

# Extended Zeilberger's Algorithm for Identities on Bernoulli and Euler Polynomials

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**Abstract.** We present a computer algebra approach to proving identities on Bernoulli polynomials and Euler polynomials by using the extended Zeilberger's algorithm given by Chen, Hou and Mu. The key idea is to use the contour integral definitions of the Bernoulli and Euler numbers to establish recurrence relations on the integrands. Such recurrence relations have certain parameter free properties which lead to the required identities without computing the integrals. Furthermore two new identities on Bernoulli numbers are derived.

**Keywords:** Bernoulli number, Euler number, Bernoulli polynomial, Euler polynomial, Zeilberger's algorithm

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## 1. Introduction

Bernoulli polynomials and Euler polynomials play fundamental roles in various branches of mathematics including combinatorics, number theory, special functions and analysis, see for example [5, 15, 22]. At first glance, the Bernoulli numbers, Euler numbers, and the corresponding polynomials do not seem to fall in the framework of hypergeometric identities. The powerful algorithm of Zeilberger [26] does not look like the right mechanism to handle the Bernoulli and Euler numbers or polynomials.

However, as will be seen, the Cauchy contour integral representations of the Bernoulli numbers and Euler numbers make it possible to transform identities on these numbers and polynomials into identities on hypergeometric sums. In order to avoid the computation of the contour integrals, it is desirable to derive recurrence relations of the hypergeometric summands with certain parameter free properties. At the first trial, one finds it quite disappointing that the recurrence relations given by Zeilberger's algorithm seldom have the desired parameter free properties. Nevertheless, this drawback can be overcome by using an extended version of Zeilberger's algorithm. Paule [18] first noticed that Zeilberger's algorithm can be extended to derive mixed recurrence relations for a hypergeometric term  $F(n, m_1, m_2, \dots, m_r, k)$ , where  $r \geq 1$  and  $m_1, m_2, \dots, m_r$  are parameters. Chen, Hou and

Mu [4] further extended Paule's algorithm to the case with additional parameter free properties. In this paradigm, many identities on Bernoulli and Euler numbers and polynomials can be verified. Moreover, some new identities can be discovered from the recurrences generated by the original Zeilberger's algorithm without the consideration of the parameter free properties.

## 2. Background

Let us recall the background on Bernoulli and Euler numbers and polynomials. Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ . The well-known Bernoulli numbers and Euler numbers are defined by the generating functions

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \frac{2e^z}{e^{2z} + 1}.$$

By the Cauchy integral formula, we have the contour integral definitions of the Bernoulli numbers and the Euler numbers

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}, \quad (2.1)$$

$$E_n = \frac{n!}{2\pi i} \oint \frac{2e^z}{e^{2z} + 1} \frac{dz}{z^{n+1}}, \quad (2.2)$$

where the contour encloses the origin, has radius less than  $2\pi$  (to avoid the poles at  $\pm 2\pi i$ ), and is traversed in a counterclockwise direction. Actually, as will be seen, there will be no need to compute the contour integrals, and one can formally treat the contour integrals as linear operators. The integral representation plays a crucial role in connecting the Bernoulli numbers and Euler numbers to hypergeometric terms.

The Bernoulli numbers are also given by the following recursion

$$\sum_{k=0}^n \binom{n+1}{k} B_k = 0, \quad n > 0, \quad (2.3)$$

with  $B_0 = 1$ . The Bernoulli numbers are rational and it is well known that  $B_{2n+1} = 0$  for  $n \geq 1$ . The first few values of the Bernoulli numbers are as follows

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}.$$

For the Euler numbers,  $E_{2n+1} = 0$  for  $n \geq 0$ .

The Bernoulli polynomials and Euler polynomials can be defined by the generating functions

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^t + 1}.$$

Clearly  $B_n = B_n(0)$  and  $E_n = 2^n E_n(\frac{1}{2})$ . The polynomials  $B_n(x)$  and  $E_n(x)$  obey the following relations

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} x^{n-k} B_k, \quad (2.4)$$

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \left(x - \frac{1}{2}\right)^{n-k} \frac{E_k}{2^k}. \quad (2.5)$$

We will need the following basic properties of  $B_n(x)$  and  $E_n(x)$ . Lehmer [14] showed that the Bernoulli polynomials satisfy the relations  $B_n(1) = (-1)^n B_n(0)$  and

$$B_n(1-x) = (-1)^n B_n(x). \quad (2.6)$$

Similarly, the Euler polynomials satisfy the relation

$$E_n(1-x) = (-1)^n E_n(x). \quad (2.7)$$

It is well known that the Bernoulli and Euler polynomials have the following binomial expansions

$$B_n(x+y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k} \quad \text{and} \quad E_n(x+y) = \sum_{k=0}^n \binom{n}{k} E_k(x) y^{n-k}. \quad (2.8)$$

These basic properties will be needed for the computation of initial values for the recurrence relations derived by our algorithm.

### 3. The Algorithm

In this section, we present an approach to proving Bernoulli number identities by using the extended Zeilberger's algorithm, and we will use an example to describe the four steps of our algorithm. The original Zeilberger's algorithm is devised to find recurrence relations of the summation  $\sum_k F(n, k)$  by solving the equation

$$a_0(n)F(n, k) + a_1(n)F(n+1, k) + \cdots + a_J(n)F(n+J, k) = G(n, k+1) - G(n, k),$$

where  $F(n, k)$  is a hypergeometric term in  $n$  and  $k$ ,  $a_i(n)$  are polynomials in  $n$  and are  $k$ -free,  $G(n, k)/F(n, k)$  is a rational function in  $n$  and  $k$ . It is known that Zeilberger's algorithm can be applied to summands with parameters in order to establish multiple index recurrence relations, for example, see [2, Section 4.3.1] and [18]. Recently, Chen, Hou and Mu [4] have found an extension of Zeilberger's algorithm to summations of hypergeometric terms  $\sum_k F(n, m_1, m_2, \dots, m_r, k)$ , where  $r \geq 1$  and  $m_1, m_2, \dots, m_r$  are parameters. In fact, there are cases when the extended Zeilberger's algorithm becomes more efficient than the original form, see [4]. We will not give a rigorous description of the extended Zeilberger's algorithm, since it will become apparent when it is being used.

For example, let us consider an identity of Gessel [8, Lemma 7.2].

**Theorem 3.1.** *We have*

$$\sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^n \binom{n}{k} B_{m+k}, \quad (3.1)$$

where  $m$  and  $n$  are nonnegative integers.

**Proof.** To justify the above identity, we aim to find recurrence relations for both sides. If they agree with each other, then the equality is established by considering the initial values. There are three steps to compute the recurrence relations for the above summations. We will give detailed steps for the left hand side of (3.1).

