The Reverse Ultra Log-Concavity of the Boros-Moll Polynomials

William Y. C. Chen and Cindy C. Y. Gu

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China
emails: chen@nankai.edu.cn, guchunyan@cfc.nankai.edu.cn

Abstract. We prove the reverse ultra log-concavity of the Boros-Moll polynomials. We further establish an inequality which implies the log-concavity of the sequence \( \{i!d_i(m)\} \) for any \( m \geq 2 \), where \( d_i(m) \) are the coefficients of the Boros-Moll polynomials \( P_m(a) \). This inequality also leads to the fact that in the asymptotic sense, the Boros-Moll sequences are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We propose two conjectures on the log-concavity and reverse ultra log-concavity of the sequence \( \{d_{i-1}(m)d_{i+1}(m)/d_i(m)^2\} \) for \( m \geq 2 \).

Keywords: log-concavity, reverse ultra log-concavity, Boros-Moll polynomials.

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1 Introduction

This paper is concerned with the reverse ultra log-concavity of the Boros-Moll polynomials. A sequence \( \{a_k\}_{k \geq 0} \) of real numbers is said to be log-concave if \( a_k^2 \geq a_{k+1}a_{k-1} \) holds for all \( k \geq 1 \). A polynomial is said to be log-concave if the sequence of its coefficients is log-concave, see Brenti [7] and Stanley [10]. Furthermore, a sequence \( \{a_k\}_{0 \leq k \leq n} \) is called ultra log-concave if \( \{a_k/\binom{n}{k}\} \) is log-concave, see Liggett [9]. This condition can be restated as

\[
k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \tag{1.1}
\]
It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. As noticed by Liggett [9], if a sequence \( \{a_k\}_{0 \leq k \leq n} \) is ultra log-concave, then the sequence \( \{k!a_k\}_{0 \leq k \leq n} \) is log-concave.

A sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (1.1), that is,

\[
k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0.
\]  

For example, it is easy to verify that for \( n \geq 2 \), the Bessel polynomial [11]

\[
y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^k
\]

is log-concave and reverse ultra log-concave.

The Boros and Moll polynomials, denoted \( P_m(a) \), arise in the following evaluation of a quartic integral

\[
\int_0^{\infty} \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),
\]

where

\[
P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k,
\]

see, [1, 2, 3, 5]. Write

\[
P_m(a) = \sum_{i=0}^{m} d_i(m) a^i.
\]

The sequence \( \{d_i(m)\}_{0 \leq i \leq m} \) is called a Boros-Moll sequence. The expression (1.3) gives the following formula for the coefficients \( d_i(m) \),

\[
d_i(m) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{i}.
\]

Clearly, the coefficients \( d_i(m) \) are positive. Moll conjectured that the sequence \( \{d_i(m)\}_i \) is log-concave for \( m \geq 2 \), that is, \( d_i(m)^2 \geq d_{i-1}(m)d_{i+1}(m) \) \((1 \leq i \leq m-1)\). This conjecture has been proved by Kauers and Paule [8].

Despite the log-concavity of \( \{d_i(m)\} \), we find that the inverse ultra log-concavity holds.
Theorem 1.1 For $m \geq 2$ and $1 \leq i \leq m - 1$, we have
\[
\left( \frac{d_{i-1}(m)}{i-1} \right) \cdot \left( \frac{d_{i+1}(m)}{i+1} \right) > \left( \frac{d_i(m)}{i} \right)^2,
\] (1.4)
or, equivalently,
\[
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m - i + 1)(i + 1)}{(m - i)i}.
\] (1.5)

On the other hand, it can be shown that the coefficients $d_i(m)$ satisfy an inequality stronger than the log-concavity. To be more specific, we will give a lower bound of $d_i(m)^2 / (d_{i-1}(m)d_{i+1}(m))$, which is very close to the above upper bound in (1.5).

Theorem 1.2 For $m \geq 2$ and $1 \leq i \leq m - 1$, we have
\[
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m - i + 1)(i + 1)(m + i)}{(m - i)i(m + i + 1)}.
\] (1.6)

This paper is organized as follows. We establish an upper bound of $d_i(m+1)/d_i(m)$ in Section 2, which leads to the reverse ultra log-concavity of $\{d_i(m)\}$. In Section 4 we give the proof of Theorem 1.2. We conclude this paper with two conjectures concerning the log-concavity and the reverse ultra log-concavity of the sequence $\{d_{i-1}(m)d_{i+1}(m)/d_i^2(m)\}$ for $m \geq 2$.

