The Reverse Ultra Log-Concavity of the Boros-Moll Polynomials

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Abstract. We prove the reverse ultra log-concavity of the Boros-Moll polynomials. We further establish an inequality which implies the log-concavity of the sequence \{ild_i(m)\} for any \(m \geq 2\), where \(d_i(m)\) are the coefficients of the Boros-Moll polynomials \(P_m(a)\). This inequality also leads to the fact that in the asymptotic sense, the Boros-Moll sequences are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We propose two conjectures on the log-concavity and reverse ultra log-concavity of the sequence \\{d_{i-1}(m)d_{i+1}(m)/d_i(m)^2\} for \(m \geq 2\).

Keywords: log-concavity, reverse ultra log-concavity, Boros-Moll polynomials.

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1 Introduction

This paper is concerned with the reverse ultra log-concavity of the Boros-Moll polynomials. A sequence \(\{a_k\}_{k \geq 0}\) of real numbers is said to be log-concave if \(a_k^2 \geq a_{k+1}a_{k-1}\) holds for all \(k \geq 1\). A polynomial is said to be log-concave if the sequence of its coefficients is log-concave, see Brenti [7] and Stanley [10]. Furthermore, a sequence \(\{a_k\}_{0 \leq k \leq n}\) is called ultra log-concave if \(\{a_k/\binom{n}{k}\}\) is log-concave, see Liggett [9]. This condition can be restated as

\[
k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0.
\] (1.1)

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. As noticed by Liggett [9], if a sequence \(\{a_k\}_{0 \leq k \leq n}\) is ultra log-concave, then the sequence \(\{k!a_k\}_{0 \leq k \leq n}\) is log-concave.

A sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (1.1), that is,

\[
k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0.
\] (1.2)

For example, it is easy to verify that for \(n \geq 2\), the Bessel polynomial [11]

\[y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^kk!(n-k)!} x^k\]
is log-concave and reverse ultra log-concave.

The Boros and Moll polynomials, denoted \( P_m(a) \), arise in the following evaluation of a quartic integral
\[
\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),
\]
where
\[
P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m - 2k}{m - k} \binom{m + k}{k} (a + 1)^k,
\]
see, [1, 2, 3, 5]. Write
\[
P_m(a) = \sum_{i=0}^{m} d_i(m) a^i.
\]
The sequence \( \{d_i(m)\}_{0 \leq i \leq m} \) is called a Boros-Moll sequence. The expression (1.3) gives the following formula for the coefficients \( d_i(m) \),
\[
d_i(m) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m - 2k}{m - k} \binom{m + k}{k} \binom{k}{i}.
\]
Clearly, the coefficients \( d_i(m) \) are positive. Moll conjectured that the sequence \( \{d_i(m)\} \) is log-concave for \( m \geq 2 \), that is, \( d_i(m)^2 \geq d_{i-1}(m)d_{i+1}(m) \) (\( 1 \leq i \leq m - 1 \)). This conjecture has been proved by Kauers and Paule [8].

Despite the log-concavity of \( \{d_i(m)\} \), we find that the inverse ultra log-concavity holds.

\begin{theorem}
For \( m \geq 2 \) and \( 1 \leq i \leq m - 1 \), we have
\[
\left( \frac{d_{i-1}(m)}{\binom{m}{i-1}} \right) \cdot \left( \frac{d_{i+1}(m)}{\binom{m}{i+1}} \right) > \left( \frac{d_i(m)}{\binom{m}{i}} \right)^2,
\]
or, equivalently,
\[
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m - i + 1)(i + 1)}{(m - i)i}.
\]
\end{theorem}

On the other hand, it can be shown that the coefficients \( d_i(m) \) satisfy an inequality stronger than the log-concavity. To be more specific, we will give a lower bound of \( d_i(m)^2 / (d_{i-1}(m)d_{i+1}(m)) \), which is very close to the above upper bound in (1.5).

\begin{theorem}
For \( m \geq 2 \) and \( 1 \leq i \leq m - 1 \), we have
\[
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m - i + 1)(i + 1)(m + i)}{(m - i)i(m + i + 1)}.
\]
\end{theorem}
This paper is organized as follows. We establish an upper bound of \(d_i(m+1)/d_i(m)\) in Section 2, which leads to the reverse ultra log-concavity of \(\{d_i(m)\}\). In Section 4 we give the proof of Theorem 1.2. We conclude this paper with two conjectures concerning the log-concavity and the reverse ultra log-concavity of the sequence \(\{d_{i-1}(m)d_{i+1}(m)/d_i^2(m)\}\) for \(m \geq 2\).

