k}} \sum_{i=2}^{n} \left((n+i)^{2k} - i^{2k}\right) \to 0$$

uniformly for t from any bounded set, where B_j's are the Bernoulli numbers and σ^2 is the variance of ξ_n as given in (2.1).

Proof. The second summation can be expanded as follows:

$$\sum_{i=2}^{n}\left((n+i)^{2k}-i^{2k}\right)=\sum_{i=2}^{n}\sum_{j=1}^{2k}\binom{2k}{j}n^j i^{2k-j}=\sum_{j=1}^{2k}\binom{2k}{j}\left(\sum_{i=2}^{n}n^j i^{2k-j}\right).$$

For $k>1$, the second factor in the preceding summation is bounded by the following integral:

$$\sum_{i=2}^{n}n^j i^{2k-j}<n^j\int_1^{n+1}t^{2k-j}dt=n^j\cdot\frac{(n+1)^{2k-j+1}-1}{2k-j+1}.$$

Consequently,

$$\sum_{i=2}^{n}\left((n+i)^{2k}-i^{2k}\right)<2^{2k}(n+1)^{2k+1}<8^{2k}n^{2k+1}.$$

Since $\sigma^2=\frac{n^3-n}{6}>\frac{n^3}{8}$ when n is sufficiently large, we have

$$\sigma^{-2k}\sum_{i=2}^{n}\left((n+i)^{2k}-i^{2k}\right)<64^{2k}n^{1-k}\le n^{-1/3}64^{2k}n^{-k/3},$$

for large n and $k>1$. Thus

$$\begin{aligned}\left|\sum_{2\nmid k,k\ge 3}B_{2k}\frac{t^{2k}}{2k(2k)!\sigma^{2k}}\sum_{i=2}^{n}\left((n+i)^{2k}-i^{2k}\right)\right|\\ <\quad n^{-1/3}\sum_{2\nmid k,k\ge 3}|B_{2k}|\frac{t^{2k}}{2k(2k)!}64^{2k}n^{-k/3}\\ =\quad n^{-1/3}\sum_{2\nmid k,k\ge 3}|B_{2k}|\frac{(64tn^{-\frac{1}{6}})^{2k}}{2k(2k)!}.\end{aligned}$$

In view of the following asymptotic expansion of the Bernoulli numbers [1],

$$|B_{2n}|\sim\frac{2(2n)!}{(2\pi)^{2n}},$$

the convergent radius R of the series $\sum_{2\nmid k,k\ge 3}|B_{2k}|\frac{t^{2k}}{2k(2k)!}$ equals 2π. Since t is from a bounded set, when n is large enough, the series

$$\sum_{2\nmid k,k\ge 3}|B_{2k}|\frac{(64tn^{-\frac{1}{6}})^{2k}}{2k(2k)!}$$

converges. Moreover, it is evident that $64tn^{-\frac{1}{6}} < 1$, we can bound the above summation by the constant

$$M_1 = \sum_{2\nmid k, k\geq 3} |B_{2k}| \frac{1}{2k(2k)!}.$$

Similarly, it can be deduced that

$$\sum_{2|k, k\geq 2} B_{2k} \frac{t^{2k}}{2k(2k)!\sigma^{2k}} \sum_{i=2}^{n} \left((n+i)^{2k} - i^{2k}\right) < \frac{M_2}{n^{\frac{1}{3}}},$$

where $M_2 = \sum\limits_{2|k, k\geq 2} B_{2k} \frac{1}{2k(2k)!}$ is a constant. Hence

$$\sum_{k=2}^{\infty} B_{2k} \frac{t^{2k}}{2k(2k)!\sigma^{2k}} \sum_{i=2}^{n} \left((n+i)^{2k} - i^{2k}\right) < \frac{M_1 + M_2}{n^{1/3}},$$

which tends to zero as $n \to \infty$. This completes the proof. ∎

In [7], Margolius applied Bernoulli numbers to show that the distribution of the number of inversions in a random permutation is asymptotically normal. In [6], Louchard and Prodinger used the saddle point method to derive some stronger results. Based on Lemma 2.2, we obtain the following theorem.

Theorem 2.3. *When $n \to \infty$, the random variable*

$$\eta_n = \frac{\xi_n - E(\xi_n)}{\mathrm{Var}(\xi_n)^{\frac{1}{2}}}$$

has the standard normal distribution.

