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Schur Positivity and the q-Log-convexity of the Narayana Polynomials

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Abstract. Using Schur positivity and the principal specialization of Schur functions, we provide a proof of a recent conjecture of Liu and Wang on the q-log-convexity of the Narayana polynomials, and a proof of the second conjecture that the Narayana transformation preserves the log-convexity. Based on a formula of Brändén which expresses the q-Narayana numbers as the specializations of Schur functions, we derive several symmetric function identities using the Littlewood-Richardson rule for the product of Schur functions, and obtain the strong q-log-convexity of the Narayana polynomials and the strong q-log-concavity of the q-Narayana numbers.

Keywords: q-log-concavity, q-log-convexity, q-Narayana number, Narayana polynomial, lattice permutation, Schur positivity, Littlewood-Richardson rule.

AMS Classification: 05E05, 05E10

Suggested Running Title: Schur positivity

1 Introduction

The main objective of this paper is to provide proofs of two recent conjectures of Liu and Wang [19] on the q-log-convexity of the Narayana polynomials by using Schur positivity derived from the Littlewood-Richardson rule. Moreover, we prove that the Narayana polynomials are strongly q-log-convex. We also study the q-log-concavity of the q-Narayana numbers, and prove that for fixed n or k the q-Narayana numbers $N_q(n,k)$ are strongly q-log-concave.

Unimodal and log-concave sequences and polynomials often arise in combinatorics, algebra and geometry, see, for example, Brenti [4, 5], Stanley [28], and Stembridge [32]. A sequence $(a_n)_{n\geq 0}$ of real numbers is said to be unimodal if there exists an integer $m \geq 0$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq a_{m+2} \geq \cdots,$$

and is said to be log-concave if

$$a_m^2 \geq a_{m+1}a_{m-1}$$

holds for all $m \geq 1$.

It has been noticed that sometimes the reciprocals of a combinatorial sequence form a log-concave sequence. For example, the sequence

$$\binom{n}{0}^{-1}, \binom{n}{1}^{-1}, \ldots, \binom{n}{n}^{-1}$$

satisfies this condition for a given positive integer n. Such sequences are called log-convex, see [19].

For polynomials, Stanley introduced the notion of q-log-concavity, which has been studied by Butler [6], Krattenthaler [16], Leroux [20], and Sagan [25]. A sequence of polynomials $(f_n(q))_{n\geq 0}$ over the field of real numbers is called q-log-concave if the difference

$$f_m(q)^2 - f_{m+1}(q)f_{m-1}(q)$$

has nonnegative coefficients as a polynomial of q for all $m \geq 1$. Sagan [26] also introduced the notion of strong q-log-concavity. We say that a sequence of polynomials $(f_n(q))_{n\geq 0}$ is strongly q-log-concave if

$$f_m(q)f_n(q) - f_{m+1}(q)f_{n-1}(q)$$

has nonnegative coefficients for any $m \geq n \geq 1$.

Based on the q-log-concavity, it is natural to define the q-log-convexity. We say that the polynomial sequence $(f_n(q))_{n\geq 0}$ is q-log-convex if the difference

$$f_{m+1}(q)f_{m-1}(q) - f_m(q)^2$$

has nonnegative coefficients as a polynomial of q for all $m \geq 1$. The notion of strong q-log-convexity is a natural counterpart of that of strong q-log-concavity. We say that a sequence of polynomials $(f_n(q))_{n\geq 0}$ is strongly q-log-convex if

$$f_{m+1}(q)f_{n-1}(q) - f_m(q)f_n(q)$$

has nonnegative coefficients for any $m \geq n \geq 1$.

As realized by Sagan [26], the strong q-log-concavity is not equivalent to the q-log-concavity, although for a sequence of positive numbers the strong log-concavity is equivalent to the log-concavity. Analogously, the strong q-log-convexity is not equivalent to the q-log-convexity. For example, the sequence

$$2q + q^2 + 3q^3, q + 2q^2 + 2q^3, q + 2q^2 + 2q^3, 2q + q^2 + 3q^3$$

is q-log-convex, but not strongly q-log-convex.

Just recently, Liu and Wang [19] have shown that some well-known polynomials such as the Bell polynomials and the Eulerian polynomials are q-log-convex, and proposed some conjectures on the Narayana polynomials based on numerical evidence.

To describe the conjectures of Liu and Wang [19], we begin with the classical Catalan numbers, as given by

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

which count the number of Dyck paths from $(0, 0)$ to $(2n, 0)$ with up steps $(1, 1)$ and down steps $(1, -1)$ but never going below the x-axis, see, Stanley [29]. It is known that the Catalan numbers C_n form a log-convex sequence. Recall that a peak of a Dyck path is defined as a point where an up step is immediately followed by a down step. Then the Narayana number

$$N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

equals the number of Dyck paths of length $2n$ with exactly $k+1$ peaks, see [3, 10, 33, 34]. The Narayana polynomials are given by

$$N_n(q) = \sum_{k=0}^{n} N(n, k)q^k.$$

Liu and Wang [19] have shown that for a given positive real number q the sequence $(N_n(q))_{n\geq 0}$ is log-convex. Note that the sequence of the Catalan numbers becomes a special case for $q = 1$. The first conjecture of Liu and Wang is as follows.

Conjecture 1.1 *The Narayana polynomials $N_n(q)$ form a q-log-convex sequence.*

We will prove the above conjecture by studying the Schur positivity of certain sums of symmetric functions. Our proof heavily relies on the Littlewood-Richardson rule for the product of Schur functions of certain shapes with only two columns. It is the formula of Brändén [3] that enables us to represent the Narayana polynomials in terms of Schur functions.

To prove the desired Schur positivity, we need to verify several identities on Schur functions, and we would acknowledge the powerful role of the Maple packages for symmetric functions, ACE [35] and SF [30].

The second conjecture of Liu and Wang [19] is concerned with the Narayana transformation on sequences of positive real numbers. The Davenport-Pólya theorem [9] states that if $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are log-convex then their binomial convolution

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}, \quad n \geq 0$$

is also log-convex. It is known that the binomial convolution also preserves the log-concavity [36]. However, it is not generally true that a log-convexity preserving transformation also preserves the log-concavity. The componentwise sum is a simple example. On the other hand, Liu and Wang [19] have realized that the componentwise sum preserves log-convexity. Moreover, it is not true that a transformation which preserves log-concavity necessarily preserves log-convexity. The ordinary convolution is such an example [19, 36].

Given combinatorial numbers $(t(n,k))_{0\leq k\leq n}$ such as the binomial coefficients, one can define a linear operator which transforms a sequence $(a_n)_{n\geq 0}$ into another sequence $(b_n)_{n\geq 0}$ as given by

$$b_n = \sum_{k=0}^{n} t(n,k) a_k, \quad n \geq 0.$$

Liu and Wang [19] have shown that the log-convexity is preserved by linear transformations associated with the binomial coefficients, the Stirling numbers of the first kind and the second kind. The following conjecture is due to Liu and Wang [19].

Conjecture 1.2 *The Narayana transformation $b_n = \sum_{k=0}^{n} N(n,k) a_k$ preserves log-convexity.*

We will give a proof of this conjecture based on the monotone property of certain quartic polynomials and the q-log-convexity of Narayana polynomials.

In addition, we further prove the strong q-log-concavity of the q-Narayana numbers. The q-Narayana numbers, as a natural q-analogue of the Narayana numbers $N(n,k)$, arise from the study of q-Catalan numbers [13]. The q-Narayana number $N_q(n,k)$ is given by

$$N_q(n,k) = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{k^2+k}, \tag{1.1}$$

where we use the standard notation

$$[k] := (1-q^k)/(1-q), \quad [k]! = [1][2]\cdots[k], \quad \begin{bmatrix} n \\ j \end{bmatrix} := \frac{[n]!}{[j]![n-j]!}$$

for the q-analogues of the integer k, the q-factorial, and the q-binomial coefficient.

It is known that the q-Narayana number $N_q(n,k)$ is the natural refinement of the q-Catalan number $c_n(1) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}$ defined in [13]. Brändén [3] studied several Narayana statistics and bi-statistics on Dyck paths, and noticed that the q-Narayana number $N_q(n,k)$ has a Schur function expression by specializing the variables.

Theorem 1.3 ([3, Theorem 6]) *For all $n, k \in \mathbb{N}$ we have*

$$N_q(n,k) = s_{(2^k)}(q, q^2, \ldots, q^{n-1}). \tag{1.2}$$

It was known that the q-analogues of many well-known combinatorial numbers are strongly q-log-concave. Bulter [6] and Krattenthaler [16] proved the q-log-concavity of the q-binomial coefficients, and Leroux [20] and Sagan [25] studied the q-log-concavity of the q-Stirling numbers of the first kind and the second kind. It was also known that the Narayana numbers $N(n,k)$ are log-concave for given n or k. Based on some symmetric function identities, we will show that $N_q(n,k)$ are strongly q-log-concave for given n or k.

This paper is organized as follows. In Section 2, we give a brief review of relevant background on symmetric functions. In Section 3, we give several symmetric function identities involving Schur functions indexed by two-column shapes, and derive the Schur positivity needed to prove the two conjectures of Liu and Wang. Section 4 deals with the strong q-log-convexity of Narayana polynomials. The notion of strong q-log-convexity is analogous to that of strong q-log-concavity as given by Sagan [26]. In Section 5, we show that the Narayana transformation preserves log-convexity. Finally, in Section 6 we derive the strong q-log-concavity of the q-Narayana numbers.

2 Background on Symmetric Functions

In this section we review some relevant background on symmetric functions and present several recurrence formulas for computing the principal specializations of Schur functions indexed by certain two-column shapes, which will be used later in the proofs of the main theorems. More specifically, the hook-content formula plays an important role in reducing the log-convexity preserving property of the Narayana transformation to the monotone property of certain polynomials, and the recurrence formulas enable us to reduce the q-log-convexity for Narayana polynomials to the Schur positivity for certain sums of symmetric functions.

Throughout this paper we will adopt the notation and terminology on partitions and symmetric functions in Stanley [29]. Given a nonnegative integer n, a partition λ of n is a weakly decreasing nonnegative integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^k \lambda_i = n$. The number of nonzero components λ_i is called the length of λ, denoted $\ell(\lambda)$. We also denote the partition λ by $(\ldots, 2^{m_2}, 1^{m_1})$ if i appears m_i times in λ. For example, $\lambda = (4,2,2,1,1,1) = (4^1, 2^2, 1^3)$, where we omit i^{m_i} if $m_i = 0$. Let $\mathrm{Par}(n)$ denote the set of all partitions of n. The Young diagram of λ is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i. By transposing the diagram of λ, we get the conjugate partition of λ, denoted λ'. A square (i,j) in the diagram of λ is the square in row i from the top and column j from the left. The hook length of (i,j), denoted $h(i,j)$, is given by $\lambda_i + \lambda'_j - i - j + 1$. The content of (i,j), denoted $c(i,j)$, is given by $j - i$. Given two partitions λ and μ, we say that λ contains μ, denoted $\mu \subseteq \lambda$, if $\lambda_i \geq \mu_i$ holds for each i. When $\mu \subseteq \lambda$, we can define a skew partition λ/μ as the diagram obtained from the diagram of λ by removing the diagram of μ at the top-left corner.

A semistandard Young tableau of shape λ/μ is an array $T = (T_{ij})$ of positive integers of shape λ/μ that is weakly increasing in every row and strictly increasing in every column. The type of T is defined as the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T. Let x denote the set of variables $\{x_1, x_2, \ldots\}$. If T has type $\mathrm{type}(T) = \alpha$, then we write

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

The skew Schur function $s_{\lambda/\mu}(x)$ of shape λ/μ is defined as the generating function

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

summed over all semistandard Young tableaux T of shape λ/μ filled with positive integers. When μ is the empty partition $\emptyset$, we call $s_\lambda(x)$ the Schur function of shape λ. In particular, we set $s_\emptyset(x) = 1$. It is well known that the Schur functions s_λ form a basis for the ring of symmetric functions.

Let $y = \{y_1, y_2, \ldots\}$ be another set of variables, and let $s_{\lambda/\mu}(x, y)$ denote the Schur function in $x \cup y$. Note that

$$s_{\lambda/\mu}(x,y) = \sum_\nu s_{\lambda/\nu}(x) s_{\nu/\mu}(y), \tag{2.3}$$

where the sum ranges over all partitions ν satisfying $\mu \subseteq \nu \subseteq \lambda$, see [21, 29].

For a symmetric function $f(x)$, its principle specialization $\mathrm{ps}_n(f)$ and specialization $\mathrm{ps}_n^1(f)$ of order n are defined by

$$\begin{aligned}\mathrm{ps}_n(f) &= f(1, q, \ldots, q^{n-1}),\\ \mathrm{ps}_n^1(f) &= \mathrm{ps}_n(f)|_{q=1} = f(1^n).\end{aligned}$$

For notational convenience, we often omit the variable set x and simply write s_λ for the Schur function $s_\lambda(x)$ if no confusion arises in the context. The following formula is called the hook-content formula due to Stanley [27].

