On the Combinatorics of the Boros-Moll Polynomials

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Abstract. The Boros-Moll polynomials arise in the evaluation of a quartic integral. The original double summation formula does not imply the fact that the coefficients of these polynomials are positive. Boros and Moll proved the positivity by using Ramanujan's Master Theorem to reduce the double sum to a single sum. Based on the structure of reluctant functions introduced by Mullin and Rota along with an extension of Foata's bijection between Meixner endofunctions and bi-colored permutations, we find a combinatorial proof of the positivity. In fact, from our combinatorial argument one sees that it is essentially the binomial theorem that makes it possible to reduce the double sum to a single sum.

Keywords: Jacobi polynomials, Boros-Moll polynomials, reluctant function, Meixner endofunction, bi-colored permutation.

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1 Introduction

The objective of this paper is to give a combinatorial proof of the positivity of the coefficients of the Boros-Moll polynomials. Boros and Moll [3–7, 17] explored the following integral which is closely related to a special class of Jacobi polynomials. They have

shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \tag{1.1}$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

The polynomials $P_m(a)$ are called the Boros-Moll polynomials [9]. Write

$$P_m(a) = \sum_{i=0}^m d_i(m) a^i.$$

Boros and Moll found a remarkable proof of the fact that the coefficients $d_i(m)$ are positive by employing Ramanujan's Master Theorem, see [6] or [7, Theorem 7.9.1]. In fact, they have shown that

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k. \tag{1.3}$$

It follows from (1.3) that

$$d_i(m) = 2^{-2m} \sum_{k=i}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}.$$

There are several proofs of this formula, see the survey of Amdeberhan and Moll [1]. By the above formula (1.3), one can express $P_m(a)$ in terms of a hypergeometric series

$$P_m(a) = 2^{-2m} \binom{2m}{m} {}_2F_1\left(-m, m+1; \frac{1}{2} - m; \frac{a+1}{2}\right).$$

Recall that ${}_2F_1$ denotes the hypergeometric series

$$_2F_1(a, b; c; x) = \sum_k \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{x^k}{k!},$$

where $(a)_k$ stands for the rising factorial defined by $(a)_k = a(a+1)\cdots(a+k-1)$ for $k > 0$ and $(a)_k = 1$ for $k = 0$. Consequently, $P_m(a)$ can be viewed as the Jacobi polynomial $P_m^{(\alpha,\beta)}(a)$ with

$$\alpha = m + \frac{1}{2}, \quad \beta = -m - \frac{1}{2}.$$

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Recall that $P_m^{(\alpha,\beta)}(a)$ is defined by

$$P_m^{(\alpha,\beta)}(a) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left(\frac{1+a}{2}\right)^k.$$

Much progress has been made since Boros and Moll proved the positivity of the coefficients of $P_m(a)$. Boros and Moll [5] have shown that the sequence $\{d_i(m)\}_{0 \le i \le m}$ is unimodal, that is, there exists an index i such that $d_0(m) \le \cdots \le d_i(m)$ and $d_i(m) \ge \cdots \ge d_m(m)$. Moll conjectured that the coefficients $d_i(m)$ form a log-concave sequence, that is, $d_i(m)^2 \ge d_{i-1}(m)d_{i+1}(m)$ for $1 \le i \le m-1$. This conjecture has been confirmed by Kauers and Paule [20]. Recently, Chen and Xia [9] have proved a stronger property of $d_i(m)$, called the ratio monotone property, which implies the log-concavity and the spiral property. The combinatorial aspects of the 2-adic valuation of the number $i!m!2^{m+i}d_i(m)$ have been studied by Amdeberhan, Manna and Moll [2], and Sun and Moll [22].

From the combinatorial point of view, it is always interesting to find combinatorial reasons for the coefficients to be positive when the direct expansion contains negative terms. It is also desirable to find combinatorial proofs of unimodal and log-concave properties. Furthermore, it would be interesting to find combinatorial interpretations of the recurrence relations of $d_i(m)$ given by Kauers and Paule [20] and Moll [18].