**Step 1.** Extract the hypergeometric sum from the Cauchy integral formula.

Denote the left hand side of (3.1) by  $L(n, m)$ . By the contour integral formula for  $B_n$ , we have

$$L(n, m) = \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \sum_{k=0}^m \binom{m}{k} \frac{(n+k)!}{z^{n+k}} dz.$$

Denote the summand in the above integral by

$$C(n, m, k) = \binom{m}{k} \frac{(n+k)!}{z^{n+k}},$$

and let

$$S(n, m) = \sum_{k=0}^m C(n, m, k).$$

**Step 2.** Construct an extended telescoping equation with a shift on the parameter  $m$  of the summand  $C(n, m, k)$ , and solve this equation by the extended Zeilberger's algorithm.

Set the hypergeometric term

$$F(n, m, k) = b_0 C(n, m, k) + b_1 C(n, m+1, k) + b_2 C(n+1, m, k) + b_3 C(n+1, m+1, k), \quad (3.2)$$

where  $b_i$ 's are  $k$ -free rational functions of  $n$  and  $m$ , namely,  $k$  does not appear in  $b_i$ 's. Moreover, we require that the rational functions  $b_i$ 's are independent of the variable  $z$ .

By Gosper's algorithm, it is easy to check that  $C(n, m, k) = z_{k+1} - z_k$  has no hypergeometric solution for  $z_k$ . Moreover, since the Bernoulli numbers are not  $P$ -recursive, one sees that Zeilberger's algorithm does not work in this case. Instead, we will try to solve the equation

$$F(n, m, k) = G(n, m, k+1) - G(n, m, k) \quad (3.3)$$

where  $G(n, m, k)$  is a hypergeometric term. By Gosper's algorithm, we get

$$r(k) = \frac{F(n, m, k+1)}{F(n, m, k)} = \frac{a(k) c(k+1)}{b(k) c(k)}, \quad (3.4)$$

where

$$\begin{aligned} a(k) &= (m - k + 1)(n + k + 1), \\ b(k) &= z(k + 1), \\ c(k) &= b_2k^2 + (b_2n - b_2m + b_0z - b_3m - b_3)k - b_0z(m + 1) \\ &\quad - b_1z(m + 1) - b_2(n + 1)(m + 1) - b_3(n + 1)(m + 1). \end{aligned}$$

Assume that  $G(n, m, k) = y(k)F(n, m, k)$ , where  $y(k)$  is an unknown rational function of  $k$ . Substituting  $y(k)F(n, m, k)$  for  $G(n, m, k)$  in (3.3) reveals that  $y(k)$  satisfies

$$r(k)y(k + 1) - y(k) = 1.$$

Substituting the factorization (3.4) into the above equation, and setting

$$x(k) = \frac{y(k)c(k)}{b(k - 1)},$$

then Zeilberger's algorithm reduces the problem further to that of finding polynomial solutions (see [19, Theorem 5.2.1]) of the following equation

$$a(k)x(k + 1) - b(k - 1)x(k) = c(k). \quad (3.5)$$

Notice that the coefficients  $a(k)$  and  $b(k)$  are independent of the unknowns  $b_i$ 's, and  $c(k)$  is a linear combination of  $b_i$ 's. One can estimate the degree of the polynomial  $x(k)$ , as in Gosper's algorithm. In this case,  $x(k)$  is of degree 0. Assume that  $x(k) = a_0$ . Then the equation (3.5) becomes

$$\begin{aligned} &(-a_0 - b_2)k^2 + (mb_2 - nb_2 + mb_3 - b_0z - (n - m)a_0 - a_0z + b_3)k \\ &+ ((n + 1)(m + 1)a_0 + b_0z + nb_3 + b_3nm + b_0zm + b_2nm + nb_2 \\ &+ b_1zm + b_1z + b_2 + mb_2 + mb_3 + b_3) = 0. \end{aligned}$$

By setting the coefficient of each power of  $k$  to zero, we get a system of linear equations in  $a_0$  and  $b_i$ 's. Note that in the solution of this system,  $a_0$  and  $b_i$ 's may contain the variable  $z$ . To prevent  $z$  from appearing in  $b_i$ 's, we should go one step further to impose that the coefficient of any positive power of  $z$  in  $b_i$ 's is zero. This may also lead to additional equations. Combining all these equations, if we can find a nonzero solution, then take this solution to the next step. Otherwise, we may try recurrences of higher order. In this case, we get a nonzero solution  $a_0 = -1, b_0 = 1, b_1 = -1, b_2 = 1, b_3 = 0$ . Note that in general the  $b_i$ 's are polynomials in  $n$  and  $m$ .

**Step 3.** Compute the recurrence for  $L(n, m)$ .

By Step 2, the solution of  $a_0, b_0, \dots, b_3$  leads to the following telescoping equation

$$C(n, m, k) - C(n, m + 1, k) + C(n + 1, m, k) = G(n, m, k + 1) - G(n, m, k), \quad (3.6)$$

where

$$G(n, m, k) = \frac{m!(n + k)!}{(k - 1)!(m - k + 1)!z^{n+k}}. \quad (3.7)$$

Summing the above recurrence over  $k$  from 0 to  $m + 1$ , we obtain

$$S(n, m) - S(n, m + 1) + S(n + 1, m) = 0.$$

Substituting the above recurrence relation to the contour integral definition of  $B_n$ , we find that  $L(n, m)$  satisfies

$$L(n, m) - L(n, m + 1) + L(n + 1, m) = 0.$$

By the same procedure, we see that the right hand side of (3.1), denoted by  $R(n, m)$ , satisfies the same recurrence relation as  $L(n, m)$ , namely,

$$R(n, m) - R(n, m + 1) + R(n + 1, m) = 0.$$

**Step 4.** Verify initial values.

By considering the parity of  $B_m$ , we see that  $(-1)^m B_m = B_m$  unless  $m = 1$ . Therefore  $L(0, 1) = R(0, 1) = 1/2$  and  $L(0, m) = R(0, m) = B_m$  for  $m \neq 1$ . This completes the proof. ■

It is known that the Bernoulli numbers and Euler numbers are not  $P$ -recursive, see [6]. Roughly speaking, this fact implies that the original Zeilberger's algorithm is not applicable to derive a recurrence relation of any order for summations involving Bernoulli numbers. For this reason, the extended Zeilberger's algorithm becomes necessary, and it also suggests that in the study of  $P$ -recursiveness of a polynomial sequence with parameters it is likely that one can get a recurrence relation with polynomial coefficients even for the sequence is not  $P$ -recursive, as long as one allows shifts on the parameters.

## 4. Bernoulli Number Identities

In this section, we give several examples of proving identities on Bernoulli numbers by using the extended Zeilberger's algorithm.

The first example is the extension of Kaneko's identity given by Momiyama [16]. It was proved by using a  $p$ -adic integral over  $\mathbb{Z}_p$ . The Kaneko identity is stated as follows [11]

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \tilde{B}_{n+k} = 0, \quad (4.1)$$

where  $\tilde{B}_n = (n + 1)B_n$ .

While our approach does not directly apply to Kaneko's identity because it has no parameters, we can deal with Momiyama's identity which reduces Kaneko's identity by setting  $m = n$ .

