The q-WZ Method for Infinite Series

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Abstract. Motivated by the telescoping proofs of two identities of Andrews and Warnaar, we find that infinite q-shifted factorials can be incorporated into the implementation of the q-Zeilberger algorithm in the approach of Chen, Hou and Mu to prove nonterminating basic hypergeometric series identities. This observation enables us to extend the q-WZ method to identities on infinite series. We give the q-WZ pairs for some classical identities such as the q-Gauss sum, the ${}_6\phi_5$ sum, the Ramanujan's ${}_1\psi_1$ sum and Bailey's ${}_6\psi_6$ sum.

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1 Introduction

The objective of this paper is to give an extension of the q-WZ method to nonterminating basic hypergeometric series identities. We will follow the standard notation on q-series [9] and always assume $|q| < 1$. The q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ are defined by

$$
\begin{aligned}
(a;q)_n &= \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1, \end{cases} \\
(a;q)_{-n} &= \frac{1}{(aq^{-n};q)_n}, \\
(a;q)_\infty &= (1-a)(1-aq)(1-aq^2)\cdots, \\
(a_1,a_2,\ldots,a_k;q)_n &= (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n.
\end{aligned}
$$

An ${}_r\phi_s$ *basic hypergeometric series* is defined by

$$
{}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_s \end{matrix}; q, z\right] := \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1.1}
$$

where $q \neq 0$ when $r > s+1$. Further, an ${}_r\psi_s$ *bilateral basic hypergeometric series* is defined by

$$ {}_r\psi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right] := \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{s-r} z^n. \tag{1.2} $$

It is assumed that q, z and the parameters are such that each term of the series is well-defined. We say that an ${}_r\phi_s$ series terminates if only a finite number of terms contribute. Otherwise, we say that the series ${}_r\phi_s$ is nonterminating.

For the ordinary nonterminating hypergeometric identities, Gessel [10] and Koornwinder [14] provided computer proofs of Gauss' summation formula and Saalschütz' summation formula by means of a combination of Zeilberger's algorithm and asymptotic estimates. Vidunas [19] (see also Koepf [12] and Koornwinder [15]) presented a method to evaluate ${}_2F_1\left({a,b \atop c} \middle| -1 \right)$ when $c-a+b$ is an integer. Recently, Chen, Hou and Mu [8] developed an approach to proving nonterminating basic hypergeometric identities based on the q-Zeilberger algorithm [13]. In this paper we will show how to apply the q-WZ method to prove nonterminating basic hypergeometric summation formulas by finding the q-WZ pairs. We will give some examples including the q-Gauss sum, the very-well-poised ${}_6\phi_5$ sum, the Ramanujan's ${}_1\psi_1$ sum and Bailey's very-well-poised series ${}_6\psi_6$ sum [9].

2 The Andrews-Warnaar Identities

In this paper, we give telescoping proofs of the following two identities on partial theta functions:

$$ \left(\sum_{n=0}^{\infty} (-1)^n a^n q^{\binom{n}{2}} \right) \left(\sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}} \right) = (q, a, b; q)_\infty \sum_{n=0}^{\infty} \frac{(abq^{n-1}; q)_n}{(q, a, b; q)_n} q^n, \tag{2.1} $$

$$ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} (a^n + b^n) = (a, b, q; q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q; q)_{2n}}{(q, a, b, ab; q)_n} q^n. \tag{2.2} $$

The identity (2.2) was first proved by Warnaar in [20]. Andrews and Warnaar [6] derived the identity (2.1) and used it to prove (2.2).

As will be seen, the telescoping proofs suggest that the approach developed Chen, Hou and Mu [8] for proving nonterminating basic hypergeometric identities can be extended so that infinite q-shifted factorials can be allowed in a q-hypergeometric term. This idea immediately leads to an extension of the q-WZ method to identities on infinite series.

Note that the formula (2.2) is a generalization of the well-known Jacobi's triple product identity. When $b = q/a$, we get the Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} (-1)^n a^n q^{\binom{n}{2}} = (a, q/a, q; q)_\infty, \tag{2.3}$$

where $|q| < 1$ and $a \neq 0$.

