Set Systems with $\mathcal{L}$-intersections modulo a Prime Number

William Y. C. Chen[1]
Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin, 300071, P. R. China

Jiuqiang Liu[2]
Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin, 300071, P. R. China and
Department of Mathematics
Eastern Michigan University
Ypsilanti, MI 48197, USA

Email: [1]chen@nankai.edu.cn, [2]jliu@emich.edu

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Abstract

Let p be a prime and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two subsets of $\{0, 1, 2, \ldots, p-1\}$ satisfying $\max l_j < \min k_i$. We will prove the following results: If $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of $[n] = \{1, 2, \ldots, n\}$ such that $|F_i \cap F_j|\ (mod\ p) \in \mathcal{L}$ for every pair $i \neq j$ and $|F_i|\ (mod\ p) \in K$ for every $1 \leq i \leq m$, then

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

If either K is a set of r consecutive integers or $\mathcal{L} = \{1, 2, \ldots, s\}$, then

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

We will also prove similar results which involve two families of subsets of $[n]$. These results improve the existing upper bounds substantially.

1 Introduction

Throughout the paper, we use X for the set $[n] = \{1, 2, \ldots, n\}$. A family $\mathcal{F}$ of subsets of $X = [n]$ is called *intersecting* if every pair of distinct subsets $E, F \in \mathcal{F}$ have a nonempty intersection. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. A family $\mathcal{F}$ of subsets of $X = [n]$ is called $\mathcal{L}$-*intersecting* if $|E \cap F| \in \mathcal{L}$ for every pair of distinct subsets $E, F \in \mathcal{F}$. A family $\mathcal{F}$ is k-*uniform* if it is a collection of k-subsets of X. Thus, a k-uniform intersecting family is $\mathcal{L}$-intersecting for $\mathcal{L} = \{1, 2, \ldots, k-1\}$.

In 1961, Erdös-Ko-Rado [4] proved the following classical result.

Theorem 1.1 *Let $n \geq 2k$ and let $\mathcal{F}$ be a k-uniform intersecting family of subsets of $[n]$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ consists of all k-subsets containing a common element.*

The following is an intersection theorem of de Bruijin and Erdös [3], which drops the condition for the subsets to be k-uniform, but requires that the intersections to have only one element.

Theorem 1.2 *If $\mathcal{F}$ is a family of subsets of X satisfying $|E \cap F| = 1$ for every pair of distinct subsets $E, F \in \mathcal{F}$, then $|\mathcal{F}| \leq n$.*

A year later, Bose [2] obtained the following more general intersection theorem which requires the intersections to have exactly λ elements.

Theorem 1.3 *If $\mathcal{F}$ is a family of subsets of X satisfying $|E \cap F| = \lambda$ for every pair of distinct subsets $E, F \in \mathcal{F}$, then $|\mathcal{F}| \leq n$.*

In 1975, Ray-Chaudhuri and Wilson [10] made a major progress by deriving the following upper bound for a k-uniform $\mathcal{L}$-intersecting family.

Theorem 1.4 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. If $\mathcal{F}$ is a k-uniform $\mathcal{L}$-intersecting family of subsets of X, then $|\mathcal{F}| \leq \binom{n}{s}$.*

In terms of the parameters n and s, this inequality is best possible, as shown by the set of all s-subsets of an n-set with $\mathcal{L} = \{0, 1, \ldots, s-1\}$. As to non-uniform $\mathcal{L}$-intersecting families, in 1981, Frankl and Wilson [6] obtain the following tight upper bound.

Theorem 1.5 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

This result is best possible in terms of the parameters n and s, as shown by the set of all subsets of size at most s of an n-set. J. Qian and Ray-Chaudhuri [9] have characterized the extremal case of this theorem.

In 1991, Alon, Babai, and Suzuki [1] considered the problem of how large a set system with specific intersection sizes and subset sizes can be, and they obtain the following theorem which is a generalization of both Theorems 1.4 and 1.5.