**Theorem 4.1 (Momiyama's identity).**

$$(-1)^m \sum_{k=0}^m \binom{m+1}{k} (n+k+1)B_{n+k} = -(-1)^n \sum_{k=0}^n \binom{n+1}{k} (m+k+1)B_{m+k}, \quad (4.2)$$

where  $m$  and  $n$  are integers and  $m + n > 0$ .

*Proof.* Denote the left hand side and the right hand side of (4.2) by  $L(n, m)$  and  $R(n, m)$ , respectively. By the contour integral definition of the Bernoulli numbers, we have

$$L(n, m) = \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=0}^m (-1)^m \binom{m+1}{k} (n+k+1) \frac{(n+k)!}{z^{n+k}} \right) dz.$$

Denote the summand in the above summation by  $F(n, m, k)$ , that is,

$$F(n, m, k) = (-1)^m \binom{m+1}{k} (n+k+1) \frac{(n+k)!}{z^{n+k}}.$$

Applying the extended Zeilberger's algorithm to  $F(n, m, k)$  and assuming that the output is independent of  $z$ , we obtain

$$F(n, m, k) + F(n, m+1, k) + F(n+1, m, k) = G(n, m, k+1) - G(n, m, k), \quad (4.3)$$

where

$$G(n, m, k) = \frac{(-1)^m (m+1)! (n+k+1)!}{(k-1)! (m+2-k)! z^{n+k}}.$$

Summing the telescoping equation (4.3) over  $k$  from 0 to  $m$ , we are led to the following recurrence relation for  $L(n, m)$

$$L(n, m) + L(n, m+1) + L(n+1, m) = -(-1)^m (n+m+2) B_{n+m+1}.$$

Similarly, we find that  $R(n, m)$  also satisfies

$$R(n, m) + R(n, m+1) + R(n+1, m) = (-1)^n (n+m+2) B_{n+m+1}.$$

Considering the parity of  $B_n$ , it is easy to see that

$$((-1)^m + (-1)^n) (n+m+2) B_{n+m+1} = 0.$$

Therefore, both sides of Momiyama's identity (4.2) satisfy the same recurrence relation.

To compute the initial values, setting  $m = 0$  we get  $L(n, 0) = (n+1)B_n$ . It follows from the recursion (2.3) that

$$\sum_{k=0}^n \binom{n+1}{k} B_k = \sum_{k=0}^n \binom{n}{k} B_k + \sum_{k=0}^n \binom{n}{k-1} B_k = 0.$$

On the other hand, for  $n \neq 1$ , we have

$$\begin{aligned} R(n, 0) &= -(-1)^n \sum_{k=0}^n \binom{n+1}{k} (k+1) B_k \\ &= -(-1)^n \left( \sum_{k=0}^n \binom{n+1}{k} k B_k + \sum_{k=0}^n \binom{n+1}{k} B_k \right) \\ &= -(-1)^n (n+1) \sum_{k=0}^n \binom{n}{k-1} B_k \\ &= (-1)^n (n+1) B_n = (n+1) B_n. \end{aligned}$$

It is easily checked that  $L(1, 0) = R(1, 0) = -1$ . So we deduce that  $L(n, 0) = R(n, 0)$  for all  $n \geq 0$ . This completes the proof.  $\blacksquare$

The following identity is due to Gessel and Viennot [9].

**Theorem 4.2 (Gessel-Viennot).**

$$\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \frac{1}{k-j} \binom{2k-2j}{k+1} \binom{2n+1}{2j+1} B_{2n-2j} = \frac{2n+1}{2k-2n+1} \binom{2k-2n+1}{k+1}, \quad n < k. \quad (4.4)$$

*Proof.* Denote the left hand side and the right hand side of the above identity by  $L(n, k)$  and  $R(n, k)$ , respectively. So we get

$$L(n, k) = \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \frac{1}{k-j} \binom{2k-2j}{k+1} \binom{2n+1}{2j+1} \frac{(2n-2j)!}{z^{2n-2j}} \right) dz.$$

Let

$$F(n, k, j) = \frac{1}{k-j} \binom{2k-2j}{k+1} \binom{2n+1}{2j+1} \frac{(2n-2j)!}{z^{2n-2j}}.$$

Applying the extended Zeilberger's algorithm, we get the following recurrence

$$\begin{aligned} 2(n+1)(2n+3)F(n, k, j) + 2(k+2)(2k+3)F(n+1, k+1, j) \\ - (k+2)(k+3)F(n+1, k+2, j) = G(n, k, j+1) - G(n, k, j), \end{aligned}$$

where

$$G(n, k, j) = \frac{4j(2n+3)!(2k-2j+1)!}{(k-2j+1)!(2j)!(k+1)!z^{2n-2j+2}}.$$

By summing the above telescoping equation over  $j$ , we obtain the following recurrence relation for  $L(n, k)$

$$2(n+1)(2n+3)L(n, k) + 2(k+2)(2k+3)L(n+1, k+1) - (k+2)(k+3)L(n+1, k+2) = 0. \quad (4.5)$$

It is easy to check that  $R(n, k)$  also satisfies the above recurrence relation. Since  $n < k$ , we can define  $L(n, n) = R(n, n) = 0$  for  $n \neq 0$ . It is also easy to verify the initial conditions

$$L(0, k) = R(0, k) = \frac{1}{2k+1} \binom{2k+1}{k+1}.$$

This completes the proof. ■

It should be noted that the recurrence relation (4.5) for  $L(n, k)$  was derived by Jacobi [10] in 1834, see Gessel and Viennot [9].

The next identity is due to Gelfand [7].

**Theorem 4.3.** *We have*

$$(-1)^{n-1} \sum_{k=1}^m \binom{m}{k-1} \frac{B_{n+k}}{n+k} + (-1)^{m-1} \sum_{k=1}^n \binom{n}{k-1} \frac{B_{m+k}}{m+k} = \frac{m!n!}{(m+n+1)!}, \quad (4.6)$$

*provided that the integers  $m, n \geq 0$  are not both zero.*



*Proof.* Denote the left and right hand sides of the above identity (4.6) by  $L(n, m) = S(n, m) + T(n, m)$  and  $R(n, m)$ , respectively, where  $S(n, m)$  and  $T(n, m)$  are the first and second sums of  $L(n, m)$ . Note that

$$S(n, m) = \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=1}^m (-1)^{n-1} \binom{m}{k-1} \frac{(n+k)!}{(n+k)z^{n+k}} \right) dz,$$

$$T(n, m) = \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=1}^n (-1)^{m-1} \binom{n}{k-1} \frac{(m+k)!}{(m+k)z^{m+k}} \right) dz.$$

