The \(q\)-WZ Method for Infinite Series

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Abstract. Motivated by the telescoping proofs of two identities of Andrews and
Warnaar, we find that infinite \(q\)-shifted factorials can be incorporated into the imple-
mentation of the \(q\)-Zeilberger algorithm in the approach of Chen, Hou and Mu to prove
nonterminating basic hypergeometric series identities. This observation enables us to
extend the \(q\)-WZ method to identities on infinite series. We give the \(q\)-WZ pairs for
some classical identities such as the \(q\)-Gauss sum, the \(6\phi_5\) sum, the Ramanujan’s \(1\psi_1\)
sum and Bailey’s \(6\psi_6\) sum.

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gorithm, the \(q\)-WZ method.

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1 Introduction

The objective of this paper is to give an extension of the \(q\)-WZ method to nontermi-
nating basic hypergeometric series identities. We will follow the standard notation on
\(q\)-series [9] and always assume \(|q| < 1\). The \(q\)-shifted factorials \((a; q)_n\) and \((a; q)_\infty\) are
defined by

\[
(a; q)_n = \begin{cases} 
1, & \text{if } n = 0, \\
(1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & \text{if } n \geq 1,
\end{cases}
\]

\[
(a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n},
\]

\[
(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots,
\]

\[
(a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_k; q)_n.
\]

An \(r\phi_s\) \textit{basic hypergeometric series} is defined by

\[
\[a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z\] := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n,
\]

(1.1)

where \(q \neq 0\) when \(r > s + 1\). Further, an \(r\psi_s\) \textit{bilateral basic hypergeometric series} is defined by

\[
\[a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z\] := \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{s-r} z^n.
\]

(1.2)

It is assumed that \(q, z\) and the parameters are such that each term of the series is well-defined. We say that an \(r\phi_s\) series terminates if only a finite number of terms contribute. Otherwise, we say that the series \(r\phi_s\) is nonterminating.

For the ordinary nonterminating hypergeometric identities, Gessel [10] and Koornwinder [14] provided computer proofs of Gauss’ summation formula and Saalschütz’ summation formula by means of a combination of Zeilberger’s algorithm and asymptotic estimates. Vidunas [19] (see also Koepf [12] and Koornwinder [15]) presented a method to evaluate \(\binom{a}{b}\) when \(c - a + b\) is an integer. Recently, Chen, Hou and Mu [8] developed an approach to proving nonterminating basic hypergeometric identities based on the \(q\)-Zeilberger algorithm [13]. In this paper we will show how to apply the \(q\)-WZ method to prove nonterminating basic hypergeometric summation formulas by finding the \(q\)-WZ pairs. We will give some examples including the \(q\)-Gauss sum, the very-well-poised \(6\phi_5\) sum, the Ramanujan’s \(1\psi_1\) sum and Bailey’s very-well-poised series \(6\psi_6\) sum [9].
2 The Andrews-Warnaar Identities

In this paper, we give telescoping proofs of the following two identities on partial theta functions:

\[
\left( \sum_{n=0}^{\infty} (-1)^n a^n q^{\binom{n}{2}} \right) \left( \sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}} \right) = (q, a, b; q) \sum_{n=0}^{\infty} \frac{(abq^{n-1}; q)_n}{(q, a, b; q)_n} q^n, \quad (2.1)
\]

\[
1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} (a^n + b^n) = (a, b, q) \sum_{n=0}^{\infty} \frac{(abq; q)_{2n}}{(q, a, b, ab; q)_n} q^n, \quad (2.2)
\]

The identity (2.2) was first proved by Warnaar in [20]. Andrews and Warnaar [6] derived the identity (2.1) and used it to prove (2.2).

As will be seen, the telescoping proofs suggest that the approach developed Chen, Hou and Mu [8] for proving nonterminating basic hypergeometric identities can be extended so that infinite $q$-shifted factorials can be allowed in a $q$-hypergeometric term. This idea immediately leads to an extension of the $q$-WZ method to identities on infinite series.

Note that the formula (2.2) is a generalization of the well-known Jacobi’s triple product identity. When $b = q/a$, we get the Jacobi’s triple product identity

\[
\sum_{n=-\infty}^{\infty} (-1)^n a^n q^{\binom{n}{2}} = (a, q/a, q; q) \infty, \quad (2.3)
\]

where $|q| < 1$ and $a \neq 0$.