2 An Upper Bound for $d_i(m+1)/d_i(m)$

In this section, we establish an upper bound for the ratio $d_i(m+1)/d_i(m)$ that will lead to the reverse ultra log-concavity of the sequence of $\{d_i(m)\}$. For $m \geq 1$ and $0 \leq i \leq m$, set
\[
T(m, i) = \frac{4m^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1} - 2i^2}{2(m - i + 1)(m + 1)}.
\] (2.1)

Theorem 2.1 For all $m \geq 2$, $1 \leq i \leq m - 1$, we have
\[
\frac{d_i(m+1)}{d_i(m)} < T(m, i),
\] (2.2)
and for $m \geq 1$, we have
\[
\frac{d_0(m+1)}{d_0(m)} = T(m, 0), \quad \frac{d_m(m+1)}{d_m(m)} = T(m, m).
\] (2.3)
The following lemma will be needed in the proof of Theorem 2.1.

**Lemma 2.2** For $m \geq 2$ and $1 \leq i \leq m - 1$,

$$T(m,i) < F(m,i),$$

(2.4)

where

$$F(m,i) = \frac{(m + i + 1)(4m + 3)(4m + 5)}{2(m + 1)(4m^2 - 2i^2 + 9m + 5 - i\sqrt{4m + 4i^2 + 5})}.$$

**Proof.** Let $A = \sqrt{4m + 4i^2 + 1}$ and $B = \sqrt{4m + 4i^2 + 5}$. It is easy to check that

$$F(m,i) - T(m,i) = \frac{i(X - Y)}{2(m + 1)(m - i + 1)(4m^2 + 9m + 5 - 2i^2 - iB)},$$

(2.5)

where

$$X = (i - 4i^3) + iAB$$

$$Y = (5 + 4m^2 + 9m - 2i^2)A - (3 + 4m^2 + 7m - 2i^2)B.$$

Since $(4m^2 + 9m + 5 - 2i^2)^2 - (iB)^2 = (4m + 5)^2(m + i + 1)(m - i + 1) > 0$, it remains to show that the numerator of (2.5) is also positive. We claim that $X > 0$ and $X^2 > Y^2$.

Since $m > i$, we have $A > 2i + 1$ and $B > 2i + 1$. Moreover, since $i \geq 1$, we find that

$$X = (i - 4i^3) + iAB \geq i - 4i^3 + i(2i + 1)^2 = 4i^2 + 2i > 0.$$

It is routine to check $X^2 - Y^2 = G(m,i) - H(m,i)$, where

$$G(m,i) = (32m^4 - 32m^2i^2 + 128m^3 - 64mi^2 + 190m^2 - 30i^2 + 124m + 30)AB,$$

$$H(m,i) = 128m^5 + 608m^4 + 1128m^3 + 1014m^2 + 436m + 128m^4i^2 + 384m^3i^2$$

$$+ 408m^2i^2 - 128m^2i^4 + 200mi^2 - 256mi^4 - 120i^4 + 50i^2 + 70.$$

Since $i < m$, it is easily seen that $G(m,i) > 0$ and $H(m,i) > 0$. To prove $G(m,i) > H(m,i)$, it suffices to show that $G(m,i)^2 > H(m,i)^2$. In fact, for $1 \leq i \leq m - 1$,

$$G(m,i)^2 - H(m,i)^2 = 16(4m + 5)^2(16mi^2 + 12i^2 - 1)(m + i + 1)^2(m - i + 1)^2 > 0.$$

This yields $X^2 > Y^2$. Since $X > 0$, we see that $X > Y$, and hence (2.4) holds for $1 \leq i \leq m - 1$.

**Proof of Theorem 2.1.** It is easy to check (2.3). To prove (2.2), we proceed by induction on $m$. For $m = 2$ and $i = 1$, we have $d_1(3)/d_1(2) = 43/15 < T(2,1) = (31 + \sqrt{13})/12$. We now assume that (2.2) is true for $m$, that is,

$$d_i(m + 1) < T(m,i)d_i(m), \quad 1 \leq i \leq m - 1.$$

(2.6)
It will be shown that
\[ d_i(m + 2) < T(m + 1, i) d_i(m + 1), \quad 1 \leq i \leq m - 1. \] (2.7)

Using the recurrence (3.3), we may write (2.7) in the following form
\[ \frac{-4i^2 + 8m^2 + 24m + 19}{2(m - i + 2)(m + 2)} d_i(m + 1) - \frac{(m + i + 1)(4m + 3)(4m + 5)}{4(m + 1)(m + 2)(m - i + 2)} d_i(m) < T(m + 1, i) d_i(m + 1). \] (2.8)

Since \( m > i \), we have \( 4m + 4i^2 + 5 < 12m + 4m^2 + 9 \). It follows that
\[
R(m, i) = \frac{-4i^2 + 8m^2 + 24m + 19}{2(m - i + 2)(m + 2)} - T(m + 1, i)
\]
\[ = \frac{4m^2 + 9m + 5 - 2i^2 - i\sqrt{4m + 4i^2 + 5}}{2(m - i + 2)(m + 2)} \geq \frac{4m^2 + 9m + 5 - 2i^2 - i(2m + 3)}{2(m - i + 2)(m + 2)} > 0. \]

Therefore, (2.8) is equivalent to the following inequality
\[ \frac{d_i(m + 1)}{d_i(m)} < F(m, i), \] (2.9)
which is a consequence of (2.6) and Lemma 2.2.

