

## The Reverse Ultra Log-Concavity of the Boros-Moll Polynomials

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**Abstract.** We prove the reverse ultra log-concavity of the Boros-Moll polynomials. We further establish an inequality which implies the log-concavity of the sequence  $\{i!d_i(m)\}$  for any  $m \geq 2$ , where  $d_i(m)$  are the coefficients of the Boros-Moll polynomials  $P_m(a)$ . This inequality also leads to the fact that in the asymptotic sense, the Boros-Moll sequences are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We propose two conjectures on the log-concavity and reverse ultra log-concavity of the sequence  $\{d_{i-1}(m)d_{i+1}(m)/d_i(m)^2\}$  for  $m \geq 2$ .

**Keywords:** log-concavity, reverse ultra log-concavity, Boros-Moll polynomials.

**AMS Classification:** 05A20; 33F10

## 1 Introduction

This paper is concerned with the reverse ultra log-concavity of the Boros-Moll polynomials. A sequence  $\{a_k\}_{k \geq 0}$  of real numbers is said to be log-concave if  $a_k^2 \geq a_{k+1}a_{k-1}$  holds for all  $k \geq 1$ . A polynomial is said to be log-concave if the sequence of its coefficients is log-concave, see Brenti [7] and Stanley [10]. Furthermore, a sequence  $\{a_k\}_{0 \leq k \leq n}$  is called ultra log-concave if  $\{a_k/\binom{n}{k}\}$  is log-concave, see Liggett [9]. This condition can be restated as

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \quad (1.1)$$

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. As noticed by Liggett [9], if a sequence  $\{a_k\}_{0 \leq k \leq n}$  is ultra log-concave, then the sequence  $\{k!a_k\}_{0 \leq k \leq n}$  is log-concave.

A sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (1.1), that is,

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0. \quad (1.2)$$

For example, it is easy to verify that for  $n \geq 2$ , the Bessel polynomial [11]

$$y_n(x) = \sum_{k=0}^n \frac{(n+k)!}{2^k k! (n-k)!} x^k$$

is log-concave and reverse ultra log-concave.

The Boros and Moll polynomials, denoted  $P_m(a)$ , arise in the following evaluation of a quartic integral

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),$$

where

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k, \quad (1.3)$$

see, [1, 2, 3, 5]. Write

$$P_m(a) = \sum_{i=0}^m d_i(m) a^i.$$

The sequence  $\{d_i(m)\}_{0 \leq i \leq m}$  is called a Boros-Moll sequence. The expression (1.3) gives the following formula for the coefficients  $d_i(m)$ ,

$$d_i(m) = 2^{-2m} \sum_{k=0}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{i}.$$

Clearly, the coefficients  $d_i(m)$  are positive. Moll conjectured that the sequence  $\{d_i(m)\}_i$  is log-concave for  $m \geq 2$ , that is,  $d_i(m)^2 \geq d_{i-1}(m)d_{i+1}(m)$  ( $1 \leq i \leq m-1$ ). This conjecture has been proved by Kauers and Paule [8].

Despite the log-concavity of  $\{d_i(m)\}$ , we find that the inverse ultra log-concavity holds.

**Theorem 1.1** For  $m \geq 2$  and  $1 \leq i \leq m - 1$ , we have

$$\binom{d_{i-1}(m)}{\binom{m}{i-1}} \cdot \binom{d_{i+1}(m)}{\binom{m}{i+1}} > \binom{d_i(m)}{\binom{m}{i}}^2, \quad (1.4)$$

or, equivalently,

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i}. \quad (1.5)$$

On the other hand, it can be shown that the coefficients  $d_i(m)$  satisfy an inequality stronger than the log-concavity. To be more specific, we will give a lower bound of  $d_i(m)^2 / (d_{i-1}(m)d_{i+1}(m))$ , which is very close to the above upper bound in (1.5).

**Theorem 1.2** For  $m \geq 2$  and  $1 \leq i \leq m - 1$ , we have

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)}. \quad (1.6)$$

This paper is organized as follows. We establish an upper bound of  $d_i(m+1)/d_i(m)$  in Section 2, which leads to the reverse ultra log-concavity of  $\{d_i(m)\}$ . In Section 4 we give the proof of Theorem 1.2. We conclude this paper with two conjectures concerning the log-concavity and the reverse ultra log-concavity of the sequence  $\{d_{i-1}(m)d_{i+1}(m)/d_i^2(m)\}$  for  $m \geq 2$ .