Proof. Let $M_n(q)$ denote the moment generating function of ξ_n. Then we have $M_n(q) = \phi_n(e^q)$, see Sachkov [8]. Hence

$$\begin{aligned} M_n(q) &= \frac{n+1}{\binom{2n}{n}} \frac{1-e^q}{1-e^{(n+1)q}} \cdot \prod_{i=1}^{n} \frac{1-e^{(n+i)q}}{1-e^{iq}} \\ &= \prod_{i=2}^{n} \frac{i}{n+i} \cdot \prod_{i=2}^{n} \frac{1-e^{(n+i)q}}{1-e^{iq}} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=2}^{n} \frac{(1-e^{(n+i)q})/(n+i)}{(1-e^{iq})/i} \\
&= \exp\left\{\frac{1}{2}\sum_{i=2}^{n}((n+i)q-iq)\right\}\prod_{i=2}^{n}\frac{(e^{(n+i)q/2}-e^{-(n+i)q/2})/\frac{n+i}{2}}{(e^{iq/2}-e^{-iq/2})/\frac{i}{2}} \\
&= \exp\left\{\frac{n(n-1)q}{2}\right\}\prod_{i=2}^{n}\frac{\sinh\left((n+i)q/2\right)/\frac{n+i}{2}}{\sinh\left(iq/2\right)/\frac{i}{2}}.
\end{aligned}
$$

Recalling the following relation on the Bernoulli numbers [7]

$$
\ln\left(\frac{\sinh(x/2)}{x/2}\right) = \sum_{k=1}^{\infty} B_{2k}\frac{x^{2k}}{2k(2k)!}, \tag{2.2}
$$

we find that

$$
\begin{aligned}
\ln M_n(q) &= \frac{n(n-1)}{2}q + \sum_{i=2}^{n}\left(\ln\left(\frac{\sinh((n+i)q/2)}{(n+i)/2}\right) - \ln\left(\frac{\sinh(iq/2)}{i/2}\right)\right) \\
&= \frac{n(n-1)}{2}q + \sum_{k=1}^{\infty} B_{2k}\frac{q^{2k}}{2k(2k)!}\sum_{i=2}^{n}\left((n+i)^{2k}-i^{2k}\right).
\end{aligned}
$$

Setting $q = t/\sigma$, where σ is the standard deviation of ξ_n as given in Theorem 2.1, we are led to the expansion

$$
\ln M_n(t/\sigma) = \frac{n(n-1)t}{2\sigma} + \sum_{k=1}^{\infty} B_{2k}\frac{t^{2k}}{2k(2k)!\sigma^{2k}}\sum_{i=2}^{n}\left((n+i)^{2k}-i^{2k}\right).
$$

Applying Lemma 2.2, we have, when $n\to\infty$,

$$
\sum_{k=2}^{\infty} B_{2k}\frac{t^{2k}}{2k(2k)!\sigma^{2k}}\sum_{i=2}^{n}((n+i)^{2k}-i^{2k}) \to 0
$$

uniformly for t from any bounded set. Finally,

$$
\lim_{n\to\infty} M_n(t/\sigma)\exp\left\{-\frac{n(n-1)t}{2\sigma}\right\}
$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \exp\left\{\sum_{k=1}^{\infty} B_{2k}\frac{t^{2k}}{2k(2k)!\sigma^{2k}}\sum_{i=2}^{n}\left((n+i)^{2k}-i^{2k}\right)\right\}\\
&= \lim_{n\to\infty} \exp\left\{B_2\frac{t^2}{2(2)!\sigma^2}\sum_{i=2}^{n}\left((n+i)^2-i^2\right)\right\}\\
&= e^{t^2/2},
\end{aligned}
$$

which coincides with the moment generating function of the standard normal distribution. Employing Curtiss's theorem [8], we reach the conclusion that η_n has the standard normal distribution when n approaches infinity. ∎

3 A General Setting

In this section, we will determine the limiting distribution of the coefficients of a quotient of products and will give two special cases.

Theorem 3.1. *Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be two sequences of positive numbers, and let*

$$\phi_n(x) = \sum_k p_n(k)x^k = \frac{(1-q^{a_1})(1-q^{a_2})\cdots(1-q^{a_n})}{(1-q^{b_1})(1-q^{b_2})\cdots(1-q^{b_n})}.$$

Suppose that ξ_n is the random variable corresponding to the generating function $\phi_n(x)$, that is,

$$P(\xi_n = k) = \frac{p_n(k)}{\sum\limits_k p_n(k)}.$$

Then ξ_n is normally distributed as $n \to \infty$, if and only if

$$\sum_{k=1}^{\infty} B_{2k}\frac{t^{2k}}{2k(2k)!}\left(\sum_{i=1}^{n}(a_i^{2k}-b_i^{2k})\right)\frac{1}{\left(\sum\limits_{i=1}^{n}(a_i^2-b_i^2)\right)^k} \to 0 \quad \textit{as} \quad n\to\infty.$$