We now describe how to prove the identities (2.1) and (2.2) by the telescoping method. Let us consider (2.1) first. Put

\[
f(a) = \left( \sum_{n=0}^{\infty} (-1)^n a^n q^{\binom{n}{2}} \right) \left( \sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}} \right).
\]

Note that the second factor does not contain the parameter $a$. It is easily verified that

\[
f(a) = (1 - a)f(aq) + aqf(aq^2), \quad (2.4)
\]

We proceed to show that the right hand side of (2.1) satisfies the same recurrence relation. Of course, we still need to verify the boundary conditions. Let

\[
g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where} \quad D_n(a) = (q, a, b; q) \sum_{n=0}^{\infty} \frac{(abq^{n-1}; q)_n q^n}{(q, a, b; q)_n}.
\]
Then it is necessary to show that
\[ g(a) - (1 - a)g(aq) - aqg(aq^2) = 0. \tag{2.5} \]

Here comes the key step of finding a telescoping relation for \( D_n(a) \). Note that, for any \( n \geq 0 \), we have
\[
D_n(a) - (1 - a)D_n(aq) - aqD_n(aq^2)
= \frac{(abq^n; q)_n(q, a, b; q)_{\infty}q^n}{(q, a, b; q)_n} \left( \frac{1 - abq^{n-1}}{1 - abq^{2n-1}} - \frac{1 - a}{1 - aq^n} - \frac{aq(1 - abq^{2n})}{(1 - aq^{n+1})(1 - aq^n)(1 - abq^n)} \right)
= \frac{(abq^n; q)_n(q, a, b; q)_{\infty}q^n}{(q, a, b; q)_n} \left( \frac{a(1 - q^n)(1 - bq^{n-1})}{(1 - aq^n)(1 - abq^{2n-1})} - \frac{aq(1 - abq^{2n})}{(1 - aq^{n+1})(1 - aq^n)(1 - abq^n)} \right)
= z_{n+1} - z_n, \tag{2.6}
\]
where
\[ z_n = \frac{(1 - q^n)(1 - bq^{n-1})(abq^{n}; q)_n(q, a, b; q)_{\infty}aq^n}{(1 - aq^n)(1 - abq^{2n-1})(q, a, b; q)_n}. \]

The above relation reveals that the infinite \( q \)-shifted factorial \((q, a, b; q)_{\infty}\) can be incorporated into the telescoping relation and this step can be automated by the \( q \)-Gosper algorithm. Moreover, one sees that infinite \( q \)-shifted factorials can be incorporated into the \( q \)-Zeilberger algorithm so that the approach of Chen, Hou and Mu [8] can be extended to terms containing infinite \( q \)-shifted factorials. In particular, one can make the \( q \)-WZ method work for nonterminating hypergeometric series.

Now, let us return our attention to the proof of (2.1). Clearly, \( z_0 = 0 \). It is also easily seen that \( \lim_{n \to +\infty} z_n = 0 \). Summing (2.6) over the non-negative integers, we obtain the recurrence relation (2.5). In order to show that \( f(a) = g(a) \), we will use the recurrence relation of \( f(a) - g(a) \) to reach this goal.

Let \( H(a) = f(a) - g(a) \). From the recurrence relations for \( f(a) \) and \( g(a) \), it follows that \( H(a) \) satisfies the recurrence relation
\[ H(a) = (1 - a)H(aq) + aqH(aq^2). \tag{2.7} \]

Iterating the above relation yields that
\[ H(a) = A_nH(aq^{n+1}) + B_nH(aq^{n+2}), \tag{2.8} \]
where \( A_n \) and \( B_n \) are given by
\[ A_0 = (1 - a), \quad B_0 = aq, \quad A_1 = (1 - a)(1 - aq) + aq, \quad B_1 = (1 - a)aq^2, \]
and
\[ A_{n+1} = (1 - aq^{n+1})A_n + aq^{n+1}A_{n-1}, \quad B_{n+1} = aq^{n+2}A_n, \quad n \geq 1. \]
Hence we have
\[ A_{n+1} - A_n = -aq^{n+1}(A_n - A_{n-1}), \]
which implies that
\[ |A_{n+1} - A_n| = |(-1)^n a^n q^{(n+2)/2} - 1||A_1 - A_0|| \]
\[ \leq |a^n q^{(n+2)/2} - 1||A_1| + |A_0| \cdot \]
So, for fixed \( a \) and \( |q| < 1 \), the limit \( \lim_{n \to +\infty} A_n \) exists. Since \( B_{n+1} = aq^{n+2}A_n \), the limit \( \lim_{n \to +\infty} B_n \) also exists. Again, by the relation (2.8), we find
\[ H(a) = H(0) \left( \lim_{n \to +\infty} A_n + \lim_{n \to +\infty} B_n \right). \]
It remains to show that \( H(0) = 0 \), that is,
\[ \sum_{n=0}^{\infty} (-1)^n b^n q^{(n/2)} = (q, b; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}. \tag{2.9} \]
We can use the telescoping method to prove (2.9). Let
\[ G(b) = \sum_{n=0}^{\infty} (-1)^n b^n q^{(n/2)} - (q, b; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}. \]
It is easy to check that
\[ G(b) = (1 - b)G(bq) + bqG(bq^2). \]
We aim to show that \( G(b) = 0 \). Since \( G(b) \) satisfies the same recurrence relation as \( H(a) \), it is sufficient to confirm \( G(0) = 0 \), that is,
\[ (q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} = 1, \]
which is special case of Euler’s identity [9, P. 354]
\[ \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1. \]
Indeed, the relation (2.9) is a limiting case of Heine’s transformation of $_2\phi_1$. For completeness, we give a proof based on Euler’s identities:

\[
(q, b; q) \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n} = (q; q) \sum_{m=0}^{\infty} \frac{q^m}{(q; q)_m} \sum_{n=0}^{\infty} \frac{(-bq^m)^n}{(q; q)_n}
\]

\[
= (q; q) \sum_{n=0}^{\infty} \frac{(-1)^n b^n q^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(q^{n+1})^m}{(q; q)_m}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n b^n q^\binom{n}{2}.
\]

Thus, we have verified that $H(a) = 0$. This completes the proof.

We remark that once the recurrence relation (2.7) is derived, one can also use the theorem of Chen, Hou and Mu [8, Theorem 3.1] to prove the existence of the limits of $A_n$ and $B_n$.

We next present a telescoping proof of (2.2). Let

\[
f(a) = 1 + \sum_{n=1}^{\infty} (-1)^n q^\binom{n}{2} (a^n + b^n).
\]

It is easily seen that

\[
(1 + aq)f(a) - (1 - a^2q)f(aq) - (aq + a^2q)f(aq^2) = (q - 1)a.
\]

(2.11)

Let

\[
g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where} \quad D_n(a) = (q, a, b; q)_\infty \frac{(ab/q; q)_n q^n}{(q, a, b, ab; q)_n}.
\]

It will be shown that

\[
(1 + aq)g(a) - (1 - a^2q)g(aq) - (aq + a^2q)g(aq^2) = (q - 1)a.
\]

(2.12)

Since

\[
\frac{q^n - abq^{n-1}}{1 - abq^{2n-1}} = \frac{(1 - a^2q)(1 - ab)q^n}{(1 + aq)(1 - aq^n)(1 - abq^n)}
\]

\[
- \frac{(a^2q + aq)(1 - abq^{2n})(1 - abq^n)}{(1 + aq)(1 - aq^n)(1 - aq^{n+1})(1 - abq^n)(1 - abq^{n+1})}
\]

\[
= \frac{(1 - abq^{2n})(-1 + q + abq^{n+1} + a^2bq^{n+2} - aq^{n+2} - q^{n+1} - a^2bq^{2n+2} + a^2bq^{2n+3})a}{(1 - aq^{n+1})(1 - aq^n)(1 - abq^n)(1 + aq)(1 - abq^{n+1})}
\]

6
\[- \frac{(-1 + q + abq^n + a^2bq^{n+1} - aq^{n+1} - q^n - a^2bq^{2n} + a^2bq^{2n+1})a(1 - bq^{-1})(1 - q^n)}{(1 - aq^n)(1 - abq^{2n-1})(1 + aq)(1 - abq^n)},\]

multiplying both sides by
\[
\frac{(ab; q)_{2n}(a, b, q; q)}{(q, a, b, ab; q)_n},
\]
we deduce that
\[
D_n(a) - \frac{(1 - a^2q)}{1 + aq}D_n(aq) - \frac{(a^2q + aq)}{1 + aq}D_n(aq^2) = z_{n+1} - z_n, \quad (2.13)
\]
where
\[
z_n = \frac{(-1 + q - aq^{n+1} - q^n + a^2bq^{n+1} + abq^n - a^2bq^{2n} + a^2bq^{2n+1})a}{(1 - aq^n)(1 - abq^{2n-1})}
\times \frac{(1 - bq^{n-1})(1 - q^n)(ab; q)_{2n}(q, a, b; q)_\infty}{(1 + aq)(1 - abq^n)(ab; q)_n(q, a, b; q)_n}.
\]

Clearly, \(z_0 = 0\) and \(\lim_{n \to +\infty} z_n = \frac{(q-1)a}{1+aq}\). Summing (2.13) over nonnegative integers, we obtain the recurrence relation (2.12).