We now describe how to prove the identities (2.1) and (2.2) by the telescoping method. Let us consider (2.1) first. Put

$$f(a) = \left(\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} a^n\right)\left(\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} b^n\right).$$

Note that the second factor does not contain the parameter a. It is easily verified that

$$f(a) = (1-a) f(aq) + aq f(aq^2). \tag{2.4}$$

We proceed to show that the right hand side of (2.1) satisfies the same recurrence relation. Of course, we still need to verify the boundary conditions. Let

$$g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where} \quad D_n(a) = (q, a, b; q)_\infty \frac{(abq^{n-1}; q)_n q^n}{(q, a, b; q)_n}.$$

Then it is necessary to show that

$$g(a) - (1-a) g(aq) - aq g(aq^2) = 0. \tag{2.5}$$

Here comes the key step of finding a telescoping relation for $D_n(a)$. Note that, for any $n \geq 0$, we have

$$\begin{aligned}
& D_n(a) - (1-a) D_n(aq) - aq D_n(aq^2) \\
=& \frac{(abq^n; q)_n (q, a, b; q)_\infty q^n}{(q, a, b; q)_n} \left(\frac{1 - abq^{n-1}}{1 - abq^{2n-1}} - \frac{1-a}{1-aq^n} - \frac{aq(1 - abq^{2n})}{(1-aq^{n+1})(1-aq^n)(1-abq^n)} \right) \\
=& \frac{(abq^n; q)_n (q, a, b; q)_\infty q^n}{(q, a, b; q)_n} \left(\frac{a(1-q^n)(1-bq^{n-1})}{(1-aq^n)(1-abq^{2n-1})} - \frac{aq(1 - abq^{2n})}{(1-aq^{n+1})(1-aq^n)(1-abq^n)} \right) \\
=& z_{n+1} - z_n,
\end{aligned} \tag{2.6}$$

where

$$z_n = -\frac{(1-q^n)(1-bq^{n-1})(abq^n; q)_n (q, a, b; q)_\infty a q^n}{(1-aq^n)(1-abq^{2n-1})(q, a, b; q)_n}.$$

The above relation reveals that the infinite q-shifted factorial $(q, a, b; q)_\infty$ can be incorporated into the telescoping relation and this step can be automated by the q-Gosper algorithm. Moreover, one sees that infinite q-shifted factorials can be incorporated into the q-Zeilberger algorithm so that the approach of Chen, Hou and Mu [8] can be extended to terms containing infinite q-shifted factorials. In particular, one can make the q-WZ method work for nonterminating hypergeometric series.

Now, let us return our attention to the proof of (2.1). Clearly, $z_0 = 0$. It is also easily seen that $\lim_{n\to+\infty} z_n = 0$. Summing (2.6) over the non-negative integers, we obtain the recurrence relation (2.5). In order to show that $f(a) = g(a)$, we will use the recurrence relation of $f(a) - g(a)$ to reach this goal.

Let $H(a) = f(a) - g(a)$. From the recurrence relations for $f(a)$ and $g(a)$, it follows that $H(a)$ satisfies the recurrence relation

$$H(a) = (1-a)H(aq) + aqH(aq^2). \tag{2.7}$$

Iterating the above relation yields that

$$H(a) = A_n H(aq^{n+1}) + B_n H(aq^{n+2}), \tag{2.8}$$

where A_n and B_n are given by

$$A_0 = (1-a),\ B_0 = aq,\ A_1 = (1-a)(1-aq) + aq,\ B_1 = (1-a)aq^2,$$

and

$$A_{n+1} = (1-aq^{n+1})A_n + aq^{n+1}A_{n-1},\ B_{n+1} = aq^{n+2}A_n, \quad n \geq 1.$$

Hence we have

$$A_{n+1} - A_n = -aq^{n+1}(A_n - A_{n-1}),$$

which implies that

$$\begin{aligned} |A_{n+1} - A_n| &= |(-1)^n a^n q^{\binom{n+2}{2}-1}||(A_1 - A_0)| \\ &\leq |a^n q^{\binom{n+2}{2}-1}| \left(|A_1| + |A_0|\right). \end{aligned}$$

So, for fixed a and $|q| < 1$, the limit $\lim\limits_{n\to+\infty} A_n$ exists. Since $B_{n+1} = aq^{n+2}A_n$, the limit $\lim\limits_{n\to+\infty} B_n$ also exists. Again, by the relation (2.8), we find

$$H(a) = H(0)\left(\lim_{n\to+\infty} A_n + \lim_{n\to+\infty} B_n\right).$$

It remains to show that $H(0) = 0$, that is,

$$\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}} = (q, b; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}. \tag{2.9}$$

We can use the telescoping method to prove (2.9). Let

$$G(b) = \sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}} - (q, b; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}.$$