Proof. The expectation of ξ_n is easy to compute, as given below:

$$E(\xi_n) = \phi_n(x)'_{q=1} = \frac{1}{2}\sum_{i=1}^{n}(a_i - b_i).$$

Proceeding analogously as in the proof of Theorem 2.1, we find

$$\sigma^2 = \mathrm{Var}(\xi_n) = \frac{1}{12}\sum_{i=1}^{n}\left(a_i^2 - b_i^2\right). \tag{3.1}$$

Hence,

$$B_2 \frac{t^2}{2(2)!\sigma^2}\left(\sum_{i=1}^{n}(a_i^2 - b_i^2)\right) = \frac{1}{6}\cdot\frac{t^2}{4\cdot\frac{1}{12}\left(\sum\limits_{i=1}^{n}(a_i^2-b_i^2)\right)}\cdot\left(\sum_{i=1}^{n}(a_i^2 - b_i^2)\right) = \frac{t^2}{2}.$$

By the same procedure as in the proof of Theorem 2.3, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} M_n(t/\sigma)\exp\left\{\frac{1}{2}\sum_{i=1}^{n}\left(a_i^{2k} - b_i^{2k}\right)\right\} \\
&= e^{t^2/2}\lim_{n\to\infty}\exp\left\{\sum_{k=2}^{\infty} B_{2k}\frac{t^{2k}}{2k(2k)!\sigma^{2k}}\left(\sum_{i=1}^{n}\left(a_i^{2k} - b_i^{2k}\right)\right)\right\}.
\end{aligned}$$

It follows that the limiting distribution of $p_n(k)$ is normal if and only if

$$\sum_{k=2}^{\infty} B_{2k}\frac{t^{2k}}{2k(2k)!\sigma^{2k}}\left(\sum_{i=1}^{n}\left(a_i^{2k} - b_i^{2k}\right)\right) \to 0 \quad \text{as} \quad n\to\infty, \tag{3.2}$$

for t from any bounded set. By virtue of the variance formula (3.1), the condition (3.2) is equivalent to

$$\sum_{k=1}^{\infty} B_{2k}\frac{t^{2k}}{2k(2k)!}\frac{\sum\limits_{i=1}^{n}\left(a_i^{2k} - b_i^{2k}\right)}{\left(\sum\limits_{i=1}^{n}\left(a_i^2 - b_i^2\right)\right)^k} \to 0 \quad \text{as} \quad n\to\infty \tag{3.3}$$

for t from any bounded set. Thus (3.2) is verified. This completes the proof. ∎

Corollary 3.2. *Let $p_n(k)$ be given as in the above theorem. Suppose that for $k \geq 2$, there exist constants $\alpha > 0$, $\beta < 0$ and $\gamma < 0$ such that*

$$\frac{\sum\limits_{i=1}^{n}\left(a_i^{2k} - b_i^{2k}\right)}{\left(\sum\limits_{i=1}^{n}(a_i^2 - b_i^2)\right)^k} < n^{\gamma}(\alpha n^{\beta})^{2k}, \tag{3.4}$$

for t from any bounded set. Then the limiting distribution of $p_n(k)$ is normal.

Proof. Note that the convergent radius R of the series

$$\sum_{2\nmid k, k\geq 3} |B_{2k}| \frac{x^{2k}}{2k(2k)!}$$

is 2π. If (3.4) holds for $k > 1$, then for t from any bounded set, and for sufficiently large n, we have

$$\left| t^{2k} \sum_{i=1}^{n} \left(a_i^{2k} - b_i^{2k}\right) / \sigma^{2k} \right| \leq n^{\gamma} (t\alpha n^{\beta})^{2k},$$

where $t\alpha n^{\beta} < 2\pi$. It is clear that $n^{\gamma} \to 0$ since $\gamma < 0$. ∎

If we choose $\alpha = 32\sqrt{3}/3$, $2\beta = \gamma = -\frac{1}{3}$, Theorem 3.2 contains Theorem 2.3 as a special case. We now give two more examples. One is the following q-analog of the Catalan numbers

$$c_n(q) = \frac{[2]}{[2n]} \begin{bmatrix} 2n \\ n-1 \end{bmatrix},$$

which are symmetric and unimodal, see Stanley [9].

Using Theorem 3.1, we reach the following assertion.