It remains to consider the case \( i = m \). We aim to show that
\[ \frac{d_m(m + 2)}{d_m(m + 1)} < T(m + 1, m). \] (2.10)

By easy computation, we find that
\[ \frac{d_m(m + 2)}{d_m(m + 1)} = \frac{(m + 1)(4m^2 + 18m + 21)}{2(2m + 3)(m + 2)}, \]
\[ T(m + 1, m) = \frac{2m^2 + 15m + 14 + m\sqrt{4m^2 + 4m + 5}}{4(m + 2)}. \]

Thus (2.10) can be rewritten as
\[ (2m^2 + 3m)\sqrt{4m^2 + 4m + 5} > 4m^3 + 8m^2 + 5m. \] (2.11)

Denote by \( U \) and \( V \) the left hand side and the right hand side of (2.11), respectively. Then, \( U^2 - V^2 = 4m^2(4m + 5) > 0 \), and so (2.10) is verified. This completes the proof.
3 The Reverse Ultra Log-concavity

In this section, we give the proof of Theorem 1.1. Our approach can be described as follows. Let \(f(x) = ax^2 + bx + c\) be a quadratic function with \(a > 0\). Suppose that the equation \(f(x) = 0\) has two distinct real zeros \(x_1\) and \(x_2\), where \(x_1 < x_2\). Then \(f(x) > 0\) if \(x > x_2\) or \(x < x_1\) and \(f(x) < 0\) if \(x_1 < x < x_2\). The key step is to transform the inequality (1.5), that is,

\[
\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m - i + 1)(i + 1)}{(m - i)i},
\]

into a quadratic inequality in the ratio \(d_i(m+1)/d_i(m)\).

We will need the following recurrence relations for the coefficients \(d_i(m)\). For \(m \geq 1\) and \(0 \leq i \leq m + 1\),

\[
2(m + 1)d_i(m + 1) = 2(m + i)d_{i-1}(m) + (4m + 2i + 3)d_i(m), \tag{3.1}
\]

\[
2(m + 1)(m + 1 - i)d_i(m + 1) = (4m - 2i + 3)(m + i + 1)d_i(m) - 2i(i + 1)d_{i+1}(m), \tag{3.2}
\]

\[
4(m + 2 - i)(m + 1)(m + 2)d_i(m + 2) = 2(m + 1)(-4i^2 + 8m^2 + 24m + 19)d_i(m + 1) - (m + i + 1)(4m + 3)(4m + 5)d_i(m). \tag{3.3}
\]

These recurrences are derived by Kauers and Paule [8]. The relation (3.3) is also derived independently by Moll [6]. Based on these recurrence relations, Kauers and Paule [8] derived the following lower bound of \(d_i(m+1)/d_i(m)\) in their proof of the log-concavity of Boros-Moll polynomials

\[
\frac{d_i(m + 1)}{d_i(m)} \geq Q(m, i), \quad 0 \leq i \leq m, \tag{3.4}
\]

where

\[
Q(m, i) = \frac{4m^2 + 7m + i + 3}{2(m + 1 - i)(m + 1)}. \tag{3.5}
\]

Note that Chen and Xia [4] have shown that the above inequality (3.4) becomes strict for \(m \geq 2\) and \(1 \leq i \leq m - 1\), that is,

\[
\frac{d_i(m + 1)}{d_i(m)} > Q(m, i). \tag{3.6}
\]

Now we are ready to prove the reverse ultra log-concavity of \(\{d_i(m)\}\).
Proof of Theorem 1.1. Applying (3.1) and (3.2), we may reformulate (1.5) in the following form

\[4(m - i + 1)^2(m + 1)^2 \left( \frac{d_i(m + 1)}{d_i(m)} \right)^2 - 4(m - i + 1)(m + 1)(4m^2 - 2i^2 + 7m + 3) \left( \frac{d_i(m + 1)}{d_i(m)} \right) - (32mi^2 - 56m^3 - 73m^2 - 42m + 13i^2 - 9 - 16m^4 + 16i^2m^2) < 0. \]  

(3.7)

For \(1 \leq i \leq m - 1\), the discriminant of the above quadratic function in \(d_i(m + 1)/d_i(m)\) equals

\[\Delta = 16i^2(m + 1)^2(4i^2 + 4m + 1)(m - i + 1)^2 > 0.\]