It is easy to check that

$$G(b) = (1-b)G(bq) + bqG(bq^2).$$

We aim to show that $G(b) = 0$. Since $G(b)$ satisfies the same recurrence relation as $H(a)$, it is suffices to confirm $G(0) = 0$, that is,

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} = 1,$$

which is special case of Euler's identity [9, P. 354]

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \qquad |z| < 1.$$

Indeed, the relation (2.9) is a limiting case of Heine's transformation of ${}_2\phi_1$. For completeness, we give a proof based on Euler's identities:

$$\begin{aligned} (q,b;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q,b;q)_n} &= (q;q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q;q)_m} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-bq^m)^n}{(q;q)_n} \\ &= (q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n b^n q^{\binom{n}{2}}}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(q^{n+1})^m}{(q;q)_m} \\ &= \sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}}. \end{aligned} \tag{2.10}$$

Thus, we have verified that $H(a) = 0$. This completes the proof. ∎

We remark that once the recurrence relation (2.7) is derived, one can also use the theorem of Chen, Hou and Mu [8, Theorem 3.1] to prove the existence of the limits of A_n and B_n.

We next present a telescoping proof of (2.2). Let

$$f(a) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} (a^n + b^n).$$

It is easily seen that

$$(1+aq)f(a) - (1-a^2q)f(aq) - (aq + a^2q)f(aq^2) = (q-1)a. \tag{2.11}$$

Let

$$g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where} \quad D_n(a) = (q,a,b;q)_\infty \frac{(ab/q;q)_{2n} q^n}{(q,a,b,ab;q)_n}.$$

It will be shown that

$$(1+aq)g(a)-(1-a^2q)g(aq)-(aq+a^2q)g(aq^2)=(q-1)a. \tag{2.12}$$

Since

$$\begin{aligned}
&\frac{q^n-abq^{n-1}}{1-abq^{2n-1}}-\frac{(1-a^2q)(1-ab)q^n}{(1+aq)(1-aq^n)(1-abq^n)}\\
&\quad-\frac{(a^2q+aq)(1-abq^{2n})(1-abq)q^n}{(1+aq)(1-aq^n)(1-aq^{n+1})(1-abq^n)(1-abq^{n+1})}\\
&=\frac{(1-abq^{2n})(-1+q+abq^{n+1}+a^2bq^{n+2}-aq^{n+2}-q^{n+1}-a^2bq^{2n+2}+a^2bq^{2n+3})a}{(1-aq^{n+1})(1-aq^n)(1-abq^n)(1+aq)(1-abq^{n+1})}\\
&\quad-\frac{(-1+q+abq^n+a^2bq^{n+1}-aq^{n+1}-q^n-a^2bq^{2n}+a^2bq^{2n+1})a(1-bq^{n-1})(1-q^n)}{(1-aq^n)(1-abq^{2n-1})(1+aq)(1-abq^n)},
\end{aligned}$$

multiplying both sides by

$$\frac{(ab;q)_{2n}(a,b,q;q)_\infty}{(q,a,b,ab;q)_n},$$

we deduce that

$$D_n(a)-\frac{(1-a^2q)}{1+aq}D_n(aq)-\frac{(a^2q+aq)}{1+aq}D_n(aq^2)=z_{n+1}-z_n, \tag{2.13}$$

where

$$\begin{aligned}
z_n=&\frac{(-1+q-aq^{n+1}-q^n+a^2bq^{n+1}+abq^n-a^2bq^{2n}+a^2bq^{2n+1})a}{(1-aq^n)(1-abq^{2n-1})}\\
&\times\frac{(1-bq^{n-1})(1-q^n)(ab;q)_{2n}(q,a,b;q)_\infty}{(1+aq)(1-abq^n)(ab;q)_n(q,a,b;q)_n}.
\end{aligned}$$

Clearly, $z_0=0$ and $\lim_{n\to+\infty}z_n=\frac{(q-1)a}{1+aq}$. Summing (2.13) over nonnegative integers, we obtain the recurrence relation (2.12).

