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Faulhaber's Theorem on Power Sums

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Abstract. We observe that the classical Faulhaber's theorem on sums of odd powers also holds for an arbitrary arithmetic progression, namely, the odd power sums of any arithmetic progression $a+b, a+2b, \ldots, a+nb$ is a polynomial in $na+n(n+1)b/2$. While this assertion can be deduced from the original Fauhalber's theorem, we give an alternative formula in terms of the Bernoulli polynomials. Moreover, by utilizing the central factorial numbers as in the approach of Knuth, we derive formulas for r-fold sums of powers without resorting to the notion of r-reflexive functions. We also provide formulas for the r-fold alternating sums of powers in terms of Euler polynomials.

Keywords: Faulhaber's theorem, power sum, alternating sum, r-fold power sum, r-fold alternating power sum, Bernoulli polynomial, Euler polynomial

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1 Introduction

The classical theorem of Faulhaber states that the sums of odd powers

$$1^{2m-1} + 2^{2m-1} + \ldots + n^{2m-1}$$

can be expressed as a polynomial of the triangular number $T_n = n(n+1)/2$; See Beardon [2], Knuth [12]. Moreover, Faulhaber observed that the r-fold summation of n^m is a polynomial in $n(n+r)$ when m is positive and $m-r$ is even [12]. The classical

Faulhaber theorem for odd power sums was proved by Jacobi [11]; See also Edwards [4]. Let us recall the notation on the r-fold power sums: $\sum^0 n^m = n^m$, and

$$\sum^r n^m = \sum^{r-1} 1^m + \sum^{r-1} 2^m + \cdots + \sum^{r-1} n^m. \tag{1.1}$$

For example, $\sum^1 n^m = 1^m + 2^m + \cdots + n^m$, and

$$\sum^2 n^m = \sum^1 1^m + \sum^1 2^m + \cdots + \sum^1 n^m = \sum_{i=1}^{n} (n+1-i) i^m.$$

For even powers, it has been shown that the sum $1^{2m} + 2^{2m} + \cdots + n^{2m}$ is a polynomial in the triangular number T_n multiplied by a linear factor in n. Gessel and Viennot [6] had a remarkable discovery that the alternating sum $\sum_{i=1}^{n} (-1)^{n-i} i^{2m}$ is also a polynomial in the triangular number T_n.

Faulhaber's theorem has drawn much attention from various points of view. Grosset and Veselov [7] investigated a generalization of the Faulhaber polynomials related to elliptic curves. Warnaar [15], Schlosser [14] and Zhao and Feng [16] studied the q-analogues of the formulas for the first few power sums. Garrett [5] found a combinatorial proof of the formula for sums of q-cubes. Guo and Zeng [8] obtained the q-analogue formula in the general case. Furthermore, Guo, Rubey and Zeng [9] have shown that the q-Faulhaber and q-Salié coefficients are nonnegative and symmetric in a combinatorial setting of nonintersecting lattice paths.

In this paper, we first formulate Faulhaber's theorem in a more general framework, that is, in terms of power sums of an arithmetic progression. Given an arithmetic progression:

$$a + b, \quad a + 2b, \quad \ldots, \quad a + nb,$$

Faulhaber's theorem implies that odd power sums of the above series are polynomials in $na + n(n+1)b/2$. In particular, an odd power sum of the first n odd numbers

$$1^{2m-1} + 3^{2m-1} + \cdots + (2n-1)^{2m-1}$$

is a polynomial in n^2, and the sum

$$1^{2m-1} + 4^{2m-1} + 7^{2m-1} + \cdots + (3n-2)^{2m-1}$$

is a polynomial in the pentagonal number $n(3n-1)/2$.

Because of the relation $(a+bi)^m = b^m (a/b + i)^m$, there is no loss of generality to consider the series

$$x+1, x+2, \ldots, x+n. \tag{1.2}$$

Let

$$\lambda = n(n+2x+1) \tag{1.3}$$

be the sum of the sequence $x+1, x+2, \ldots, x+n$. The the power sums

$$S_{2m-1} = (x+1)^{2m-1} + (x+2)^{2m-1} + \cdots + (x+n)^{2m-1}$$

is a polynomial in λ. For exmaple,

$$\begin{aligned}
S_3 &= \frac{\lambda^2}{4} + \frac{(x^2+x)}{2}\lambda; \\
S_5 &= \frac{\lambda^3}{6} + \frac{1}{12}(6x^2+6x-1)\lambda^2 + \frac{1}{6}(3x^4+6x^3+2x^2-x)\lambda; \\
S_7 &= \frac{\lambda^4}{8} + \frac{1}{6}(3x^2+3x-1)\lambda^3 + \frac{1}{12}(9x^4+18x^3+3x^2-6x+1)\lambda^2 \\
&\quad + \frac{1}{6}(3x^6+9x^5+6x^4-3x^3-2x^2+x)\lambda.
\end{aligned}$$