Lemma 2.1 ([29, Corollary 7.21.4]) *For any partition λ and $n \geq 1$, we have*

$$\mathrm{ps}_n(s_\lambda) = q^{\sum_{k\geq 1}(k-1)\lambda_k} \prod_{(i,j)\in\lambda} \frac{[n + c(i,j)]}{[h(i,j)]} \tag{2.4}$$

and

$$\mathrm{ps}_n^1(s_\lambda) = \prod_{(i,j)\in\lambda} \frac{n + c(i,j)}{h(i,j)}. \tag{2.5}$$

On the other hand, in view of (2.3), we deduce the following formulas for the principle specializations of the Schur functions s_λ indexed by two-column shapes.

Lemma 2.2 *Let k be a positive integer and $n > 1$. For any $a < 0$ or $b < 0$, set $s_{(2^a,1^b)} = 0$ by convention. Then we have*

$$\begin{aligned}\mathrm{ps}_n\left(s_{(2^k)}\right) = \mathrm{ps}_{n-1}\left(s_{(2^k)}\right) &+ q^{n-1}\mathrm{ps}_{n-1}\left(s_{(2^{k-1},1)}\right)\\ &+ q^{2(n-1)}\mathrm{ps}_{n-1}\left(s_{(2^{k-1})}\right)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\mathrm{ps}_n\left(s_{(2^k,1)}\right) = \mathrm{ps}_{n-1}\left(s_{(2^k,1)}\right) &+ q^{n-1}\mathrm{ps}_{n-1}\left(s_{(2^k)} + s_{(2^{k-1},1^2)}\right)\\ &+ q^{2(n-1)}\mathrm{ps}_{n-1}\left(s_{(2^{k-1},1)}\right).\end{aligned} \tag{2.7}$$

Furthermore,

$$\mathrm{ps}_n^1\left(s_{(2^k)}\right) = \mathrm{ps}_{n-1}^1\left(s_{(2^k)} + s_{(2^{k-1},1)} + s_{(2^{k-1})}\right), \tag{2.8}$$

$$\mathrm{ps}_n^1\left(s_{(2^k,1)}\right) = \mathrm{ps}_{n-1}^1\left(s_{(2^k,1)} + s_{(2^k)} + s_{(2^{k-1},1^2)} + s_{(2^{k-1},1)}\right). \tag{2.9}$$

Lemma 2.3 *For any $m \geq n \geq 1$ and $k \geq 0$, we have*

$$\mathrm{ps}_m^1\left(s_{(2^k)}\right) = \sum_{0 \leq a \leq b \leq m-n} \mathrm{ps}_n^1(s_{(2^{k-b},1^{b-a})})\mathrm{ps}_{m-n}^1(s_{(2^a,1^{b-a})}). \tag{2.10}$$

The Littlewood-Richardson rule enables us to expand a product of Schur functions in terms of Schur functions. There are several versions of the Littlewood-Richardson rule; see [29, Chapter 7, Appendix A1.3] and [12, Part I, Chapter 5]. These settings have their own advantages when applied to various problems. For example, Knutson and Tao [15] used the honeycomb model to prove the saturation conjecture. A well-known version is the combinatorial interpretation of the Littlewood-Richardson coefficients in terms of lattice permutations, which we will adopt for our purpose.

Recall that a lattice permutation of length n is a sequence $w_1w_2\cdots w_n$ such that for any i and j in the subsequence $w_1w_2\cdots w_j$ the number of i's is greater than or equal to the number of $i+1$'s. Let T be a semistandard Young tableau. The reverse reading word T^{rev} is a sequence of entries of T obtained by first reading each row from right to left and then concatenating the rows from top to bottom. If the reverse reading word T^{rev} is a lattice permutation, we call T a Littlewood-Richardson tableau. Given two Schur functions s_μ and s_ν, Littlewood-Richardson coefficients $c_{\mu\nu}^\lambda$ can be defined by the following relation

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda. \tag{2.11}$$

Theorem 2.4 ([29, Theorem A1.3.3]) *The Littlewood-Richardson coefficient $c_{\mu\nu}^\lambda$ is equal to the number of Littlewood-Richardson tableaux of shape λ/μ and type ν.*

Take $\lambda = (9,5,3,3,1)$, $\mu = (4,2,1)$, $\nu = (7,4,3)$. By using the Maple package for symmetric functions we find that $c_{\mu\nu}^\lambda = 3$. Indeed, there are three Littlewood-Richardson tableaux of shape λ/μ and type ν as shown in Figure 2.1.

$*$	$*$	$*$	$*$	1	1	1	1	1
$*$	$*$	1	1	2				
$*$	2	2						
2	3	3						
3								

$*$	$*$	$*$	$*$	1	1	1	1	1
$*$	$*$	1	2	2				
$*$	2	2						
1	3	3						
3								

$*$	$*$	$*$	$*$	1	1	1	1	1
$*$	$*$	1	2	2				
$*$	1	2						
2	3	3						
3								

Figure 2.1: Skew Littlewood-Richardson tableaux

When taking $\nu = (n)$ or $\nu = (1^n)$ in (2.11), the Littlewood-Richardson rule has a simpler description, known as Pieri's rule. We need the notion of horizontal and vertical strips. A skew partition λ/μ is called a horizontal (or vertical) strip if there are no two squares in the same column (resp. in the same row).

Theorem 2.5 ([29, Theorem 7.15.7, Corollary 7.15.9]) *We have*

$$s_\mu s_{(n)} = \sum_\lambda s_\lambda$$

summed over all partitions λ such that λ/μ is a horizontal strip of size n, and

$$s_\mu s_{(1^n)} = \sum_\lambda s_\lambda$$

summed over all partitions λ such that λ/μ is a vertical strip of size n.

3 Schur positivity

The main goal of this section is to prove the Schur positivity of certain sums of symmetric functions, which will be needed in the proof of the q-log-convexity of the Narayana polynomials in Section 4. Given a symmetric function f, recall that f is called s-positive (or s-negative) if the coefficients a_λ in the expansion $f = \sum_\lambda a_\lambda s_\lambda$ of f in terms of Schur functions are all nonnegative (resp. nonpositive).

The Schur positivity we establish is deduced from several symmetric function identities which will be proved by induction based on the Littlewood-Richardson rule. More specifically, the identities we consider will involve the products of Schur functions indexed by partitions with only two-columns. These Schur functions are of particular interest for their own sake, see, for example, Rosas [24], and Remmel and Whitehead [23].

It is time to mention that throughout this paper, the Littlewood-Richardson coefficients are either one or two, and we will consider those shapes that will occur in the expansion of the product of Schur functions. It is worth mentioning that the Schur expansion of the product of two Schur functions would be multiplicity free when one factor is indexed by a rectangular shape, see Stembridge [31].

Let us first introduce certain classes of products of Schur functions that will be the ingredients to establish the desired Schur positivity. Given $m \in \mathbb{N}$ and $0 \le i \le m$, let

$$\begin{aligned} D^{(1)}_{m,i} &= s_{(2^i)} s_{(2^{m-i-1})}, \\ D^{(2)}_{m,i} &= s_{(2^{i-1},1^2)} s_{(2^{m-i-1})}, \\ D^{(3)}_{m,i} &= s_{(2^{i-1},1)} s_{(2^{m-i-1},1)}, \end{aligned}$$

and let

$$D_{m,i} = D^{(1)}_{m,i} + D^{(2)}_{m,i} - D^{(3)}_{m,i}, \tag{3.12}$$

where $s_{(2^i,1)} = s_{(2^i,1^2)} = 0$ for $i < 0$ by convention. It is clear that $D_{m,m} \equiv 0$.

For two partitions λ and μ, let $\lambda \cup \mu$ be the partition obtained by taking the union of all parts of λ and μ and then rearranging them in the weakly decreasing order. For $k \in \mathbb{N}$ we use λ^k to represent the union of k λ's, and in particular put $\lambda^k = \emptyset$ if $k = 0$. In this notation, we introduce an operator Δ^μ on the ring of symmetric functions defined by a partition μ. For a symmetric function f, suppose that f has the expansion

$$f = \sum_\lambda a_\lambda s_\lambda,$$

and then the action of Δ^μ on f is given by

$$\Delta^\mu(f) = \sum_\lambda a_\lambda s_{\lambda\cup\mu}.$$

For example, if

$$f = s_{(4,3,2)} + 3s_{(2,2,1)} + 2s_{(5)},$$

then

$$\Delta^{(3,1)} f = s_{(4,3,3,2,1)} + 3s_{(3,2,2,1,1)} + 2s_{(5,3,1)}.$$

Lemma 3.1 *For any $n \geq k \geq 1$, we have*

$$\begin{aligned} s_{(2^k)} s_{(2^{n+1})} &= \Delta^{(2)}\big(s_{(2^k)} s_{(2^n)}\big), & (3.13)\\ s_{(2^{k-1},1^2)} s_{(2^{n+1})} &= \Delta^{(2)}\big(s_{(2^{k-1},1^2)} s_{(2^n)}\big), & (3.14)\\ s_{(2^k)} s_{(2^{n+1},1^2)} &= \Delta^{(2)}\big(s_{(2^k)} s_{(2^n,1^2)}\big), & (3.15)\\ s_{(2^{k-1},1)} s_{(2^{n+1},1)} &= \Delta^{(2)}\big(s_{(2^{k-1},1)} s_{(2^n,1)}\big). & (3.16) \end{aligned}$$

Proof. Define a_λ by

$$s_{(2^k)} s_{(2^n)} = \sum_\lambda a_\lambda s_\lambda.$$

By Theorem 2.4, the coefficient a_λ is equal to the number of Littlewood-Richardson tableaux of shape $\lambda/(2^n)$ and type (2^k). We claim that $a_\lambda = 0$ if the diagram of λ contains the square $(n+1, 3)$; Otherwise, we get a contradiction to the assumption $n \geq k$ since the column strictness of Young tableaux requires that there should be at least $n+1$ distinct numbers in the tableau. Therefore, for a Littlewood-Richardson tableau T of shape $\lambda/(2^n)$ and type (2^k) we can construct a Littlewood-Richardson tableau T' of shape $\lambda \cup (2)/(2^{n+1})$ and of the same type by moving all rows of T to the next row except for the first n rows and inserting two empty squares at the $(n+1)$-th row. Clearly, the construction of T' is reversible, as illustrated in Figure 3.2. Hence the

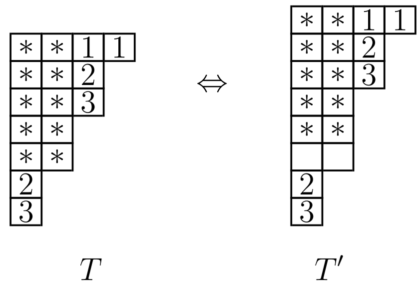

Figure 3.2: Bijection between Littlewood-Richardson tableaux

first formula is verified. The other identities can be proved based on similar arguments. ∎

Sometimes it is convenient to regard a tableau T of type $(2^k, 1^l)$ as a semistandard tableau $\tilde{T}$ filled with distinct numbers in the ordered set

$$\{1 < 1' < 2 < 2' < \cdots < n < n' < \cdots\}.$$

For this purpose, let $\tilde{T}$ be the tableau such that $\tilde{T}^{\mathrm{rev}}$ is the word obtained from T^{rev} by replacing the first occurrence of i in T^{rev} by i' for each i and keeping rest elements unchanged, as shown in Figure 3.3.