Let \(H(a) = f(a) - g(a)\). Then \(H(a)\) satisfies the following recurrence relation
\[
H(a) = \frac{1 - a^2q}{1 + aq}H(aq) + \frac{aq + a^2q}{1 + aq}H(aq^2). \quad (2.14)
\]
By iteration, we obtain
\[
H(a) = A_nH(aq^{n+1}) + B_nH(aq^{n+2}), \quad (2.15)
\]
where \(A_n\) and \(B_n\) are given by
\[
A_0 = \frac{1 - a^2q}{1 + aq}, \quad A_1 = \frac{1 + a^3q^3}{1 + aq^2},
\]
\[
B_0 = \frac{aq + a^2q}{1 + aq}, \quad B_1 = \frac{aq^2(1 - a^2q)}{(1 + aq^2)},
\]
and for \(n \geq 1\),
\[
A_{n+1} = \frac{1 - a^2q^{n+3}}{1 + aq^{n+2}}A_n + \frac{aq^{n+1} + a^2q^{2n+1}}{1 + aq^{n+1}}A_{n-1}, \quad (2.16)
\]
\[
B_{n+1} = \frac{aq^{n+2} + a^2q^{2n+3}}{1 + aq^{n+2}}A_n. \quad (2.17)
\]
Based on the above recurrence relations, one can deduce that both \( \lim_{n \to +\infty} A_n \) and \( \lim_{n \to +\infty} B_n \) exist. We note that Zeilberger [26] has shown that

\[
A_n = \frac{1 + (-1)^{n+1}a^{n+2}q^{(n+2)}}{1 + aq^{n+1}}
\]

and

\[
B_n = \frac{aq^{n+1}(1 + (-1)^n a^{n+1}q^{(n+1)})}{1 + aq^{n+1}}.
\]

Now we see that the limits \( \lim_{n \to +\infty} A_n \) and \( \lim_{n \to +\infty} B_n \) exist. By the relation (2.15), we deduce that

\[
H(a) = H(0)\left( \lim_{n \to +\infty} A_n + \lim_{n \to +\infty} B_n \right).
\]

The identity (2.10) implies that \( f(0) = g(0) \). So we have \( H(a) = 0 \). This completes the proof. \( \blacksquare \)

We also note that once the recurrence relation (2.14) is established, one may assume that \( |a| < 1 \) and may use the theorem in Chen, Hou and Mu [8, Theorem 3.1)] to the existence of the limits of \( A_n \) and \( B_n \). Moreover, we may drop the assumption \( |a| < 1 \) by analytic continuation.

### 3 The \( q \)-WZ Pairs for Infinite Series

Our approach to the \( q \)-WZ method for infinite series can be described as follows. The key step is to construct \( q \)-WZ pairs for infinite sums. Suppose that we aim to prove an identity of the form:

\[
\sum_{k=N_0}^{\infty} F_k(a_1, a_2, \ldots, a_t) = R(a_1, a_2, \ldots, a_t), \tag{3.1}
\]

where \( t \) is a positive integer, and the sum is either a unilateral or bilateral basic hypergeometric series, namely, \( N_0 = 0 \) or \( N_0 = -\infty \), \( R(a_1, a_2, \ldots, a_t) \) is either zero or a quotient of two products of infinite \( q \)-shifted factorials.

First, we set some parameters, say, \( a_1, \ldots, a_p, (1 \leq p \leq t) \) to \( a_1q^n, \ldots, a_pq^n \), so that we get

\[
\sum_{k=N_0}^{\infty} F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t) = R(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t). \tag{3.2}
\]
If \( R(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t) \neq 0 \), set
\[
F(n, k) = \frac{F_k(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t)}{R(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t)}.
\]
Otherwise, set
\[
F(n, k) = F_k(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_t).
\]
Our goal is to show that
\[
\sum_{k=N_0}^{\infty} F(n, k) = \text{constant, } n = 0, 1, 2, \ldots
\] (3.3)
The constant can be determined by setting \( n = 0 \) and setting \( a_1, a_2, \ldots, a_t \) to special values. We claim that the above goal can be achieved by adopting the \( q \)-WZ method for finite sums.

Let us recall the boundary and limit conditions for the \( q \)-WZ-method. Let \( f(n) \) denote the left hand side of (3.3), i.e.,
\[
f(n) = \sum_{k=N_0}^{\infty} F(n, k)
\]
and we aim to show that
\[
f(n) = \text{constant}
\]
for every nonnegative integer \( n \). To this end, it suffices to show that \( f(n+1) - f(n) = 0 \) for every nonnegative integer \( n \). This can be done by finding \( G(n, k) \) such that
\[
F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k).
\] (3.4)
A pair of functions \((F(n, k), G(n, k))\) that satisfy (3.4) is called a \( q \)-WZ pair. Once a \( q \)-WZ pair is found, one can check the boundary and limit conditions to ensure that \( f(n) \) equals the claimed constant. Here are the conditions:

(C1) For each integer \( n \geq 0 \), \( \lim_{k \to \pm \infty} G(n, k) = 0 \).