Let $H(a)=f(a)-g(a)$. Then $H(a)$ satisfies the following recurrence relation

$$H(a)=\frac{1-a^2q}{1+aq}H(aq)+\frac{aq+a^2q}{1+aq}H(aq^2). \tag{2.14}$$

By iteration, we obtain

$$H(a)=A_nH(aq^{n+1})+B_nH(aq^{n+2}), \tag{2.15}$$

where A_n and B_n are given by

$$A_0=\frac{1-a^2q}{1+aq},\quad A_1=\frac{1+a^3q^3}{1+aq^2},$$

$$B_0 = \frac{aq + a^2 q}{1 + aq}, \quad B_1 = \frac{aq^2(1 - a^2 q)}{(1 + aq^2)},$$

and for $n \geq 1$,

$$A_{n+1} = \frac{1 - a^2 q^{2n+3}}{1 + aq^{n+2}} A_n + \frac{aq^{n+1} + a^2 q^{2n+1}}{1 + aq^{n+1}} A_{n-1}, \tag{2.16}$$

$$B_{n+1} = \frac{aq^{n+2} + a^2 q^{2n+3}}{1 + aq^{n+2}} A_n. \tag{2.17}$$

Based on the above recurrence relations, one can deduce that both $\lim_{n\to+\infty} A_n$ and $\lim_{n\to+\infty} B_n$ exist. We note that Zeilberger [26] has shown that

$$A_n = \frac{1 + (-1)^{n+1} a^{n+2} q^{\binom{n+2}{2}}}{1 + aq^{n+1}}$$

and

$$B_n = \frac{aq^{n+1}\left(1 + (-1)^n a^{n+1} q^{\binom{n+1}{2}}\right)}{1 + aq^{n+1}}.$$

Now we see that the limits $\lim\limits_{n\to+\infty} A_n$ and $\lim\limits_{n\to+\infty} B_n$ exist. By the relation (2.15), we deduce that

$$H(a) = H(0)(\lim_{n\to+\infty} A_n + \lim_{n\to+\infty} B_n).$$

The identity (2.10) implies that $f(0) = g(0)$. So we have $H(a) = 0$. This completes the proof. ∎

We also note that once the recurrence relation (2.14) is established, one may assume that $|a| < 1$ and may use the the theorem in Chen, Hou and Mu [8, Theorem 3.1]) to the existence of the limits of A_n and B_n. Moreover, we may drop the assumption $|a| < 1$ by analytic continuation.

3 The q-WZ Pairs for Infinite Series

Our approach to the q-WZ method for infinite series can be described as follows. The key step is to construct q-WZ pairs for infinite sums. Suppose that we aim to prove an identity of the form:

$$\sum_{k=N_0}^{\infty} F_k(a_1, a_2, \ldots, a_t) = R(a_1, a_2, \ldots, a_t), \tag{3.1}$$

where t is a positive integer, and the sum is either a unilateral or bilateral basic hypergeometric series, namely, $N_0 = 0$ or $N_0 = -\infty$, $R(a_1, a_2, \ldots, a_t)$ is either zero or a quotient of two products of infinite q-shifted factorials.

First, we set some parameters, say, $a_1, \ldots, a_p$, $(1 \le p \le t)$ to $a_1 q^n, \ldots, a_p q^n$, so that we get

$$\sum_{k=N_0}^{\infty} F_k(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t) = R(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t). \tag{3.2}$$

If $R(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t) \neq 0$, set

$$F(n,k) = \frac{F_k(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t)}{R(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t)}.$$

Otherwise, set

$$F(n,k) = F_k(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t).$$

Our goal is to show that

$$\sum_{k=N_0}^{\infty} F(n,k) = \text{constant}, \qquad n = 0, 1, 2, \ldots. \tag{3.3}$$

The constant can be determined by setting $n = 0$ and setting $a_1, a_2, \ldots, a_t$ to special values. We claim that the above goal can be achieved by adopting the q-WZ method for finite sums.

Let us recall the boundary and limit conditions for the q-WZ-method. Let $f(n)$ denote the left hand side of (3.3), i.e.,

$$f(n) = \sum_{k=N_0}^{\infty} F(n,k)$$

and we aim to show that

$$f(n) = \text{constant}$$

for every nonnegative integer n. To this end, it suffices to show that $f(n+1) - f(n) = 0$ for every nonnegative integer n. This can be done by finding $G(n,k)$ such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{3.4}$$

A pair of functions $(F(n,k), G(n,k))$ that satisfy (3.4) is called a *q-WZ pair*. Once a q-WZ pair is found, one can check the boundary and limit conditions to ensure that $f(n)$ equals the claimed constant. Here are the conditions:

(C1) For each integer $n \ge 0$, $\lim_{k \to \pm\infty} G(n,k) = 0$.

(C2) For each integer k, the limit

$$f_k = \lim_{n \to \infty} F(n,k) \tag{3.5}$$

exists and is finite.