Theorem 1.6 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers and $K = \{k_1, k_2, \ldots, k_r\}$ be a set of integers satisfying $k_i > s - r$ for every i. Let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of X such that $|F| \in K$ for every $F \in \mathcal{F}$. Then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

Clearly, Theorem 1.4 is a special case of Theorem 1.6 for $r = 1$ and Theorem 1.5 is a special case of Theorem 1.6 for $r = n$ and $K = X = [n]$, under the convention that $\binom{i}{j} = 0$ if $i \geq 0$ and $j < 0$. Moreover, this result is also best possible, as demonstrated by the set of all subsets of an n-set X with cardinalities at least $s - r + 1$ and at most s.

Note that the set $\mathcal{L}$ in the above theorems may contain 0. Stronger bounds can be obtained if we restrict $\mathcal{L}$ to be a set of positive integers. To this end, the following theorem was conjectured by Frankl and Füredi in 1981 [5]. It was proved by Ramanan [11] in 1997. A different proof was given by Sankar and Vishwanathan [12].

Theorem 1.7 *Let $\mathcal{L} = \{1, 2, \ldots, s\}$. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

For a general set $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ of s positive integers, a conjecture was made by Snevily in 1994 [13], and proved by himself in 2003 [16], which is described as in the following theorem.

Theorem 1.8 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

In the same paper [16], Snevily made the following two conjectures.

Conjecture 1.9 *Let p be a prime and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, 2, \ldots, p-1\}$. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of X such that $|F_i \cap F_j| \pmod{p} \in \mathcal{L}$ for every pair $i \neq j$ and $|F_i| \pmod{p} \in K$ for every $1 \leq i \leq m$. Then*

$$|\mathcal{F}| \leq \binom{n}{s} = \binom{n-1}{s} + \binom{n-1}{s-1}.$$

Conjecture 1.10 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| \in \mathcal{L}$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. Then*

$$m \leq \binom{n}{s} = \binom{n-1}{s} + \binom{n-1}{s-1}.$$

Here, we will prove the following results which either improve the existing upper bounds substantially or confirm the above conjectures partially.

Theorem 1.11 *Let p be a prime and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two subsets of $\{0, 1, 2, \ldots, p-1\}$ satisfying $max\ l_j < min\ k_i$. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of X such that $|F_i \cap F_j|\ (mod\ p) \in \mathcal{L}$ for every pair $i \neq j$ and $|F_i|\ (mod\ p) \in K$ for every $1 \leq i \leq m$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

As an immediate consequence to this theorem, by taking $r = 1$, we have the following which shows that Conjecture 1.9 is true when $\mathcal{F}$ is a k-uniform family of subsets (i.e., a family of k-subsets) of $X = [n]$.

Corollary 1.12. Let p be a prime and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k\}$ be two subsets of $\{0, 1, 2, \ldots, p-1\}$ satisfying $max\ l_j < k$. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of k-subsets of X such that $|F_i \cap F_j|\ (mod\ p) \in \mathcal{L}$ for every pair $i \neq j$. Then

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1}.$$

Theorem 1.13. Let p be a prime and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k, k+1, \ldots, k+r-1\}$ be two subsets of $\{0, 1, 2, \ldots, p-1\}$ satisfying $max\ l_j < k$. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$

is a family of subsets of X such that $|F_i \cap F_j|$ $(mod\ p) \in \mathcal{L}$ for every pair $i \neq j$ and $|F_i|$ $(mod\ p) \in K$ for every $1 \leq i \leq m$. Then

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

Theorem 1.14. Let p be a prime and let $\mathcal{L} = \{1, 2, \ldots, s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two subsets of $\{0, 1, 2, \ldots, p-1\}$ satisfying $s < min\ k_i$. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of X such that $|F_i \cap F_j|$ $(mod\ p) \in \mathcal{L}$ for every pair $i \neq j$ and $|F_i|$ $(mod\ p) \in K$ for every $1 \leq i \leq m$. Then

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

Note that Theorem 1.14 gives an extension of the main theorem in [8] to its modular version.

Theorem 1.15. Let p be a prime and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\} \subseteq \{1, 2, \ldots, p-1\}$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j|(mod\ p) \in \mathcal{L}$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. If $max\ l_j < min\{|A_i|(mod\ p)|1 \leq i \leq m\}$, then

$$m \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1},$$

where r is the number of different set sizes in $\mathcal{A}$.