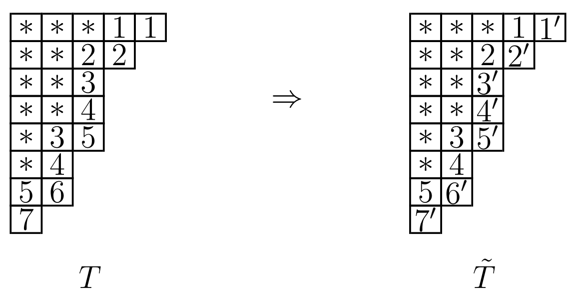

Figure 3.3: Construct $\tilde{T}$ from T

We also need the following notation to represent a set of partitions associated with a specified partition. Given a partition μ, let

$$Q_\mu(n) = \{\lambda \in \mathrm{Par}(n) : \lambda = \mu \cup (4)^a \cup (3,1)^b \cup (2,2)^c \text{ for } a, b, c \in \mathbb{N}\}.$$

Lemma 3.2 *Let $m = 2k+1$ for some $k \in \mathbb{N}$. The following statements hold.*

(i)

$$D^{(1)}_{m,k} = D^{(1)}_{2k+1,k} = s_{(2^k)} s_{(2^k)} = \sum_{\lambda \in Q_\emptyset(4k)} s_\lambda.$$

(ii)

$$D^{(1)}_{m,k+1} = D^{(1)}_{2k+1,k+1} = s_{(2^{k+1})}s_{(2^{k-1})} = \sum_{\lambda \in Q_{(2,2)}(4k)} s_\lambda.$$

(iii) Let $Q_1(n) = Q_{(3,1)}(n) \cup Q_{(2,1,1)}(n) \cup Q_{(3,3,2)}(n)$*. Then*

$$D^{(2)}_{m,k} = D^{(2)}_{2k+1,k} = s_{(2^{k-1},1^2)}s_{(2^k)} = \sum_{\lambda \in Q_1(4k)} s_\lambda.$$

(iv) Let $Q_2(n) = Q_{(2,1,1)}(n) \cup Q_{(3,2,2,1)}(n) \cup Q_{(3,3,2,2,2)}(n)$*. Then*

$$D^{(2)}_{m,k+1} = D^{(2)}_{2k+1,k+1} = s_{(2^k,1^2)}s_{(2^{k-1})} = \sum_{\lambda \in Q_2(4k)} s_\lambda.$$

(v) Let $Q_3(n) = Q_{(3,1)}(n) \cup Q_{(2,2)}(n) \cup Q_{(2,1,1)}(n) \cup Q_{(3,3,2)}(n)$*. Then*

$$D^{(3)}_{m,k} = D^{(3)}_{2k+1,k} = s_{(2^{k-1},1)}s_{(2^k,1)} = \sum_{\lambda \in Q_3(4k)} a_\lambda s_\lambda,$$

where $a_\lambda = 2$ *if* $\lambda \in Q_{(3,2,2,1)}(4k)$*, otherwise* $a_\lambda = 1$*.*

(vi) We have

$$D^{(3)}_{m,k+1} = D^{(3)}_{2k+1,k+1} = s_{(2^k,1)}s_{(2^{k-1},1)} = \sum_{\lambda \in Q_3(4k)} a_\lambda s_\lambda,$$

where $a_\lambda = 2$ *if* $\lambda \in Q_{(3,2,2,1)}(4k)$*, otherwise* $a_\lambda = 1$*.*

Proof.

(i) Use induction on k. Clearly, the assertion holds for $k = 0$ since $s_\emptyset = 1$, and it also holds for $k = 1$ by applying Pieri's rule; see Theorem 2.5. From the Littlewood-Richardson rule it follows that if s_λ appears in the Schur expansion of $s_{(2^k)}s_{(2^k)}$, then λ does not contain any part greater than 4. So we need to show that for each Littlewood-Richardson tableau T of shape $\mu/(2^k)$ and type (2^k), subject to the conditions on the shapes and types, there are uniquely three Littlewood-Richardson tableaux of type (2^{k+1}), which are T_1 of shape $\mu \cup (4)/(2^{k+1})$, T_2 of shape $\mu \cup (3,1)/(2^{k+1})$ and T_3 of shape $\mu \cup (2,2)/(2^{k+1})$.

Let T_1 be the tableau obtained from T by increasing all numbers by 1 and then inserting a four-square row on top of T such that the rightmost two squares are filled with 1's.

Suppose that T has r rows of length greater than 2, and that the largest number in the first r rows is j and we set $j = 0$ if $r = 0$. Consider the relabeled tableau

$\tilde{T}$ corresponding to T. Let $\tilde{T}'$ be the tableau obtained from $\tilde{T}$ by increasing all numbers below the r-th row by 1 (i.e., changing i to i' and i' to $i+1$), inserting a three-square row at the $(r+1)$-th row such that the rightmost square is filled with $(j+1)'$, and appending a single square row at the bottom filled with $k+1$. Let T_2 be the tableau obtained from $\tilde{T}'$ by replacing each i' with i.

To construct the tableau T_3, note that the tableau T does not contain the square $(k+1, 3)$. Consider the numbers in the first k rows. Let j_1 and j_2 be the smallest and largest numbers which appear only once in the first k rows of T. Starting with the tableau $\tilde{T}$, let $\tilde{T}'$ be the tableau obtained from $\tilde{T}$ by increasing all numbers below the k-th row by 2 (i.e., changing i to $i+1$ and i' to $(i+1)'$), inserting a row of two empty squares below the k-th row, and then inserting a two-square row filled with $(j_1, (j_2+1)')$ immediately below the row that has been inserted. If no number appears only once in the first k rows, consider the largest number j which appears twice in these rows (taking $j = 0$ if no such number exists). Then let $\tilde{T}'$ be the tableau obtained from $\tilde{T}$ by increasing all numbers below the k-th row by 2, inserting a row of two empty squares below the k-th row, and then inserting a two-square row filled with $(j+1, (j+1)')$ immediately below the row just inserted. Let T_3 be the tableau obtained from $\tilde{T}'$ by replacing each i' with i.

Note that if T is a Littlewood-Richardson tableau of shape $\mu/(2^k)$ and type (2^k), then there exist some nonnegative integers r, s, t such that the reverse reading word $\tilde{T}^{\text{rev}}$ is of the form (w_a, w_b, w_c, w_d), where

$$\begin{aligned} w_a &= 1', 1, \ldots, r', r \\ w_b &= (r+1)', \ldots, (r+s)' \\ w_c &= (r+s+1)', (r+1), \ldots, (r+s+t)', (r+s) \\ w_d &= (r+s+1), \ldots, (r+s+t) \end{aligned}$$

and $r + s + t = k$. From $\tilde{T}^{\text{rev}}$ we can write out ${T_1}^{\text{rev}}, {T_2}^{\text{rev}}, {T_3}^{\text{rev}}$ explicitly according to the above constructions. Now it is easy to verify that they are lattice permutations. Figure 3.4 is an illustration of the constructions of T_1, T_2, T_3.

On the other hand, it is also necessary to show that for each Littlewood-Richardson tableau T' of shape $\lambda/(2^{k+1})$ and type (2^{k+1}), we can find a Littlewood-Richardson tableau T of shape $\mu/(2^k)$ and of type 2^k such that $\lambda = \mu \cup (4)$, $\lambda = \mu \cup (3, 1)$ or $\lambda = \mu \cup (2, 2)$. It is easy to see that if λ contains at least one row of length 4, then T can be obtained from T' by reversing the construction of T_1. If T' has a two-square row fully filled with numbers and all rows of T' contain at most three squares, then T can be obtained by reversing the construction of T_3. Otherwise, T' contains at least one row of length 1 and one row of length 3 in view of the type of T'. In this case, we reverse the construction of T_2 to obtain T. Note that T is not uniquely determined by T'. Nevertheless, there exists a unique

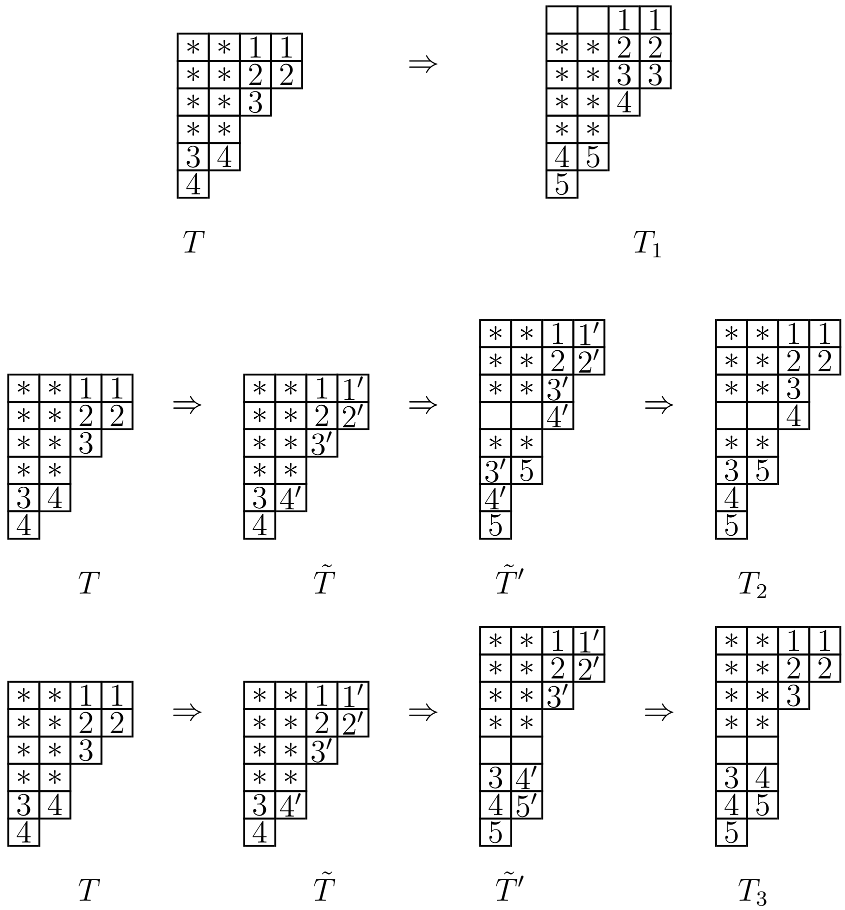

Figure 3.4: Construction of T_1, T_2, T_3 from T of shape $(4^2, 3, 2^2, 1)/(2^4)$

Littlewood-Richardson tableau of shape $\lambda/(2^k)$ and type (2^k) if s_λ appears in the expansion of $s_{(2^k)}s_{(2^k)}$. Thus, $s_{(2^k)}s_{(2^k)}$ is multiplicity free. The proof is completed by induction.

(ii) Clearly, the assertion holds for $k = 0, 1$, and $D^1_{3,2} = s_{(2,2)}$. The proof is similar to that of (i). Here we consider the Littlewood-Richardson tableau of shape $\lambda/(2^k)$ and type (2^{k-1}) if s_λ appears in $s_{(2^{k+1})}s_{(2^{k-1})}$.

(iii) Notice that $D^{(2)}_{2k+1,k} = 0$ for $k = 0$, and $D^{(2)}_{2k+1,k} = s_{(3,1)} + s_{(2,1,1)}$ for $k = 1$. For $k = 2$, we have

$$\begin{aligned} D^{(2)}_{2k+1,k} &= s_{(4,3,1)} + s_{(4,2,1^2)} + s_{(3^2,2)} + s_{(3^2,1^2)} \\ &\quad + s_{(3,2^2,1)} + s_{(3,2,1^3)} + s_{(2^3,1^2)}. \end{aligned}$$

We now use induction on k. If s_λ appears in the expansion of $D^{(2)}_{2k+1,k}$, then λ does not contain the square $(k+1,3)$, because there exists no Littlewood-Richardson tableau of shape $\lambda/(2^k)$ and type $(2^{k-1},1,1)$, or equivalently, there is no filling of the $(k+1)$-th row satisfying the lattice permutation condition. Then we can proceed as in the proof of (i).

(iv) For $k=0$, it is clear that $D^{(2)}_{2k+1,k+1}=0$. For $k=1$, we have $D^{(2)}_{2k+1,k+1}=s_{2,1,1}$. For $k=2$, we find

$$D^{(2)}_{2k+1,k+1}=s_{(4,2,1^2)}+s_{(3,2^2,1)}+s_{(3,2,1^3)}+s_{(2^3,1^2)}.$$

For $k=3$, we find

$$\begin{aligned} D^{(2)}_{2k+1,k+1} &= s_{(4^2,2,1^2)}+s_{(4,3,2^2,1)}+s_{(4,3,2,1^3)}+s_{(4,2^3,1^2)}+s_{(3^2,2^3)} \\ &\quad +s_{(3^2,2^2,1^2)}+s_{(3,2^4,1)}+s_{(3^2,2,1^4)}+s_{(3,2^3,1^3)}+s_{(2^5,1^2)}. \end{aligned}$$

Then we use induction on $k\geq 4$ and consider Littlewood-Richardson tableaux of shape $\lambda/(2^k,1^2)$ and type (2^{k-1}).

(v) For $k=0$, $D^{(3)}_{2k+1,k}=0$. For $k=1$, we get

$$D^{(3)}_{2k+1,k}=s_{(3,1)}+s_{(2^2)}+s_{(2,1^2)}.$$

For $k=2$, we have

$$\begin{aligned} D^{(3)}_{2k+1,k} &= s_{(4,3,1)}+s_{(4,2^2)}+s_{(4,2,1^2)}+s_{(3^2,2)} \\ &\quad +2s_{(3,2^2,1)}+s_{(3^2,1^2)}+s_{(3,2,1^3)}+s_{(2^4)}+s_{(2^3,1^2)} \end{aligned}$$

To use induction on k, we consider Littlewood-Richardson tableaux of shape $\lambda/(2^k,1)$ and type $(2^{k-1},1)$. If $\lambda\in Q_{(3,2,2,1)}(4k)$ there are exactly two such Littlewood-Richardson tableaux, see Figure 3.5 for the case of $\lambda=(4,3^3,2^2,1^3)$. The rest of the proof is similar to that of (i).

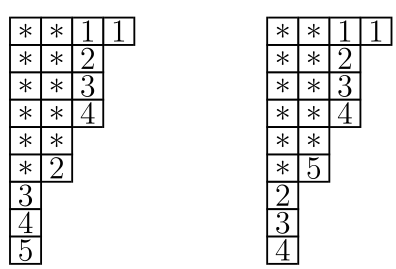

Figure 3.5: Littlewood-Richardson tableaux of shape $(4,3^3,2^2,1^3)/(2^5,1)$ and type $(2^4,1)$

(vi) It is immediate from (v).