In this paper, we will take the first step in this direction. We will give a combinatorial interpretation of the positivity of the coefficients of the Boros-Moll polynomials based on the structure of reluctant functions introduced by Mullin and Rota [19] along with an extension of Foata's bijection between Meixner endofunctions and bi-colored permutations. It should noted that the structure of reluctant functions and Meixner endofuntions have also been used in the combinatorial study of the Pfaff identity by Chen and Pang [8].

More specifically, we will give a combinatorial proof of the following identity which implies the equivalence of the two expressions (1.2) and (1.3) for $P_m(a)$:

$$\sum_{j,k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}$$

$$= 2^{-2m} \sum_{k} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k. \tag{1.4}$$

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2 The Combinatorial Proof

In order to give a combinatorial interpretation of the relation (1.4) that implies the positivity of the coefficients of the Boros-Moll identities, we need to use a variant of the identity by multiplying both sides by $m!$. The following reformulation of the identity after the multiplication by $m!$ is straightforward and can be made purely combinatorial in principle. Let us denote the left hand side and the right hand side of (1.4) by L and R, respectively. Throughout this paper, we will use the notation $(x)_n$ to for rising factorials, that is, $(x)_n = x(x+1)\cdots(x+n-1)$ for $n > 0$ and $(x)_n = 1$ for $n = 0$. On one hand, we have

$$
m! \cdot L = m! \sum_{i+j+k=m} \frac{(2m+1)!}{(2j)!(2m+1-2j)!} \cdot \frac{(m-j)!}{k!i!} \cdot \frac{(2m-2i)!}{(m-i)!(m-i)!} \cdot \frac{(a+1)^j(a-1)^k}{2^{3m-3i}}
$$

$$
= m! \sum_{i+j+k=m} \frac{2^{2m+1}m!(m+\frac{1}{2})!}{2^{2j}j!(j-\frac{1}{2})!2^{2m+1-2j}(m+\frac{1}{2}-j)!(m-j)!} \cdot \frac{(m-j)!}{k!i!}
$$

$$
\cdot \frac{2^{2m-2i}(m-i)!(m-i-\frac{1}{2})!}{(m-i)!(m-i)!} \cdot \frac{(a+1)^j(a-1)^k}{2^{3m-3i}}
$$

$$
= \sum_{i+j+k=m} \binom{m}{i,j,k} \frac{m!(m+\frac{1}{2})!}{(j-\frac{1}{2})!(m+\frac{1}{2}-j)!} \cdot \frac{(m-i-\frac{1}{2})!}{(m-i)!} \cdot \left(\frac{a+1}{2}\right)^j \left(\frac{a-1}{2}\right)^k
$$

$$
= \sum_{i+j+k=m} \binom{m}{i,j,k} (m-i+1)_i \left(m-j+\frac{3}{2}\right)_j \left(j+\frac{1}{2}\right)_k \left(\frac{a+1}{2}\right)^j \left(\frac{a-1}{2}\right)^k
$$

$$
= \sum_{i+j+k=m} \binom{m}{i,j,k} (-m)_i(-1)^i \left(-m-\frac{1}{2}\right)_j (-1)^j \left(j+\frac{1}{2}\right)_k \left(\frac{a+1}{2}\right)^j \left(\frac{-a+1}{2}\right)^k (-1)^k
$$

$$
= (-1)^m \sum_{i+j+k=m} \binom{m}{i,j,k} (-m)_i \left(-m-\frac{1}{2}\right)_j \left(j+\frac{1}{2}\right)_k \left(\frac{a+1}{2}\right)^j \left(\frac{-a+1}{2}\right)^k.
$$

On the other hand, we have

$$
m! \cdot R = m! \cdot \sum_{i+j=m} 2^{-2m+i} \binom{2j}{j} \binom{m+i}{i} (a+1)^i
$$

$$
= m! \cdot \sum_{i+j=m} 2^{-2m+i} \frac{(2j)!}{j!j!} \frac{(m+i)!}{i!m!} (a+1)^i
$$

$$
= \sum_{i+j=m} \binom{m}{i,j} 2^{-2m+i} \frac{(2j)!(m+i)!}{j!m!} (a+1)^i
$$

$$
= \sum_{i+j=m} \binom{m}{i,j} 2^{-i} \left(j-\frac{1}{2}\right)! \frac{(m+i)!}{m!} (a+1)^i
$$