Denote the summand in  $S(n, m)$  by  $F(n, m, k)$ , and by the extended Zeilberger's algorithm, we obtain

$$F(n, m, k) - F(n, m+1) - F(n+1, m) = G(n, m, k+1) - G(n, m, k),$$

where

$$G(n, m, k) = (-1)^{n-1} \frac{m!(n+k-1)!}{(k-2)!(m+2-k)!z^{n+k}}.$$

Summing the above telescoping equation over  $k$  from 1 to  $m$ , we get a recurrence for  $S(n, m)$

$$S(n, m) - S(n, m+1) - S(n+1, m) = (-1)^n \frac{B_{m+n+1}}{m+n+1}.$$

By the same procedure, or by the symmetric property  $T(n, m) = S(m, n)$ , we find that

$$T(n, m) - T(n, m+1) - T(n+1, m) = (-1)^m \frac{B_{m+n+1}}{m+n+1}.$$

With the aid of the property  $B_{2n+1} = 0$  for  $n \geq 1$ , we have

$$L(n, m) - L(n, m+1) - L(n+1, m) = ((-1)^m + (-1)^n) \frac{B_{m+n+1}}{m+n+1} = 0.$$

It is easy to verify that  $R(n, m)$  also satisfies the above recurrence relation. To check the initial values, we have

$$L(n, 0) = 0 - \sum_{k=1}^n \binom{n}{k-1} \frac{B_k}{k} = -\frac{1}{n+1} \sum_{k=1}^n \binom{n+1}{k} B_k = \frac{1}{n+1} = R(n, 0).$$

This completes the proof. ■

Agoh and Dilcher [1, Theorem 2.1] obtained a convolution identity for Bernoulli numbers. By the extended Zeilberger's algorithm and Woodcock's identity (4.10), we can give a direct proof of this result which is restated in the following equivalent form.

**Theorem 4.4.** *Let  $m, n, k \geq 0$  be integers, with  $m$  and  $k$  not both zero. Then*

$$\begin{aligned} \sum_{j=0}^n \binom{n}{j} B_{k+j} B_{m+n-j} &= -\frac{k!m!}{(m+k+1)!} (n + \delta(m, k)(m+k+1)) B_{m+n+k} \\ &+ \sum_{r=0}^{m+k} (-1)^r \frac{B_{m+k+1-r}}{m+k+1-r} (-1)^k \binom{k+1}{r} \left( \frac{k+1-r}{k+1} n - \frac{rm}{k+1} \right) B_{n+r-1} \\ &+ \sum_{r=0}^{m+k} (-1)^r \frac{B_{m+k+1-r}}{m+k+1-r} (-1)^m \binom{m+1}{r} \left( \frac{m+1-r}{m+1} n - \frac{rk}{m+1} \right) B_{n+r-1}, \end{aligned} \tag{4.7}$$

where  $\delta(m, k) = 0$  when  $m = 0$  or  $k = 0$ , and  $\delta(m, k) = 1$  otherwise.

*Proof.* Let  $L(n, m, k)$  and  $R(n, m, k)$  denote the left hand side and the right hand side of the above identity (4.7), respectively. Our approach leads to the recurrence relation

$$S(n, m + 1, k) - S(n + 1, m, k) + S(n, m, k + 1) = 0, \quad (4.8)$$

where  $m \neq 0$  and  $k \neq 0$ . Considering the parity of the Bernoulli numbers, we have  $(-1)^k B_k = B_k$  for  $k \neq 1$ . The known convolution identity on Bernoulli numbers

$$\sum_{k=0}^n \binom{n}{k} B_k B_{n-k} = -n B_{n-1} - (n-1) B_n, \quad n \geq 1 \quad (4.9)$$

yields that

$$\begin{aligned} L(0, m, 1) &= B_1 B_m = -\frac{1}{2} B_m, \\ R(0, m, 1) &= -\frac{1}{m+1} B_{m+1} + \sum_{r=0}^{m+1} (-1)^r \frac{B_{m+2-r}}{m+2-r} (-1) \binom{2}{r} \left(-\frac{rm}{2}\right) B_{r-1} \\ &\quad + \sum_{r=0}^{m+1} (-1)^r \frac{B_{m+2-r}}{m+2-r} (-1)^m \binom{m+1}{r} \left(-\frac{r}{m+1}\right) B_{r-1} \\ &= -\frac{1}{m+1} B_{m+1} - \frac{m}{m+1} B_{m+1} + B_m B_1 + \frac{(-1)^m}{m+1} \sum_{r=0}^m (-1)^r \binom{m+1}{r} B_{m+1-r} B_r \\ &= -B_{m+1} + B_m B_1 + \frac{(-1)^m}{m+1} \left( \sum_{r=0}^{m+1} \binom{m+1}{r} B_{m+1-r} B_r - 2(m+1) B_m B_1 - B_{m+1} \right) \\ &= -B_{m+1} + B_m B_1 - (-1)^m B_{m+1} \\ &= -\frac{1}{2} B_m. \end{aligned}$$

This gives the proof for (4.7) when  $m \neq 0$  and  $k \neq 0$ .

Moreover, if  $m = 0$  or  $k = 0$ , we can simplify the identity to an equivalent form of a known identity discovered by Woodcock [24]

$$\frac{1}{m} \sum_{k=1}^m (-1)^k \binom{m}{k} B_{m-k} B_{n-1+k} = \frac{1}{n} \sum_{k=1}^n (-1)^k \binom{n}{k} B_{n-k} B_{m-1+k}. \quad (4.10)$$

This completes the proof. ■

## 5. Bernoulli Polynomial Identities

In this section, we show that our approach is also valid for proving identities on Bernoulli polynomials. We will explain how this method works by considering an identity due to Sun [21].

**Theorem 5.1.** *We have*

$$(-1)^k \sum_{j=0}^k \binom{k}{j} x^{k-j} B_{l+j}(y) = (-1)^l \sum_{j=0}^l \binom{l}{j} x^{l-j} B_{k+j}(z), \quad (5.1)$$

provided that  $x + y + z = 1$ .

*Proof.* Denote both sides of the above equation by  $L(k, l)$  and  $R(k, l)$ , respectively. We have

$$L(k, l) = \frac{1}{2\pi i} \oint \frac{1}{e^u - 1} \left( \sum_{j=0}^k \sum_{h=0}^{l+j} (-1)^k \binom{k}{j} \binom{l+j}{h} x^{k-j} y^{l+j-h} \frac{h!}{u^h} \right) du.$$

Let  $F(k, l, h, j)$  denote the summand in the above integral, that is,

$$F(k, l, h, j) = (-1)^k \binom{k}{j} \binom{l+j}{h} x^{k-j} y^{l+j-h} \frac{h!}{u^h}.$$

Applying the extended Zeilberger's algorithm to  $F(k, l, h, j)$  with the assumption that the output is independent of the variables  $u$  and  $h$ , we arrive at the relation

$$xF(k, l, h, j) + F(k+1, l, h, j) + F(k, l+1, h, j) = G(k, l, h, j+1) - G(k, l, h, j), \quad (5.2)$$

where

$$G(k, l, h, j) = \frac{xj}{k-j+1} F(k, l, h, j).$$

Summing both sides of (5.2) over  $h$  and  $j$  gives the recurrence relation

$$xL(k, l) + L(k+1, l) + L(k, l+1) = 0.$$

Similarly, it can be shown that  $R(k, l)$  satisfies the same recurrence relation. It remains to check the initial values

$$\begin{aligned} L(0, l) &= B_l(y), \\ R(0, l) &= (-1)^l \sum_{j=0}^l \binom{l}{j} x^{l-j} B_j(z) = (-1)^l \sum_{j=0}^l \binom{l}{j} x^{l-j} (B+z)^j \\ &= (-1)^l (B+x+z)^l = (-1)^l B_l(x+z) = (-1)^l B_l(1-y) = B_l(y), \end{aligned}$$

as desired. ■

It is worth noting that the extended Zeilberger's algorithm is indeed efficient in deriving recurrence relations for multiple sums. The next identity is given by Wu, Sun and Pan [25].