Clearly, by selecting a prime p greater than n, we obtain the following immediate corollary.

Corollary 1.16. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| \in \mathcal{L}$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. If $max\ l_j < min\{|A_i| : 1 \leq i \leq m\}$, then

$$m \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

where r is the number of different set sizes in $\mathcal{A}$.

As an immediate consequence to Corollary 1.16, by taking $r = 1$, we have the following which shows that Conjecture 1.10 is true when either $\mathcal{A}$ is k-uniform (or $\mathcal{B}$ is k-uniform by symmetry).

Corollary 1.17. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers and $max\ l_j < k$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| \in \mathcal{L}$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. If either $\mathcal{A}$ is k-uniform or $\mathcal{B}$ is k-uniform, then

$$m \leq \binom{n}{s} = \binom{n-1}{s} + \binom{n-1}{s-1}.$$

Note that this bound is sharp as shown by taking all k-subsets of $[n]$ for $\mathcal{A}$ and all $(n-k)$-subsets for $\mathcal{B}$.

When either the set sizes $(mod\ p)$ in $\mathcal{A}$ is a set of r consecutive integers or the set sizes $(mod\ p)$ in $\mathcal{B}$ is a set of r consecutive integers, we have the following theorem which gives a better bound than Theorem 1.15.

Theorem 1.18. Let p be a prime and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\} \subseteq \{1, 2, \ldots, p-1\}$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| (mod\ p) \in \mathcal{L}$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. If the set sizes $(mod\ p)$ in $\mathcal{A}$ (or in $\mathcal{B}$) is a set of r consecutive integers in $\{1, 2, \ldots, p-1\}$ and $max\ l_j < min\{|A_i| (mod\ p) | 1 \leq i \leq m\}$, then

$$m \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

2 Proof of Theorems 1.11, 1.13, and 1.14

We will use $x = (x_1, x_2, \ldots, x_n)$ to denote a vector of n variables with each variable x_j taking values 0 or 1. A polynomial $p(x)$ in variables x_i, $1 \leq i \leq n$, is called *multilinear* if the power of each variable x_i in each term is at most one. Clearly, if each variable x_i takes only the values 0 or 1, then any polynomial in variables x_i, $1 \leq i \leq n$, is multilinear since any positive power of a variable x_i may be replaced by one. For a subset F of $X = [n]$, we define *the characteristic vector* of F to be the vector $u = (u_1, u_2, \ldots, u_n) \in R^n$ with $u_j = 1$ if $j \in F$ and $u_j = 0$ otherwise. In what follows, we will use v_i to denote the characteristic vector of $F_i \in \mathcal{F}$.

To prove our results, we need the following lemma which is Lemma 3.6 in [1]. We say a set $H = \{h_1, h_2, \ldots, h_t\} \subseteq [n]$ has a gap of size $\geq d$ (where the h_i are arranged in increasing order) if either $h_1 \geq d-1$, or $n - h_t \geq d-1$, or $h_{i+1} - h_i \geq d$ for some i $(1 \leq i \leq t-1)$. For a subset $I \subseteq [n]$, we denote $x_I = \prod_{j \in I} x_j$.

Lemma 2.1. Let p be a prime and $H \subseteq \{0, 1, \ldots, p-1\}$ be a set of integers such that the set $(H + p\mathbf{Z}) \cap \{0, 1, \ldots, n\}$ has a gap $\geq d+1$, where $d \geq 0$. Let f denote the following polynomial in n variables

$$f(x) = \prod_{h \in H} \left(\sum_{j=1}^{n} x_j - h \right).$$

Then the set of polynomials $\{x_I f \mid |I| \leq d-1\}$ is linearly independent over $\mathbf{F}_p$.

Proof of Theorem 1.11. Let p be a prime and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two subsets of $\{0, 1, 2, \ldots, p-1\}$ satisfying $max\ l_j < min\ k_i$. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of X such that $|F_i \cap F_j|\ (mod\ p) \in \mathcal{L}$ for every pair $i \neq j$ and $|F_i|\ (mod\ p) \in K$ for every $1 \leq i \leq m$.