(C3) We have $\lim_{L\to\infty}\sum_{n\geq 0} G(n,-L)=0$.

The WZ method can be formally stated as follows.

Theorem 3.1 (Wilf and Zeilberger [22]). *Assume that $(F(n,k),G(n,k))$ is a WZ pair* (3.4)*. If* (C1) *holds, then we have*

$$\sum_k F(n,k)=\text{constant},\qquad n=0,1,2,\ldots. \tag{3.6}$$

If (C2) *and* (C3) *hold, then we have the companion identity*

$$\sum_{n=0}^{\infty} G(n,k)=\sum_{j\leq k-1}(f_j-F(0,j)), \tag{3.7}$$

where f_j is defined by (3.5).

We now explain how to compute the desired q-WZ pair for the identity (3.1). In fact, it can be produced by applying the q-Gasper algorithm to $F(n+1,k)-F(n,k)$. It should be noted that $F(n+1,k)-F(n,k)$ is a q-hypergeometric term with respect to q^k, even if $F(n,k)$ contains infinite q-shifted factorials such as $(aq^n;q)_\infty$. Obviously, $F(n+1,k)-F(n,k)$ is a q-hypergeometric term when $R(a_1,\ldots,a_t)=0$. Assume that $R(a_1,\ldots,a_t)\neq 0$. Let

$$M_1=\frac{R(a_1q^{n+1},\ldots,a_pq^{n+1},a_{p+1},\ldots,a_l)}{R(a_1q^{n},\ldots,a_pq^{n},a_{p+1},\ldots,a_t)},$$

$$M_2=\frac{F_{k+1}(a_1q^{n+1},\ldots,a_pq^{n+1},a_{p+1},\ldots,a_t)}{F_k(a_1q^{n+1},\ldots,a_pq^{n+1},a_{p+1},\ldots,a_t)},$$

$$M_3=\frac{F_{k+1}(a_1q^{n},\ldots,a_pq^{n},a_{p+1},\ldots,a_t)}{F_k(a_1q^{n+1},\ldots,a_pq^{n+1},a_{p+1},\ldots,a_t)},$$

$$M_4=\frac{F_{k}(a_1q^{n},\ldots,a_pq^{n},a_{p+1},\ldots,a_t)}{F_k(a_1q^{n+1},\ldots,a_pq^{n+1},a_{p+1},\ldots,a_t)}.$$

Since M_1 is a rational function in q^n and is independent of k, M_2,M_3,M_4 are all rational functions in q^k. Observe that

$$\frac{F(n+1,k+1)-F(n,k+1)}{F(n+1,k)-F(n,k)}=\frac{M_2-M_1M_3}{1-M_1M_4} \tag{3.8}$$

is a rational function in q^k, i.e., $F(n+1,k)-F(n,k)$ is a q-hypergeometric term with respect to q^k. It is necessary to mention that even if $F(n,k)$ contains infinite q-shifted factorials of the form $(aq^n;q)_\infty$, the quotient (3.8) no longer contains the q-shifted factorial $(aq^n;q)_\infty$ and it is still a rational function in q^k. Consequently, we can employ the q-Gosper algorithm to determine whether $G(n,k)$ exists. Nevertheless, it is also necessary to note that $G(n,k)$ contains infinite q-shifted factorials if $F(n,k)$ does.

There is another way to look at the above procedure. Suppose that $F(n,k)$ contains an infinite q-shifted factorial $(a;q)_\infty$, where a is a chosen parameter for the substitution $a \to aq^n$. If we set $G'(n,k) = R(aq^n)G(n,k)$. Then the equation (3.4) becomes

$$F(n+1,k)R(aq^n) - F(n,k)R(aq^n) = G'(n,k+1) - G'(n,k).$$

It is evident that the infinite q-shifted factorial $(aq^n;q)_\infty$ will disappear in the above equation, and one can use the q-Gosper algorithm to find a q-WZ pair if it exists.

