On Balanced Colorings of the n-Cube

William Y.C. Chen[1] and Larry X.W. Wang[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China
Email: [1]chen@nankai.edu.cn, [3]wxw@cfc.nankai.edu.cn

Abstract. A 2-coloring of the n-cube in the n-dimensional Euclidean space can be considered as an assignment of weights of 1 or 0 to the vertices. Such a colored n-cube is said to be balanced if its center of mass coincides with its geometric center. Let $B_{n,2k}$ be the number of balanced 2-colorings of the n-cube with $2k$ vertices having weight 1. Palmer, Read and Robinson conjectured that for $n \geq 1$, the sequence $\{B_{n,2k}\}_{k=0,1,\ldots,2^{n-1}}$ is symmetric and unimodal. We give a proof of this conjecture. We also propose a conjecture on the log-concavity of $B_{n,2k}$ for fixed k, and by probabilistic method we show that it holds when n is sufficiently large.

Keywords: unimodalily, n-cube, balanced coloring

AMS Classification: 05A20, 05D40

Suggested Running Title: Balanced Colorings of the n-Cube

1 Introduction

This paper is concerned with a conjecture of Palmer, Read and Robinson [5] in the n-dimensional Euclidean space. A 2-coloring of the n-cube is considered as an assignment of weights of 1 or 0 to the vertices. The black vertices are considered as having weight 1 whereas the white vertices are considered as having weight 0. We say that a 2-coloring of the n-cube is balanced if the colored n-cube is balanced, namely, the center of mass is located at its geometric center.

Let $\mathcal{B}_{n,2k}$ denote the set of balanced 2-colorings of the n-cube with exactly $2k$ black vertices and $B_{n,2k} = |\mathcal{B}_{n,2k}|$. Palmer, Read and Robinson proposed the conjecture that the sequence $\{B_{n,2k}\}_{0\leq k\leq 2^{n-1}}$ is unimodal with the maximum at $k = 2^{n-2}$ for any $n \geq 1$. For example, when $n = 4$, the sequence $\{B_{n,2k}\}$ reads

$$1, 8, 52, 152, 222, 152, 52, 8, 1.$$

A sequence $\{a_i\}_{0\leq i\leq m}$ is called unimodal if there exists k such that

$$a_0 \leq \cdots \leq a_k \geq \cdots \geq a_m,$$

and is called strictly unimodal if

$$a_0 < \cdots < a_k > \cdots > a_m.$$

A sequence $\{a_i\}_{0\leq i\leq m}$ of real numbers is said to be log-concave if

$$a_i^2 \geq a_{i+1}a_{i-1}$$

for all $1 \leq i \leq m-1$.

Palmer, Read and Robinson [5] used Pólya's theorem to derive a formula for $B_{n,2k}$, which is a sum over integer partitions of $2k$. However, the unimodality of the sequence $\{B_{n,2k}\}$ does not seem to be an easy consequence since the summation involves negative terms. In Section 2, we will establish a relation on a refinement of the numbers $\mathcal{B}_{n,2k}$ from which the unimodality easily follows. In Section 3, we conjecture that the sequence $\{B_{n,2k}\}$ is log-concave for fixed k, and shall show that it holds when n is sufficiently large.

2 The unimodality

In this section, we shall give a proof of the unimodality conjecture of Palmer, Read and Robinson. Let Q_n be the n-dimensional cube represented by a graph whose vertices are sequences of 1's and -1's of length n, where two vertices are adjacent if they differ only at one position. Let V_n denote the set of vertices of Q_n, namely,

$$V_n = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \,|\, \epsilon_i = -1 \text{ or } 1,\ 1 \leq i \leq n\}.$$

By a 2-coloring of the Q_n we mean an assignment of weights 1 or 0 to the vertices of Q_n. The weight of a 2-coloring is the sum of weights or the numbers of vertices with weight 1. The center of mass of a coloring f with $w(f) \neq 0$ is the point whose coordinates are given by

$$\frac{1}{w(f)} \sum (\epsilon_1, \epsilon_2, \ldots, \epsilon_n),$$

where the sum ranges over all black vertices. If $w(f) = 0$, we take the center of mass to be the origin. A 2-coloring is balanced if its center of mass coincides with the origin. A pair of vertices of the n-cube is called an antipodal pair if it is of the form $(v, -v)$. A 2-coloring is said to be antipodal if any vertex v and its antipodal have the same color.