**Theorem 5.2.** *We have*

$$\begin{aligned} &(-1)^m \sum_{k=0}^m \binom{m+1}{k} (n+k+1) B_{n+k}(x) \\ &+ (-1)^n \sum_{k=0}^n \binom{n+1}{k} (m+k+1) B_{m+k}(-x) \\ &= (-1)^m (n+m+1)(n+m+2) x^{n+m}. \end{aligned} \quad (5.3)$$

*Proof.* Denote the two sums on the left hand side of (5.3) by  $S(n, m)$  and  $T(n, m)$  respectively. Let  $L(n, m) = S(n, m) + T(n, m)$ , and let  $R(n, m)$  denote the right hand side of (5.3). Write

$$S(n, m) = \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=0}^m \sum_{j=0}^{n+k} (-1)^m \binom{m+1}{k} (n+k+1) \binom{n+k}{j} x^{n+k-j} \frac{j!}{z^j} \right) dz.$$

Denote the summand in the above expression by  $F(n, m, k, j)$ . Applying the extended Zeilberger's algorithm with the assumption that the output is independent of the parameters  $z$  and  $j$ , we obtain that

$$F(n, m, k, j) + F(n+1, m, k, j) + F(n, m+1, k, j) = G(n, m, k+1, j) - G(n, m, k, j),$$

where

$$G(n, m, k, j) = \frac{k}{m-k+1} F(n, m, k, j).$$

By summing the above telescoping equation over  $j$  from 0 to  $n+k$  and  $k$  from 0 to  $m+1$ , we deduce that

$$S(n, m) + S(n+1, m) + S(n, m+1) = (-1)^{m+1} (n+m+2) B_{n+m+1}(x). \quad (5.4)$$

From the symmetry property it follows that  $T(n, m)(x) = S(m, n)(-x)$ . This leads to the following recurrence relation for  $T(n, m)$

$$T(n, m) + T(n, m+1) + T(n+1, m) = (-1)^{n+1} (n+m+2) B_{n+m+1}(-x). \quad (5.5)$$

Adding (5.4) to (5.5), we derive a recurrence relation satisfied by  $L(n, m)$

$$\begin{aligned} & L(n, m) + L(n+1, m) + L(n, m+1) \\ &= (-1)^{m+1} (n+m+2) B_{n+m+1}(x) + (-1)^{n+1} (n+m+2) B_{n+m+1}(-x) \\ &= (-1)^{m+1} (n+m+2) \sum_{k=0}^{n+m+1} \binom{n+m+1}{k} x^{n+m+1-k} B_k \left( 1 + (-1)^{k+1} \right) \\ &= 2(-1)^{m+1} (n+m+2) \sum_{\substack{k=0 \\ k, \text{ odd}}}^{n+m+1} \binom{n+m+1}{k} x^{n+m+1-k} B_k \\ &= 2(-1)^{m+1} (n+m+2) (n+m+1) x^{n+m} B_1 \\ &= (-1)^m (n+m+1) (n+m+2) x^{n+m}. \end{aligned}$$

It is easy to see that  $R(n, m)$  satisfies the same recurrence relation as  $L(n, m)$ . Based on the well-known identity for Bernoulli polynomials

$$nx^{n-1} = \sum_{k=1}^n \binom{n}{k} B_{n-k}(x) = \sum_{k=0}^{n-1} \binom{n}{k} B_k(x),$$

it is straightforward to verify that

$$L(n, -1) = 0 + (-1)^n \sum_{k=0}^n \binom{n+1}{k} k B_{k-1}(-x)$$

$$\begin{aligned}
&= (-1)^n (n+1) \sum_{k=0}^{n-1} \binom{n}{k} B_k(-x) \\
&= (-1)^n (n+1) n (-x)^{n-1} = -n(n+1)x^{n-1} = R(n, -1).
\end{aligned}$$

This completes the proof. ■

Note that the above identity (5.3) reduces to Momiyama's identity (4.2) by setting  $x = 0$ . We also note that integrating the identity (5.3) over  $x$  and using the Bernoulli number identity (3.1), one can derive the following identity of Wu, Sun and Pan [25]

$$(-1)^m \sum_{i=0}^m \binom{m}{i} B_{n+i}(x) = (-1)^n \sum_{j=0}^n \binom{n}{j} B_{m+j}(-x). \quad (5.6)$$

The following identity is derived by Sun [21].

**Theorem 5.3.** *We have*

$$(-1)^k \sum_{j=0}^k \binom{k}{j} x^{k-j} \frac{B_{l+j+1}(y)}{l+j+1} + (-1)^l \sum_{j=0}^l \binom{l}{j} x^{l-j} \frac{B_{k+j+1}(z)}{k+j+1} = \frac{(-x)^{k+l+1}}{(k+l+1) \binom{k+l}{k}}, \quad (5.7)$$

provided that  $x + y + z = 1$ .

*Proof.* Let  $L(k, l)$  and  $R(k, l)$  denote the left hand side and the right hand side of (5.7), respectively. It can be shown that

$$xL(k, l) + L(k+1, l) + L(k, l+1) = 0.$$

It can also be shown that  $R(k, l)$  satisfies the same recurrence relation. To check the initial conditions, we have

$$\begin{aligned}
L(0, l) &= \frac{B_{l+1}(y)}{l+1} + (-1)^l \sum_{j=0}^l \binom{l}{j} x^{l-j} \frac{B_{j+1}(z)}{j+1} \\
&= \frac{B_{l+1}(y)}{l+1} + \frac{(-1)^l}{l+1} \sum_{j=0}^l \binom{l+1}{l-j+1} x^j B_{l-j+1}(z) \\
&= \frac{B_{l+1}(y)}{l+1} + \frac{(-1)^l}{l+1} \sum_{j=0}^l \binom{l+1}{j} x^j (B+z)^{l-j+1} \\
&= \frac{B_{l+1}(y)}{l+1} + \frac{(-1)^l}{l+1} (B+x+z)^{l+1} - \frac{(-1)^l}{l+1} x^{l+1} \\
&= \frac{B_{l+1}(y)}{l+1} + \frac{(-1)^l}{l+1} B_{l+1}(1-y) - \frac{(-1)^l}{l+1} x^{l+1} \\
&= \frac{B_{l+1}(y)}{l+1} - \frac{1}{l+1} B_{l+1}(y) - \frac{(-1)^l}{l+1} x^{l+1} \quad (\text{by (2.6)}) \\
&= \frac{(-x)^{l+1}}{l+1} = R(0, l),
\end{aligned}$$

as desired. ■

We remark that the above identity (5.7) reduces to (5.1) by viewing  $z = 1 - x - y$  as a function of  $y$  and by taking partial derivative with respect to  $y$ . It also specializes to (5.6) when setting  $y \rightarrow x$  and  $z = -y \rightarrow -x$ . Moreover, differentiating both sides of (5.7) with respect to  $y$  twice, we obtain the following identity derived by Sun [21], which can be verified by our approach. The proof is omitted.