We now take the q-binomial theorem [9, P. 354] as an example to explain the above steps:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z|<1. \tag{3.9}$$

In this case, we have

$$F_k(a) = \frac{(a;q)_k}{(q;q)_k} z^k, \quad R(a) = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

We choose the parameter a, and substitute a with aq^n. Then we set

$$F(n,k) = \frac{F_k(aq^n)}{R(aq^n)} = \frac{(aq^n;q)_k (z;q)_\infty}{(q;q)_k (azq^n;q)_\infty} z^k.$$

In order to find $G(n,k)$ such that (3.4) holds, it is easily checked that $F(n+1,k) - F(n,k)$ is a q-hypergeometric term. By examining the q-Gosper algorithm, one sees that it is capable to deal with the input $F(n+1,k) - F(n,k)$, or we can set

$$G'(n,k) = R(aq^n)G(n,k)$$

and find a solution of the equation

$$(1-azq^n)\frac{(aq^{n+1};q)_k}{(q;q)_k} z^k - \frac{(aq^n;q)_k}{(q;q)_k} z^k = G'(n,k+1) - G'(n,k). \tag{3.10}$$

Finally, we obtain the q-WZ pair

$$F(n,k) = \frac{(aq^n;q)_k (z;q)_\infty}{(q;q)_k (azq^n;q)_\infty} z^k,$$

$$G(n,k) = -\frac{(aq^n;q)_k (z;q)_\infty (a - aq^k)}{(q;q)_k (azq^n;q)_\infty (1-aq^n)} q^n z^k.$$

If $|z|<1$, it is easy to see that $F(n,k)$ and $G(n,k)$ satisfy the conditions (C1), (C2) and (C3). By (3.6),

$$\sum_{k=-\infty}^{\infty} F(n,k) = \sum_{k=0}^{\infty} F(n,k) = \text{constant}, \qquad n = 0,1,2,\ldots.$$

Setting $z = 0$ yields that the constant equals 1. Setting $n = 0$, we have

$$\sum_{k=0}^{\infty} F(0,k) = \text{constant} = 1.$$

By (3.7), we get the companion identity of (3.9)

$$\sum_{j=0}^{k} \frac{(a;q)_j}{(q;q)_j} z^j = (az;q)_\infty \sum_{j=0}^{k} \frac{z^j}{(q;q)_j} + \frac{az^{k+1}(a;q)_{k+1}}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(az;q)_n (aq^{k+1};q)_n}{(a;q)_{n+1}} q^n.$$

We remark that our algorithm depends on the choice of parameters $a_{i_1}, \ldots, a_{i_k}$. For a given choice of parameters, it is not guaranteed that one can find a q-WZ pair. Nevertheless, this approach applies to many classical identities.

We now give a few more examples.

Example 3.1. The q-Gauss sum [9, P. 354]:

$$\sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}, \qquad |c/ab| < 1. \tag{3.11}$$

By computation we get the q-WZ pair

$$F(n,k) = \frac{(b, aq^n;q)_k (c/ab, cq^n;q)_\infty}{(q, cq^n;q)_k (c/a, cq^n/b;q)_\infty} \left(\frac{c}{ab}\right)^k,$$

$$G(n,k) = -\frac{(a-aq^k)(b, aq^n;q)_k (c/ab, cq^n;q)_\infty}{(1-aq^n)(q, cq^n;q)_k (c/a, cq^n/b;q)_\infty} \left(\frac{c}{ab}\right)^k q^n.$$

If $|c/ab| < 1$, it is easy to verify that the two functions $(F(n,k), G(n,k))$ satisfy the relation (3.4) and conditions (C1), (C2) and (C3). By (3.6), we have

$$\sum_{k=-\infty}^{\infty} F(n,k) = \sum_{k=0}^{\infty} F(n,k) = \text{constant}, \qquad n = 0, 1, 2, \ldots.$$

Setting $c = 0$ and $n = 0$, we find that the constant equals 1, and we have

$$\sum_{k=0}^{\infty} F(0,k) = \text{constant} = 1.$$

After simplification, we obtain the identity (3.11).

By (3.7), we obtain the companion identity of (3.11)

$$-\sum_{n=0}^{\infty} \frac{(a-aq^k)(b, aq^n;q)_k (c/ab, cq^n;q)_\infty}{(1-aq^n)(q, cq^n;q)_k (c/a, cq^n/b;q)_\infty} \left(\frac{c}{ab}\right)^k q^n = \sum_{j \le k-1} (f_j - F(0,j)), \tag{3.12}$$

where

$$f_j = \lim_{n\to\infty} F(n,j) = \frac{(b;q)_j (c/ab;q)_\infty}{(q;q)_j (c/a;q)_\infty} \left(\frac{c}{ab}\right)^j,$$

which can be restated as

$$\begin{aligned}\sum_{j=0}^{k} \frac{(a,b;q)_j}{(q,c;q)_j}\left(\frac{c}{ab}\right)^j =& \frac{(c/b;q)_\infty}{(c;q)_\infty}\sum_{j=0}^{k}\frac{(b;q)_j}{(q;q)_j}\left(\frac{c}{ab}\right)^j \\ &+ \frac{(a,b;q)_{k+1}c^{k+1}}{(q;q)_k (c;q)_{k+1} a^k b^{k+1}} \sum_{n=0}^{\infty} \frac{(aq^{k+1}, c/b;q)_n}{(a;q)_{n+1}(cq^{k+1};q)_n} q^n.\end{aligned}$$