This completes the proof of the lemma. ∎

Theorem 3.3 *Let $m = 2k+1$ for some $k \in \mathbb{N}$.*

(i) We have

$$\begin{aligned} D_{m,k} &= s_{(3^k)}s_{(1^k)}, & (3.17)\\ D_{m,k+1} &= s_{(4^k)} - s_{(3^k)}s_{(1^k)} - \Delta^{(2)}(s_{(3^k)}s_{1^{(k-2)}}). & (3.18)\end{aligned}$$

(ii) For any $0 \le i \le k-1$, we have

$$\begin{aligned} D_{m,i} &= \Delta^{(2)}(D_{m-1,i}), & (3.19)\\ D_{m,m-i} &= \Delta^{(2)}(D_{m-1,m-1-i}). & (3.20)\end{aligned}$$

Proof. (i) To prove (3.17), we need (i), (iii) and (v) of Lemma 3.2. If $\lambda \in Q_{(3,2,2,1)}(4k)$, then s_λ appears in the expansion of both $D^{(1)}_{m,k}$ and $D^{(2)}_{m,k}$, and therefore vanishes in $D_{m,k}$. If $\lambda \in Q_{(3,3,2)}(4k) \cup Q_{(2,1,1)}(4k)$, then s_λ appears in both $D^{(2)}_{m,k}$ and $D^{(3)}_{m,k}$, and also vanishes in $D_{m,k}$. If $\lambda \in Q_{(2,2)}(4k)$ but $\lambda \notin Q_{(3,1)}(4k)$, then s_λ appears in both $D^{(1)}_{m,k}$ and $D^{(3)}_{m,k}$, and also vanishes in $D_{m,k}$. Therefore, for a term s_λ which does not vanish in $D_{m,k}$, the index partition λ belongs to the set $Q_\emptyset(4k)$ but 2 does not appear as a part. By virtue of Pieri's rule, the Schur functions not vanishing in $D_{m,k}$ coincide with the terms in the Schur expansion of $s_{(3^k)}s_{(1^k)}$. Similarly, we can prove (3.18) using (ii), (iv) and (vi) of Lemma 3.2.

(ii) These are direct consequences of Lemma 3.1.

This completes the proof of the theorem. ∎

Using the same argument in the proof of Lemma 3.2, we can deduce the following expansion formulas when m is even.

Lemma 3.4 *Let $m = 2k$ for $k \in \mathbb{N}$. The following statements hold.*

(i)

$$D^{(1)}_{m,k} = D^{(1)}_{2k,k} = s_{(2^k)}s_{(2^{k-1})} = \sum_{\lambda \in Q_{(2)}(4k-2)} s_\lambda.$$

(ii)

$$D^{(1)}_{m,k-1} = D^{(1)}_{2k,k-1} = s_{(2^{k-1})}s_{(2^k)} = \sum_{\lambda\in Q_{(2)}(4k-2)} s_\lambda.$$

(iii) Let $R_1(n) = Q_{(1,1)}(n) \cup Q_{(3,3,2,2)}(n) \cup Q_{(3,2,1)}(n)$. *Then*

$$D^{(2)}_{m,k} = D^{(2)}_{2k,k} = s_{(2^{k-1},1^2)}s_{(2^{k-1})} = \sum_{\lambda\in R_1(4k-2)} s_\lambda.$$

(iv) Let $R_2(n) = Q_{(3,3)}(n) \cup Q_{(3,2,1)}(n) \cup Q_{(2,2,1,1)}(n)$. *Then*

$$D^{(2)}_{m,k-1} = D^{(2)}_{2k,k-1} = s_{(2^{k-2},1^2)}s_{(2^k)} = \sum_{\lambda\in R_2(4k-2)} s_\lambda.$$

(v) Let $R_3(n) = Q_{(3,3)}(n) \cup Q_{(2)}(n) \cup Q_{(1,1)}(n)$. *Then*

$$D^{(3)}_{m,k} = D^{(3)}_{2k,k} = s_{(2^{k-1},1)}s_{(2^{k-1},1)} = \sum_{\lambda\in R_3(4k-2)} a_\lambda s_\lambda,$$

where $a_\lambda = 2$ *if* $\lambda \in Q_{(3,2,1)}(4k-2)$, *otherwise* $a_\lambda = 1$.

(vi) Let $R_4(n) = Q_{(3,3,2,2)}(n) \cup Q_{(3,2,1)}(n) \cup Q_{(2,2,2)}(n) \cup Q_{(2,2,1,1)}(n)$. *Then*

$$D^{(3)}_{m,k-1} = s_{(2^{k-2},1)}s_{(2^k,1)} = \sum_{\lambda\in R_4(4k-2)} a_\lambda s_\lambda,$$

where $a_\lambda = 2$ *if* $\lambda \in Q_{(3,2,2,2,1)}(4k)$, *otherwise* $a_\lambda = 1$.

In view of Lemmas 3.1 and 3.4, we deduce the following theorem for even m. The proof is similar to that of Theorem 3.3 and is omitted.

Theorem 3.5 *Let* $m = 2k$ *for some* $k \in \mathbb{N}$.

(i) We have

$$\begin{aligned} D_{m,k-1} &= s_{(3^k)}s_{(1^{k-2})} + \Delta^{(2)}(s_{(3^{k-1})}s_{(1^{k-1})}), && (3.21)\\ D_{m,k} &= -s_{(3^k)}s_{(1^{k-2})}. && (3.22)\end{aligned}$$

(ii) For any $0 \le i \le k-2$, *we have*

$$\begin{aligned} D_{m,i} &= \Delta^{(2)}(D_{m-1,i}), && (3.23)\\ D_{m,m-i} &= \Delta^{(2)}(D_{m-1,m-1-i}), && (3.24)\\ D_{m,m-k+1} &= \Delta^{(2)}(D_{m-1,m-k}). && (3.25)\end{aligned}$$

Corollary 3.6 *Assume $k \geq 1$.*

(i) If $m = 2k+1$, then $D_{m,i}$ is s-positive for $0 \leq i \leq k$, and $D_{m,i}$ is s-negative for $k+1 \leq i \leq m-1$.

(ii) If $m = 2k$, then $D_{m,i}$ is s-positive for $0 \leq i \leq k-1$, and $D_{m,i}$ is s-negative for $k \leq i \leq m-1$.

Proof. Use induction on m. It is easy to verify that the result holds for $k = 1$. For $m = 2k+1$, we see that $D_{m,k}$ is s-positive and $D_{m,k+1}$ is s-negative in view of (i) of Theorem 3.3. For $0 \leq i \leq k-1$, using (ii) of Theorem 3.3 we see that $D_{m,i} = \Delta^{(2)} D_{2k,i}$ is s-positive by induction. Similarly, for $k+2 \leq i \leq 2k$ we find that $D_{m,i} = \Delta^{(2)} D_{2k,i-1}$ is s-negative by induction. For $m = 2k$, from (i) of Theorem 3.5 it follows that $D_{m,k-1}$ is s-positive and $D_{m,k}$ is s-negative. For $0 \leq i \leq k-2$, using (ii) of Theorem 3.5, by induction we obtain that $D_{m,i} = \Delta^{(2)} D_{2k-1,i}$ is s-positive. Similarly, for $k+1 \leq i \leq 2k-1$, by induction we deduce that $D_{m,i} = \Delta^{(2)} D_{2k-1,i-1}$ is s-negative. ∎

Theorems 3.3 and 3.5 lead to a construction for the underlying partitions of the Schur expansion. Table 3.1 is an illustration.

Given a set S of positive integers, let $\mathrm{Par}_S(n)$ denote the set of partitions of n whose parts belong to S. We are now ready to present the following theorem on Schur positivity.

Theorem 3.7 *For any $m \geq 0$, we have*

$$\begin{aligned}\textstyle\sum_{i=0}^{m} \left(s_{(2^{i-1})} s_{(2^{m-i})} + s_{(2^{i-2},1^2)} s_{(2^{m-i})} - s_{(2^{i-1},1)} s_{(2^{m-i-1},1)} \right) \\ = \textstyle\sum_{\lambda \in \mathrm{Par}_{\{2,4\}}(2m-2)} s_\lambda. \end{aligned} \tag{3.26}$$

Consequently, the summation on the left-hand side of the above identity is s-positive.

Before proving the above theorem, let us give some examples. Taking $m = 3, 4, 5$ and using the Maple package, we observe that

$$\begin{aligned}\textstyle\sum_{k=0}^{3} \left(s_{(2^{k-1})} s_{(2^{3-k})} + s_{(2^{k-2},1^2)} s_{(2^{3-k})} - s_{(2^{k-1},1)} s_{(2^{3-k-1},1)} \right) \\ = s_{(4)} + s_{(2,2)}. \\ \textstyle\sum_{k=0}^{4} \left(s_{(2^{k-1})} s_{(2^{4-k})} + s_{(2^{k-2},1^2)} s_{(2^{4-k})} - s_{(2^{k-1},1)} s_{(2^{4-k-1},1)} \right) \\ = s_{(4,2)} + s_{(2,2,2)}. \\ \textstyle\sum_{k=0}^{5} \left(s_{(2^{k-1})} s_{(2^{5-k})} + s_{(2^{k-2},1^2)} s_{(2^{5-k})} - s_{(2^{k-1},1)} s_{(2^{5-k-1},1)} \right) \\ = s_{(4,4)} + s_{(4,2,2)} + s_{(2,2,2,2)}.\end{aligned}$$

	$m=7$
$D_{7,0}$	$s_{(2^6)}$
$D_{7,1}$	$s_{(4,2^4)} + s_{(3^2,2^3)} + s_{(3,2^4,1)}$
$D_{7,2}$	$s_{(3^2,2^2,1^2)} + s_{(4,3^2,2)} + s_{(4^2,2^2)} + s_{(3^3,2,1)} + s_{(4,3,2^2,1)}$
$D_{7,3}$	$s_{(4,3^2,1^2)} + s_{(3^3,1^3)} + s_{(4^2,3,1)} + s_{(4^3)}$
$D_{7,4}$	$-s_{(4,3^2,2)} - s_{(4,3^2,1^2)} - s_{(3^3,2,1)} - s_{(3^3,1^3)} - s_{(4^2,3,1)}$
$D_{7,5}$	$-s_{(3^2,2^3)} - s_{(3^2,2^2,1^2)} - s_{(4,3,2^2,1)}$
$D_{7,6}$	$-s_{(3,2^4,1)}$
$D_{7,7}$	0

	$m=8$
$D_{8,0}$	$s_{(2^7)}$
$D_{8,1}$	$s_{(4,2^5)} + s_{(3^2,2^4)} + s_{(3,2^5,1)}$
$D_{8,2}$	$s_{(3^2,2^3,1^2)} + s_{(4,3^2,2^2)} + s_{(4^2,2^3)} + s_{(3^3,2^2,1)} + s_{(4,3,2^3,1)}$
$D_{8,3}$	$s_{(4,3^2,2,1^2)} + s_{(3^3,2,1^3)} + s_{(4^2,3,2,1)} + s_{(4^3,2)}$ $+s_{(3^4,1^2)} + s_{(4^2,3^2)} + s_{(4,3^3,1)}$
$D_{8,4}$	$-s_{(3^4,1^2)} - s_{(4^2,3^2)} - s_{(4,3^3,1)}$
$D_{8,5}$	$-s_{(4^2,3,2,1)} - s_{(3^3,2^2,1)} - s_{(3^3,2,1^3)} - s_{(4,3^2,2,1^2)} - s_{(4,3^2,2^2)}$
$D_{8,6}$	$-s_{(3^2,2^4)} - s_{(3^2,2^3,1^2)} - s_{(4,3,2^3,1)}$
$D_{8,7}$	$-s_{(3,2^5,1)}$
$D_{8,8}$	0

	$m=9$
$D_{9,0}$	$s_{(2^8)}$
$D_{9,1}$	$s_{(4,2^6)} + s_{(3^2,2^5)} + s_{(3,2^6,1)}$
$D_{9,2}$	$s_{(3^2,2^4,1^2)} + s_{(4,3^2,2^3)} + s_{(4^2,2^4)} + s_{(3^3,2^3,1)} + s_{(4,3,2^4,1)}$
$D_{9,3}$	$s_{(4,3^2,2^2,1^2)} + s_{(3^3,2^2,1^3)} + s_{(4^2,3,2^2,1)} + s_{(4^3,2^2)}$ $+s_{(3^4,2,1^2)} + s_{(4^2,3^2,2)} + s_{(4,3^3,2,1)}$
$D_{9,4}$	$s_{(4,3^3,1^3)} + s_{(4^2,3^2,1^2)} + s_{(4^4)} + s_{(4^3,3,1)} + s_{(3^4,1^4)}$
$D_{9,5}$	$-s_{(4,3^3,1^3)} - s_{(4^2,3^2,1^2)} - s_{(4^4)} - s_{(4^3,3,1)} - s_{(3^4,1^4)}$ $-s_{(3^4,2,1^2)} - s_{(4^2,3^2,2)} - s_{(4,3^3,2,1)}$
$D_{9,6}$	$-s_{(4^2,3,2^2,1)} - s_{(3^3,2^3,1)} - s_{(3^3,2^2,1^3)} - s_{(4,3^2,2^2,1^2)} - s_{(4,3^2,2^3)}$
$D_{9,7}$	$-s_{(3^2,2^5)} - s_{(3^2,2^4,1^2)} - s_{(4,3,2^4,1)}$
$D_{9,8}$	$-s_{(3,2^6,1)}$
$D_{9,9}$	0

Table 3.1: Schur function expansions of $D_{m,k}$ for $m = 7, 8, 9$

Proof of Theorem 3.7. By convention, for $i=0$ or $i=m+1$, it is natural to set

$$s_{(2^{i-1})}s_{(2^{m-i})}+s_{(2^{i-2},1^2)}s_{(2^{m-i})}=0.$$

Therefore,

$$\sum_{i=0}^{m}\left(s_{(2^{i-1})}s_{(2^{m-i})}+s_{(2^{i-2},1^2)}s_{(2^{m-i})}-s_{(2^{i-1},1)}s_{(2^{m-i-1},1)}\right)=\sum_{i=0}^{m}D_{m,i}.$$

It suffices to prove that

$$\sum_{i=0}^{m+1}D_{m+1,i}=\begin{cases}\Delta^{(2)}\left(\sum_{i=0}^{m}D_{m,i}\right), & \text{if } m=2k-1\\ s_{(4^k)}+\Delta^{(2)}\left(\sum_{i=0}^{m}D_{m,i}\right), & \text{if } m=2k\end{cases} \tag{3.27}$$

for $m\geq 0$. The case for $m=0$ is obvious. We now assume $m\geq 1$.