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$$= \sum_{i+j=m} \binom{m}{i,j} 2^{-i}(-1)^j \left(\frac{1}{2}-j\right)_j (m+1)_i(a+1)^i$$

$$= (-1)^m \sum_{i+j=m} \binom{m}{i,j} \left(\frac{1}{2}-j\right)_j (m+1)_i \left(\frac{-a-1}{2}\right)^i.$$

So the identity (1.4) can be converted into the following equivalent form

$$\sum_{i+j+k=m} \binom{m}{i,j,k}(-m)_i \left(-m-\frac{1}{2}\right)_j \left(j+\frac{1}{2}\right)_k \left(\frac{a+1}{2}\right)^j \left(\frac{-a+1}{2}\right)^k$$

$$= \sum_{i+j=m} \binom{m}{i,j} \left(\frac{1}{2}-j\right)_j (m+1)_i \left(\frac{-a-1}{2}\right)^i. \tag{2.1}$$

Our combinatorial approach to the above identity consists of three steps. The first step is to give combinatorial interpretations of the sums on both sides of (2.1). We will show that the left hand side is the sum of weights of Meixner bi-endofunctions, and the right hand side is the sum of weights of Meixner endofunctions with a different weight assignments. The second step is to transform the sum of weights of Meixner bi-endofunctions to the sum of weights of 3-colored permutations. This is achieved by a weight preserving bijection between Meixner bi-endofunctions and 3-colored permutations, which is a natural extension of Foata's bijection. Meanwhile, the sum of weights for the right hand side can be transformed to the sum of weights of bi-colored permutations by the original bijection of Foata. The third step is to compare the weights of 3-colored permutations and bi-colored permutations. One sees that the equality follows from the weight distribution on a cycle. Roughly speaking, if there are two ways to give a weight w_1 or w_2 to a cycle, then it is equivalent to assigning only one weight $w_1 + w_2$ to the cycle. This step yields a combinatorial interpretation of why the double sum (1.2) reduces to a single sum (1.3).

Note that a basic ingredient of the combinatorial settings for the above hypergeometric identity is the interpretation of the rising factorial $(x)_n$, or, in general, of $(x+k)_n$. It is well known that $(x)_n$ can be expanded in terms of the signless Stirling numbers of the first kind. Note that $(x)_n$ can also be interpreted as the number of dispositions from $[n] = \{1, 2, \ldots, n\}$ to a set X with x elements, see Joni, Rota and Sagan [13] for more details.

In general, the rising factorial $(a+j)_i$ can be explained as the sum of the weights of reluctant functions from A to B, where A and B are disjoint, and $|A| = i$ and $|B| = j$. Recall that the notion of reluctant functions was introduced by Mullin and Rota [19] in their theory of sequences of polynomials of binomial type. A reluctant function f from A to B, where A and B are two disjoint finite sets, is defined as an injective map
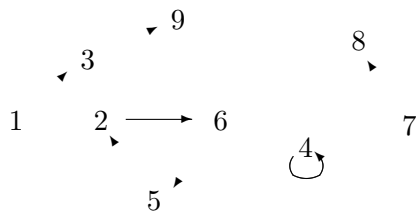
Figure 2.1: The digraph of a reluctant function.

from A to $A \cup B$. The functional digraph of f is a digraph on $A \cup B$ with arcs $(k, f(k))$ for $k \in A$. The weight of f is defined as a^k, where k is the number of cycles in the functional digraph of f.