The key idea of our proof relies on the following further classification of the set $\mathcal{B}_{n,2k}$ of balanced 2-colorings.

Theorem 2.1 *Let $\mathcal{B}_{n,2k,i}$ denote the set of the balanced 2-colorings in $\mathcal{B}_{n,2k}$ containing exactly i antipodal pairs of black vertices. Then we have*

$$(2^{n-1} - 2k + i)|\mathcal{B}_{n,2k,i}| = (i + 1)|\mathcal{B}_{n,2k+2,i+1}|, \tag{2.1}$$

for $0 \leq i \leq k$ and $0 \leq k \leq 2^{n-2} - 1$.

Proof. We aim to show that both sides of (2.1) count the number of ordered pairs (F, G), where $F \in \mathcal{B}_{n,2k,i}$ and $G \in \mathcal{B}_{n,2k+2,i+1}$, such that G can be obtained by changing a pair of antipodal white vertices of F to black vertices. Equivalently, F can be obtained from G by changing a pair of antipodal black vertices to white vertices.

First, for each $F \in \mathcal{B}_{n,2k,i}$, we wish to obtain G in $\mathcal{B}_{n,2k+2,i+1}$ by changing a pair of antipodal white vertices to black. By the definition of $\mathcal{B}_{n,2k,i}$, for each F there are i antipodal pairs of black vertices and $2k-2i$ black vertices whose antipodal vertices are colored by white. Since $k \leq 2^{n-2} - 1$, that is, $2^{n-1} - 2(k - i) - i > 0$, there are exactly $2^{n-1} - 2(k - i) - i$ antipodal pairs of white vertices in F. Thus from each $F \in \mathcal{B}_{n,2k,i}$, we can obtain $2^{n-2} - 2k + i$ different 2-coloring in $\mathcal{B}_{n,2k+2,i+1}$ by changing a pair of antipodal white vertices of F to black. Hence the number of ordered pair (F, G) equals $(2^{n-1} - 2k + i)|\mathcal{B}_{n,2k,i}|$.

On the other hand, for each $G \in \mathcal{B}_{n,2k+2,i+1}$, since there are $i+1$ antipodal pairs of black vertices in G, we see that from G we can obtain $i+1$ different 2-colorings in $\mathcal{B}_{n,2k,i}$ by changing a pair of antipodal black vertices to white. So the number of ordered pairs (F, G) equals $(i+1)|\mathcal{B}_{n,2k+2,i+1}|$. This completes the proof. ∎

Theorem 2.2 *For $n \geq 1$, the sequence $\{B_{n,2k}\}_{0 \leq k \leq 2^{n-1}}$ is strictly unimodal with the maximum attained at $k = 2^{n-2}$.*

Proof. It is easily seen that $\{B_{n,2k}\}_{0 \leq k \leq 2^{n-1}}$ is symmetric for any $n \geq 1$. Given a balanced coloring of the n-cube, if we exchange the colors on all vertices, the complementary coloring is still balanced. Thus it is sufficient to prove $B_{n,2k} < B_{n,2k+2}$ for $0 \leq k \leq 2^{n-2} - 1$.