(C2) For each integer \( k \), the limit
\[
f_k = \lim_{n \to \infty} F(n, k)
\] (3.5)
exists and is finite.

(C3) We have \( \lim_{L \to \infty} \sum_{n=0}^{\infty} G(n, -L) = 0 \).
The WZ method can be formally stated as follows.

**Theorem 3.1** (Wilf and Zeilberger [22]). Assume that \((F(n, k), G(n, k))\) is a WZ pair (3.4). If (C1) holds, then we have

\[
\sum_k F(n, k) = \text{constant}, \quad n = 0, 1, 2, \ldots. \quad (3.6)
\]

If (C2) and (C3) hold, then we have the companion identity

\[
\sum_{n=0}^{\infty} G(n, k) = \sum_{j \leq k-1} (f_j - F(0, j)), \quad (3.7)
\]

where \(f_j\) is defined by (3.5).

We now explain how to compute the desired \(q\)-WZ pair for the identity (3.1). In fact, it can be produced by applying the \(q\)-Gasper algorithm to \(F(n+1, k) - F(n, k)\). It should be noted that \(F(n+1, k) - F(n, k)\) is a \(q\)-hypergeometric term with respect to \(q^k\), even if \(F(n, k)\) contains infinite \(q\)-shifted factorials such as \((aq^n; q)_\infty\). Obviously, \(F(n+1, k) - F(n, k)\) is a \(q\)-hypergeometric term when \(R(a_1, \ldots, a_t) = 0\). Assume that \(R(a_1, \ldots, a_t) \neq 0\). Let

\[
M_1 = \frac{R(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)}{R(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)},
\]

\[
M_2 = \frac{F_{k+1}(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)}{F_k(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)},
\]

\[
M_3 = \frac{F_{k+1}(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)}{F_k(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)},
\]

\[
M_4 = \frac{F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)}{F_k(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)}.
\]

Since \(M_1\) is a rational function in \(q^n\) and is independent of \(k\), \(M_2, M_3, M_4\) are all rational functions in \(q^k\). Observe that

\[
\frac{F(n+1, k+1) - F(n, k+1)}{F(n+1, k) - F(n, k)} = \frac{M_2 - M_1M_3}{1 - M_1M_4} \quad (3.8)
\]

is a rational function in \(q^k\), i.e., \(F(n+1, k) - F(n, k)\) is a \(q\)-hypergeometric term with respect to \(q^k\). It is necessary to mention that even if \(F(n, k)\) contains infinite \(q\)-shifted factorials of the form \((aq^n; q)_\infty\), the quotient (3.8) no longer contains the \(q\)-shifted
factorial \((aq^n; q)_\infty\) and it is still a rational function in \(q^k\). Consequently, we can employ the \(q\)-Gosper algorithm to determine whether \(G(n, k)\) exists. Nevertheless, it is also necessary to note that \(G(n, k)\) contains infinite \(q\)-shifted factorials if \(F(n, k)\) does.

There is another way to look at the above procedure. Suppose that \(F(n, k)\) contains an infinite \(q\)-shifted factorial \((a; q)_\infty\), where \(a\) is a chosen parameter for the substitution \(a \to aq^n\). If we set \(G'(n, k) = R(aq^n)G(n, k)\). Then the equation (3.4) becomes

\[
F(n + 1, k)R(aq^n) - F(n, k)R(aq^n) = G'(n, k + 1) - G'(n, k).
\]

It is evident that the infinite \(q\)-shifted factorial \((aq^n; q)_\infty\) will disappear in the above equation, and one can use the \(q\)-Gosper algorithm to find a \(q\)-WZ pair if it exists.