If $m=2k-1$ for some $k\geq 1$, then

$$\begin{aligned}
\sum_{i=0}^{m+1}D_{m+1,i} &= \sum_{i=0}^{2k}D_{2k,i}\\
&= \sum_{i=0}^{k-2}D_{2k,i}+D_{2k,k-1}+D_{2k,k}+D_{2k,k+1}+\sum_{i=0}^{k-2}D_{2k,2k-i}\\
&= \sum_{i=0}^{k-2}\Delta^{(2)}(D_{2k-1,i})+\left(s_{(3^k)}s_{(1^{k-2})}+\Delta^{(2)}\left(s_{(3^{k-1})}s_{(1^{k-1})}\right)\right)\\
&\quad+\left(-s_{(3^k)}s_{(1^{k-2})}\right)+\Delta^{(2)}(D_{2k-1,k})\\
&\quad+\sum_{i=0}^{k-2}\Delta^{(2)}(D_{2k-1,2k-1-i}) \qquad \text{(by Theorem 3.5)}\\
&= \sum_{i=0}^{k-2}\Delta^{(2)}(D_{2k-1,i})+\Delta^{(2)}(D_{2k-1,k-1})+\Delta^{(2)}(D_{2k-1,k})\\
&\quad+\sum_{i=0}^{k-2}\Delta^{(2)}(D_{2k-1,2k-1-i}) \qquad \text{(by (3.17))}\\
&= \sum_{i=0}^{2k-1}\Delta^{(2)}(D_{2k-1,i})=\Delta^{(2)}\left(\sum_{i=0}^{m}D_{m,i}\right).
\end{aligned}$$

If $m=2k$ for some $k\geq 1$, then

$$\begin{aligned}
\sum_{i=0}^{m+1}D_{m+1,i} &= \sum_{i=0}^{2k+1}D_{2k+1,i}\\
&= \sum_{i=0}^{k-1}D_{2k+1,i}+D_{2k+1,k}+D_{2k+1,k+1}+\sum_{i=0}^{k-1}D_{2k+1,2k+1-i}\\
&= \sum_{i=0}^{k-1}\Delta^{(2)}(D_{2k,i})+s_{(3^k)}s_{(1^k)}\\
&\quad+\left(s_{(4^k)}-s_{(3^k)}s_{(1^k)}-\Delta^{(2)}\left(s_{(3^k)}s_{1^{(k-2)}}\right)\right)\\
&\quad+\sum_{i=0}^{k-1}\Delta^{(2)}(D_{2k,2k-i}) \qquad \text{(by Theorem 3.3)}\\
&= s_{(4^k)}+\sum_{i=0}^{2k}\Delta^{(2)}(D_{2k,i}) \qquad \text{(by (3.22))}\\
&= s_{(4^k)}+\Delta^{(2)}\left(\sum_{i=0}^{m}D_{m,i}\right).
\end{aligned}$$

Based on (3.27), we obtain the desired assertion by induction on m. ∎

Now we consider other products of Schur functions, which are necessary to prove the strong q-log-convexity of the Narayana polynomials.

Given $a, b, r \in \mathbb{N}$ and $0 \le k \le r$, let

$$\begin{aligned} D_1(a,b,k,r) &= s_{(2^{k-b-1},1^{b+2-a})}s_{(2^{r-k-1})}, \\ D_2(a,b,k,r) &= s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1})}, \\ D_3(a,b,k,r) &= s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k-1},1)}. \end{aligned}$$

and let

$$D(a,b,k,r) = D_1(a,b,k,r) + D_2(a,b,k,r) - D_3(a,b,k,r), \tag{3.28}$$

where $s_{(2^i,1^j)} = 0$ for $i < 0$ or $j < 0$. It is easy to see that $D(a,b,r,r) \equiv 0$. For $i = 1, 2, 3$, it is also clear that

$$D_i(a,b,k,r) = D_i(a-1,b-1,k-1,r-1),$$

hence

$$D(a,b,k,r) = D(a-1,b-1,k-1,r-1). \tag{3.29}$$

Some values of $D(a,b,k,r)$ are given in Table 3.2.

Given a pair (λ,μ) of partitions and a pair (f_1,f_2) of symmetric functions, we define the product $\tilde{\Delta}^{\lambda,\mu}(f_1,f_2)$ of f_1 and f_2 as follows. Suppose that

$$\Delta^{\lambda}(f_1) = \sum_{\nu} a_\nu s_\nu, \tag{3.30}$$

$$\Delta^{\mu}(f_2) = \sum_{\nu} b_\nu s_\nu. \tag{3.31}$$

Define

$$\tilde{\Delta}^{\lambda,\mu}(f_1,f_2) = \sum_{\nu} \max(a_\nu, b_\nu) s_\nu. \tag{3.32}$$

Lemma 3.8 *For any $r \ge k \ge b \ge a \ge 0$ and $i = 1, 2, 3$, we have the following recurrence relations*

$$D_i(a,b,k,r) = \tilde{\Delta}^{(1),(3)}(D_i(a,b-1,k-1,r-1), D_i(a,b-1,k-1,r-2)). \tag{3.33}$$

Proof. We first prove that

$$\begin{aligned} &s_{(2^{k-b-1},1^{b+2-a})}s_{(2^{r-k-1})} = \\ &\quad \tilde{\Delta}^{(1),(3)}\big(s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k-1})}, s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k-2})}\big). \end{aligned}$$

	$a=0,b=1,r=8$
$D(a,b,0,r)$	0
$D(a,b,1,r)$	$s_{(3,2^5)} + s_{(2^6,1)}$
$D(a,b,2,r)$	$s_{(3^3,2^2)} + s_{(3^2,2^3,1)} + s_{(3,2^4,1^2)} + s_{(4,3,2^3)} + s_{(4,2^4,1)}$
$D(a,b,3,r)$	$s_{(4,3^2,2,1)} + s_{(4,3,2^2,1^2)} + s_{(3^3,2,1^2)} + s_{(3^2,2^2,1^3)}$ $+s_{(4^2,3,2)} + s_{(4^2,2^2,1)} + s_{(4,3^3)} + s_{(3^4,1)}$
$D(a,b,4,r)$	$s_{(4^2,3,1^2)} - s_{(4,3^3)} + s_{(4,3^2,1^3)} - s_{(3^4,1)} + s_{(3^3,1^4)} + s_{4^3,1}$
$D(a,b,5,r)$	$-s_{(4,3^2,2,1)} - s_{(3^3,2^2)} - s_{(3^3,2,1^2)} - s_{(4^2,3,1^2)} - s_{(4,3^2,1^3)} - s_{(3^3,1^4)}$
$D(a,b,6,r)$	$-s_{(4,3,2^2,1^2)} - s_{(3^2,2^3,1)} - s_{(3^2,2^2,1^3)}$
$D(a,b,7,r)$	$-s_{(3,2^4,1^2)}$
$D(a,b,8,r)$	0

	$a=0,b=1,r=9$
$D(a,b,0,r)$	0
$D(a,b,1,r)$	$s_{(3,2^6)} + s_{(2^7,1)}$
$D(a,b,2,r)$	$s_{(3^3,2^3)} + s_{(3^2,2^4,1)} + s_{(3,2^5,1^2)} + s_{(4,3,2^4)} + s_{(4,2^5,1)}$
$D(a,b,3,r)$	$s_{(4,3^3,2)} + s_{(3^4,2,1)} + s_{4^2,2^3,1)} + s_{(4,3,2^3,1^2)}$ $+s_{(3^2,2^3,1^3)} + s_{(4^2,3,2^2)} + s_{(4,3^2,2^2,1)} + s_{(3^3,2^2,1^2)}$
$D(a,b,4,r)$	$s_{(4^3,3)} + s_{(4^2,3^2,1)} + s_{(4,3^3,1^2)} + s_{(3^4,1^3)} + s_{(4^3,2,1)}$ $+s_{(4^2,3,2,1^2)} + s_{(4,3^2,2,1^3)} + s_{(3^3,2,1^4)}$
$D(a,b,5,r)$	$-s_{(4^2,3^2,1)} - s_{(4,3^3,1^2)} - s_{(3^4,1^3)} - s_{(4,3^3,2)} - s_{(3^4,2,1)}$
$D(a,b,6,r)$	$-s_{(4,3^2,2^2,1)} - s_{(3^3,2^3)} - s_{(3^3,2^2,1^2)} - s_{(4^2,3,2,1^2)}$ $-s_{(4,3^2,2,1^3)} - s_{(3^3,2,1^4)}$
$D(a,b,7,r)$	$-s_{(4,3,2^3,1^2)} - s_{(3^2,2^4,1)} - s_{(3^2,2^3,1^3)}$
$D(a,b,8,r)$	$-s_{(3,2^5,1^2)}$

	$a=0,b=2,r=10$
$D(a,b,1,r)$	0
$D(a,b,2,r)$	$s_{(3^2,2^5)} + s_{(3,2^6,1)} + s_{(2^7,1^2)}$
$D(a,b,3,r)$	$s_{(3^4,2^2)} + s_{(4,3^2,2^3)} + s_{(4,2^5,1^2)}$ $+s_{(3^3,2^3,1)} + s_{(3^2,2^4,1^2)} + s_{(3,2^5,1^3)} + s_{(4,3,2^4,1)}$
$D(a,b,4,r)$	$s_{(3^5,1)} + s_{(4,3^4)} + s_{(4^2,3^2,2)} + s_{(4,3^3,2,1)} + s_{(3^4,2,1^2)} + s_{(4^2,2^3,1^2)}$ $+s_{(4,3,2^3,1^3)} + s_{(3^2,2^3,1^4)} + s_{(4^2,3,2^2,1)} + s_{(4,3^2,2^2,1^2)} + s_{(3^3,2^2,1^3)}$
$D(a,b,5,r)$	$-s_{(3^5,1)} - s_{(4,3^4)} + s_{(4^3,3,1)} + s_{(4^2,3^2,1^2)} + s_{(4,3^3,1^3)} + s_{(3^4,1^4)}$ $+s_{(4^3,2,1^2)} + s_{(4^2,3,2,1^3)} + s_{(4,3^2,2,1^4)} + s_{(3^3,2,1^5)}$
$D(a,b,6,r)$	$-s_{(4^2,3^2,1^2)} - s_{(4,3^3,1^3)} - s_{(3^4,1^4)} - s_{(4,3^3,2,1)} - s_{(3^4,2,1^2)} - s_{(3^4,2^2)}$
$D(a,b,7,r)$	$-s_{(4,3^2,2^2,1^2)} - s_{(3^3,2^2,1^3)} - s_{(3^3,2^3,1)} - s_{(3^3,2,1^5)}$ $-s_{(4,3^2,2,1^4)} - s_{(4^2,3,2,1^3)}$
$D(a,b,8,r)$	$-s_{(4,3,2^3,1^3)} - s_{(3^2,2^4,1^2)} - s_{(3^2,2^3,1^4)}$
$D(a,b,9,r)$	$-s_{(3,2^5,1^3)}$

Table 3.2: Schur function expansion of $D(a,b,k,r)$ for $r=8,9,10$

We claim that there exists an injective map between the set of Littlewood-Richardson tableaux of shape $\mu/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+1-a})$ and the set of Littlewood-Richardson tableaux of shape $\mu \cup (1)/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+2-a})$. This means that if s_λ appears in the Schur expansion of $s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k-1})}$, then $s_{\lambda\cup(1)}$ appears in $s_{(2^{k-b-1},1^{b+2-a})}s_{(2^{r-k-1})}$. This injective map can be constructed as follows. Given a Littlewood-Richardson tableau T of shape $\mu/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+1-a})$, let T' be the tableau obtained from T by appending one row composed of a single square filled with $k+1-a$. Clearly, T' is a Littlewood-Richardson tableau of $\lambda \cup (1)/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+2-a})$. See the first two tableaux in Figure 3.6.