### 2 An Upper Bound for \(d_i(m+1)/d_i(m)\)

In this section, we establish an upper bound for the ratio \(d_i(m+1)/d_i(m)\) that will lead to the reverse ultra log-concavity of the sequence \(\{d_i(m)\}\). For \(m \geq 1\) and \(0 \leq i \leq m\), set

\[
T(m, i) = \frac{4m^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1} - 2i^2}{2(m - i + 1)(m + 1)}.
\]  

(2.1)

**Theorem 2.1** For all \(m \geq 2\), \(1 \leq i \leq m - 1\), we have

\[
\frac{d_i(m+1)}{d_i(m)} < T(m, i),
\]

(2.2)

and for \(m \geq 1\), we have

\[
\frac{d_0(m+1)}{d_0(m)} = T(m, 0), \quad \frac{d_m(m+1)}{d_m(m)} = T(m, m).
\]

(2.3)

The following lemma will be needed in the proof of Theorem 2.1.

**Lemma 2.2** For \(m \geq 2\) and \(1 \leq i \leq m - 1\),

\[
T(m, i) < F(m, i),
\]

(2.4)

where

\[
F(m, i) = \frac{(m + i + 1)(4m + 3)(4m + 5)}{2(m + 1)(4m^2 - 2i^2 + 9m + 5 - i\sqrt{4m + 4i^2 + 5})}.
\]

Proof. Let \(A = \sqrt{4m + 4i^2 + 1}\) and \(B = \sqrt{4m + 4i^2 + 5}\). It is easy to check that

\[
F(m, i) - T(m, i) = \frac{i(X - Y)}{2(m + 1)(m - i + 1)(4m^2 + 9m + 5 - 2i^2 - iB)},
\]

(2.5)

where

\[
X = (i - 4i^3) + iAB
\]

\[
Y = (5 + 4m^2 + 9m - 2i^2)A - (3 + 4m^2 + 7m - 2i^2)B.
\]
Since \((4m^2 + 9m + 5 - 2i^2)^2 - (iB)^2 = (4m + 5)^2(m + 1)(m - i + 1) > 0\), it remains to show that the numerator of \((2.5)\) is also positive. We claim that \(X > 0\) and \(X^2 > Y^2\).

Since \(m > i\), we have \(A > 2i + 1\) and \(B > 2i + 1\). Moreover, since \(i \geq 1\), we find that

\[
X = (i - 4i^3) + iAB \geq i - 4i^3 + (2i + 1)^2 = 4i^2 + 2i > 0.
\]

It is routine to check \(X^2 - Y^2 = G(m, i) - H(m, i)\), where

\[
G(m, i) = (32m^4 - 32m^2i^2 + 128m^3 - 64mi^2 + 190m^2 - 30i^2 + 124m + 30)iB,
\]

\[
H(m, i) = 128m^5 + 608m^4 + 1128m^3 + 1014m^2 + 436m + 128m^4i^2 + 384m^3i^2
\]

\[\quad + 408m^2i^2 - 128m^2i^4 + 200mi^2 - 256mi^4 - 120i^4 + 50i^2 + 70.\]

Since \(i < m\), it is easily seen that \(G(m, i) > 0\) and \(H(m, i) > 0\). To prove \(G(m, i) > H(m, i)\), it suffices to show that \(G(m, i)^2 > H(m, i)^2\). In fact, for \(1 \leq i \leq m - 1\),

\[
G(m, i)^2 - H(m, i)^2 = 16(4m + 5)^2(16m^2 - 12m^2 - 1)(m + i + 1)^2(m - i + 1)^2 > 0.
\]

This yields \(X^2 > Y^2\). Since \(X > 0\), we see that \(X > Y\), and hence \((2.4)\) holds for \(1 \leq i \leq m - 1\).

\[\text{Proof of Theorem 2.1.}\] It is easy to check \((2.3)\). To prove \((2.2)\), we proceed by induction on \(m\). For \(m = 2\) and \(i = 1\), we have \(d_1(3)/d_1(2) = 43/15 < T(2, 1) = (31 + \sqrt{13})/12\).

We now assume that \((2.2)\) is true for \(m\), that is,

\[
d_i(m + 1) < T(m, i)d_i(m), \quad 1 \leq i \leq m - 1. \quad (2.6)
\]

It will be shown that

\[
d_i(m + 2) < T(m + 1, i)d_i(m + 1), \quad 1 \leq i \leq m - 1. \quad (2.7)
\]

Using the recurrence \((3.3)\), we may write \((2.7)\) in the following form

\[
\frac{-4i^2 + 8m^2 + 24m + 19}{2(m - i + 2)(m + 2)}d_i(m + 1) - \frac{(m + i + 1)(4m + 3)(4m + 5)}{4(m + 1)(m + 2)(m - i + 2)}d_i(m)
\]

\[\quad < T(m + 1, i)d_i(m + 1). \quad (2.8)
\]