We now take the \(q\)-binomial theorem [9, P. 354] as an example to explain the above steps:

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k z^k}{(q; q)_k} = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1. \quad (3.9)
\]

In this case, we have

\[
F_k(a) = \frac{(a; q)_k z^k}{(q; q)_k}, \quad R(a) = \frac{(az; q)_\infty}{(z; q)_\infty}.
\]

We choose the parameter \(a\), and substitute \(a\) with \(aq^n\). Then we set

\[
F(n, k) = \frac{F_k(aq^n)}{R(aq^n)} = \frac{(aq^n; q)_k (z; q)_\infty}{(q; q)_k (azq^n; q)_\infty} z^k.
\]

In order to find \(G(n, k)\) such that (3.4) holds, it is easily checked that \(F(n + 1, k) - F(n, k)\) is a \(q\)-hypergeometric term. By examining the \(q\)-Gosper algorithm, one sees that it is capable to deal with the input \(F(n + 1, k) - F(n, k)\), or we can set

\[
G'(n, k) = R(aq^n)G(n, k)
\]

and find a solution of the equation

\[
(1 - azq^n)\frac{(aq^{n+1}; q)_k z^k}{(q; q)_k} - \frac{(aq^n; q)_k z^k}{(q; q)_k} = G'(n, k + 1) - G'(n, k). \quad (3.10)
\]

Finally, we obtain the \(q\)-WZ pair

\[
F(n, k) = \frac{(aq^n; q)_k (z; q)_\infty}{(q; q)_k (azq^n; q)_\infty} z^k,
\]

\[
G(n, k) = -\frac{(aq^n; q)_k (z; q)_\infty (a - aq^k)}{(q; q)_k (azq^n; q)_\infty (1 - aq^n)} q^n z^k.
\]
If \(|z| < 1\), it is easy to see that \(F(n, k)\) and \(G(n, k)\) satisfy the conditions (C1), (C2) and (C3). By (3.6),

\[
\sum_{k=-\infty}^{\infty} F(n, k) = \sum_{k=0}^{\infty} F(n, k) = \text{constant, } n = 0, 1, 2, \ldots.
\]

Setting \(z = 0\) yields that the constant equals 1. Setting \(n = 0\), we have

\[
\sum_{k=0}^{\infty} F(0, k) = \text{constant} = 1.
\]

By (3.7), we get the companion identity of (3.9)

\[
\sum_{j=0}^{k} \frac{(a; q)_j}{(q; q)_j} z^j = (az; q) \sum_{j=0}^{k} \frac{z^j}{(q; q)_j} + \frac{az^{k+1}(a; q)_{k+1}}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(az; q)_n(q^{k+1}; q)_n}{(a; q)_{n+1}} q^n.
\]

We remark that our algorithm depends on the choice of parameters \(a_i, \ldots, a_{i_k}\). For a given choice of parameters, it is not guaranteed that one can find a \(q\)-WZ pair. Nevertheless, this approach applies to many classical identities.

We now give a few more examples.

**Example 3.1.** The \(q\)-Gauss sum [9, P. 354]:

\[
\sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} \left( \frac{c}{ab} \right)^k = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}, \quad |c/ab| < 1. \tag{3.11}
\]

By computation we get the \(q\)-WZ pair

\[
F(n, k) = \frac{(b, aq^n; q)_k(c/ab, cq^n; q)_{\infty}}{(q, cq^n; q)_k(c/a, cq^n/b; q)_{\infty}} \left( \frac{c}{ab} \right)^k,
\]

\[
G(n, k) = -\frac{(a - aq^n)(b, aq^n; q)_k(c/ab, cq^n; q)_{\infty}}{(1 - aq^n)(q, cq^n; q)_k(c/a, cq^n/b; q)_{\infty}} \left( \frac{c}{ab} \right)^k q^n.
\]

If \(|c/ab| < 1\), it is easy to verify that the two functions \((F(n, k), G(n, k))\) satisfy the relation (3.4) and conditions (C1), (C2) and (C3). By (3.6), we have

\[
\sum_{k=-\infty}^{\infty} F(n, k) = \sum_{k=0}^{\infty} F(n, k) = \text{constant, } n = 0, 1, 2, \ldots.
\]
Setting \( c = 0 \) and \( n = 0 \), we find that the constant equals 1, and we have

\[
\sum_{k=0}^{\infty} F(0, k) = \text{constant} = 1.
\]

After simplification, we obtain the identity (3.11).

By (3.7), we obtain the companion identity of (3.11)

\[
-\sum_{n=0}^{\infty} \frac{(a - a q^k)(b, a q^n; q)_k (c/ab, c q^n; q)_\infty}{(1 - a q^n)(q, c q^n; q)_k (c/a, c q^n/b; q)_\infty} \left( \frac{c}{ab} \right)^k q^n = \sum_{j \leq k-1} (f_j - F(0, j)),
\]

where

\[
f_j = \lim_{n \to \infty} F(n, j) = \frac{(b; q)_j (c/ab; q)_\infty}{(q; q)_j (c/a; q)_\infty} \left( \frac{c}{ab} \right)^j,
\]

which can be restated as

\[
\sum_{j=0}^{k} \frac{(a, b; q)_j}{(q, c; q)_j} \left( \frac{c}{ab} \right)^j q^n = \frac{(b; q)_j (c/ab; q)_\infty}{(q; q)_j (c/a; q)_\infty} \left( \frac{c}{ab} \right)^j
\]

\[
+ \frac{(a, b; q)_{k+1} c^{k+1}}{(q; q)_k (c; q)_{k+1} a^{k+1}} \sum_{n=0}^{\infty} \frac{(a q^{k+1}, c/b; q)_n}{(a; q)_{n+1} (c q^{k+1}; q)_n} q^n.
\]