It will be shown that there exists an injective map between the set of Littlewood-Richardson tableaux of shape $\mu/(2^{r-k-2})$ and type $(2^{k-b-1}, 1^{b+1-a})$ and the set of Littlewood-Richardson tableaux of shape $\mu \cup (3)/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+2-a})$. This means that if s_μ appears in the expansion of $s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k-2})}$, then $s_{\mu\cup(3)}$ appears in $s_{(2^{k-b-1},1^{b+2-a})}s_{(2^{r-k-1})}$. Given a Littlewood-Richardson tableau T of shape $\mu/(2^{r-k-2})$ and type $(2^{k-b-1}, 1^{b+1-a})$, we consider the corresponding tableau $\tilde{T}$. Suppose that T has m rows of length 4 (taking $m = 0$ if no such row exists). Let $\tilde{T}'$ be the tableau obtained from $\tilde{T}$ by inserting one row of three squares at the $(m+1)$-th row in which the rightmost square is filled with $(m+1)'$, and then increasing all numbers below the $(m+1)$-th row by 1 (i.e., changing i to i' and i' to $i+1$). Let T' be the tableau obtained from $\tilde{T}'$ by replacing i' with i for each i. It is routine to verify that T' is a Littlewood-Richardson tableau of shape $\mu \cup (3)/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+2-a})$, as desired. See the last four tableaux in Figure 3.6.

Moreover, it remains to prove that any Littlewood-Richardson tableau T' of shape $\lambda/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+2-a})$ can be constructed from a Littlewood-Richardson tableau T, which is either of shape $\mu/(2^{r-k-2})$ and type $(2^{k-b-1}, 1^{b+1-a})$ with $\lambda = \mu \cup (3)$, or of shape $\mu/(2^{r-k-1})$ and type $(2^{k-b-1}, 1^{b+1-a})$ with $\lambda = \mu \cup (1)$. If 3 is a part of λ, then we can reverse the map in the pervious paragraph to obtain T. If 3 does not appear as a part of λ, then the lattice permutation property requires that λ should contain a part of size 1 and the bottom square should be filled with $k+1-a$. In this case, let T be the tableau obtained from T' by removing the bottom row.

Thus we complete the proof of the recurrence of $D_1(a, b, k, r)$, and the rest can be proved in the same manner. ∎

Theorem 3.9 *For any $b \geq a \geq 0$ and $r \geq 0$, the symmetric function $\sum_{k=0}^{r} D(a, b, k, r)$ is s-positive.*

Proof. We use induction on the difference $b - a$. When $a = b$, note that

$$\sum_{k=0}^{r} D(a, b, k, r) = \sum_{k=a}^{r} D(0, 0, k-a, r-a) = \sum_{i=0}^{r-a} D_{r-a,i}.$$

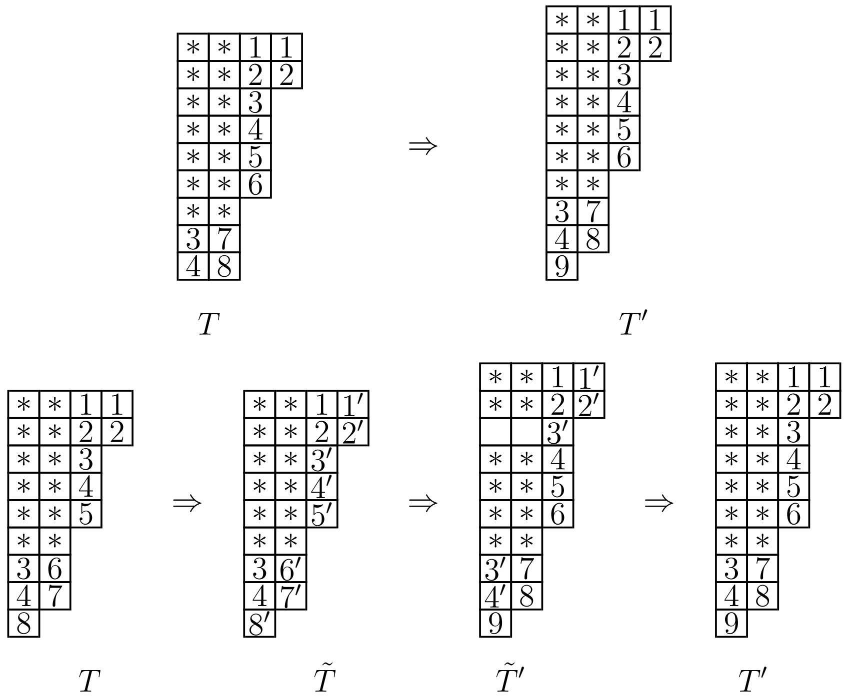

Figure 3.6: Two ways to construct T'

According to Theorem 3.7, it is s-positive. Now suppose $b - a \geq 1$. The negative terms of $D(a, b, k, r)$ come from either $\Delta^{(1)}(D(a, b-1, k-1, r-1))$ or $\Delta^{(3)}(D(a, b-1, k-1, r-2))$ by Lemma 3.8. They always vanish in $\sum_{k=0}^{r} D(a, b, k, r)$ since both $\sum_{k=0}^{r} D(a, b-1, k-1, r-1)$ and $\sum_{k=0}^{r} D(a, b-1, k-1, r-2)$ are s-positive by induction. This completes the proof. ∎

4 The q-Log-convexity

The main objective of this section is to show that the Narayana polynomials form a strongly q-log-convex sequence. This is a stronger version of the conjecture of Liu and Wang.

Theorem 4.1 *The Narayana polynomials $N_n(q)$ form a strongly q-log-convex sequence.*

Proof. Note that, for $0 \leq k \leq n$, we have

$$N(n, k) = N_q(n, k)|_{q=1} = s_{(2^k)}(1^{n-1}) = \mathrm{ps}^1_{n-1}\left(s_{(2^k)}\right). \tag{4.34}$$

For $k>n$, we see that $N(n,k)=0=\mathrm{ps}_{n-1}^{1}\left(s_{(2^k)}\right)$.

For any $m\geq n\geq 1$ and $r\geq 0$, the coefficient of q^r in $N_{m+1}(q)N_{n-1}(q)$ equals

$$C_1=\sum_{k=0}^{r}\mathrm{ps}_{m}^{1}\left(s_{(2^k)}\right)\mathrm{ps}_{n-2}^{1}\left(s_{(2^{r-k})}\right), \tag{4.35}$$

and the coefficient of q^r in $N_m(q)N_n(q)$ equals

$$C_2=\sum_{k=0}^{r}\mathrm{ps}_{m-1}^{1}\left(s_{(2^k)}\right)\mathrm{ps}_{n-1}^{1}\left(s_{(2^{r-k})}\right). \tag{4.36}$$

According to Lemma 2.3, we have

$$\begin{aligned}
\mathrm{ps}_{m}^{1}\left(s_{(2^k)}\right) &= \sum_{0\leq a\leq b\leq m-n+2}\mathrm{ps}_{n-2}^{1}\big(s_{(2^{k-b},1^{b-a})}\big)\mathrm{ps}_{m-n+2}^{1}\big(s_{(2^a,1^{b-a})}\big),\\
\mathrm{ps}_{m-1}^{1}\left(s_{(2^k)}\right) &= \sum_{0\leq a\leq b\leq m-n+1}\mathrm{ps}_{n-2}^{1}\big(s_{(2^{k-b},1^{b-a})}\big)\mathrm{ps}_{m-n+1}^{1}\big(s_{(2^a,1^{b-a})}\big),\\
\mathrm{ps}_{n-1}^{1}\left(s_{(2^{r-k})}\right) &= \mathrm{ps}_{n-2}^{1}\left(s_{(2^{r-k})}+s_{(2^{r-k-1},1)}+s_{(2^{r-k-1})}\right).
\end{aligned}$$

By expansion we obtain that

$$\begin{aligned}
&C_1-C_2=\\
&\quad\textstyle\sum_{k=0}^{r}\sum_{0\leq a\leq b\leq m-n+2}\mathrm{ps}_{m-n+2}^{1}\big(s_{(2^a,1^{b-a})}\big)\mathrm{ps}_{n-2}^{1}\big(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}\big)\\
&-\textstyle\sum_{k=0}^{r}\sum_{0\leq a\leq b\leq m-n+1}\mathrm{ps}_{m-n+1}^{1}\big(s_{(2^a,1^{b-a})}\big)\mathrm{ps}_{n-2}^{1}\big(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}\big)\\
&-\textstyle\sum_{k=0}^{r}\sum_{0\leq a\leq b\leq m-n+1}\mathrm{ps}_{m-n+1}^{1}\big(s_{(2^a,1^{b-a})}\big)\mathrm{ps}_{n-2}^{1}\big(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1},1)}\big)\\
&-\textstyle\sum_{k=0}^{r}\sum_{0\leq a\leq b\leq m-n+1}\mathrm{ps}_{m-n+1}^{1}\big(s_{(2^a,1^{b-a})}\big)\mathrm{ps}_{n-2}^{1}\big(s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1})}\big).
\end{aligned}$$

To simplify the notation, let $d=m-n+1$. Note that

$$\begin{aligned}
\mathrm{ps}_{d+1}^{1}\big(s_{(2^a,1^{b-a})}\big) &= \mathrm{ps}_{d}^{1}\big(s_{(2^a,1^{b-a})}\big)+\mathrm{ps}_{d}^{1}\big(s_{(2^a,1^{b-a-1})}\big)\\
&\quad+\mathrm{ps}_{d}^{1}\big(s_{(2^{a-1},1^{b-a})}\big)+\mathrm{ps}_{d}^{1}\big(s_{(2^{a-1},1^{b-a+1})}\big).
\end{aligned}$$

Therefore, the double summation

$$\sum_{k=0}^{r}\sum_{0\leq a\leq b\leq d+1}\mathrm{ps}_{d+1}^{1}\big(s_{(2^a,1^{b-a})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}$$

can be divided into four parts

$$\begin{aligned} A1 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k})} \\ A2 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a-1})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k})} \\ A3 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^{a-1},1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k})} \\ A4 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^{a-1},1^{b-a+1})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k})}. \end{aligned}$$

Let

$$\begin{aligned} B1 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k})}, \\ B2 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k-1},1)}, \\ B3 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k-1})}. \end{aligned}$$

The equality $A1 = B1$ holds because

$$A1 = B1 + \sum_{k=0}^{r} \sum_{0\le a\le d+1} \mathrm{ps}_d^1\big(s_{(2^a,1^{d+1-a})}\big) s_{(2^{k-d-1},1^{d+1-a})} s_{(2^{r-k})},$$

but $\mathrm{ps}_d^1\big(s_{(2^a,1^{d+1-a})}\big) \equiv 0$.