We see that the quadratic function on the left hand side of (3.7) has two real roots

\[x_1 = \frac{4m^2 - 2i^2 + 7m + 3 - i\sqrt{4m + 4i^2 + 1}}{2(m - i + 1)(m + 1)},\]
\[x_2 = \frac{4m^2 - 2i^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1}}{2(m - i + 1)(m + 1)}.\]

Clearly, \(Q(m, i) > x_1\). In view of (3.4), we deduce that \(d_i(m + 1)/d_i(m) \geq Q(m, i) > x_1\). Observe that \(x_2\) coincides with the upper bound \(T(m, i)\) in Theorem 2.1. Thus we have \(d_i(m + 1)/d_i(m) < x_2\). So we have shown that for \(1 \leq i \leq m - 1\),

\[x_1 < \frac{d_i(m + 1)}{d_i(m)} < x_2,\]

which implies (3.7). This completes the proof of Theorem 1.1.

\[\text{\textbf{4 A Lower Bound for } } d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))\]

In this section, we give the proof of Theorem 1.2 on a lower bound of \(d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))\). As will be seen, the lower bound for \(d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))\) is very close to the upper bound (1.5) for the reverse ultra log-concavity. So in the asymptotic sense, we may say that the Boros-Moll polynomials are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We conclude this paper with two conjectures.

Proof of Theorem 1.2. Utilizing the recurrence relations (3.1) and (3.2), the inequality
(1.6) can be restated as

\[4(m + 1)^2(m - i + 1)^2 \left( \frac{d_i(m+1)}{d_i(m)} \right)^2 - 4(m - i + 1)(m + 1)(4m^2 + 7m - 2i^2 + 3) \frac{d_i(m+1)}{d_i(m)} + (4m^2 + 7m + 3)(-4i + 3 + 4m)(m + i + 1) > 0.\]

For \(1 \leq i \leq m-1\), the discriminant of the above quadratic function in \(d_i(m+1)/d_i(m)\) equals

\[\delta = 16i^2(2i + 1)^2(m+1)^2(m - i + 1)^2 > 0. \quad (4.1)\]

Hence the above quadratic function has two real roots,

\[x_1 = \frac{4m^2 + 7m - 4i^2 - i + 3}{2(m + 1)(m - i + 1)}, \quad x_2 = \frac{4m^2 + 7m + i + 3}{2(m + 1)(m - i + 1)}.\]

As \(x_2 = Q(m, i)\), it follows from (3.6) that \(d_i(m+1)/d_i(m) > x_2\). So we arrive at (1.6).

This completes the proof.

Notice that for \(1 \leq i \leq m-1\),

\[\frac{(m - i + 1)(i + 1)(m + i)}{(m - i)i(m + i + 1)} > \frac{i + 1}{i}.\]

As a consequence of Theorem 1.2, we obtain the log-concavity of the sequence \(\{i!d_i(m)\}\).

**Corollary 4.1** For \(m \geq 2\) and \(1 \leq i \leq m-1\),

\[\frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} > \frac{i + 1}{i}, \quad (4.2)\]

or equivalently, the sequences \(\{i!d_i(m)\}\) is log-concave.

**Corollary 4.2** For \(1 \leq i \leq m-1\), let

\[c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} \quad \text{and} \quad u_i(m) = \left(1 + \frac{1}{i}\right) \left(1 + \frac{1}{m - i}\right).\]

Then for any \(i \geq 1\),

\[\lim_{m \to \infty} \frac{c_i(m)}{u_i(m)} = 1. \quad (4.3)\]
Proof. By Theorems 1.1 and 1.2, we find that
\[ \frac{m + i}{m + i + 1} < \frac{c_i(m)}{u_i(m)} < 1, \]
which implies (4.3).

We remark that even when \( m \) is small, \( c_i(m) \) is quite close to \( u_i(m) \) for any \( 1 \leq i \leq m - 1 \). Numerical evidence indicates that \( c_i(m)/u_i(m) \) is increasing for given \( m \). For example, when \( m = 8 \), the values of \( c_i(m)/u_i(m) \) for \( 1 \leq i \leq 7 \) are given below

\[ 0.956593, \ 0.969751, \ 0.978293, \ 0.983956, \ 0.987811, \ 0.990507, \ 0.992445. \]

We propose the following two conjectures on the log-concavity and reverse ultra log-concavity of the sequence \( \{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\} \).

**Conjecture 4.3** For \( m \geq 2 \), the sequence \( \{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2} \) is log-concave.

**Conjecture 4.4** For \( m \geq 2 \), the sequence \( \{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2 \leq i \leq m-2} \) is reverse ultra log-concave.

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