It should be noticed that the above more general setting of Faulhaber's theorem can be deduced on the the original version of Faulhaber's theorem. When x is a positive integer, we have the relation

$$\sum_{i=1}^{n}(x+i)^m = \sum_{i=1}^{n+x} i^m - \sum_{i=1}^{x} i^m.$$

By Faulhaber's theorem, $\sum_{i=1}^{n+x} i^m$ and $\sum_{i=1}^{x} i^m$ are polynomials in $(n+x)(n+x+1)$ and $x(x+1)$, respectively. Using the following simple but important relation

$$(n+x)(n+x+1) = n(n+2x+1) + x(x+1), \tag{1.4}$$

we see that

$$[(n+x)(n+x+1)]^i - [x(x+1)]^i = \sum_{k=1}^{i} \binom{i}{k} [n(n+2x+1)]^k [x(x+1)]^{i-k}, \tag{1.5}$$

which is a polynomial in $n(n+2x+1)$. Clearly, one sees that the above assertion holds for all real numbers x.

Although in principle Faulhaber's theorem is valid for any arithmetic progression, from a computational point of view it still seems worthwhile to find a formula for the coefficients in terms Bernoulli polynomials. The main result of this paper is an approach to the study of the r-fold sums of powers without resorting to the properties of r-reflective functions as in the approach of Knuth [12]. In the last section, we obtain formulas for the r-fold alternating sums of powers in terms of the Euler polynomials.

2 An Alternative Formula

In this section, we give an explicit formula for the coefficients regarding Faulhaber's theorem for the series $x+1, x+2, \ldots, x+n$, which reduces to an alternative formula to the Gessel-Viennot formula when setting $x=0$. We first recall some basic facts about Bernoulli polynomials $B_n(x)$ which are defined by the following generating function:

$$\sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} = \frac{te^{xt}}{e^t - 1}. \tag{2.1}$$

The power sums of the first n positive integers can be expressed in terms of $B_i(x)$:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}(B_{m+1}(n+1) - B_{m+1}(1)).$$

Moreover, we have

$$\sum_{i=1}^{n} (x+i)^{2m-1} = \frac{1}{2m}\left(B_{2m}(x+n+1) - B_{2m}(x+1)\right). \tag{2.2}$$

The Bernoulli numbers B_n are given by $B_n = B_n(0)$. Note that the Bernoulli polynomials satisfy the following relations

$$\begin{aligned} B_n(x+1) - B_n(x) &= nx^{n-1}, & (2.3)\\ B_n(1-x) &= (-1)^n B_n(x), & (2.4)\\ \frac{d}{dx} B_n(x) &= nB_{n-1}(x), & (2.5)\\ B_n(x+y) &= \sum_{i=0}^{n} \binom{n}{i} B_i(x) y^{n-i}. & (2.6) \end{aligned}$$

The evaluation of Bernoulli polynomials at $1/2$ is of special interest. For $n \geq 0$, we have

$$B_{2n+1}(1/2) = 0, \qquad B_{2n}(1/2) = (2^{1-2n} - 1)B_{2n}. \tag{2.7}$$

From (2.6) and (2.7), we can deduce the following form of Faulhaber's theorem.

Theorem 2.1 *Let $\lambda = n(n+2x+1)$. Then we have*

$$\sum_{i=1}^{n} (x+i)^{2m-1} = \sum_{k=1}^{m} F_k^{(m)}(x) \lambda^k, \tag{2.8}$$

where

$$F_k^{(m)}(x) = \frac{1}{2m} \sum_{i=k}^{m} \binom{2m}{2i} \binom{i}{k} \left(x + \frac{1}{2}\right)^{2i-2k} B_{2m-2i}\left(\frac{1}{2}\right). \tag{2.9}$$

Proof. From the binomial expansion (2.6), we get

$$B_{2m}(x+n+1) = \sum_{i=0}^{2m} \binom{2m}{i} B_{2m-i}\left(\frac{1}{2}\right) \cdot \left(x+n+\frac{1}{2}\right)^i. \tag{2.10}$$