Clearly, for each $F \in \mathcal{B}_{n,2k}$, there are at most k antipodal pairs of black vertices. It follows that

$$B_{n,2k} = \sum_{i=0}^{k} |\mathcal{B}_{n,2k,i}|.$$

We wish to establish the inequality

$$|\mathcal{B}_{n,2k,i}| < |\mathcal{B}_{n,2k+2,i+1}|. \tag{2.2}$$

If it is true, then

$$B_{n,2k} = \sum_{i=0}^{k} |\mathcal{B}_{n,2k,i}| < \sum_{i=1}^{k+1} |\mathcal{B}_{n,2k+2,i}| \leq \sum_{i=0}^{k+1} |\mathcal{B}_{n,2k+2,i}| = B_{n,2k+2},$$

for $0 \leq k \leq 2^{n-2} - 1$, as claimed in the theorem. Thus it remains to prove (2.2). Since $0 \leq k \leq 2^{n-2} - 1$, it is clear that

$$(2^{n-1} - 2k + i) - (i + 1) = 2^{n-1} - 2k - 1 \geq 1.$$

Applying Theorem 2.1, we find that

$$|\mathcal{B}_{n,2k,i}| < |\mathcal{B}_{n,2k+2,i+1}|,$$

for $0 \leq i \leq k$ and $1 \leq k \leq 2^{n-2} - 1$, and hence (2.2) holds. This completes the proof. ∎

3 The log-concavity for fixed k

Log-concave sequences and polynomials often arise in combinatorics, algebra and geometry, see for example, Brenti [1] and Stanley [6]. While $\{B_{n,2k}\}_k$ is not log-concave in general, we shall show that the sequence $\{B_{n,2k}\}_n$ is log-concave for fixed k and sufficiently large n, and we conjecture that the log-concavity holds for any given k.

Conjecture 3.1 *When $0 \leq k \leq 2^{n-1}$, we have*

$$B_{n,2k}^2 \geq B_{n-1,2k} B_{n+1,2k}.$$

Palmer, Read and Robinson [5] have shown that

$$B_{n,2} = 2^{n-1}$$

and

$$B_{n,4} = \frac{1}{4^n}((4!)^{n-1} - 2^{3n-3}).$$

It is easy to verify that the sequences $\{B_{n,2}\}_{n \geq 1}$ and $\{B_{n,4}\}_{n \geq 2}$ are both log-concave. In the remaining of this paper, we shall be concerned with the case $k \geq 3$. To be more specific, we shall show that Conjecture 3.1 is true for $n > 5 \log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$. Our proof utilizes the well-known Bonferroni inequality, which can be stated as follows. Let $P(E_i)$ be the probability of the event E_i, and let $P\left(\bigcup_{i=1}^{n} E_i\right)$ be the probability that at least one of the events $E_1, E_2, \ldots, E_n$ will occur. Then

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i).$$

Before we present the proof of the asymptotic log-concavity of the sequence $\{B_{n,2k}\}$ for fixed k, let us introduce the $(0,1)$-matrices associated with a balanced 2-coloring of the n-cube with $2k$ vertices having weight 1. Since such a 2-coloring is uniquely determined by the set of vertices having weight 1, we may represent a 2-coloring by these vertices with weight 1. This leads us to consider the set $\mathcal{M}_{n,2k}$ of $n \times 2k$ matrices such that each row contains k $+1$'s and k -1's without two identical columns. Let $M_{n,2k} = |\mathcal{M}_{n,2k}|$. It is clear that

$$M_{n,2k} = (2k)! B_{n,2k}.$$

Hence the log-concavity of the sequence $\{M_{n,2k}\}_{n \geq \log_2 k+1}$ is equivalent to the log-concavity of the sequence $\{B_{n,2k}\}_{n \geq \log_2 k+1}$.

Canfield, Gao, Greenhill, McKay and Robinson [2] obtained the following estimate.

Theorem 3.2 *If* $0 \leq k \leq o(2^{n/2})$, *then*

$$M_{n,2k} = \binom{2k}{k}^n \left(1 - O\left(\frac{k^2}{2^n}\right)\right).$$

To prove the asymptotic log-concavity of $M_{n,2k}$ for fixed k, we need the following monotone property which implies Theorem 3.2.