Since \(m > i\), we have \(4m + 4i^2 + 5 < 12m + 4m^2 + 9\). It follows that

\[
R(m, i) = \frac{-4i^2 + 8m^2 + 24m + 19}{2(m - i + 2)(m + 2)} - T(m + 1, i)
\]

\[\quad = \frac{4m^2 + 9m + 5 - 2i^2 - i\sqrt{4m + 4i^2 + 5}}{2(m - i + 2)(m + 2)}
\]

\[\quad \geq \frac{4m^2 + 9m + 5 - 2i^2 - i(2m + 3)}{2(m - i + 2)(m + 2)} > 0.
\]
Therefore, (2.8) is equivalent to the following inequality
\[
\frac{d_i(m + 1)}{d_i(m)} < F(m, i),
\]
which is a consequence of (2.6) and Lemma 2.2.

It remains to consider the case \(i = m\). We aim to show that
\[
\frac{d_m(m + 2)}{d_m(m + 1)} < T(m + 1, m).
\]
By easy computation, we find that
\[
\frac{d_m(m + 2)}{d_m(m + 1)} = \frac{(m + 1)(4m^2 + 18m + 21)}{2(2m + 3)(m + 2)},
\]
\[
T(m + 1, m) = \frac{2m^2 + 15m + 14 + m\sqrt{4m^2 + 4m + 5}}{4(m + 2)}.
\]
Thus (2.10) can be rewritten as
\[
(2m^2 + 3m)\sqrt{4m^2 + 4m + 5} > 4m^3 + 8m^2 + 5m.
\]
Denote by \(U\) and \(V\) the left hand side and the right hand side of (2.11), respectively. Then, \(U^2 - V^2 = 4m^2(4m + 5) > 0\), and so (2.10) is verified. This completes the proof. 

3 The Reverse Ultra Log-concavity

In this section, we give the proof of Theorem 1.1. Our approach can be described as follows. Let \(f(x) = ax^2 + bx + c\) be a quadratic function with \(a > 0\). Suppose that the equation \(f(x) = 0\) has two distinct real zeros \(x_1\) and \(x_2\), where \(x_1 < x_2\). Then \(f(x) > 0\) if \(x > x_2\) or \(x < x_1\) and \(f(x) < 0\) if \(x_1 < x < x_2\). The key step is to transform the inequality (1.5), that is,
\[
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m - i + 1)(i + 1)}{(m - i)i},
\]
into a quadratic inequality in the ratio \(d_i(m + 1)/d_i(m)\).

We will need the following recurrence relations for the coefficients \(d_i(m)\). For \(m \geq 1\) and \(0 \leq i \leq m + 1,\)
\[
2(m + 1)d_i(m + 1) = 2(m + i)d_i(m) + (4m + 2i + 3)d_i(m), \quad (3.1)
\]
\[
2(m + 1)(m + 1 - i)d_i(m + 1) = (4m - 2i + 3)(m + i + 1)d_i(m) - 2i(i + 1)d_{i+1}(m), \quad (3.2)
\]
\[
4(m + 2 - i)(m + 1)(m + 2)d_i(m + 2) = 2(m + 1)(-4i^2 + 8m^2 + 24m + 19)d_i(m + 1) - \left(m + i + 1\right)(4m + 3)(4m + 5)d_i(m). \quad (3.3)
\]
These recurrences are derived by Kauers and Paule [8]. The relation (3.3) is also derived independently by Moll [6]. Based on these recurrence relations, Kauers and Paule [8] derived the following lower bound of \( \frac{d_i(m+1)}{d_i(m)} \) in their proof of the log-concavity of Boros-Moll polynomials

\[
\frac{d_i(m+1)}{d_i(m)} \geq Q(m, i), \quad 0 \leq i \leq m, \tag{3.4}
\]

where

\[
Q(m, i) = \frac{4m^2 + 7m + i + 3}{2(m + 1 - i)(m + 1)}. \tag{3.5}
\]

Note that Chen and Xia [4] have shown that the above inequality (3.4) becomes strict for \( m \geq 2 \) and \( 1 \leq i \leq m - 1 \), that is,

\[
\frac{d_i(m+1)}{d_i(m)} > Q(m, i). \tag{3.6}
\]

Now we are ready to prove the reverse ultra log-concavity of \( \{d_i(m)\} \).