It follows from (2.2) and (2.7) that

$$\sum (x+n)^{2m-1} = \frac{1}{2m} \sum_{i=0}^{m} \binom{2m}{2i} B_{2m-2i}\left(\frac{1}{2}\right) \cdot \left(x+n+\frac{1}{2}\right)^{2i} - \frac{1}{2m} B_{2m}(x+1).$$

Since

$$\left(x+n+\frac{1}{2}\right)^{2i} = \left(\lambda + \left(x+\frac{1}{2}\right)^2\right)^i = \sum_{k=0}^{i} \binom{i}{k} \left(x+\frac{1}{2}\right)^{2i-2k} \lambda^k, \tag{2.11}$$

we immediately get (2.9) for $k \geqslant 1$. For $k = 0$, we have

$$F_0^{(m)}(x) = \frac{1}{2m} \sum_{i=0}^{m} \binom{2m}{2i} B_{2m-2i}\left(\frac{1}{2}\right) \left(x+\frac{1}{2}\right)^{2i} = \frac{1}{2m} B_{2m}(x+1).$$

This completes the proof. ∎

The above formula can be viewed as an alternative form of the formula of Gessel and Viennot [6] for the coefficients $A_k^{(m)} = F_k^{(m)}(0)$:

$$A_k^{(m)} = (-1)^{m-k} \sum_j \binom{2m}{m-k-j} \binom{m-k+j}{j} \frac{m-k-j}{m-k+j} B_{m+k+j}, \quad 0 \le k < m.$$

Note that $B_0 = 1$ and $B_1 = -1/2$ are used in the above formula whereas the formula (2.9) does not involve B_1. The equivalence between the formulas for $F_k^{(m)}(0)$ and $A_k^{(m)}$ can be established via the following generating function for the coefficients $F_k^{(m+1)}(x)$. The proof is analogous to that given by Gessel and Viennot [6] for the case $x = 0$.

Theorem 2.2 *We have*

$$\sum_{m=0}^{\infty} \sum_{k=1}^{\infty} F_k^{(m+1)}(x) t^k \frac{y^{2m+1}}{(2m+1)!} = \frac{\cosh y\sqrt{(x+\frac{1}{2})^2 + t} - \cosh y(x+\frac{1}{2})}{2\sinh(\frac{y}{2})}. \tag{2.12}$$

Similarly, the theorem of Gessel and Viennot on the alternating sums of even powers of the first n natural numbers can be extended to an arithmetic progression $x+1, x+2, \ldots, x+n$. It turns out that the Euler polynomials play the same role as the Bernoulli polynomials for sums of odd powers.

The Euler polynomials $E_n(x)$ are defined by

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1}. \tag{2.13}$$

The following expansion formula holds:

$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}. \tag{2.14}$$

For positive even number n, we have $E_n(1) = 0$. The Euler numbers E_n and the Euler polynomials are related by

$$E_{2n+1} = 0, \quad E_n = 2^n E_n(1/2), \quad n \geq 0. \tag{2.15}$$

Theorem 2.3 *Let $\lambda = n(n+2x+1)$. Then we have*

$$\sum_{i=1}^{n} (-1)^{n-i} (x+i)^{2m} = \sum_{k=0}^{m} G_k^{(m)}(x) \lambda^k,$$

where

$$\begin{aligned} G_k^{(m)}(x) &= \frac{1}{2} \sum_{i=k}^{m} \binom{2m}{2i} \binom{i}{k} E_{2m-2i} \left(\frac{1}{2}\right) \left(x+\frac{1}{2}\right)^{i-k}, \qquad 1 \leq k \leq m; \\ G_0^{(m)}(x) &= \frac{1}{2}(1-(-1)^n) E_{2m}(x+1). \end{aligned}$$

The generating function for $G_k^{(m)}(x)$ is given below, which is a straightforward extension of the formula of Gessel and Viennot [6].

Theorem 2.4 *We have*

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} G_k^{(m)}(x) t^k \frac{y^{2m}}{(2m)!} = \frac{\cosh y \sqrt{(x+\frac{1}{2})^2 + t}}{2 \cosh(\frac{y}{2})}. \tag{2.16}$$

3 r-Fold Sums of Powers

In this section, we derive a formula for the r-fold sums of powers of the series $x+1, x+2, \ldots, x+n$ in terms of the central factorial numbers as used in the approach of Knuth [12]. A key step in our approach is the r-fold summation formula for the lower factorials. It can be seen from our formula that if r and m have the same parity then the r-fold power sum $\sum^r (n+x)^m$ is a polynomial in $n(n+2x+r)$ plus a term that vanishes when $x=0$. The cases when r and m have different parities can be dealt with some care, and the details are omitted.