Corollary 3.3. *The distribution of the coefficients in $c_n(q)$ is asymptotically normal.*

Proof. First, we write $c_n(q)$ in the following form:

$$\frac{\prod_{i=3}^{n} (1 - q^{n+i-1})}{(1-q) \prod_{i=3}^{n-1} (1 - q^i)},$$

Set $a_1 = a_2 = 1,\ a_i = n + i - 1,\ 3 \leq i \leq n$, and $b_1 = b_2 = 1,\ b_3 = 1,\ b_i = i - 1,\ 4 \leq i \leq n$. Then we have

$$\begin{aligned} \sum_{i=1}^{n} \left(a_i^{2k} - b_i^{2k}\right) &= \left(a_3^{2k} - b_3^{2k}\right) + \sum_{i=4}^{n} \left(a_i^{2k} - b_i^{2k}\right) \\ &= (n+2)^{2k} - 1 + \sum_{i=3}^{n-1} \left((n+i)^{2k} - i^{2k}\right) \end{aligned}$$

and

$$\left(\sum_{i=1}^{n}(a_i^2-b_i^2)\right)^k = \left((n+2)^2-1+\sum_{i=3}^{n-1}\left((n+i)^2-i^2\right)\right)^k$$
$$= (n-1)^k(n+1)^k(2n-3)^k.$$

By the same arguments as in the proof of Lemma 2.2, we may set $\alpha = 32\sqrt{3}/3$ and $2\beta = \gamma = -\frac{1}{3}$ such that the condition (3.4) is satisfied. Therefore, Theorem 3.1 implies the limiting distribution of the coefficients of $c_n(q)$. ∎

The *m-Catalan numbers* are defined by

$$C_{n,m} = \frac{1}{(m-1)n+1}\binom{mn}{n},$$

for $n \geq 1$. Accordingly, the generalized q-Catalan numbers are given by

$$C_{n,m}(q) = \frac{1}{[(m-1)n+1]}\begin{bmatrix} mn \\ n \end{bmatrix}.$$

Theorem 3.1 has the following consequence.

Corollary 3.4. *The coefficients of the generalized q-Catalan numbers $C_{n,m}(q)$ are normally distributed when $n \to \infty$.*

Proof. First, express $C_{n,m}(q)$ as follows

$$\prod_{i=2}^{n}\frac{1-q^{(m-1)n+i}}{1-q^i}.$$

Set $a_1 = 1,\ a_i = (m-1)n+i,\ 2 \leq i \leq n$, and $b_1 = 1,\ \ b_i = i,\ 2 \leq i \leq n$. Then we have

$$\sum_{i=1}^{n}\left(a_i^{2k}-b_i^{2k}\right) = \sum_{i=2}^{n}\left(a_i^{2k}-b_i^{2k}\right) = \sum_{i=2}^{n}\sum_{j=1}^{2k}\binom{2k}{j}\left((m-1)n\right)^{2k-j} i^j.$$

The same argument as in the proof of Lemma 2.2 yields the following bound

$$\sum_{i=1}^{n}\left(a_i^{2k}-b_i^{2k}\right) < 8^{2k}\left((m-1)\,n\right)^{2k+1}.$$

Now,

$$\begin{aligned}\left(\sum_{i=1}^{n}(a_i^2-b_i^2)\right)^k &= \left(\sum_{i=2}^{n}\left(((m-1)n+i)^2-i^2\right)\right)^k\\ &> (m-1)^{2k}n^{2k}(n-1)^k\\ &> (m-1)^{2k+1}n^{3k}/(2m)^k.\end{aligned}$$

It follows that

$$\frac{\sum_{i=1}^{n}\left(a_i^{2k}-b_i^{2k}\right)}{\left(\sum_{i=1}^{n}(a_i^2-b_i^2)\right)^k} < (8\sqrt{2m})^{2k}n^{1-k}.$$

Again, by the same arguments as in the proof of Lemma 2.2, we may set $\alpha = 8\sqrt{2m}$ and $2\beta = \gamma = -\frac{1}{3}$ such that the condition (3.4) holds. Finally, we may use Theorem 3.1 to get the desired distribution. ∎

4 Open Problems

While the q-Catalan numbers are not unimodal for small n, see Stanley [9], the limiting distribution suggests that the coefficients are almost unimodal in certain sense for sufficiently large n. Obviously, the first and the last term should not be taken into account; otherwise one can never expect to have unimodality. In fact, an easy computation indicates that $C_n(q)$ are unimodal for $n \geq 16$.

Conjecture 4.1. *The sequence $\{m_n(1), \ldots, m_n(n(n-1)-1)\}$ is unimodal when n is sufficiently large.*

When $n > 70$, numerical evidence is suggestive of a stronger conjecture:

Conjecture 4.2. *There exists an integer t such that when n is sufficiently large, the sequence $\{m_n(t), \ldots, m_n(n(n-1)-t)\}$ is log-concave, namely,*

$$(m_n(k))^2 \geq m_n(k+1)m_n(k-1)$$

for $t+1 \leq k \leq n(n-1)-t-1$. Moreover, the minimum value of t seems to be 75.

We also conjecture that similar properties hold for the generalized q-Catalan numbers.

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