**Theorem 5.4.** *Suppose that  $x + y + z = 1$ , then*

$$\begin{aligned} & (-1)^k \sum_{j=0}^k \binom{k+1}{j} x^{k-j+1} (l+j+1) B_{l+j}(y) \\ & + (-1)^l \sum_{j=0}^l \binom{l+1}{j} x^{l-j+1} (k+j+1) B_{k+j}(z) \\ & = (-1)^k (k+l+2) (B_{k+l+1}(x+y) - B_{k+l+1}(y)). \end{aligned} \quad (5.8)$$

In [22, Theorem 1.1], Sun and Pan find a symmetric relation between products of the Bernoulli polynomials.

**Theorem 5.5.** *Let  $n \in \mathbb{Z}^+$  and  $x + y + z = 1$ . If  $r + s + t = n$ , then*

$$\begin{aligned} & r \sum_{k=0}^n (-1)^k \binom{s}{k} \binom{t}{n-k} B_{n-k}(x) B_k(y) \\ & + s \sum_{k=0}^n (-1)^k \binom{t}{k} \binom{r}{n-k} B_{n-k}(y) B_k(z) \\ & + t \sum_{k=0}^n (-1)^k \binom{r}{k} \binom{s}{n-k} B_{n-k}(z) B_k(x) = 0. \end{aligned} \quad (5.9)$$

*Proof.* Denote the three sums on the left hand side of the above identity by  $S(n, r, s)$ ,  $T(n, r, s)$ ,  $R(n, r, s)$  respectively. Since  $n = r + s + t$ ,  $S(n, r, s)$  can be expressed as

$$\left(\frac{1}{2\pi i}\right)^2 \oint \frac{1}{e^u - 1} \oint \frac{1}{e^v - 1} \left( \sum_{k,j,h} (-1)^k \binom{s}{k} \binom{n-r-s}{n-k} \binom{n-k}{j} \binom{k}{h} \frac{j!}{u^j} \frac{h!}{v^h} r x^{n-k-j} y^{k-h} \right) dudv.$$

Our approach yields the following recurrence relation

$$(s+1)S(n, r+1, s) + (r+1)S(n, r, s+1) + (n-r-s-1)S(n, r+1, s+1) = 0.$$

Similarly, it can be shown that  $T(n, r, s)$  and  $R(n, r, s)$  satisfy the same recurrence relation. Since  $r + s + t = n$ , we obtain that

$$S(n, 0, s) + T(n, 0, s) + R(n, 0, s) = (-1)^n s \binom{n-s}{n} B_n(z) + (n-s) \binom{s}{n} B_n(z) = 0,$$

and

$$S(n, r, 0) + T(n, r, 0) + R(n, r, 0) = r \binom{n-r}{n} B_n(x) + (n-r) (-1)^n \binom{r}{n} B_n(x) = 0.$$

It follows that  $S(n, r, s) + T(n, r, s) + R(n, r, s)$  is identically zero. This completes the proof for all integers  $r, s$  and  $t$ . Then by taking the left hand side of (5.9) as a polynomial in  $r, s, t$ , it follows that (5.9) is true for all  $r, s, t$  such that  $r + s + t = n$ .  $\blacksquare$

## 6. Euler Number and Polynomial Identities

In this section, we show how to prove identities on Euler numbers and polynomials by using our approach. As the first example, we consider the following identity due to Wu, Sun and Pan [25].

**Theorem 6.1.** *We have*

$$(-1)^m \sum_{k=0}^m \binom{m}{k} \frac{E_{n+k}}{2^{n+k}} = (-1)^n \sum_{j=0}^n \binom{n}{j} E_{m+j} \left(-\frac{1}{2}\right), \quad (6.1)$$

where  $m$  and  $n$  are nonnegative integers.

*Proof.* Denote the left and right hand sides of (6.1) by  $L(n, m)$  and  $R(n, m)$ , respectively. By the contour integral definition of the Euler numbers (2.2) and the relation (2.5), we have

$$L(n, m) = \frac{1}{2\pi i} \oint \frac{2e^z}{e^{2z} + 1} \left( \sum_{k=0}^m (-1)^m \binom{m}{k} \frac{(n+k)!}{2^{n+k} z^{n+k+1}} \right) dz,$$

$$R(n, m) = \frac{1}{2\pi i} \oint \frac{2e^z}{e^{2z} + 1} \left( \sum_{j=0}^n \sum_{k=0}^{m+j} (-1)^n \binom{n}{j} \binom{m+j}{k} (-1)^{m+j-k} \frac{k!}{2^k z^{k+1}} \right) dz.$$

Denote the summands in the above two integrands by

$$S(m, n, k) = (-1)^m \binom{m}{k} \frac{(n+k)!}{2^{n+k} z^{n+k+1}},$$

$$T(m, n, k, j) = (-1)^n \binom{n}{j} \binom{m+j}{k} (-1)^{m+j-k} \frac{k!}{2^k z^{k+1}}.$$

Applying the extended Zeilberger's algorithm, we obtain

$$S(n, m, k) + S(n, m+1, k) + S(n+1, m, k) = G(n, m, k+1) - G(n, m, k),$$

$$T(n, m, k, j) + T(n, m+1, k, j) + T(n+1, m, k, j) = H(n, m, k, j+1) - H(n, m, k, j),$$

where

$$G(n, m, k) = \frac{k}{m+1-k} S(n, m, k), \quad H(n, m, k, j) = \frac{j}{n+1-j} T(n, m, k, j).$$

Therefore,  $L(n, m)$  and  $R(n, m)$  satisfy the same recurrence

$$L(n, m) + L(n, m+1) + L(n+1, m) = 0.$$

Consequently, the identity (6.1) can be verified by computing the initial values

$$L(0, m) = (-1)^m \sum_{k=0}^m \binom{m}{k} \frac{E_k}{2^k} = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \frac{E_k}{2^k} = E_m \left( -\frac{1}{2} \right) = R(0, m),$$

as desired. ■

Wu, Sun and Pan [25] also derived an identity by substituting the Bernoulli polynomials in (5.6) with Euler polynomials. This identity can be verified by our approach. The proof is omitted.