Recall the notation for the lower factorials $(x)_k = x(x-1)\cdots(x-k+1)$. The definition of the central factorials $x^{[k]}$ [13, p. 213], is given by

$$x^{[k]} = x(x+k/2-1)_{k-1}.$$

The central factorial numbers $T(m,k)$ are determined by the following relation:

$$x^m = \sum_{k=1}^{m} T(m,k) x^{[k]}, \quad m \geq 1. \tag{3.1}$$

Note that $T(m,k)=0$ when $m-k$ is odd. In particular, we need the following relation

$$x^{2m-1} = \sum_{k=1}^{m} T(2m,2k)(x+k-1)_{2k-1}. \tag{3.2}$$

We first give a formula for the r-fold sums of lower factorials. From the recursive definition (1.1) of r-fold summations, we have

$$\sum\nolimits^{r} (x+n)_l = \sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_r\leqslant n} (x+i_1)_l. \tag{3.3}$$

The above multiple summation can be simplified to a single sum.

Theorem 3.1 *We have*

$$\sum\nolimits^{r} (x+n)_l \;=\; \frac{(x+n+r)_{l+r}}{(l+r)_r} - \sum_{i=1}^{r} \binom{n+r-i-1}{r-i} \frac{(x+i)_{l+i}}{(l+i)_i}.$$

Proof. Setting

$$F(n) = \frac{1}{l+1}(x+n+1)_{l+1}$$

gives $F(i)-F(i-1)=(x+i)_l$. Hence we get

$$\begin{aligned}\sum_{1\leqslant i_1\leqslant i_2}(x+i_1)_l &= \sum_{i_1=1}^{i_2}\Big(F(i_1)-F(i_1-1)\Big)\\ &= \frac{1}{l+1}(x+i_2+1)_{l+1}-\frac{1}{l+1}(x+1)_{l+1}. \qquad (3.4)\end{aligned}$$

Iterating (3.4) $r-1$ times and using the following identity

$$\sum_{i=1}^{n}\binom{l+i-1}{l}=\binom{l+n}{l+1},$$

we obtain the desired formula. ∎

From Theorem 3.1 and the relation (3.2), we derive two formulas for the r-fold sums of the m-th powers when r and m have the same parity.

Theorem 3.2 *For $m\geq 1$, we have*

$$\begin{aligned}\sum{}^{2r+1}(x+n)^{2m-1} &= \sum_{k=1}^{m}T(2m,2k)\Bigg\{\frac{(x+n+k+2r)_{2k+2r}}{(2k+2r)_{2r+1}}\\ &\qquad -\sum_{i=1}^{2r+1}\binom{n+2r-i}{2r-i+1}\frac{(x+k+i-1)_{2k+i-1}}{(2k+i-1)_i}\Bigg\}.\end{aligned}$$

We remark that the second summation in the above formula vanishes when $x=0$, and the lower factorial $(n+k+2r)_{2k+2r}$ can be rewritten as

$$\prod_{i=1}^{k+r}(n+k+2r+1-i)(n-k+i)=\prod_{i=1}^{k+r}[n(n+2r+1)-(k+2r-i+1)(k-i)].$$

Hence we obtain Faulhaber's theorem for the $(2r+1)$-fold sums of odd powers of the first n positive integers.

Applying Theorem 3.1 together with the following relation

$$(x+n)(x+n+k-1)_{2k-1}=\frac{1}{2}(x+n+k)_{2k}+\frac{1}{2}(x+n+k-1)_{2k}, \qquad (3.5)$$

we arrive at the following formula for the $(2r)$-fold summation of even powers.