## 2 An Upper Bound for $d_i(m+1)/d_i(m)$

In this section, we establish an upper bound for the ratio  $d_i(m+1)/d_i(m)$  that will lead to the reverse ultra log-concavity of the sequence of  $\{d_i(m)\}$ . For  $m \geq 1$  and  $0 \leq i \leq m$ , set

$$T(m, i) = \frac{4m^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1} - 2i^2}{2(m-i+1)(m+1)}. \quad (2.1)$$

**Theorem 2.1** For all  $m \geq 2$ ,  $1 \leq i \leq m - 1$ , we have

$$\frac{d_i(m+1)}{d_i(m)} < T(m, i), \quad (2.2)$$

and for  $m \geq 1$ , we have

$$\frac{d_0(m+1)}{d_0(m)} = T(m, 0), \quad \frac{d_m(m+1)}{d_m(m)} = T(m, m). \quad (2.3)$$

The following lemma will be needed in the proof of Theorem 2.1.

**Lemma 2.2** For  $m \geq 2$  and  $1 \leq i \leq m - 1$ ,

$$T(m, i) < F(m, i), \quad (2.4)$$

where

$$F(m, i) = \frac{(m+i+1)(4m+3)(4m+5)}{2(m+1)(4m^2-2i^2+9m+5-i\sqrt{4m+4i^2+5})}.$$

*Proof.* Let  $A = \sqrt{4m+4i^2+1}$  and  $B = \sqrt{4m+4i^2+5}$ . It is easy to check that

$$F(m, i) - T(m, i) = \frac{i(X-Y)}{2(m+1)(m-i+1)(4m^2+9m+5-2i^2-iB)}, \quad (2.5)$$

where

$$X = (i - 4i^3) + iAB$$

$$Y = (5 + 4m^2 + 9m - 2i^2)A - (3 + 4m^2 + 7m - 2i^2)B.$$

Since  $(4m^2+9m+5-2i^2)^2 - (iB)^2 = (4m+5)^2(m+i+1)(m-i+1) > 0$ , it remains to show that the numerator of (2.5) is also positive. We claim that  $X > 0$  and  $X^2 > Y^2$ .

Since  $m > i$ , we have  $A > 2i+1$  and  $B > 2i+1$ . Moreover, since  $i \geq 1$ , we find that

$$X = (i - 4i^3) + iAB \geq i - 4i^3 + i(2i+1)^2 = 4i^2 + 2i > 0.$$

It is routine to check  $X^2 - Y^2 = G(m, i) - H(m, i)$ , where

$$G(m, i) = (32m^4 - 32m^2i^2 + 128m^3 - 64mi^2 + 190m^2 - 30i^2 + 124m + 30)AB,$$

$$H(m, i) = 128m^5 + 608m^4 + 1128m^3 + 1014m^2 + 436m + 128m^4i^2 + 384m^3i^2 \\ + 408m^2i^2 - 128m^2i^4 + 200mi^2 - 256mi^4 - 120i^4 + 50i^2 + 70.$$

Since  $i < m$ , it is easily seen that  $G(m, i) > 0$  and  $H(m, i) > 0$ . To prove  $G(m, i) > H(m, i)$ , it suffices to show that  $G(m, i)^2 > H(m, i)^2$ . In fact, for  $1 \leq i \leq m - 1$ ,

$$G(m, i)^2 - H(m, i)^2 = 16(4m+5)^2(16mi^2 + 12i^2 - 1)(m+i+1)^2(m-i+1)^2 > 0.$$

This yields  $X^2 > Y^2$ . Since  $X > 0$ , we see that  $X > Y$ , and hence (2.4) holds for  $1 \leq i \leq m - 1$ .  $\blacksquare$

*Proof of Theorem 2.1.* It is easy to check (2.3). To prove (2.2), we proceed by induction on  $m$ . For  $m = 2$  and  $i = 1$ , we have  $d_1(3)/d_1(2) = 43/15 < T(2, 1) = (31 + \sqrt{13})/12$ . We now assume that (2.2) is true for  $m$ , that is,

$$d_i(m+1) < T(m, i)d_i(m), \quad 1 \leq i \leq m - 1. \quad (2.6)$$

It will be shown that

$$d_i(m+2) < T(m+1, i)d_i(m+1), \quad 1 \leq i \leq m-1. \quad (2.7)$$

Using the recurrence (3.3), we may write (2.7) in the following form

$$\begin{aligned} \frac{-4i^2 + 8m^2 + 24m + 19}{2(m-i+2)(m+2)}d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+1)(m+2)(m-i+2)}d_i(m) \\ < T(m+1, i)d_i(m+1). \end{aligned} \quad (2.8)$$

Since  $m > i$ , we have  $4m + 4i^2 + 5 < 12m + 4m^2 + 9$ . It follows that

$$\begin{aligned} R(m, i) &= \frac{-4i^2 + 8m^2 + 24m + 19}{2(m-i+2)(m+2)} - T(m+1, i) \\ &= \frac{4m^2 + 9m + 5 - 2i^2 - i\sqrt{4m + 4i^2 + 5}}{2(m-i+2)(m+2)} \\ &\geq \frac{4m^2 + 9m + 5 - 2i^2 - i(2m+3)}{2(m-i+2)(m+2)} > 0. \end{aligned}$$

Therefore, (2.8) is equivalent to the following inequality

$$\frac{d_i(m+1)}{d_i(m)} < F(m, i), \quad (2.9)$$

which is a consequence of (2.6) and Lemma 2.2.