**Theorem 6.2.** *We have*

$$(-1)^m \sum_{k=0}^m \binom{m}{k} E_{n+k}(x) = (-1)^n \sum_{k=0}^n \binom{n}{k} E_{m+k}(-x). \quad (6.2)$$

Note that differentiating both sides of the identity (6.2) with respect to  $x$  leads to the following identity also due to Wu, Sun and Pan [25]:

$$\begin{aligned} & (-1)^m \sum_{k=0}^m \binom{m+1}{k} (n+k+1) E_{n+k}(x) \\ & + (-1)^n \sum_{k=0}^n \binom{n+1}{k} (m+k+1) E_{m+k}(-x) \\ & = (-1)^m 2(n+m+2)(x^{n+m+1} - E_{n+m+1}(x)). \end{aligned} \quad (6.3)$$

The following identity is derived by Sun [21].

**Theorem 6.3.**

$$(-1)^k \sum_{j=0}^k \binom{k}{j} x^{k-j} \frac{E_{l+j+1}(y)}{l+j+1} + (-1)^l \sum_{j=0}^l \binom{l}{j} x^{l-j} \frac{E_{k+j+1}(z)}{k+j+1} = \frac{(-x)^{k+l+1}}{(k+l+1) \binom{k+l}{k}}, \quad (6.4)$$

provided that  $x + y + z = 1$ .

*Proof.* Denote the two sums in the left hand side of the above identity by  $S(k, l)$  and  $T(k, l)$  respectively. Let  $L(k, l) = S(k, l) + T(k, l)$ , and let  $R(k, l)$  denote the right hand side of (6.4). By computation, we find

$$xS(k, l) + S(k+1, l) + S(k, l+1) = 0.$$

Since  $T(k, l) = S(l, k)$ ,

$$xT(k, l) + T(k+1, l) + T(k, l+1) = 0.$$

Therefore,

$$xL(k, l) + L(k+1, l) + L(k, l+1) = 0.$$



It is easy to check that  $R(k, l)$  satisfies the same recurrence relation. To check the initial values, we have

$$\begin{aligned}
L(0, l) &= \frac{E_{l+1}}{l+1} + (-1)^l \frac{1}{l+1} \sum_{j=0}^l \binom{l+1}{j+1} x^{l-j} E_{j+1}(z) \\
&= \frac{E_{l+1}}{l+1} + (-1)^l \frac{1}{l+1} \sum_{j=1}^{l+1} \binom{l+1}{j} x^{l+1-j} E_j(z) \\
&= \frac{E_{l+1}}{l+1} + (-1)^l \frac{1}{l+1} (E_{l+1}(x+z) - x^{l+1}) \\
&= \frac{(-x)^{l+1}}{l+1} = R(0, l),
\end{aligned}$$

as desired. ■

Our approach can also be applied to identities involving products of the Euler polynomials and the Bernoulli polynomials. We take the following identity of Sun and Pan [22, Theorem 1.1] as an example. Note that in Sun and Pan's identity, the variables  $r, s$  and  $t$  should be real numbers.

**Theorem 6.4.** *Let  $n \in \mathbb{Z}^+$ ,  $r + s + t = n - 1$  and  $x + y + z = 1$ , then*

$$\begin{aligned}
&\sum_{k=0}^n (-1)^k \binom{r}{k} \binom{s}{n-k} B_k(x) E_{n-k}(z) \\
&- (-1)^n \sum_{k=0}^n (-1)^k \binom{r}{k} \binom{t}{n-k} B_k(y) E_{n-k}(z) \\
&= \frac{r}{2} \sum_{l=0}^{n-1} (-1)^l \binom{s}{l} \binom{t}{n-1-l} E_l(y) E_{n-1-l}(x). \tag{6.5}
\end{aligned}$$

*Proof.* Denote the two sums on the left hand side of the above identity (6.5) by  $S(n, r, s)$  and  $T(n, r, s)$  respectively. Let  $L(n, r, s) = S(n, r, s) - T(n, r, s)$ , and denote the right hand side of (6.5) by  $R(n, r, s)$ . Note that

$$\begin{aligned}
S(n, r, s) &= \left( \frac{1}{2\pi i} \right)^2 \oint \frac{1}{e^u - 1} \oint \frac{2e^v}{e^{2v} + 1} \left( \sum_{k=0}^n \sum_{j=0}^k \sum_{h=0}^{n-k} (-1)^k \binom{r}{k} \binom{s}{n-k} \binom{k}{j} \right. \\
&\quad \left. \times x^{k-j} \frac{j!}{u^j} \binom{n-k}{h} \left( z - \frac{1}{2} \right)^{n-k-h} \frac{h!}{2^h v^{h+1}} \right) dudv.
\end{aligned}$$

Applying the extended Zeilberger's algorithm, we have

$$(s+1)S(n, r+1, s) + (r+1)S(n, r, s+1) + (n-s-r-2)S(n, r+1, s+1) = 0.$$

It can also be shown that  $T(n, r, s)$  and  $R(n, r, s)$  satisfy the same recurrence relation. Setting  $r = 0$ , since  $r + s + t = n - 1$ , it follows that

$$L(n, 0, s) = \left( \binom{s}{n} - (-1)^n \binom{n-1-s}{n} \right) E_n(z) = 0 = R(n, 0, s).$$

To show that  $L(n, r, 0) = R(n, r, 0)$ , it is equivalent to verify

$$\sum_{k=0}^n (-1)^{n+k} \binom{r}{k} \binom{n-1-r}{n-k} B_k(y) E_{n-k}(z) = (-1)^n \binom{r}{n} B_n(x) - \frac{r}{2} \binom{n-1-r}{n-1} E_{n-1}(x). \quad (6.6)$$

It is easy to see that both sides of the above identity satisfy the following recurrence relation

$$(r+1)S(n, r) + (n-1-r)S(n, r+1) = 0.$$

Then (6.6) can be proved by checking the initial case  $r = 0$ . This completes the proof of (6.5) for all integers  $r, s$  and  $t$ . Considering both sides of (6.5) as polynomials in  $r, s, t$ , we deduce that (6.5) holds for all  $r, s, t$  such that  $r + s + t = n - 1$ .  $\blacksquare$

## 7. Deriving New Identities

Applying the original Zeilberger's algorithm to a Bernoulli number summation, we may obtain a recurrence relation for the summand which contains the integral variable  $z$ . Although such a recurrence cannot be used to prove the Bernoulli number identity itself, it may lead to a new identity. For example, let us consider Kaneko's identity (4.1)

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \tilde{B}_{n+k} = 0,$$

where  $\tilde{B}_n = (n+1)B_n$ . From the recurrence obtained by Zeilberger's algorithm, we can get the following generalization of this identity.