Theorem 3.3 *For $m \geq 1$, we have,*

$$\begin{aligned}\sum\nolimits^{2r}(x+n)^{2m} &= \sum_{k=1}^{m} T(2m,2k)\Bigg\{\frac{(x+n+r)(x+n+k+2r-1)_{2k+2r-1}}{(2k+2r)_{2r}} \\ &\quad -\sum_{i=1}^{2r}\binom{n+2r-i-1}{2r-i}\frac{(2x+i)(x+k+i-1)_{2k+i-1}}{2(2k+i)_i}\Bigg\}.\end{aligned}$$

Setting $x=0$ in the above formula, the second summation vanishes. One sees that the $(2r)$-fold summation becomes a polynomial in $n(n+2r)$ because $(n+r)(n+k+2r-1)_{2k+2r-1}$ can be rewritten as

$$\prod_{i=1}^{k+r}(n+k+2r-i)(n-k+i) = \prod_{i=1}^{k+r}[n(n+2r)+(k+2r-i)(i-k)].$$

4 r-Fold Alternating Sums of Powers

In this section, we investigate the r-fold alternating sums of powers. Following the notation of Faulhaber, we define

$$\sum\nolimits^{r}(-1)^n(n+x)^m := \sum_{1\leq i_1\leq\cdots\leq i_r\leq n}(-1)^{i_1}(i_1+x)^m. \tag{4.1}$$

We will show that the $2r$-fold alternating sum of even powers $\sum^{2r}(-1)^n n^{2m}$ is a polynomial in $n(n+r)$. For other cases concerning the parities of r, m, we will outline the results without proofs.

Define

$$E_m^{(r)}(x_1,\ldots,x_r) := \sum_{i_1+\cdots+i_r=m}\binom{m}{i_1,\ldots,i_r}E_{i_1}(x_1)\cdots E_{i_r}(x_r). \tag{4.2}$$

The following lemma holds. The proof is based on induction and is omitted.

Lemma 4.1 *Let r, m be positive integers. Then*

$$\begin{aligned}&\sum\nolimits^{r}(-1)^n(n+x)^m \\ &\quad = (-1)^n 2^{-r}E_m^{(r)}\Big(1/2,\ldots,1/2,(x+n+r/2+1/2)\Big) \\ &\qquad +\sum_{k=1}^{r}\binom{n+r-k-1}{r-k}2^{-k}E_m^{(k)}\Big(1/2,\ldots,1/2,x+(k+1)/2\Big).\end{aligned}$$

We now give recursive formulas for $E_{2m}^{(k)}$ in order to compute the multiple sums in the above lemma.

Lemma 4.2 *Let k, m be positive integers. Then $E_{2m}^{(k)}[1/2,\ldots,1/2,(k+1)/2]$ equals*

$$\sum_{i=0}^{k/2}\binom{k}{2i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j E_{2m}^{(2j)}\left(1/2,\ldots,1/2\right), \tag{4.3}$$

and $E_{2m+1}^{(k)}\left(1/2,\ldots,1/2,(k+1)/2\right)$ equals

$$\sum_{i=0}^{k/2}\binom{k}{2i+1}\sum_{j=0}^{i}\binom{i}{j}(-1)^{j+1} E_{2m+1}^{(2j+1)}\left(1/2,\ldots,1/2,1\right). \tag{4.4}$$

Proof. From the generating function of $E_m(x)$, one sees that the exponential generating function of $E_m^{(k)}\Big(1/2,\ldots,1/2,(k+1)/2\Big)$ equals

$$\sum_{m=0}^{\infty} E_m^{(k)}\Big(1/2,\ldots,1/2,(k+1)/2\Big)\frac{t^m}{m!} = \left(\frac{2e^t}{1+e^t}\right)^k.$$

Observe that

$$\left(\frac{2e^t}{1+e^t}\right)^2 = 2\cdot\frac{2e^t}{1+e^t} - \left(\frac{2e^{\frac{t}{2}}}{1+e^t}\right)^2.$$

Denote the exponential generating functions of $E_n(1)$ and $E_n(1/2)$ by $A(t)$ and $B(t)$, respectively. Then the above relation implies that $A(t)^2 = 2A(t) - B(t)^2$, which yields

$$A(t) = 1 - \sqrt{1-B(t)^2}.$$

Taking the k-th power, we obtain

$$\begin{aligned} A(t)^k &= \sum_{i=0}^{k/2}\binom{k}{2i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j B(t)^{2j} \\ &\quad + \sum_{i=0}^{(k-1)/2}\binom{k}{2i+1}\sum_{j=0}^{i}\binom{i}{j}(-1)^j B(t)^{2j}(1-A(t)). \end{aligned}$$

Since $E_{2m+1}(1/2) = 0$ and $E_{2m} = 0$ for $m \geq 1$, equating the coefficients of both sides of above identity yields the desired recurrence relations. ∎

The following theorem is concerned with $2r$-fold alternating sums of even powers.