Example 3.2. The sum of a very-well-poised ${}_6\phi_5$ series [9, P. 356]:

$$\begin{aligned}\sum_{k=0}^{\infty} & \frac{(1-aq^{2k})(a,b,c,d;q)_k}{(1-a)(q,aq/b,aq/c,aq/d;q)_k}\left(\frac{aq}{bcd}\right)^k \\ &= \frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}, \qquad |aq/bcd|<1. \end{aligned} \tag{3.13}$$

By computation we get the following q-WZ pair:

$$\begin{aligned} F(n,k) =& \frac{(1-aq^{n+2k})(c,d,aq^n,bq^n;q)_k}{(1-aq^n)(q,aq/b,aq^{n+1}/c,aq^{n+1}/d;q)_k} \\ &\times \frac{(aq/b,aq/bcd,aq^{n+1}/c,aq^{n+1}/d;q)_\infty}{(aq/bc,aq/bd,aq^{n+1},aq^{n+1}/cd;q)_\infty}\left(\frac{aq}{bcd}\right)^k, \\ G(n,k) =& \frac{(c,d;q)_k (a/b,a/bcd;q)_\infty}{(q,a/b;q)_k (aq^n,aq^n/cd;q)_\infty} \\ &\times \frac{(aq^n,bq^n;q)_k (aq^n/c,aq^n/d;q)_\infty}{(aq^n/c,aq^n/d;q)_k (a/bd,a/bc;q)_\infty} \\ &\times \frac{(a-bc)(a-bd)(aq^n-cd)(1-q^k)}{(a-bcd)(bq^n-1)(aq^{n+k}-c)(aq^{n+k}-d)}\left(\frac{aq}{bcd}\right)^k q^n. \end{aligned}$$

It is easily seen that $F(n,k)$ and $G(n,k)$ satisfy the conditions (C1), (C2) and (C3). Therefore, by (3.6), we have $\sum_{k=0}^{\infty} F(n,k)$ is a constant. Setting $n=0$ and $a=0$, we find that the constant equals 1. Thus we have

$$\sum_{k=0}^{\infty} F(0,k) = \text{constant} = 1,$$

which is nothing but (3.13). Since

$$f_k = \frac{(c,d;q)_k}{(q,aq/b;q)_k}\frac{(aq/b,aq/bcd;q)_\infty}{(aq/bc,aq/bd;q)_\infty}\left(\frac{aq}{bcd}\right)^k$$

and

$$F(0,j) = \frac{(1-aq^{2j})(a,b,c,d;q)_j(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}{(1-a)(q,aq/b,aq/c,aq/d;q)_j(aq,aq/bc,aq/bd,aq/cd;q)_\infty}\left(\frac{aq}{bcd}\right)^j,$$

by (3.7), we obtain the companion identity

$$\begin{aligned}
&\sum_{j=0}^{k}\frac{(1-aq^{2j})(a,b,c,d;q)_j}{(1-a)(q,aq/b,aq/c,aq/d;q)_j}\left(\frac{aq}{bcd}\right)^k \\
&\qquad = \frac{(aq,aq/cd;q)_\infty}{(aq/c,aq/d;q)_\infty}\sum_{j=0}^{k}\frac{(c,d;q)_j}{(q,aq/b;q)_j}\left(\frac{aq}{bcd}\right)^j \\
&\qquad + \frac{b(aq;q)_k(b,c,d;q)_{k+1}}{(q,aq/b;q)_k(aq/c,aq/d;q)_{k+1}}\left(\frac{aq}{bcd}\right)^{k+1} \\
&\qquad \times \sum_{n=0}^{\infty}\frac{(aq/cd;q)_n(aq^{k+1},bq^{k+1};q)_n}{(b;q)_{n+1}(aq^{k+2}/c,aq^{k+2}/d;q)_n}q^n.
\end{aligned}$$