We also have the equality $A3 = B3$ since

$$\begin{aligned} A3 &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^{a-1},1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k})} \\ &= \sum_{k=0}^{r} \sum_{1\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^{a-1},1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k})} \\ &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b-1},1^{b-a})} s_{(2^{r-k})} \\ &= \sum_{k=1}^{r} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b-1},1^{b-a})} s_{(2^{r-k})} \\ &= \sum_{k=0}^{r-1} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k-1})} \\ &= \sum_{k=0}^{r} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big) s_{(2^{k-b},1^{b-a})} s_{(2^{r-k-1})} \\ &= B3. \end{aligned}$$

Moreover, we have

$$\begin{aligned}
A2 &= \textstyle\sum_{k=0}^{r}\sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a-1})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=0}^{r}\sum_{0\le a< b\le d+1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a-1})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=0}^{r}\sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big)s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=0}^{r}\sum_{0\le a< b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big)s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k})}\\
&\quad + \textstyle\sum_{k=0}^{r}\sum_{0\le a\le d} \mathrm{ps}_d^1\big(s_{(2^a)}\big)s_{(2^{k-a-1},1)}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=1}^{r}\sum_{0\le a< b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big)s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k})}\\
&\quad + \textstyle\sum_{k=0}^{r}\sum_{0\le a\le d} \mathrm{ps}_d^1\big(s_{(2^a)}\big)s_{(2^{k-a-1},1)}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=1}^{r}\sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b+1-a})}\big)s_{(2^{k-b-2},1^{b+2-a})}s_{(2^{r-k})}\\
&\quad + \textstyle\sum_{k=0}^{r}\sum_{0\le a\le d} \mathrm{ps}_d^1\big(s_{(2^a)}\big)s_{(2^{k-a-1},1)}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=0}^{r-1}\sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b+1-a})}\big)s_{(2^{k-b-1},1^{b+2-a})}s_{(2^{r-k-1})}\\
&\quad + \textstyle\sum_{k=0}^{r}\sum_{0\le a\le d} \mathrm{ps}_d^1\big(s_{(2^a)}\big)s_{(2^{k-a-1},1)}s_{(2^{r-k})}
\end{aligned}$$

and

$$\begin{aligned}
A4 &= \textstyle\sum_{k=0}^{r}\sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1\big(s_{(2^{a-1},1^{b-a+1})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=0}^{r}\sum_{1\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^{a-1},1^{b-a+1})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=1}^{r}\sum_{1\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^{a-1},1^{b-a+1})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=1}^{r}\sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b+1-a})}\big)s_{(2^{k-b-1},1^{b-a})}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=0}^{r-1}\sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b+1-a})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1})}
\end{aligned}$$

and

$$\begin{aligned}
B2 &= \textstyle\sum_{k=0}^{r}\sum_{0\le a\le b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1},1)}\\
&= \textstyle\sum_{k=0}^{r}\sum_{0\le a< b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1},1)}\\
&\quad + \textstyle\sum_{k=0}^{r}\sum_{0\le a\le d} \mathrm{ps}_d^1\big(s_{(2^a)}\big)s_{(2^{k-a})}s_{(2^{r-k-1},1)}\\
&= \textstyle\sum_{k=0}^{r}\sum_{0\le a< b\le d} \mathrm{ps}_d^1\big(s_{(2^a,1^{b-a})}\big)s_{(2^{k-b},1^{b-a})}s_{(2^{r-k-1},1)}\\
&\quad + \textstyle\sum_{k=0}^{r}\sum_{0\le a\le d} \mathrm{ps}_d^1\big(s_{(2^a)}\big)s_{(2^{k-a-1},1)}s_{(2^{r-k})}\\
&= \textstyle\sum_{k=0}^{r-1}\sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1\big(s_{(2^a,1^{b+1-a})}\big)s_{(2^{k-b-1},1^{b+1-a})}s_{(2^{r-k-1},1)}\\
&\quad + \textstyle\sum_{k=0}^{r}\sum_{0\le a\le d} \mathrm{ps}_d^1\big(s_{(2^a)}\big)s_{(2^{k-a-1},1)}s_{(2^{r-k})}.
\end{aligned}$$

Therefore,

$$\begin{aligned} C_1 - C_2 &= \mathrm{ps}^1_{n-2}((A_1 + A_2 + A_3 + A_4) - (B_1 + B_2 + B_3)) \\ &= \mathrm{ps}^1_{n-2}(A_2 + A_4 - B_2) \\ &= \mathrm{ps}^1_{n-2}\left(\textstyle\sum_{0\le a\le b\le d-1} \mathrm{ps}^1_d(s_{(2^a,1^{b+1-a})}) \sum_{k=0}^{r} D(a,b,k,r)\right) \end{aligned}$$

From Theorem 3.9 we deduce that

$$\sum_{0\le a\le b\le d-1} \mathrm{ps}^1_d(s_{(2^a,1^{b+1-a})}) \sum_{k=0}^{r} D(a,b,k,r)$$

is s-positive, hence $C_1 - C_2$ is nonnegative, as desired. ∎

As a corollary, we are led to an affirmative answer to Conjecture 1.1.

Corollary 4.2 *The Narayana polynomials $N_n(q)$ form a q-log-convex sequence.*

Remark. Butler and Flanigan [7] defined a different q-analogue of log-convexity. In their definition, a sequence of polynomials $(f_k(q))_{k\ge 0}$ is called q-log-convex if

$$f_{m-1}(q)f_{n+1}(q) - q^{n-m+1}f_m(q)f_n(q)$$

has nonnegative coefficients for $n \ge m \ge 1$. They proved that the q-Catalan numbers of Carlitz and Riordan [8] form a q-log-convex sequence. However, the Narayana polynomial sequence $(N_n(q))_{n\ge 0}$ is not q-log-convex by the definition of Butler and Flanigan.

5 The Narayana transformation

In [19] Liu and Wang studied several log-convexity preserving transformations, and they also realized the connection between the q-log-convexity and the linear transformations preserving the log-convexity. They conjectured that if the sequence $(a_k)_{k\ge 0}$ of positive real numbers is log-convex then the sequence

$$b_n = \sum_{k=0}^{n} N(n,k)a_k, \quad n \ge 0$$

is also log-convex. In this section we will provide a proof of this conjecture. We first give two lemmas.

For any $n \geq 1$ and $0 \leq r \leq 2n$, we define the following polynomials in x with integer coefficients:

$$\begin{aligned} f_1(x) &= (n+1)(n-x+1)(n-x)^2(n-x-1), \\ f_2(x) &= (n+1)(n-(r-x)+1)(n-(r-x))^2(n-(r-x)-1), \\ f_3(x) &= (n-1)(n-x)(n-x+1)(n-(r-x))(n-(r-x)+1). \end{aligned}$$

Let

$$f(x) = f_1(x) + f_2(x) - 2f_3(x).$$

Lemma 5.1 *For fixed integers $n \geq 1$ and $0 \leq r < 2n$, the polynomial $f(x)$ is monotone decreasing in x on the interval $(-\infty, \frac{r}{2}]$.*

Proof. Taking the derivative $f'(x)$ of $f(x)$ with respect to x, we obtain that

$$f'(x) = 2(2x-r)g(x),$$

where

$$g(x) = 4x^2 - 4xr - 2n + r + 2r^2 - 5nr - 8n^2r - 2 + 2nr^2 + 6n^2 + 8n^3.$$

Note that the discriminant of the quadratic polynomial $g(x)$ equals

$$\begin{aligned} &(-4r)^2 - 16(-2n + r + 2r^2 - 5nr - 8n^2r - 2 + 2nr^2 + 6n^2 + 8n^3) \\ &= 16(-r^2 + 2n - r + 5nr + 8n^2r + 2 - 2nr^2 - 6n^2 - 8n^3). \end{aligned} \tag{5.37}$$

Let us consider the following polynomial

$$g_1(y) = -y^2 + 2n - y + 5ny + 8n^2y + 2 - 2ny^2 - 6n^2 - 8n^3$$

in y on the interval $(-\infty, 2n)$. The derivative of $g_1(y)$ with respect to y is

$$g_1'(y) = -2y - 1 + 5n + 8n^2 - 4ny = (4n+2)(2n-y) + n - 1.$$

Therefore, $g_1'(y) > 0$ for $y \in (-\infty, 2n)$. Then for any $0 \leq r < 2n$ and $n \geq 1$ we have

$$g_1(r) \leq g_1(2n-1) = -3n + 2 < 0.$$

This implies that $g(x) > 0$ and $f'(x) = 2(2x-r)g(x) < 0$ for $x \in (-\infty, \frac{r}{2})$. Therefore, $f(x)$ is monotone decreasing on the interval $(-\infty, \frac{r}{2}]$. ∎

Lemma 5.2 *For any $n \geq 1$, $0 \leq r \leq 2n$ and $0 \leq k \leq \lfloor \frac{r}{2} \rfloor$, let*

$$\begin{aligned}\alpha(n,r,k) &= N(n+1,k)N(n-1,r-k) + N(n+1,r-k)N(n-1,k) \\ &\quad -2N(n,r-k)N(n,k).\end{aligned}$$

Then, for given n and r, there always exists an integer $k' = k'(n,r)$ such that $\alpha(n,r,k) \geq 0$ for $k \leq k'$ and $\alpha(n,r,k) \leq 0$ for $k > k'$.

Proof. Assume that n and r are given. Clearly, if $k \leq r-n-1$, then $n \leq (r-k)-1$ and $\alpha(n,r,k) = 0$. We only need to determine the sign of $\alpha(n,r,k)$ for $r-n-1 < k \leq \lfloor \frac{r}{2} \rfloor$.

Note that $N(m,k) = \mathrm{ps}^1_{m-1}(s_{(2^k)})$ for any $m \in \mathbb{N}$. By Lemma 2.1 we find that

$$N(m,k) = \frac{((n-1)(n-2)\cdots(n-r+k)) \cdot (n(n-1)\cdots(n-r+k+1))}{k!(k+1)!}.$$

Let

$$\begin{aligned}C &= (n-1)(n-2)^2(n-3)^2\cdots(n-k+2)^2(n-k+1), \\ C' &= (n-1)(n-2)^2(n-3)^2\cdots(n-(r-k)+2)^2(n-(r-k)+1).\end{aligned}$$

Then we have

$$\alpha(n,r,k) = \frac{C}{k!(k+1)!} \cdot \frac{C'}{(r-k)!(r-k+1)!} \cdot f(k).$$

Now let us consider the value of $f(k)$ for fixed r. We have the following three cases.

(i) When $r = 2m+1$ for some $0 \leq m < n$, by Lemma 5.1 we have

$$f(0) \geq f(1) \geq \cdots \geq f(m),$$

where

$$f(0) = 2(2m+1)(n+1)((4m+1)(n-m)(n-m-1) + m(m+1)) \geq 0.$$

(ii) When $r = 2m$ for some $0 \leq m < n$, by Lemma 5.1 we have

$$f(0) \geq f(1) \geq \cdots \geq f(m),$$

where

$$f(0) = 4m(n+1)((4m-1)(n-m)^2 + m(m-1) \geq 0.$$

(iii) When $r = 2n$, we have

$$f(k) = 4(n-k+1)(n-k-1)(n-k)^2.$$

Therefore, $f(k) > 0$ for any $k < n$ and $f(n) = 0$.

Notice that there always exists an integer k' such that $f(k) \geq 0$ for $k \leq k'$ and $f(k) \leq 0$ for $k > k'$. Because both C and C' are nonnegative, we reach the desired conclusion. ∎

Theorem 5.3 *If the sequence $(a_k)_{k\geq 0}$ of positive real numbers is log-convex, then the sequence*

$$b_n = \sum_{k=0}^{n} N(n,k)a_k, \quad n \geq 0$$

is log-convex.

In general, the Narayana transformation does not preserve the log-convexity, and the condition that $(a_k)_{k\geq 0}$ is a positive sequence is necessary for the above theorem. For example, if we take $a_k = (-1)^k$ for $k \geq 0$, then it is easy to see that $(a_k)_{k\geq 0}$ is log-convex, but $(b_n)_{n\geq 0}$ is not log-convex.

Proof of Theorem 5.3. For any $n, r, k \geq 0$, let

$$\alpha'(n,r,k) = \begin{cases} \alpha(n,r,k)/2, & \text{if } r \text{ is even and } k = r/2, \\ \alpha(n,r,k), & \text{otherwise.} \end{cases}$$

Note that for $n \geq 1$

$$b_{n-1}b_{n+1} - b_n^2 = \sum_{r=0}^{2n} \left(\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) a_k a_{r-k} \right)$$

and

$$N_{n-1}(q)N_{n+1}(q) - N_n(q)^2 = \sum_{r=0}^{2n} \left(\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) \right) q^r.$$

By Corollary 4.2, we see that

$$\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) \geq 0$$

for any $r \geq 0$. Since the sequence $(a_k)_{k\geq 0}$ is a log-convex sequence of positive real numbers, we obtain that

$$a_0 a_r \geq a_1 a_{r-1} \geq a_2 a_{r-2} \geq \cdots .$$

Lemma 5.2 implies that there exists an integer $k' = k'(n, r)$ such that

$$\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n, r, k) a_k a_{r-k} \geq \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n, r, k) a_{k'} a_{r-k'} \geq 0.$$

Therefore, $(b_n)_{n\geq 0}$ is log-convex. ∎

6 The q-log-concavity

This section is devoted to the q-log-concavity of the q-Narayana numbers $N_q(n, k)$ for given n or k. First we apply Brändén's formula (1.2) to express the q-Narayana numbers in terms of specializations of Schur functions. This formulation enables us to reduce the q-log-concavity of the q-Narayana numbers to the Schur positivity of some differences between the products of Schur functions indexed by two-column shapes. Notice that much work has been done on the Schur positivity of the differences of products of Schur functions, see, for example, Bergeron, Biagioli and Rosas [2], Fomin, Fulton, Li and Poon [11] and Okounkov [22].

We now proceed to prove the q-log-concavity of q-Narayana numbers $N_q(n, k)$ for fixed n.