Theorem 3.3 *Let* $c_{n,k}$ *be the real number such that*

$$M_{n,2k} = \binom{2k}{k}^n \left(1 - c_{n,k}\left(\frac{k^2}{2^n}\right)\right). \tag{3.3}$$

Then we have

$$c_{n,k} > c_{n+1,k},$$

for $k \geq 3$ *and* $n \geq 5 \log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$.

Proof. Let $\mathcal{L}_{n,2k}$ be the set of matrices with every row consisting of k -1's and k $+1$'s that do not belong to $\mathcal{M}_{n,2k}$ and $L_{n,2k} = |\mathcal{L}_{n,2k}|$. In other words, any matrix in $\mathcal{L}_{n,2k}$ has two identical columns. Since the number of $n \times 2k$

matrices with each row consisting of k $+1$'s and k -1's equals $\binom{2k}{k}^n$. From (3.3) it is easily checked that

$$L_{n,2k} = c_{n,k}\frac{k^2}{2^n}\binom{2k}{k}^n. \tag{3.4}$$

We now proceed to give an upper bound on the cardinality of $\mathcal{L}_{n+1,2k}$. For each $M \in \mathcal{L}_{n+1,2k}$, it is easy to see that the matrix M' obtained from M by deleting the $(n+1)$-st row contains two identical columns as well. Therefore, every matrix in $\mathcal{L}_{n+1,2k}$ can be obtained from a matrix in $\mathcal{L}_{n,2k}$ by adding a suitable row to a matrix in $\mathcal{L}_{n,2k}$ as the $(n+1)$-st row. This observation enables us to construct three classes of matrices M from $\mathcal{L}_{n+1,2k}$ by the properties of M'. It is obvious that any matrix in $\mathcal{L}_{n+1,2k}$ belongs to one of these three classes. Note that the classes are not necessarily exclusive.

Class 1: There exist at least three identical columns in M'. For each row of M', the probability that the three prescribed positions of this row are identical equals

$$2\binom{2k-3}{k}\bigg/\binom{2k}{k}.$$

Here the factor 2 indicates that there are two choices for the values at the prescribed positions. Consequently, the probability that the three prescribed columns in M' are identical equals

$$\left(2\binom{2k-3}{k}\bigg/\binom{2k}{k}\right)^n = \left(\frac{k-2}{2(2k-1)}\right)^n < \frac{1}{4^n}.$$

By the Bonferroni inequality, the probability that there are at least three identical columns in M' is bounded by $\frac{8k^3}{4^n}$. Because the number of $(n+1)\times 2k$ matrices with each row consisting of k $+1$'s and k -1's is $\binom{2k}{k}^{n+1}$, the number of matrices M in $\mathcal{L}_{n+1,2k}$ with M' containing at least three identical columns is bounded by

$$\frac{8k^3}{4^n}\binom{2k}{k}^{n+1}.$$

Class 2: There exist at least two pairs of identical columns in M'. For any two prescribed pairs (i_1, i_2) and (j_1, j_2) of columns, let us estimate the probability that in M' the i_1-th column is identical to the i_2-th column and the j_1-th column is identical to the j_2-th column, that is, for any row of M', the value of the i_1-th (respectively, j_1-th) position is equal to the value of the i_2-th (respectively, j_2-th) position. We have two cases for each row of M'. The first case is that the values at the positions i_1, i_2, j_1 and j_2 are all identical. The probability for any given row to be in this case equals

$$2\binom{2k-4}{k-4}\bigg/\binom{2k}{k}.$$

Again, the factor 2 comes from the two choices for the values at the prescribed positions.