Example 3.3. The Ramanujan's ${}_1\psi_1$ sum [9, P. 357]

$${}_1\psi_1(a;b;q,z) = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \quad |b/a|<|z|<1. \tag{3.14}$$

In this case, we find that

$$F(n,k) = \frac{(aq^n;q)_k(z,b/az,bq^n,q^{1-n}/a;q)_\infty}{(bq^n;q)_k(q,b/a,azq^n,q^{1-n}/az;q)_\infty}z^k,$$

$$G(n,k) = \frac{(z,b/az,bq^n,q^{-n}/a;q)_\infty(aq^n;q)_k(1-azq^n)}{(q,b/a,azq^n,q^{-n}/az;q)_\infty(bq^n;q)_k(z-azq^n)}z^k.$$

If $|b/a|<|z|<1$, utilizing the following identity

$$(a;q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \qquad n=0,1,2,\ldots, \tag{3.15}$$

we can verify that $G(n,k)$ satisfies the condition (C1). It follows that (3.6),

$$\sum_{k=-\infty}^{\infty}F(n,k) = \text{constant}, \qquad n=0,1,2,\ldots. \tag{3.16}$$

Setting $n = 0$, $b = q$ and utilizing q-binomial theorem (3.9), we see that the constant equals 1. Setting $n = 0$, we obtain the identity (3.14). However, we note that the conditions for the companion identity do not hold in this case.

Example 3.4. The sum of a very-well-poised ${}_6\psi_6$ series [9, P. 357]:

$$\sum_{k=-\infty}^{\infty} \frac{(1-aq^{2k})(b,c,d,e;q)_k}{(1-a)(aq/b,aq/c,aq/d,aq/e;q)_k} \left(\frac{a^2q}{bcde}\right)^k$$
$$= \frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,a^2q/bcde;q)_\infty}. \tag{3.17}$$

We obtain the following q-WZ pair:

$$F(n,k) = \frac{(1-aq^{n+2k})(d,e,bq^n,cq^n;q)_k(aq/b,aq/c;q)_\infty}{(1-aq^n)(aq/b,aq/c,aq^{n+1}/d,aq^{n+1}/e;q)_k(aq/bd,aq/be;q)_\infty}$$
$$\times \frac{(q/d,q/e,a^2q/bcde,aq^{n+1}/d,aq^{n+1}/e,q^{1-n}/b,q^{1-n}/c;q)_\infty}{(q,aq/cd,aq/ce,aq^{n+1},aq^{n+1}/de,q^{1-n}/a,aq^{1-n}/bc;q)_\infty} \left(\frac{a^2q}{bcde}\right)^k,$$

$$G(n,k) = \frac{(d,e,bq^n,cq^n;q)_k(a/b,a/c,1/e,a^2/bcde,1/d;q)_\infty}{(a/b,a/c,aq^n/d,aq^n/e;q)_k(q,a/bd,a/be,a/cd,a/ce;q)_\infty}$$
$$\times \frac{(aq^n/d,aq^n/e,q^{-n}/b,q^{-n}/c;q)_\infty(-1+aq^n)}{(aq^n,aq^n/de,aq^{-n}/bc,q^{-n}/a;q)_\infty(1-bq^n)(1-cq^n)}$$
$$\times \frac{(a-bd)(a-be)(a-cd)(a-ce)(aq^n-de)q^n}{(aq^{n+k}-d)(aq^{n+k}-e)(a-ad)(1-e)(a^2-bcde)} \left(\frac{a^2q}{bcde}\right)^k.$$

Since $|a^2q/bcde| < 1$, from the identity (3.15) it follows that $G(n,k)$ satisfies the condition (C1). By (3.6), we find

$$\sum_{k=-\infty}^{\infty} F(n,k) = \text{constant}, \qquad n = 0,1,2,\ldots. \tag{3.18}$$

In order to determine the constant, we set $n = 0$ and $b = a$. From the ${}_6\phi_5$ summation formula (3.13), we see that the constant equals

$$\sum_{k=-\infty}^{\infty} F(0,k) = \sum_{k=0}^{\infty} \frac{(1-aq^{2k})(a,c,d,e;q)_k}{(1-a)(aq/c,aq/d,aq/e;q)_k}$$
$$\times \frac{(aq,aq/cd,aq/ce,aq/de;q)_\infty}{(aq/c,aq/d,aq/e,aq/cde;q)_\infty} \left(\frac{aq}{cde}\right)^k = 1,$$

which can be restated as (3.17). Nevertheless, we note that the conditions for the companion identity do not hold in this case.

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