**Example 3.2.** The sum of a very-well-poised \( \phi_5 \) series [9, P. 356]:

\[
\sum_{k=0}^{\infty} \frac{(1 - a q^{2k})(a, b, c, d; q)_k}{(1 - a)(q, a q/b, a q/c, a q/d; q)_k} \left( \frac{a q}{b c d} \right)^k
\]

\[
= \frac{(a q, a q/b c, a q/b d, a q/c d; q)_\infty}{(a q/b, a q/c, a q/d, a q/b c d; q)_\infty}, \quad |a q/b c d| < 1.
\]

By computation we get the following \( q \)-WZ pair:

\[
F(n, k) = \frac{(1 - a q^{n+2k})(c, d, a q^n, b q^n; q)_k}{(1 - a q^n)(q, a q/b, a q^{n+1}/c, a q^{n+1}/d; q)_k}
\]

\[
\times \frac{(a q/b, a q/b c d, a q^{n+1}/c, a q^{n+1}/d; q)_\infty}{(a q/b c, a q/b d, a q^{n+1}, a q^{n+1}/c d; q)_\infty} \left( \frac{a q}{b c d} \right)^k,
\]

\[
G(n, k) = \frac{(c, d; q)_k (a/b, a/b c d; q)_\infty}{(q, a/b; q)_k (a q^n, a q^n/c d; q)_\infty}
\]
\[
\times \frac{\left(aq^n, bq^n; q\right)_k (aq^n/c, aq^n/d; q)_{\infty}}{(aq^n/c, aq^n/d; q)_k (a/bd, a/bc; q)_{\infty}}
\times \frac{\left(a - bc\right)\left(a - bd\right)\left(aq^n - cd\right)\left(1 - q^k\right)}{(a - bcd) \left(bq^n - 1\right) \left(aq^n + k - c\right) \left(aq^n + k - d\right)} \left(\frac{aq}{bcd}\right)^k q^n.
\]

It is easily seen that \(F(n, k)\) and \(G(n, k)\) satisfy the conditions (C1), (C2) and (C3). Therefore, by (3.6), we have \(\sum_{k=0}^{\infty} F(n, k)\) is a constant. Setting \(n = 0\) and \(a = 0\), we find that the constant equals 1. Thus we have
\[
\sum_{k=0}^{\infty} F(0, k) = \text{constant} = 1,
\]
which is nothing but (3.13). Since
\[
f_k = \frac{(c, d; q)_k (aq/b, aq/bcd; q)_{\infty}}{(q, aq/b; q)_k (aq/bc, aq/bd; q)_{\infty}} \left(\frac{aq}{bcd}\right)^k
\]
and
\[
F(0, j) = \frac{(1 - aq^{2j}) (a, b, c, d; q)_j (aq/b, aq/c, aq/d, aq/bcd; q)_{\infty}}{(1 - a)(q, aq/b, aq/c, aq/d; q)_j} \left(\frac{aq}{bcd}\right)^j,
\]
by (3.7), we obtain the companion identity
\[
\sum_{j=0}^{k} \frac{(1 - aq^{2j}) (a, b, c, d; q)_j}{(1 - a)(q, aq/b, aq/c, aq/d; q)_j} \left(\frac{aq}{bcd}\right)^k
\]
\[
= \frac{(aq, aq/cd; q)_{\infty}}{(aq/c, aq/d; q)_{\infty}} \sum_{j=0}^{k} \frac{(c, d; q)_j}{(q, aq/b; q)_j} \left(\frac{aq}{bcd}\right)^j
\]
\[
+ \frac{b(aq; q)_k (b, c, d; q)_{k+1}}{(q, aq/b; q)_k (aq/c, aq/d; q)_{k+1}} \left(\frac{aq}{bcd}\right)^{k+1}
\]
\[
\times \sum_{n=0}^{\infty} \frac{(aq/cd; q)_n (aq^{k+1}, bq^{k+1}; q)_n}{(b; q)_{n+1} (aq^{k+2}/c, aq^{k+2}/d; q)_n} q^n.
\]