The second case is that the value of the i_1-th position is different from the value of the j_1-th position. In this case, we have either the values at the i_1-th and i_2-th positions are $+1$ and the values at the j_1-th and j_2-th positions are -1 or the values at i_1-th and i_2-th position are -1 and the values at the j_1-th and j_2-th positions are $+1$. Thus the probability for any given row to be in this case equals

$$2\binom{2k-4}{k-2}\Big/\binom{2k}{k}.$$

Combining the above two cases, we see that for $k \geq 3$, the probability that M' has two prescribed pairs of identical columns equals

$$\left(2\binom{2k-4}{k-4}\Big/\binom{2k}{k} + 2\binom{2k-4}{k-2}\Big/\binom{2k}{k}\right)^n < \frac{1}{4^n}.$$

Again, by the Bonferroni inequality, the probability that there exist at least two pairs of identical columns of M' is bounded by $\frac{16k^4}{4^n}$. It follows that the number of matrices M in $\mathcal{L}_{n+1,2k}$ with M' containing at least two pairs of identical columns is bounded by

$$\frac{16k^4}{4^n}\binom{2k}{k}^{n+1}.$$

Class 3: There exists exactly one pair of identical columns in M'. By the definition, the number of matrices M' containing exactly one pair of identical columns is bounded by $L_{n,2k}$. On the other hand, it is easy to see that for each M' containing exactly one pair of identical columns, there are

$$2\binom{2k-2}{k} = \frac{k-1}{2k-1}\binom{2k}{k} \tag{3.5}$$

matrices of $\mathcal{L}_{n+1,2k}$ which can be obtained by adding a suitable row as the $(n+1)$-th row. Combining (3.4) and (3.5), we find that the number of matrices M of $\mathcal{L}_{n+1,2k}$ such that M' contains exactly one pair of identical columns is bounded by

$$\frac{k-1}{2k-1}c_{n,k}\frac{k^2}{2^n}\binom{2k}{k}^{n+1}.$$

Clearly, $L_{n+1,2k}$ is bounded by the sum of the cardinalities of the above three classes. This yields the upper bound

$$L_{n+1,2k} < \frac{8k^3}{4^n}\binom{2k}{k}^{n+1} + \frac{16k^4}{4^n}\binom{2k}{k}^{n+1} + \frac{k-1}{2k-1}c_{n,k}\frac{k^2}{2^n}\binom{2k}{k}^{n+1},$$

for $k \geq 3$.

We claim that

$$\frac{8k^3}{4^n} + \frac{16k^4}{4^n} < \frac{1}{4k-2} c_{n,k} \frac{k^2}{2^n}, \tag{3.6}$$

when

$$n \geq 5 \log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96. \tag{3.7}$$

Notice that the probability that two specified columns in M' are identical is

$$\left(2\binom{2k-2}{k} \Big/ \binom{2k}{k} \right)^n = \left(\frac{k-1}{2k-1} \right)^n.$$

Since $c_{n,k} \frac{k^2}{2^n}$ is the probability that there exists at least two identical columns in M', for $k \geq 2$ we deduce that

$$c_{n,k} \frac{k^2}{2^n} > \left(2\binom{2k-2}{k} \Big/ \binom{2k}{k} \right)^n = \left(\frac{k-1}{2k-1} \right)^n > \frac{1}{3^n}.$$

But under the condition (3.7), we have

$$\frac{8k^3}{4^n} + \frac{16k^4}{4^n} < \frac{1}{3^n(4k-2)},$$

which implies (3.6). Since $\frac{k-1}{2k-1} + \frac{1}{4k-2} = \frac{1}{2}$, it follows from (3.6) that

$$L_{n+1,2k} < c_{n,k} \frac{k^2}{2^{n+1}} \binom{2k}{k}^{n+1}, \tag{3.8}$$

subject to the condition (3.7). Restating formula (3.4) for $n+1$, we have

$$L_{n+1,2k} = c_{n+1,k} \frac{k^2}{2^{n+1}} \binom{2k}{k}^{n+1}. \tag{3.9}$$

Combining (3.8) and (3.9) gives

$$c_{n,k} > c_{n+1,k},$$

given the condition (3.7). This completes the proof. ∎

Applying Theorem 3.3, we arrive at the following inequality.