**Proof of Theorem 1.1.** Applying (3.1) and (3.2), we may reformulate (1.5) in the following form

\[
4(m - i + 1)^2(m + 1)^2 \left( \frac{d_i(m+1)}{d_i(m)} \right)^2 - 4(m - i + 1)(m + 1)(4m^2 - 2i^2 + 7m + 3) \left( \frac{d_i(m+1)}{d_i(m)} \right) - (32m^2 - 56m^3 - 73m^2 - 42m + 13i^2 - 9 - 16m^4 + 16i^2m^2) < 0. \tag{3.7}
\]

For \( 1 \leq i \leq m - 1 \), the discriminant of the above quadratic function in \( \frac{d_i(m+1)}{d_i(m)} \) equals

\[
\Delta = 16i^2(m + 1)^2(4i^2 + 4m + 1)(m - i + 1)^2 > 0.
\]

We see that the quadratic function on the left hand side of (3.7) has two real roots

\[
x_1 = \frac{4m^2 - 2i^2 + 7m + 3 - i\sqrt{4m + 4i^2 + 1}}{2(m - i + 1)(m + 1)},
\]

\[
x_2 = \frac{4m^2 - 2i^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1}}{2(m - i + 1)(m + 1)}.
\]

Clearly, \( Q(m, i) > x_1 \). In view of (3.4), we deduce that \( \frac{d_i(m+1)}{d_i(m)} \geq Q(m, i) > x_1 \). Observe that \( x_2 \) coincides with the upper bound \( T(m, i) \) in Theorem 2.1. Thus we have \( \frac{d_i(m+1)}{d_i(m)} < x_2 \). So we have shown that for \( 1 \leq i \leq m - 1 \),

\[
x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2,
\]

which implies (3.7). This completes the proof of Theorem 1.1. \( \blacksquare \)
4 A Lower Bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$

In this section, we give the proof of Theorem 1.2 on a lower bound of $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$. As will be seen, the lower bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$ is very close to the upper bound (1.5) for the reverse ultra log-concavity. So in the asymptotic sense, we may say that the Boros-Moll polynomials are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We conclude this paper with two conjectures.

**Proof of Theorem 1.2.** Utilizing the recurrence relations (3.1) and (3.2), the inequality (1.6) can be restated as

$$4(m + 1)^2(m - i + 1)^2 \left( \frac{d_i(m + 1)}{d_i(m)} \right)^2 - 4(m - i + 1)(m + 1)(4m^2 + 7m - 2i^2 + 3) \frac{d_i(m + 1)}{d_i(m)} + (4m^2 + 7m + 3)(-4i + 3 + 4m)(m + i + 1) > 0.$$  

For $1 \leq i \leq m - 1$, the discriminant of the above quadratic function in $d_i(m + 1)/d_i(m)$ equals

$$\delta = 16i^2(2i + 1)^2(m + 1)^2(m - i + 1)^2 > 0. \quad (4.1)$$

Hence the above quadratic function has two real roots,

$$x_1 = \frac{4m^2 + 7m - 4i^2 - i + 3}{2(m + 1)(m - i + 1)},$$

$$x_2 = \frac{4m^2 + 7m + i + 3}{2(m + 1)(m - i + 1)}.$$

As $x_2 = Q(m, i)$, it follows from (3.6) that $d_i(m + 1)/d_i(m) > x_2$. So we arrive at (1.6). This completes the proof.

Notice that for $1 \leq i \leq m - 1$,

$$\frac{(m - i + 1)(i + 1)(m + i)}{(m - i)i(m + i + 1)} > \frac{i + 1}{i}.$$

As a consequence of Theorem 1.2, we obtain the log-concavity of the sequence \{i!d_i(m)\}.

**Corollary 4.1** For $m \geq 2$ and $1 \leq i \leq m - 1$,

$$\frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} > \frac{i + 1}{i}, \quad (4.2)$$

or equivalently, the sequences \{i!d_i(m)\} is log-concave.

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Corollary 4.2 For $1 \leq i \leq m - 1$, let
\[ c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} \quad \text{and} \quad u_i(m) = \left(1 + \frac{1}{i}\right) \left(1 + \frac{1}{m - i}\right). \]
Then for any $i \geq 1$,
\[ \lim_{m \to \infty} \frac{c_i(m)}{u_i(m)} = 1. \quad (4.3) \]

Proof. By Theorems 1.1 and 1.2, we find that
\[ \frac{m + i}{m + i + 1} < \frac{c_i(m)}{u_i(m)} < 1, \]
which implies (4.3).

We remark that even when $m$ is small, $c_i(m)$ is quite close to $u_i(m)$ for any $1 \leq i \leq m - 1$. Numerical evidence indicates that $c_i(m)/u_i(m)$ is increasing for given $m$. For example, when $m = 8$, the values of $c_i(m)/u_i(m)$ for $1 \leq i \leq 7$ are given below

0.956593, 0.969751, 0.978293, 0.983956, 0.987811, 0.990507, 0.992445.

We propose the following two conjectures on the log-concavity and reverse ultra log-concavity of the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}$.

Conjecture 4.3 For $m \geq 2$, the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2}$ is log-concave.

Conjecture 4.4 For $m \geq 2$, the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2}$ is reverse ultra log-concave.

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