Observe that the functional digraph of any reluctant function f has a unique decomposition into disjoint cycles on elements in A and directed paths ending with an element in B. The ending points in B are called *terminals*. Now, let us review the canonical cycle representation of a reluctant function, introduced by Chen and Pang in [8] as a natural extension of the canonical cycle representation of a permutation, see Stanley [21, Page 17]. Assume that f is a reluctant function from A to B. The functional digraph of f can be decomposed into k cycles $C_1, C_2, \ldots, C_k$ and s directed paths $P_1, P_2, \ldots, P_s$. We first write down the cycles in canonical cycle representation, that is, write a cycle $C = (i_1 i_2 \cdots i_r)$ in such a way that i_1 is the minimum element of C, then arrange the cycles $C_1, C_2, \ldots, C_k$ in accordance with the decreasing order of their minimum elements. Moreover, each path P_i is written as $(j_1 j_2 \cdots j_l)$ such that $j_1 \in B$ and $f(j_t) = j_{t-1}$ for $2 \leq t \leq l$, and $P_1, P_2, \ldots, P_s$ are arranged according to the increasing order of their first elements.

For example, the reluctant function in Figure 2.1 with $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{8, 9\}$ has the following canonical cycle representation

$$(4)(265)(\mathbf{8}7)(\mathbf{9}31).$$

It can be seen that the canonical cycle representation is in fact uniquely determined by the sequence $4265\mathbf{8}7\mathbf{9}31$. Clearly, the reluctant function f can be recovered from the canonical cycle representation. To transform a sequence $a_1 a_2 \cdots a_m$ to the canonical cycle representation, we need to consider the left-to-right minimum elements in the sequence. Recall that an element a_i in $a_1 a_2 \cdots a_m$ is called a left-to-right minimum element if $a_i < a_j$ for any $j < i$. For example, 4 and 2 are left-to-right minimum elements in the above sequence. On one hand, we can insert a left parenthesis in the sequence before each element in B, which is in boldface. On the other hand, we can insert a left parenthesis in the sequence preceding every left-to-right minimum element a_i in A as long as $a_1, a_2, \ldots, a_{i-1}$ all belong to A. After the left parentheses are placed in the sequence, the right parentheses can be added accordingly.

The following proposition is well-known, see, for example, [8, 10, 12, 14, 16]. It plays

a crucial role in the combinatorial interpretation of the identity (2.1).

Proposition 2.1. *Let A and B be two disjoint subsets of $[m]$, and let $|A| = i$ and $|B| = j$. Then the sum of weights of reluctant functions from A to B equals $(a + j)_i$.*

To show that the left hand side of the identity (2.1) equals the sum of weights of 3-colored permutations, we need an extension of Foata's bijection between Meixner endofunctions and bi-colored permutations [11,15]. To be more specific, we will extend Foata's bijection to Meixner bi-endofunctions and 3-colored permutations. Recall that a *Meixner endofunction* on a finite set S is represented by $(A, B; \pi_A; \pi_B)$, where (A, B) is a composition of S and π_A is an injective map from A to S and π_B is a permutation on B. A *bi-colored permutation* on a finite set S is represented by $(A, B; \sigma)$, where (A, B) is a composition of S, and σ is a permutation on S. Note that a composition (A, B) of a set S can be considered as a 2-coloring of S. Foata's bijection can be described as follows.

Property 2.2. *There is a bijection between the set of Meixner endofunctions on $[m]$ and the set of bi-colored permutations on $[m]$.*

We now define Meixner bi-endofunctions and 3-colored permutations. A *Meixner bi-endofunction* on a finite set S is denoted by $(A, B, C; \pi_A; \pi_B, \pi_C)$, where (A, B, C) is a composition of S, π_A is an injective map from A to $A \cup B$ and (π_B, π_C), where π_B is a permutation on B and π_C is a permutation on C. Given a 3-coloring of $[m]$, say by the three colors red, black and white, a 3-colored permutation is defined as a permutation on $[m]$ such that no red elements appears in any cycle containing a black or white element. For example,

$$(\underline{8}, \underline{7}, \underline{9})(\mathbf{2}, 5, \mathbf{4})(\mathbf{10}, 1)(\mathbf{3})(\underline{11}, \underline{12})(6)$$

is a 3-colored permutation, where the underlined elements are red, and the black elements are in boldface.