Theorem 3.4 *When* $n > 5 \log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$, *we have*

$$M_{n,2k}^2 > M_{n-1,2k} M_{n+1,2k}.$$

Proof. We only consider the case $k \geq 3$. Let

$$M_{n,2k} = \binom{2k}{k}^n \left(1 - c_{n,k}\frac{k^2}{2^n}\right).$$

Then

$$\begin{aligned}
& M_{n,2k}^2 - M_{n-1,2k}M_{n+1,2k} \\
= & \binom{2k}{k}^{2n} \left[\left(1 - c_{n,k}\frac{k^2}{2^n}\right)^2 - \left(1 - c_{n+1,k}\frac{k^2}{2^{n+1}}\right)\left(1 - c_{n-1,k}\frac{k^2}{2^{n-1}}\right)\right] \\
= & \binom{2k}{k}^{2n} \left[-c_{n,k}\frac{k^2}{2^{n-1}} + c_{n,k}^2\frac{k^4}{4^n} + c_{n+1,k}\frac{k^2}{2^{n+1}} + c_{n-1,k}\frac{k^2}{2^{n-1}} - c_{n-1,k}c_{n+1,k}\frac{k^4}{4^n}\right].
\end{aligned}$$

By Theorem 3.3, we have $c_{n-1,k} > c_{n,k}$ when $k \geq 3$ and $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$. This implies that

$$c_{n,k}\frac{k^2}{2^{n-1}} < c_{n-1,k}\frac{k^2}{2^{n-1}},$$

when $k \geq 3$ and $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$.

Now we claim $c_{n,k} < 4$ for any n. The probability that a specified pair of columns are equal is given by

$$\left(2\binom{2k-2}{k} \Big/ \binom{2k}{k}\right)^n = \left(\frac{k-1}{2k-1}\right)^n < \frac{1}{2^n}.$$

Since there are $2k$ columns in every M, by the Bonferroni inequality, the probability that there exist at least two identical columns in M is bounded by $\frac{4k^2}{2^n}$. This implies that $c_{n,k} < 4$ for any n.

Since

$$5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96 > 2\log_2 k + 3,$$

using the condition (3.7), we have

$$c_{n-1,k}c_{n+1,k}\frac{k^4}{4^n} < c_{n+1,k}\frac{k^4}{4^{n-1}} \leq c_{n+1,k}\frac{k^2}{2^{n+1}}.$$

Hence

$$M_{n,2k}^2 > M_{n-1,2k}M_{n+1,2k}.$$

This completes the proof. ∎

Since $M_{n,2k} = (2k)!B_{n,2k}$, Theorem 3.4 implies the asymptotic log-concavity of $B_{n,2k}$ for fixed k.

Corollary 3.5 *When $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$, we have*

$$B_{n,2k}^2 > B_{n-1,2k}B_{n+1,2k}.$$

Acknowledgments. We are indebted to the referee for valuable comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc., 413 (1989), 1–106.

[2] E.R. Canfield, Z. Gao, C. Greenhill, B.D. McKay and R.W. Robinson, Asymptotic enumeration of correlation-immune boolean functions, arXiv:0909.3321.

[3] E.M. Palmer and R.W. Robinson, Enumeration under two representations of the wreath product, Acta Math., 131 (1973), 123–143.

[4] E.M. Palmer and R.W. Robinson, Enumeration of self-dual configurations, Pacific J. Math., 110 (1984), 203–221.

[5] E.M. Palmer, R.C. Read and R.W. Robinson, Balancing the n-cube: A census of colorings, J. Algebraic Combin., 1 (1992), 257–273.

[6] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics and geometry, Ann. New York Acad. Sci., 576 (1989), 500–535.