For $1 \leq i \leq m$, define

$$f_i(x) = \prod_{j=1}^{s}(v_i \cdot x - l_j),$$

where $x = (x_1, x_2, \ldots, x_n)$ with each x_j taking values 0 or 1. Then each $f_i(x)$ is a multilinear polynomial of degree at most s since any positive power of a variable may be replaced by one. Moreover, since $max\ l_j < min\ k_i$, $\mathcal{L} \cap K = \emptyset$ and $f_i(v_i) \neq 0(mod\ p)$ for every $i \leq m$ and $f_i(v_j) = 0(mod\ p)$ for every pair $i \neq j$ since $|F_i \cap F_j|(mod\ p) \in \mathcal{L}$.

Let Q be the family of subsets of $X = [n]$ with size at most s which contain n. Then $|Q| = \sum_{i=0}^{s-1}\binom{n-1}{i}$. For each $L \in Q$, define

$$q_L(x) = (1 - x_n)\prod_{j \in L, j \neq n} x_j.$$

Let $H = \{k_i - 1 | k_i \in K\} \cup K$. Then $|H| \leq 2r$. Set

$$f(x) = \prod_{h \in H}\left(\sum_{j=1}^{n-1} x_j - h\right).$$

Let W be the family of subsets of $[n]$ with sizes at most $s - 2r$ which do not contain n, Then $|W| = \sum_{i=0}^{s-2r}\binom{n-1}{i}$. For each $I \in W$, define

$$A_I(x) = f(x)\prod_{j \in I} x_j.$$

Then each $A_I(x)$ is a multilinear polynomial of degree at most s.

We now proceed to show that the polynomials in

$$\{f_i(x) | 1 \leq i \leq m\} \cup \{q_L(x) | L \in Q\} \cup \{A_I(x) | I \in W\}$$

are linearly independent over $\mathbf{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero:

$$\sum_{i=1}^{m} \alpha_i f_i(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.1}$$

Claim 1. $\alpha_i = 0$ for each i with $n \in F_i$.

Suppose, to the contrary, that i_0 is a subscript such that $n \in F_{i_0}$ and $\alpha_{i_0} \neq 0$. Since $n \in F_{i_0}$, $q_L(v_{i_0}) = 0$ for every $L \in Q$. Recall that $f_j(v_{i_0}) = 0$ for $j \neq i_0$ and $f(v_j) = 0$ for every $1 \leq j \leq m$. By evaluating equation (2.1) with $x = v_{i_0}$, we obtain that $\alpha_{i_0} f_{i_0}(v_{i_0}) = 0(mod\ p)$. Since $f_{i_0}(v_{i_0}) \neq 0(mod\ p)$, we have $\alpha_{i_0} = 0$, a contradiction. Thus, Claim 1 holds.

Claim 2. $\alpha_i = 0$ for each i with $n \notin F_i$. Applying Claim 1, we get

$$\sum_{n \notin F_i} \alpha_i f_i(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.2}$$

Suppose, to the contrary, that i_0 is a subscript such that $n \notin F_{i_0}$ and $\alpha_{i_0} \neq 0$. Let $v^*_{i_0} = v_{i_0} + (0, 0, \ldots, 0, 0, 1)$ (namely, making $x_n = 1$ in $v^*_{i_0}$). Then $q_L(v^*_{i_0}) = 0$ for every $L \in Q$. Note that $f_i(v^*_{i_0}) = f_i(v_{i_0})$ for each i with $n \notin F_i$ and $A_I(v^*_{i_0}) = 0$ for each $I \in W$ as $f(v^*_{i_0}) = 0$. By evaluating equation (2.2) with $x = v^*_{i_0}$, we obtain $\alpha_{i_0} f_{i_0}(v^*_{i_0}) = \alpha_{i_0} f_{i_0}(v_{i_0}) = 0(mod\ p)$ which implies $\alpha_{i_0} = 0$, a contradiction. Thus, the claim is verified.

Claim 3. $\beta_L = 0$ for each $L \in Q$.