It remains to consider the case  $i = m$ . We aim to show that

$$\frac{d_m(m+2)}{d_m(m+1)} < T(m+1, m). \quad (2.10)$$

By easy computation, we find that

$$\begin{aligned} \frac{d_m(m+2)}{d_m(m+1)} &= \frac{(m+1)(4m^2 + 18m + 21)}{2(2m+3)(m+2)}, \\ T(m+1, m) &= \frac{2m^2 + 15m + 14 + m\sqrt{4m^2 + 4m + 5}}{4(m+2)}. \end{aligned}$$

Thus (2.10) can be rewritten as

$$(2m^2 + 3m)\sqrt{4m^2 + 4m + 5} > 4m^3 + 8m^2 + 5m. \quad (2.11)$$

Denote by  $U$  and  $V$  the left hand side and the right hand side of (2.11), respectively. Then,  $U^2 - V^2 = 4m^2(4m+5) > 0$ , and so (2.10) is verified. This completes the proof.  $\blacksquare$

### 3 The Reverse Ultra Log-concavity

In this section, we give the proof of Theorem 1.1. Our approach can be described as follows. Let  $f(x) = ax^2 + bx + c$  be a quadratic function with  $a > 0$ . Suppose that the equation  $f(x) = 0$  has two distinct real zeros  $x_1$  and  $x_2$ , where  $x_1 < x_2$ . Then  $f(x) > 0$  if  $x > x_2$  or  $x < x_1$  and  $f(x) < 0$  if  $x_1 < x < x_2$ . The key step is to transform the inequality (1.5), that is,

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i},$$

into a quadratic inequality in the ratio  $d_i(m+1)/d_i(m)$ .

We will need the following recurrence relations for the coefficients  $d_i(m)$ . For  $m \geq 1$  and  $0 \leq i \leq m+1$ ,

$$2(m+1)d_i(m+1) = 2(m+i)d_{i-1}(m) + (4m+2i+3)d_i(m), \quad (3.1)$$

$$\begin{aligned} 2(m+1)(m+1-i)d_i(m+1) &= (4m-2i+3)(m+i+1)d_i(m) \\ &\quad - 2i(i+1)d_{i+1}(m), \end{aligned} \quad (3.2)$$

$$\begin{aligned} 4(m+2-i)(m+1)(m+2)d_i(m+2) &= 2(m+1)(-4i^2+8m^2+24m+19)d_i(m+1) \\ &\quad - (m+i+1)(4m+3)(4m+5)d_i(m). \end{aligned} \quad (3.3)$$

These recurrences are derived by Kauers and Paule [8]. The relation (3.3) is also derived independently by Moll [6]. Based on these recurrence relations, Kauers and Paule [8] derived the following lower bound of  $d_i(m+1)/d_i(m)$  in their proof of the log-concavity of Boros-Moll polynomials

$$\frac{d_i(m+1)}{d_i(m)} \geq Q(m, i), \quad 0 \leq i \leq m, \quad (3.4)$$

where

$$Q(m, i) = \frac{4m^2 + 7m + i + 3}{2(m+1-i)(m+1)}. \quad (3.5)$$

Note that Chen and Xia [4] have shown that the above inequality (3.4) becomes strict for  $m \geq 2$  and  $1 \leq i \leq m-1$ , that is,

$$\frac{d_i(m+1)}{d_i(m)} > Q(m, i). \quad (3.6)$$

Now we are ready to prove the reverse ultra log-concavity of  $\{d_i(m)\}$ .

*Proof of Theorem 1.1.* Applying (3.1) and (3.2), we may reformulate (1.5) in the following form

$$\begin{aligned}
& 4(m-i+1)^2(m+1)^2 \left( \frac{d_i(m+1)}{d_i(m)} \right)^2 \\
& - 4(m-i+1)(m+1)(4m^2 - 2i^2 + 7m + 3) \left( \frac{d_i(m+1)}{d_i(m)} \right) \\
& - (32mi^2 - 56m^3 - 73m^2 - 42m + 13i^2 - 9 - 16m^4 + 16i^2m^2) < 0. \tag{3.7}
\end{aligned}$$