Theorem 4.3 *Let r, m be positive integers. Then the $2r$-fold alternating sum $\sum^{2r}(-1)^n n^{2m}$ is of the form $(-1)^n F(n(n+2r)) + G(n(n+2r))$, where F and G are polynomials of degree m and $r-1$ respectively.*

Proof. Applying Lemma 4.1 with $2r$ replaced by 2, m replaced by $2m$, and setting $x=0$, we get

$$\sum^{2r}(-1)^n n^{2m}$$
$$= (-1)^n 2^{-2r} E_{2m}^{(2r)}\Big(1/2,\ldots,1/2,(n+r+1/2)\Big) \tag{4.5}$$
$$+\sum_{k=1}^{2r}\binom{n+2r-k-1}{2r-k}2^{-k}E_{2m}^{(k)}\Big(1/2,\ldots,1/2,(k+1)/2\Big). \tag{4.6}$$

In the expansion of (4.5) indexed by $2i_1+2i_2+\cdots+2i_{2r}=2m$, each term contains a factor of the form $E_{2i_r}(n+r+1/2)$. According to (2.14), we find that

$$E_{2i_{2r}}\left(n+r+\frac{1}{2}\right)=\sum_{k=0}^{i_{2r}}\binom{2i_{2r}}{2k}E_{2i_{2r}-2k}\left(\frac{1}{2}\right)(n+r)^{2k}.$$

Since $(n+r)^{2k} = (n(n+2r)+r^2)^k$, it is easily seen that (4.5) is a polynomial in $n(n+2r)$ of degree m. We need to show that (4.6) is also a polynomial in $n(n+2r)$. Applying (4.3), we find that (4.6) equals

$$\sum_{k=1}^{2r}\binom{n+2r-k-1}{2r-k}2^{-k}\sum_{i=0}^{k/2}\binom{k}{2i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j E_{2m}^{(2j)}\,(1/2,\ldots,1/2)$$
$$=\sum_{j=0}^{r}(-1)^j E_{2m}^{(2j)}\,(1/2,\ldots,1/2)$$
$$\times\sum_{k=j}^{r}\binom{n+2r-k-1}{2r-k}2^{-k}\sum_{i=j}^{k/2}\binom{k}{2i}\binom{i}{j}. \tag{4.7}$$

Using the identity

$$\sum_{i=j}^{n/2}\binom{n}{2i}\binom{i}{j}=2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j}, \tag{4.8}$$

we deduce that the second sum in (4.7) equals

$$\sum_{k=j}^{r}\binom{n+2r-k-1}{2r-k}\binom{k-j-1}{j-1}\frac{k}{j}2^{-1-2j}. \tag{4.9}$$

In view of the following relation,

$$\sum_k \binom{n-k}{i}\binom{m+k}{j} = \binom{m+n+1}{i+j+1}.$$

the above sum (4.9) simplifies to

$$\binom{n+2r-j-1}{2r-2j-1}\frac{n+r}{r-j}2^{-1-2j}. \tag{4.10}$$

Now, $(n+r)(n+2r-j-1)_{2r-2j-1}$ can be rewritten as

$$\prod_{i=1}^{r-j}(n+i+j)(n+2r-i-j) = \prod_{i=1}^{r-j}[n(n+2r)+(2r-j-i)(i+j)].$$

which is a polynomial in $n(n+2r)$ of degree $r-j$. Since $E_{2m}^{(0)}(\cdot)=0$, (4.6) is a polynomial in $n(n+2r)$ of degree $r-1$. This completes the proof. ∎

For the remaining three cases with respect to the parities of r and m, we have the following theorem. The proof is omitted.

Theorem 4.4 *For $r, m \geq 0$, we have*

$$\begin{aligned} &\sum^{2r+1}(-1)^n n^{2m} \\ &\quad = (-1)^n F_m^{(1)}(n(n+2r+1)) + (2n+2r+1)G_{r-1}^{(1)}(n(n+2r+1)), \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\sum^{2r}(-1)^n n^{2m+1} \\ &\quad = (-1)^n (n+r) F_m^{(2)}(n(n+2r)) + (n+r)G_{r-1}^{(2)}(n(n+2r)), \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\sum^{2r+1}(-1)^n n^{2m+1} \\ &\quad = (-1)^n (n+r) F_m^{(3)}(n(n+2r+1)) + G_r^{(3)}(n(n+2r+1)), \end{aligned} \tag{4.13}$$

where $F_m^{(i)}(x)$ and $G_r^{(i)}(x)$ $(i=1,2,3)$ stand for polynomials of degrees m and r respectively, and $G_{-1}^{(i)}=0$ for $i=1,2$.

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