By Claims 1 and 2, we obtain

$$\sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.3}$$

Rewrite equation (2.3) as

$$\left[\sum_{L \in Q} \beta_L q'_L(x) + \sum_{I \in W} \mu_I A_I(x)\right] - \left(\sum_{L \in Q} \beta_L q'_L(x)\right) x_n = 0, \tag{2.4}$$

where $q'_L = \prod_{j \in L, j \neq n} x_j$. Note that x_n does not appear in the first parentheses of equation (2.4). Setting $x_n = 0$ in equation (2.4) gives us

$$\sum_{L \in Q} \beta_L q'_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0$$

and

$$\left(\sum_{L \in Q} \beta_L q'_L(x)\right) x_n = 0.$$

By setting $x_n = 1$, we obtain

$$\sum_{L \in Q} \beta_L q'_L(x) = 0.$$

It is not difficult to see that the polynomials $q'_L(x)$, $L \in Q$, are linearly independent. Therefore, we conclude that $\beta_L = 0$ for each $L \in Q$.

By Claims 1-3, we now have

$$\sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.5}$$

Since $H = \{k_i - 1 | k_i \in K\} \cup K$ and $s - 1 \leq max\ l_j < min\ k_i$, $H \subseteq \{0, 1, \ldots, p-1\}$ and H has a gap at least s. Recall that

$$f(x) = \prod_{h \in H} \left(\sum_{j=1}^{n-1} x_j - h \right).$$

By applying Lemma 2.1 with $d - 1 = s - 2r$, we conclude that the set of polynomials $\{A_I(x) = x_I f(x) | I \in W\}$ is linearly independent over $\mathbf{F}_p$, and so $\mu_I = 0$ for each $I \in W$ in equation (2.5).

In summary, we have shown that the polynomials in

$$\{f_i(x) | 1 \leq i \leq m\} \cup \{q_L(x) | L \in Q\} \cup \{A_I(x) | I \in W\}$$

are linearly independent. Since the set of all monomials in variables x_i, $1 \leq i \leq n$, of degree at most s forms a basis for the vector space of multilinear polynomials of degree at most s, it follows that

$$m + \sum_{i=0}^{s-1} \binom{n-1}{i} + \sum_{i=0}^{s-2r} \binom{n-1}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$$

which implies that

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

This completes the proof of the theorem. □

Note that if $K = \{k, k+1, \ldots, k+r-1\}$ is a set of r consecutive integers, then the set $H = \{k_i - 1 | k_i \in K\} \cup K$ has size $|H| = r+1$. Thus, with a little bit modification in the proof of Theorem 1.11, we obtain a proof for Theorem 1.13.

Proof of Theorem 1.13. The proof is almost identical to the proof of Theorem 1.11 by selecting W to be the set of all subsets of $[n]$ with sizes at most $s-r-1$ which do not contain n. $\square$

Next, we prove Theorem 1.14.

Proof of Theorem 1.14. Let p be a prime and let $\mathcal{L} = \{1, 2, \ldots, s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two subsets of $\{0, 1, 2, \ldots, p-1\}$ satisfying $\max l_j < \min k_i$. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of X such that $|F_i \cap F_j| \ (mod\ p) \in \mathcal{L}$ for every pair $i \neq j$ and $|F_i| \ (mod\ p) \in K$ for every $1 \leq i \leq m$.

For $1 \leq i \leq m$, define

$$f_i(x) = \prod_{j=1}^{s} (v_i \cdot x - l_j),$$

where $x = (x_1, x_2, \ldots, x_n)$ with each x_j taking values 0 or 1. Then $f_i(v_i) \neq 0 (mod\ p)$ for every $i \leq m$ and $f_i(v_j) = 0 (mod\ p)$ for every pair $i \neq j$.

Let Q be the family of subsets of $X = [n]$ with size at most s which contain n. Then $|Q| = \sum_{i=0}^{s-1} \binom{n-1}{i}$. For each $L \in Q$, define

$$q_L(x) = \prod_{j \in L} x_j.$$

Set

$$f(x) = \prod_{k \in K} \left(\sum_{j=1}^{n} x_j - k \right).$$

Let W be the family of subsets of $[n]$ with sizes at most $s-r$ which do contain n, Then

$|W| = \sum_{i=0}^{s-r-1} \binom{n-1}{i}$. For each $I \in W$, define

$$A_I(x) = (x_n - 1)f(x) \prod_{j \in I, j \neq n} x_j.$$

Then each $A_I(x)$ is a multilinear polynomial of degree at most s.