**Example 3.3.** The Ramanujan’s \(1\psi_1\) sum [9, P. 357]
\[
1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_{\infty}}{(b, q/a, z, b/az; q)_{\infty}}, \quad |b/a| < |z| < 1. \quad (3.14)
\]
In this case, we find that
\[
F(n, k) = \frac{(aq^n; q)_k(z, b/az, bq^n, q^{1-n}/a; q)_\infty}{(bq^n; q)_k(q, b/a, azq^n, q^{1-n}/az; q)_\infty}z^k,
\]
\[
G(n, k) = \frac{(z, b/az, bq^n, q^{n-n}/a; q)_\infty(aq^n; q)_k(1 - azq^n)}{(q, b/a, azq^n, q^{n-n}/az; q)_\infty(bq^n; q)_k(z - azq^n)}z^k.
\]

If \(|b/a| < |z| < 1\), utilizing the following identity
\[
(a; q)_n = \frac{(-q/a)^n q^{(n)}}{(q/a; q)_n}, \quad n = 0, 1, 2, \ldots, \tag{3.15}
\]
we can verify that \(G(n, k)\) satisfies the condition (C1). It follows that (3.6),
\[
\sum_{k=\infty}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \ldots. \tag{3.16}
\]

Setting \(n = 0\), \(b = q\) and utilizing \(q\)-binomial theorem (3.9), we see that the constant equals 1. Setting \(n = 0\), we obtain the identity (3.14). However, we note that the conditions for the companion identity do not hold in this case.

**Example 3.4.** The sum of a very-well-poised \(6\psi_6\) series [9, P. 357]:
\[
\sum_{k=-\infty}^{\infty} \frac{(1 - aq^{2k})(b, c, d, e; q)_k}{(1 - a)(aq/b, aq/c, aq/d, aq/e; q)_k} \left(\frac{a^2 q}{bcde}\right)^k
\]
\[
= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a^2 q/bcde; q)_\infty}. \tag{3.17}
\]

We obtain the following \(q\)-WZ pair:
\[
F(n, k) = \frac{(1 - aq^{n+2k})(d, e, bq^n, cq^n; q)_k(aq/b, aq/c; q)_\infty}{(1 - aq^n)(aq/b, aq/c, aq^{n+1}/d, aq^{n+1}/e; q)_k(aq/bd, aq/bc; q)_\infty}
\]
\[
\times \frac{(q/d, q/e, a^2 q/bcde, aq^{n+1}/d, aq^{n+1}/e, q^{1-n}/b, q^{1-n}/c; q)_\infty}{(q, aq/cd, aq/ce, aq^{n+1}/d, aq^{n+1}/e, q^{1-n}/a, q^{1-n}/bc; q)_\infty} \left(\frac{a^2 q}{bcde}\right)^k,
\]
\[
G(n, k) = \frac{(d, e, bq^n, cq^n; q)_k(a/b, a/c, 1/e, a^2/bcde, 1/d; q)_\infty}{(a/b, a/c, aq^n/d, aq^n/e; q)_k(q, a/bd, a/be, a/cd, a/ce; q)_\infty}
\]
\[
\times \frac{(aq^n/d, aq^n/e, q^{-n}/b, q^{-n}/c; q)_\infty(-1 + aq^n)}{(aq^n, aq^n/de, aq^{-n}/bc, q^{-n}/a; q)_\infty(1 - bq^n)(1 - cq^n)}.
\]
\begin{align*}
\times \frac{(a - bd)(a - be)(a - cd)(a - ce)(aq^n - de)q^n}{(aq^{n+k} - d)(aq^{n+k} - e)(a - ad)(1 - e)(a^2 - bcde)} \left( \frac{a^2 q}{bcde} \right)^k.
\end{align*}

Since \(|a^2 q/bcde| < 1\), from the identity (3.15) it follows that \(G(n, k)\) satisfies the condition (C1). By (3.6), we find

\begin{equation}
\sum_{k=\infty}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \ldots.
\end{equation}

In order to determine the constant, we set \(n = 0\) and \(b = a\). From the \(\phi_5\) summation formula (3.13), we see that the constant equals

\begin{align*}
\sum_{k=\infty}^{\infty} F(0, k) &= \sum_{k=0}^{\infty} \frac{(1 - aq^{2k})(a, c, d, e; q)_k}{(1 - a)(aq/c, aq/d, aq/e; q)_k} \\
&\times \frac{(aq, aq/cd, aq/ce, aq/de; q)_\infty}{(aq/c, aq/d, aq/e, aq/cde; q)_\infty} \left( \frac{aq}{cde} \right)^k = 1,
\end{align*}

which can be restated as (3.17). Nevertheless, we note that the conditions for the companion identity do not hold in this case.

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