Theorem 6.1 *Given a positive integer n, the sequence $(N_q(n, k))_{k\geq 0}$ is strongly q-log-concave.*

Proof. Using (1.2), for any $k \geq l \geq 1$, we get

$$N_q(n, k)N_q(n, l) - N_q(n, k + 1)N_q(n, l - 1) = s_{(2^k)} s_{(2^l)} - s_{(2^{k+1})} s_{(2^{l-1})},$$

where each Schur function on the righthand side is over the variable set $\{q, q^2, \ldots, q^{n-1}\}$. Using induction on $k-l$, we can show that the symmetric function $s_{(2^k)} s_{(2^l)} - s_{(2^{k+1})} s_{(2^{l-1})}$ is s-positive. Clearly, this statement is true for $k = l$, because by (i) and (ii) of Lemma 3.2, we have

$$s_{(2^k)} s_{(2^k)} - s_{(2^{k+1})} s_{(2^{k-1})} = \sum_{\lambda \in Q_{\emptyset}(4k)} s_\lambda - \sum_{\lambda \in Q_{(2,2)}(4k)} s_\lambda = \sum_{a=0}^{k} s_{(4^a, 3^{k-a}, 1^{k-a})}.$$

For $k > l$, by (3.13) we have

$$\begin{aligned} s_{(2^k)} s_{(2^l)} &= \Delta^{(2)}(s_{(2^{k-1})} s_{(2^l)}), \\ s_{(2^{k+1})} s_{(2^{l-1})} &= \Delta^{(2)}(s_{(2^k)} s_{(2^{l-1})}). \end{aligned}$$

It follows that

$$s_{(2^k)} s_{(2^l)} - s_{(2^{k+1})} s_{(2^{l-1})} = \Delta^{(2)}(s_{(2^{k-1})} s_{(2^l)} - s_{(2^k)} s_{(2^{l-1})}).$$

By induction, $s_{(2^{k-1})} s_{(2^l)} - s_{(2^k)} s_{(2^{l-1})}$ is s-positive, so is $s_{(2^k)} s_{(2^l)} - s_{(2^{k+1})} s_{(2^{l-1})}$. The Schur positivity of the above difference was also shown by Bergeron and McNamara [1, Remark 7.2], and Kleber [14] gave a proof for the case of $k = l$. In view of the variable set for symmetric functions, we see that the difference $N_q(n, k)N_q(n, l) - N_q(n, k + 1)N_q(n, l - 1)$ has nonnegative coefficients as a polynomial of q. This completes the proof. ∎

Next we will consider the q-log-concavity of the q-Narayana numbers $N_q(n, k)$ for fixed k. We will use a result due to Lam, Postnikov and Pylyavaskyy [18]. Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, let

$$\begin{aligned} \lambda \vee \mu &= (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \ldots), \\ \lambda \wedge \mu &= (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \ldots). \end{aligned}$$

For two skew partitions λ/μ and ν/ρ, we define

$$\begin{aligned} (\lambda/\mu) \vee (\nu/\rho) &= (\lambda \vee \nu)/(\mu \vee \rho), \\ (\lambda/\mu) \wedge (\nu/\rho) &= (\lambda \wedge \nu)/(\mu \wedge \rho). \end{aligned}$$

The following assertion was conjectured by Lam and Pylyavaskyy [17] and proved by Lam, Postnikov and Pylyavaskyy [18]. We will be interested in two special cases of this fact.

Theorem 6.2 ([18, Theorem 5]) *For any two skew partitions λ/μ and ν/ρ, the difference*

$$s_{(\lambda/\mu)\vee(\nu/\rho)} s_{(\lambda/\mu)\wedge(\nu/\rho)} - s_{\lambda/\mu} s_{\nu/\rho}$$

is s-positive.

In particular, we will need the following special cases.

Corollary 6.3 *Let k be a positive integer. If I, J are partitions with $I \subseteq (2^{k-1})$ and $J \subseteq (2^{k-1}, 1)$, then both*

$$s_{(2^{k-1})}s_{(2^k)/I} - s_{(2^{k-1})/I}s_{(2^k)} \tag{6.38}$$

and

$$s_{(2^{k-1},1)}s_{(2^k)/J} - s_{(2^{k-1},1)/J}s_{(2^k)} \tag{6.39}$$

are s-positive.

Proof. For (6.38), take $\lambda = (2^{k-1}), \mu = I, \nu = (2^k)$ and $\rho = \emptyset$ in Theorem 6.2. For (6.39), take $\lambda = (2^{k-1}, 1), \mu = J, \nu = (2^k)$ and $\rho = \emptyset$. ∎

For any $r \geq 1$, let

$$X_r = \{q, q^2, \ldots, q^{r-1}\}, \quad X_r^{-1} = \{q^{-1}, q^{-2}, \ldots, q^{-(r-1)}\}.$$

The following relations are crucial for the proof of the q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for given k.

Lemma 6.4 *For any $m \geq n \geq 1$ and $k \geq 1$, we have*

$$\begin{aligned} & q^{n-1}s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - q^m s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1}) \\ = \ & q^{k-1}\left(s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})\right) \end{aligned} \tag{6.40}$$

and

$$\begin{aligned} & q^{2(n-1)}s_{(2^{k-1})}(X_{n-1})s_{(2^k)}(X_m) - q^{2m}s_{(2^{k-1})}(X_m)s_{(2^k)}(X_{n-1}) \\ = \ & q^{2k(m+n-1)}\left(s_{(2^{k-1})}(X_{n-1}^{-1})s_{(2^k)}(X_m^{-1}) - s_{(2^{k-1})}(X_m^{-1})s_{(2^k)}(X_{n-1}^{-1})\right). \end{aligned} \tag{6.41}$$

Proof. We will adopt the following notation for q-series in the proof. For indeterminates $a, a_1, \cdots, a_s$ and integer $r \geq 0$, let

$$\begin{aligned} (a;q)_r &= (1-a)(1-aq)\cdots(1-aq^{r-1}), \\ (a_1, a_2, \cdots, a_s; q)_r &= (a_1;q)_r(a_2;q)_r\cdots(a_s;q)_r. \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} s_{(2^{k-1},1)}(X_{n-1}) &= s_{(2^{k-1},1)}(q, q^2, \cdots, q^{n-2}) \\ &= \frac{q^{k^2}(q^{n-k-1};q)_k(q^{n-k+1};q)_{k-1}}{(1-q)(q;q)_{k-1}(q^3;q)_{k-1}} \end{aligned}$$

and

$$\begin{aligned}
s_{(2^k)}(X_n) &= s_{(2^k)}(q, q^2, \cdots, q^{n-1}) \\
&= \frac{q^{k(k+1)}(q^{n-k};q)_k(q^{n-k+1};q)_k}{(q;q)_k(q^2;q)_k}.
\end{aligned}$$

Therefore, the left hand side of (6.40) equals

$$\begin{aligned}
&\frac{q^{2k^2+k+n-1}(q^{n-k+1};q)_{k-1}(q^{n-k-1},q^{m-k},q^{m-k+1};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k} \\
&- \frac{q^{2k^2+k+m}(q^{m-k+2};q)_{k-1}(q^{m-k},q^{n-k-1},q^{n-k};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k} \\
&= \frac{q^{2k^2+k+n-1}(1-q^{m-n+1})(q^{m-k+2},q^{n-k+1};q)_{k-1}(q^{m-k},q^{n-k-1};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k}
\end{aligned}$$

and the difference $s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})$ equals

$$\begin{aligned}
&\frac{q^{2k^2+k}(q^{n-k+1};q)_{k-1}(q^{n-k-1},q^{m-k},q^{m-k+1};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k} \\
&- \frac{q^{2k^2+k}(q^{m-k+2};q)_{k-1}(q^{m-k},q^{n-k-1},q^{n-k};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k} \\
&= \frac{q^{2k^2+n}(1-q^{m-n+1})(q^{m-k+2},q^{n-k+1};q)_{k-1}(q^{m-k},q^{n-k-1};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k}.
\end{aligned}$$

Comparing the above two identities, we arrive at (6.40).

Next we prove the second identity. The left hand side of (6.41) equals

$$\begin{aligned}
&\frac{q^{2(n+k^2-1)}(q^{n-k},q^{n-k+1};q)_{k-1}(q^{m-k},q^{m-k+1};q)_k}{(q,q^2;q)_{k-1}(q,q^2;q)_k} \\
&- \frac{q^{2(m+k^2)}(q^{m-k+1},q^{m-k+2};q)_{k-1}(q^{n-k-1},q^{n-k};q)_k}{(q,q^2;q)_{k-1}(q,q^2;q)_k} \\
&= \frac{f(q)(q^{n-k},q^{n-k+1},q^{m-k+1},q^{m-k+2};q)_{k-1}}{(q,q^2;q)_{k-1}(q,q^2;q)_k},
\end{aligned}$$

where

$$f(q) = q^{2k^2-k-2}(q^{m+1}-q^n)(q^{m+n+1}+q^{m+n}-q^{m+k+1}-q^{n+k}).$$

The difference $s_{(2^{k-1})}(X_{n-1})s_{(2^k)}(X_m) - s_{(2^{k-1})}(X_m)s_{(2^k)}(X_{n-1})$ equals

$$\begin{aligned}
&\frac{q^{2k^2}(q^{n-k}, q^{n-k+1}; q)_{k-1}(q^{m-k}, q^{m-k+1}; q)_k}{(q, q^2; q)_{k-1}(q, q^2; q)_k} \\
&- \frac{q^{2k^2}(q^{m-k+1}, q^{m-k+2}; q)_{k-1}(q^{n-k-1}, q^{n-k}; q)_k}{(q, q^2; q)_{k-1}(q, q^2; q)_k} \\
&= \frac{g(q)(q^{n-k}, q^{n-k+1}, q^{m-k+1}, q^{m-k+2}; q)_{k-1}}{(q, q^2; q)_{k-1}(q, q^2; q)_k},
\end{aligned}$$

where

$$g(q) = q^{2k^2-2k-1}(q^{m+1} - q^n)(q^{m+1} + q^n - q^{k+1} - q^k).$$

It is routine to verify that $g(q^{-1}) = q^{2k+1-4k^2-2m-2n} f(q)$. Then (6.41) follows from the fact that $(1 - q^{-r}) = -q^{-r}(1 - q^r)$ for any r. ∎

Now we are ready to prove the q-log-concavity of the q-Narayana numbers $(N_q(n,k))_{n\geq 0}$ for given k.

Theorem 6.5 *Given a positive integer k, the sequence $(N_q(n,k))_{n\geq 0}$ is strongly q-log-concave.*

Proof. For any $m \geq n \geq 1$, let

$$A_{m,n}(q) = N_q(m,k)N_q(n,k) - N_q(m+1,k)N_q(n-1,k).$$

By (1.2), we have

$$A_{m,n}(q) = s_{(2^k)}(X_m)s_{(2^k)}(X_n) - s_{(2^k)}(X_{m+1})s_{(2^k)}(X_{n-1}).$$

Applying (2.3) to $s_{(2^k)}(X_n)$ and $s_{(2^k)}(X_{m+1})$, the above $A_{m,n}(q)$ equals

$$\begin{aligned}
&s_{(2^k)}(X_m)\left(s_{(2^k)}(X_{n-1}) + q^{n-1}s_{(2^{k-1},1)}(X_{n-1}) + q^{2(n-1)}s_{(2^{k-1})}(X_{n-1})\right) \\
&\quad - \left(s_{(2^k)}(X_m) + q^m s_{(2^{k-1},1)}(X_m) + q^{2m}s_{(2^{k-1})}(X_m)\right)s_{(2^k)}(X_{n-1}) \\
&= \left(q^{n-1}s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - q^m s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})\right) \\
&\quad + \left(q^{2(n-1)}s_{(2^{k-1})}(X_{n-1})s_{(2^k)}(X_m) - q^{2m}s_{(2^{k-1})}(X_m)s_{(2^k)}(X_{n-1})\right).
\end{aligned}$$

By Lemma 6.4, we obtain that $A_{m,n}(q)$ equals

$$\begin{aligned}
& q^{k-1}\left(s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})\right) \\
& \quad + q^{2k(m+n-1)}\left(s_{(2^{k-1})}(X_{n-1}^{-1})s_{(2^k)}(X_m^{-1}) - s_{(2^{k-1})}(X_m^{-1})s_{(2^k)}(X_{n-1}^{-1})\right) \\
& = q^{k-1}s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(Z) \\
& \quad + q^{k-1}\textstyle\sum_{J\subseteq(2^{k-1},1)} s_J(Z)\left(s_{(2^{k-1},1)}s_{(2^k)/J} - s_{(2^{k-1},1)/J}s_{(2^k)}\right)(X_{n-1}) \\
& \quad + q^{2k(m+n-1)}s_{(2^{k-1})}(X_{n-1}^{-1})s_{(2^k)}(Z^{-1}) \\
& \quad + q^{2k(m+n-1)}s_{(2^{k-1})}(X_{n-1}^{-1})s_{(2^{k-1},1)}(Z^{-1})s_{(1)}(X_{n-1}^{-1}) \\
& \quad + q^{2k(m+n-1)}\textstyle\sum_{I\subseteq(2^{k-1})} s_I(Z)\left(s_{(2^{k-1})}s_{(2^k)/I} - s_{(2^{k-1})/I}s_{(2^k)}\right)(X_{n-1}^{-1}),
\end{aligned}$$

where $Z = \{q^{n-1}, \ldots, q^{m-1}\}$ and $Z^{-1} = \{q^{1-n}, \ldots, q^{1-m}\}$. Applying Corollary 6.3, we complete the proof. ∎

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