We now proceed to show that the polynomials in

$$\{f_i(x) | 1 \leq i \leq m\} \cup \{q_L(x) | L \in Q\} \cup \{A_I(x) | I \in W\}$$

are linearly independent over $\mathbf{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero:

$$\sum_{i=1}^{m} \alpha_i f_i(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.6}$$

Claim 1. $\alpha_i = 0$ for each i with $n \notin F_i$.

Suppose, to the contrary, that i_0 is a subscript such that $n \notin F_{i_0}$ and $\alpha_{i_0} \neq 0$. Since $n \notin F_{i_0}$, $q_L(v_{i_0}) = 0$ for every $L \in Q$. Recall that $f_j(v_{i_0}) = 0$ for $j \neq i_0$ and $f(v_j) = 0$ for every $1 \leq j \leq m$. By evaluating equation (2.6) with $x = v_{i_0}$, we obtain that $\alpha_{i_0} f_{i_0}(v_{i_0}) = 0 (mod\ p)$. Since $f_{i_0}(v_{i_0}) \neq 0 (mod\ p)$, we have $\alpha_{i_0} = 0$, a contradiction. Thus, Claim 1 holds.

Claim 2. $\beta_L = 0$ for each $L \in Q$. By Claim 1, we obtain

$$\sum_{n \in F_i} \alpha_i f_i(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.7}$$

Suppose, to the contrary, that L is a minimal subset in Q such that $\beta_L \neq 0$. Let v_L be the characteristic vector for L. Then $q_{L'}(v_L) = 0$ for each $L' \in Q$ which is not a subset of L. Since $n \in L$, $A_I(v_L) = 0$ for each $I \in W$. For each F_j with $n \in F_j$, since $|L \cap F_j| \in \mathcal{L}$, we have $f_j(v_L) = 0$. Thus, by evaluating equation (2.7) with $x = v_L$, we obtain $\beta_L = 0$, a contradiction. Therefore, $\beta_L = 0$ for each $L \in Q$.

Claim 3. $\alpha_i = 0$ for each i with $n \in F_i$. Applying Claims 1 and 2, we get

$$\sum_{n \in F_i} \alpha_i f_i(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.8}$$

Suppose, to the contrary, that i_0 is a subscript such that $n \in F_{i_0}$ and $\alpha_{i_0} \neq 0$. Note that $f(v_{i_0}) = 0$ and so $A_I(v_{i_0}) = 0$ for each $I \in W$. By evaluating equation (2.8) with $x = v_{i_0}$, we obtain $\alpha_{i_0} f_{i_0}(v_{i_0}) = 0(mod\ p)$ which implies $\alpha_{i_0} = 0$, a contradiction. Thus, the claim is verified.

By Claims 1-3, we now have

$$\sum_{I \in W} \mu_I A_I(x) = 0. \tag{2.9}$$

Since $s - 1 \leq max\ l_j < min\ k_i$, $K \subseteq \{0, 1, \ldots, p-1\}$ and K has a gap at least s. Recall that

$$f(x) = \prod_{k \in K} \left(\sum_{j=1}^{n} x_j - k \right).$$

Setting $x_n = 0$ and applying Lemma 2.1 with $d - 1 = s - r - 1$, we conclude that the set of polynomials $\{A_I(x) = x_{I'}(x_n - 1) f(x) | I \in W, I' = I - \{n\}\}$ is linearly independent over $\mathbf{F}_p$, and so $\mu_I = 0$ for each $I \in W$ in equation (2.9).

In summary, we have shown that the polynomials in

$$\{f_i(x) | 1 \leq i \leq m\} \cup \{q_L(x) | L \in Q\} \cup \{A_I(x) | I \in W\}$$

are linearly independent. Since the set of all monomials in variables x_i, $1 \leq i \leq n$, of degree at most s forms a basis for the vector space of multilinear polynomials of degree at most s, it follows that

$$m + \sum_{i=0}^{s-1} \binom{n-1}{i} + \sum_{i=0}^{s-r-1} \binom{n-1}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$$

which implies that

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

This completes the proof of the theorem. □

3 Proof of Theorems 1.15 and 1.18

We first give a proof for Theorem 1.15 which is alone the same line as the proof of Theorem 1.11 but with some differences.