For  $1 \leq i \leq m-1$ , the discriminant of the above quadratic function in  $d_i(m+1)/d_i(m)$  equals

$$\Delta = 16i^2(m+1)^2(4i^2 + 4m + 1)(m-i+1)^2 > 0.$$

We see that the quadratic function on the left hand side of (3.7) has two real roots

$$\begin{aligned}
x_1 &= \frac{4m^2 - 2i^2 + 7m + 3 - i\sqrt{4m + 4i^2 + 1}}{2(m-i+1)(m+1)}, \\
x_2 &= \frac{4m^2 - 2i^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1}}{2(m-i+1)(m+1)}.
\end{aligned}$$

Clearly,  $Q(m, i) > x_1$ . In view of (3.4), we deduce that  $d_i(m+1)/d_i(m) \geq Q(m, i) > x_1$ . Observe that  $x_2$  coincides with the upper bound  $T(m, i)$  in Theorem 2.1. Thus we have  $d_i(m+1)/d_i(m) < x_2$ . So we have shown that for  $1 \leq i \leq m-1$ ,

$$x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2,$$

which implies (3.7). This completes the proof of Theorem 1.1. ■

## 4 A Lower Bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$

In this section, we give the proof of Theorem 1.2 on a lower bound of  $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$ . As will be seen, the lower bound for  $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$  is very close to the upper bound (1.5) for the reverse ultra log-concavity. So in the asymptotic sense, we may say that the Boros-Moll polynomials are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We conclude this paper with two conjectures.

*Proof of Theorem 1.2.* Utilizing the recurrence relations (3.1) and (3.2), the inequality

(1.6) can be restated as

$$\begin{aligned}
& 4(m+1)^2(m-i+1)^2 \left( \frac{d_i(m+1)}{d_i(m)} \right)^2 \\
& - 4(m-i+1)(m+1)(4m^2+7m-2i^2+3) \frac{d_i(m+1)}{d_i(m)} \\
& + (4m^2+7m+3)(-4i+3+4m)(m+i+1) > 0.
\end{aligned}$$

For  $1 \leq i \leq m-1$ , the discriminant of the above quadratic function in  $d_i(m+1)/d_i(m)$  equals

$$\delta = 16i^2(2i+1)^2(m+1)^2(m-i+1)^2 > 0. \quad (4.1)$$

Hence the above quadratic function has two real roots,

$$\begin{aligned}
x_1 &= \frac{4m^2+7m-4i^2-i+3}{2(m+1)(m-i+1)}, \\
x_2 &= \frac{4m^2+7m+i+3}{2(m+1)(m-i+1)}.
\end{aligned}$$

As  $x_2 = Q(m, i)$ , it follows from (3.6) that  $d_i(m+1)/d_i(m) > x_2$ . So we arrive at (1.6). This completes the proof.  $\blacksquare$

Notice that for  $1 \leq i \leq m-1$ ,

$$\frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)} > \frac{i+1}{i}.$$

As a consequence of Theorem 1.2, we obtain the log-concavity of the sequence  $\{i!d_i(m)\}$ .

**Corollary 4.1** For  $m \geq 2$  and  $1 \leq i \leq m-1$ ,

$$\frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} > \frac{i+1}{i}, \quad (4.2)$$

or equivalently, the sequences  $\{i!d_i(m)\}$  is log-concave.

**Corollary 4.2** For  $1 \leq i \leq m-1$ , let

$$c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} \quad \text{and} \quad u_i(m) = \left(1 + \frac{1}{i}\right) \left(1 + \frac{1}{m-i}\right).$$

Then for any  $i \geq 1$ ,

$$\lim_{m \rightarrow \infty} \frac{c_i(m)}{u_i(m)} = 1. \quad (4.3)$$



*Proof.* By Theorems 1.1 and 1.2, we find that

$$\frac{m+i}{m+i+1} < \frac{c_i(m)}{u_i(m)} < 1,$$

which implies (4.3). ■

We remark that even when  $m$  is small,  $c_i(m)$  is quite close to  $u_i(m)$  for any  $1 \leq i \leq m-1$ . Numerical evidence indicates that  $c_i(m)/u_i(m)$  is increasing for given  $m$ . For example, when  $m=8$ , the values of  $c_i(m)/u_i(m)$  for  $1 \leq i \leq 7$  are given below

$$0.956593, \quad 0.969751, \quad 0.978293, \quad 0.983956, \quad 0.987811, \quad 0.990507, \quad 0.992445.$$

We propose the following two conjectures on the log-concavity and reverse ultra log-concavity of the sequence  $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}$ .

**Conjecture 4.3** *For  $m \geq 2$ , the sequence  $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2}$  is log-concave.*

**Conjecture 4.4** *For  $m \geq 2$ , the sequence  $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2}$  is reverse ultra log-concave.*

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