**Theorem 7.1.** *We have*

$$\sum_{k=0}^{n+3} \binom{n+3}{k} (n+k+3)(n+k+2) \tilde{B}_{n+k} = 0. \quad (7.1)$$

*Proof.* Denote the left hand side of Kaneko's identity by  $L(n)$ . By the contour integral definition of the Bernoulli numbers, we have

$$\begin{aligned} L(n) &= \sum_{k=0}^{n+1} \binom{n+1}{k} (n+k+1) B_{n+k} \\ &= \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=0}^{n+1} \binom{n+1}{k} (n+k+1) \frac{(n+k)!}{z^{n+k}} \right) dz. \end{aligned}$$

Denote the summation in the above integral by  $S(n)$ . Obviously,

$$L(n) = \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} S(n) dz = 0$$

for all  $n \geq 0$ . Applying Zeilberger's algorithm, we get

$$z^2 S(n+2) = 2(n+3)(2n+5)S(n+1) + (n+2)(n+3)S(n).$$

By integrating over  $z$  on both sides of the above recurrence, it follows that

$$\begin{aligned}
& \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} z^2 S(n+2) dz \\
&= \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=0}^{n+3} \binom{n+3}{k} (n+k+3) \frac{(n+k+2)!}{z^{n+k}} \right) dz \\
&= \sum_{k=0}^{n+3} \binom{n+3}{k} (n+k+3)(n+k+2)(n+k+1) B_{n+k} \\
&= \sum_{k=0}^{n+3} \binom{n+3}{k} (n+k+3)(n+k+2) \tilde{B}_{n+k} \\
&= 2(n+3)(2n+5) \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} S(n+1) dz + (n+2)(n+3) \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} S(n) dz = 0.
\end{aligned}$$

This completes the proof.  $\blacksquare$

It should be mentioned that the above identity (7.1) is the special case  $s = r = 3$  of an identity of K.-W. Chen [3]:

$$\sum_{k=0}^{n+r} \binom{n+r}{k} \binom{n+r+k}{s} B_{n+r+k-s} = 0, \quad n \in \mathbb{N}, r, s \in \mathbb{Z}^+ \text{ and } s \text{ is odd.} \quad (7.2)$$

It can be seen that our approach also applies to the above identity (7.2).

Gessel [8, Theorem 7.3] extended Kaneko's identity (4.1) to the following form

$$\frac{1}{n+1} \sum_{k=0}^{n+1} m^{n+1-k} \binom{n+1}{k} \tilde{B}_{n+k} = \sum_{k=1}^{m-1} \left( (2n+1)k - (n+1)m \right) k^n (k-m)^{n-1}. \quad (7.3)$$

Notice that when  $m = 1$ , the above identity becomes Kaneko's identity. From the above identity, we can deduce the following theorem by applying Zeilberger's algorithm.

**Theorem 7.2.** *We have*

$$\frac{1}{(n+3)} \sum_{k=0}^{n+3} m^{n+3-k} \binom{n+3}{k} (n+k+3)(n+k+2) \tilde{B}_{n+k} = \sum_{k=1}^{m-1} p(n, m, k) k^n (k-m)^{n-1}, \quad (7.4)$$

where

$$\begin{aligned}
p(n, m, k) = & 2(n+2)(2n+3)(2n+5)k^3 - 2m(n+2)(2n+5)(3n+5)k^2 \\
& + 3m^2(n+2)(2n^2+7n+7)k - m^3(n+1)^2(n+2).
\end{aligned}$$

*Proof.* Denote the left hand side and the right hand side of (7.3) by  $L(n, m)$  and  $R(n, m)$ , respectively. Then we have

$$L(n, m) = \frac{1}{n+1} \sum_{k=0}^{n+1} m^{n+1-k} \binom{n+1}{k} (n+k+1) B_{n+k}$$

$$= \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=0}^{n+1} m^{n+1-k} \binom{n+1}{k} \frac{(n+k+1)(n+k)!}{n+1 z^{n+k}} \right) dz.$$

Denote the summation in the above integral by  $S(n, m)$ . By Zeilberger's algorithm, we find that

$$z^2 S(n+2, m) = 2(n+2)(2n+5)S(n+1, m) + m^2(n+1)(n+2)S(n, m). \quad (7.5)$$

Integrating the left hand side of the above recurrence over  $z$ , we get

$$\begin{aligned} & \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} (z^2 S(n+2, m)) dz \\ &= \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( \sum_{k=0}^{n+3} m^{n+3-k} \binom{n+3}{k} \frac{(n+k+3)(n+2+k)!}{n+3 z^{n+k}} \right) dz \\ &= \frac{1}{(n+3)} \sum_{k=0}^{n+3} m^{n+3-k} \binom{n+3}{k} (n+k+3)(n+k+2) \tilde{B}_{n+k}. \end{aligned}$$

On the other hand, integrating the right hand side of (7.5) over  $z$  and substituting  $L(n, m)$  by  $R(n, m)$ , we obtain

$$\begin{aligned} & \frac{1}{2\pi i} \oint \frac{1}{e^z - 1} \left( 2(n+2)(2n+5)S(n+1, m) + m^2(n+1)(n+2)S(n, m) \right) dz \\ &= 2(n+2)(2n+5)L(n+1, m) + m^2(n+1)(n+2)L(n, m) \\ &= 2(n+2)(2n+5) \sum_{k=1}^{m-1} \left( (2n+3)k - (n+2)m \right) k^{n+1} (k-m)^n \\ & \quad + m^2(n+1)(n+2) \sum_{k=1}^{m-1} \left( (2n+1)k - (n+1)m \right) k^n (k-m)^{n-1} \\ &= \sum_{k=1}^{m-1} p(n, m, k) k^n (k-m)^{n-1}, \end{aligned}$$

as desired. ■

Obviously, the above identity (7.4) reduces to (7.1) by setting  $m = 1$ .

## 8. Concluding Remarks

To conclude this paper, we remark that our approach is not restricted to identities on Bernoulli and Euler polynomials. It also applies to sequences  $a_0, a_1, a_2, \dots$  whose generating functions  $f(z)$  lead to contour integral representations of  $a_n$  with hypergeometric integrands. For example, the Genocchi numbers fall into this framework. We can apply the extended Zeilberger's algorithm to prove the following identity on Genocchi numbers

$$\sum_{k=0}^n (-1)^k \binom{n}{k} G_{m+k} = \sum_{k=0}^m (-1)^k \binom{m}{k} \sum_{j=0}^{n+k} (-1)^j \binom{n+k}{j} G_j, \quad (8.1)$$

where  $m, n \in \mathbb{Z}^+$ . Recall that the Genocchi numbers can be defined by the generating function

$$\sum_{n=1}^{\infty} G_n \frac{z^n}{n!} = \frac{2z}{e^z + 1}.$$

We note that there are other approaches to proving identities related to special numbers and functions. For example, Paule and Schneider [17] used Karr's summation algorithm in difference fields [12] and Zeilberger's algorithm to prove harmonic number identities and derive new identities. Kauers [13] gave an algorithm which can be applied to verify many known identities on Stirling numbers and to discover new identities. Stan [20] applied Wegschaider's mathematica software package `MultiSum` [23] to deal with identities related to Poisson integrals. Moreover, the package `MultiSum` can establish multiple index recurrence relations for the hypergeometric terms with parameters which can also be established by using the extended Zeilberger's algorithm.

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