Proof of Theorem 1.15. Let p be a prime and $\mathcal{L} = \{l_1, l_2, \dots, l_s\} \subseteq \{1, 2, \dots, p-1\}$. Suppose that $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| (mod\ p) \in \mathcal{L}$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. Without loss of generality, let r be the number of different set sizes in $\mathcal{A}$ which is no bigger than the number of different set sizes in $\mathcal{B}$. In what follows, we will use v_I to denote the characteristic vector of I for each subset $I \subseteq [n]$.

For each $B_i \in \mathcal{B}$, define

$$f_{B_i}(x) = \prod_{j=1}^{s} (v_{B_i} \cdot x - l_j).$$

Then each $f_{B_i}(x)$ is a multilinear polynomial of degree at most s. Since $|A_i \cap B_i| = 0(mod\ p)$ for each i and $|A_i \cap B_j| (mod\ p) \in \mathcal{L}$ for $i \neq j$, $f_{B_i}(v_{A_i}) = \prod_{j=1}^{s}(-l_j) \neq 0(mod\ p)$ for every $i \leq m$ and $f_{B_i}(v_{A_j}) = 0(mod\ p)$ for every pair $i \neq j$.

Let Q be the family of subsets of $X = [n]$ with size at most s which contain n. Then $|Q| = \sum_{i=0}^{s-1} \binom{n-1}{i}$. For each $L \in Q$, define

$$q_L(x) = (\prod_{j \in L} x_j).$$

Let $H = \{|A_i| - 1(mod\ p) | A_i \in \mathcal{A}\} \cup \{|A_i| (mod\ p) | A_i \in \mathcal{A}\}$. Then $|H| \leq 2r$. Set

$$f(x) = \prod_{h \in H} \left(\sum_{j=1}^{n-1} x_j - h \right).$$

Let W be the family of subsets of $[n]$ with sizes at most $s - 2r$ which do not contain n, Then $|W| = \sum_{i=0}^{s-2r} \binom{n-1}{i}$. For each $I \in W$, define

$$K_I(x) = \left(\prod_{j \in I} x_j \right) f(x).$$

Then each $K_I(x)$ is a multilinear polynomial of degree at most s.

We now proceed to show that the polynomials in

$$\{f_{B_i}(x)|1 \le i \le m\} \cup \{q_L(x)|L \in Q\} \cup \{K_I(x)|I \in W\}$$

are linearly independent over $\mathbf{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero:

$$\sum_{i=1}^{m} \alpha_i f_{B_i}(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I K_I(x) = 0. \tag{3.1}$$

Claim 1. $\alpha_i = 0$ for each i with $n \notin A_i$.

Suppose, to the contrary, that i' is a subscript such that $n \notin A_{i'}$ and $\alpha_{i'} \neq 0$. Since $n \notin A_{i'}$, $q_L(v_{A_{i'}}) = 0$ for every $L \in Q$. Recall that $f_{B_j}(v_{A_{i'}}) = 0$ for $j \neq i'$ and $f(v_{A_{i'}}) = 0$. By evaluating equation (3.1) with $x = v_{A_{i'}}$, we obtain that $\alpha_{i'} f_{B_{i'}}(v_{A_{i'}}) = 0(mod\ p)$. Since $f_{B_{i'}}(v_{A_{i'}}) \neq 0(mod\ p)$, we have $\alpha_{i'} = 0$, a contradiction. Thus, Claim 1 holds.

Claim 2. $\alpha_i = 0$ for each i with $n \in A_i$. Applying Claim 1, we get

$$\sum_{n \in A_i} \alpha_i f_{B_i}(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I K_I(x) = 0. \tag{3.2}$$

Suppose, to the contrary, that i' is a subscript such that $n \in A_{i'}$ and $\alpha_{i'} \neq 0$. Since $|A_i \cap B_i| = 0$ for every i, $n \notin B_i$ whenever $n \in A_i$. Let $v'_{A_{i'}} = v_{A_{i'}} - (0, 0, \ldots, 0, 0, 1)$ (namely, making $x_n = 0$ in $v'_{A_{i'}}$). Note that $f_{B_j}(v'_{A_{i'}}) = f_{B_j}(v_{A_{i'}})$ for each B_j with $n \notin B_j$, and $K_I(v'_{A_{i'}}) = 0$ for each $I \in W$. By evaluating equation (3.2) with $x = v'_{A_{i'}}$, we obtain $\alpha_{i'} f_{B_{i'}}(v'_{A_{i'}}) = \alpha_{i'} f_{B_{i'}}(v_{A_{i'}}) = 0(mod\ p)$ which implies $\alpha_{i'} = 0$, a contradiction. Thus, the claim is verified.

Claim 3. $\beta_L = 0$ for each $L \in Q$.

By Claims 1 and 2, we obtain

$$\sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I K_I(x) = 0. \tag{3.3}$$

Note that the first sum has a factor x_n while x_n does not appear in the second sum in equation (3.3). Setting $x_n = 0$ in equation (3.3) gives us

$$\sum_{I \in W} \mu_I K_I(x) = 0$$

and so

$$\sum_{L \in Q} \beta_L q_L(x) = 0.$$

It is not difficult to see that the polynomials $q_L(x)$, $L \in Q$, are linearly independent. Therefore, we conclude that $\beta_L = 0$ for each $L \in Q$.

By Claims 1-3, we now have

$$\sum_{I \in W} \mu_I K_I(x) = 0. \tag{3.4}$$

Since $H = \{|A_i| - 1(mod\ p)|A_i \in \mathcal{A}\} \cup \{|A_i|(mod\ p)|A_i \in \mathcal{A}\}$ and $s \leq max\ l_j < min\{|A_i| : 1 \leq i \leq m\}$, $H \subseteq \{0, 1, \ldots, p-1\}$ and H has a gap at least s. Recall that

$$f(x) = \prod_{h \in H} \left(\sum_{j=1}^{n-1} x_j - h \right).$$

By applying Lemma 2.1 with $d - 1 = s - 2r$, we conclude that the set of polynomials $\{K_I(x) = x_I f(x) | I \in W\}$ is linearly independent over $\mathbf{F}_p$, and so $\mu_I = 0$ for each $I \in W$ in equation (3.4).

In summary, we have shown that the polynomials in

$$\{f_{B_i}(x) | 1 \leq i \leq m\} \cup \{q_L(x) | L \in Q\} \cup \{K_I(x) | I \in W\}$$

are linearly independent. Since the set of all monomials in variables x_i, $1 \leq i \leq n$, of degree at most s forms a basis for the vector space of multilinear polynomials of degree at most s, it follows that

$$m + \sum_{i=0}^{s-1} \binom{n-1}{i} + \sum_{i=0}^{s-2r} \binom{n-1}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$$

which implies that

$$m \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

This completes the proof of the theorem. $\square$

We remark that with exactly the same proof as above, we can obtain the following stronger result than Theorem 1.15.

Theorem 3.1. Let p be a prime and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\} \subseteq \{1, 2, \ldots, p-1\}$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| (mod\ p) \in \mathcal{L}$ for $i \neq j$, $|A_i \cap B_i| (mod\ p) \notin \mathcal{L}$ and $n \notin A_i \cap B_i$ for every i. If $max\ l_j < min\{|A_i| (mod\ p) | 1 \le i \le m\}$, then

$$m \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1},$$

where r is the number of different set sizes in $\mathcal{A}$.

Note that if the set sizes $(mod\ p)$ in $\mathcal{A}$ (or in $\mathcal{B}$) is a set of r consecutive integers in $\{1, 2, \ldots, p-1\}$, then $H = \{|A_i| - 1 (mod\ p) | A_i \in \mathcal{A}\} \cup \{|A_i| (mod\ p) | A_i \in \mathcal{A}\}$ has size $|H| = r + 1$. Thus, with a little bit modification in the proof of Theorem 1.15, we obtain a proof for Theorem 1.18.

Proof of Theorem 1.18. The proof is almost identical to the proof of Theorem 1.15 by selecting W to be the set of all subsets of $[n]$ with sizes at most $s - r - 1$ which do not contain n. $\square$

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