Notice that a Meixner bi-endofunction $(A, B, C; \pi_A; \pi_B, \pi_C)$ reduces to a Meixner endofunction when $C = \emptyset$. Applying Foata's bijection to the cycles composed of elements in B, we obtain the following extension of Proposition 2.2.

Property 2.3. *There is a bijection between the set of Meixner bi-endofunctions on $[m]$ and the set of 3-colored permutations on $[m]$.*

Proof. Given a Meixner bi-endofunction $(A, B, C; \pi_A; \pi_B, \pi_C)$, we color the elements in A, B and C by white, black and red, respectively. Consider the cycle representation of π_A, π_B, π_C. We may view a Meixner bi-endofunction as a union of disjoint cycles on
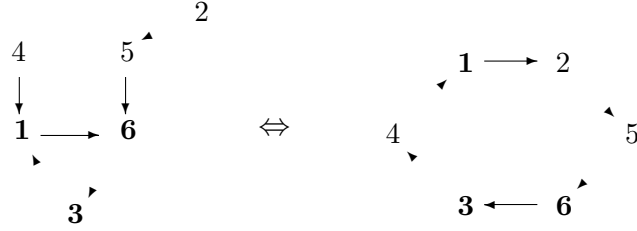
Figure 2.2: Foata's bijection.

A, B, C along with some directed paths on A attached to some element in B. Since π_A is injective, two directed paths on A cannot be incident to the same element in B.

The bijection will involve only the components consisting of cycles on a subset of B attached with some paths on A. Let D be such a cycle, and P be a directed path attached to D. Assume that x is the terminal element of P that is on D. Let (y, x) be an arc on on D. Then we can break this arc from y to x and connect y to the starting point of P. Considering the colors of the elements on the path P, we see that the above operation is reversible. Taking all the paths attached to D into account, we obtain the desired bijection. ∎

For example, as illustrated in Figure 2.2, the Meixner endofunction

$$(\{2, 4, 5\}, \{1, 3, 6\}; (1, 4)(6, 5, 2); (1, 6, 3)),$$

corresponds to the bi-colored permutation $(\mathbf{3}, 4, \mathbf{1}, 2, 5, \mathbf{6})$, where the black elements are in boldface.

We are now ready to give a combinatorial proof of the identity (2.1). First, we define the weights of Meixner bi-endofunctions and Meixner endofunctions. Let $(A, B, C; \pi_A; \pi_B, \pi_C)$ be a Meixner bi-endofunction on $[m]$. An element in A, B or C is assigned the weight

$$(-a + 1)/2, \quad (a + 1)/2, \quad 1,$$

respectively. Similarly, the weight of a cycle in π_A, π_B or π_C is given by

$$1/2, \quad -m - 1/2, \quad -m.$$

Then the weight of a Meixner bi-endofunctions is the product of the weights of the elements and the weights of the cycles.

Next, we define the weight of a Meixner endofunction $(A, B; \pi_A; \pi_B)$ on $[m]$. The weight of an element in A is given by 1, the weight of an element in B is given by $(-a - 1)/2$, the weight of a cycle in π_A is given by $1/2 - m$, and the weight of a cycle in π_B is given by $1 + m$. Then the weight of a Meixner endofunction is the product of the weights of the elements and the weights of the cycles. Given the above weight assignments for Meixner bi-endofunctions and Meixner endofunctions, the identity (2.1) is equivalent to the following statement.

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Theorem 2.4. *The sum of weights of Meixner bi-endofunctions on $[m]$ equals the sum of weights of Meixner endofunctions on $[m]$.*

Proof. By Proposition 2.1, it is not check to see that the sum of weights of Meixner bi-endofunctions $(A, B, C; \pi_A; \pi_B, \pi_C)$ on $[m]$ equals the summation on the left hand side of (2.1):

$$\sum_{i+j+k=m} \binom{m}{i,j,k} (-m)_i \left(-m - \frac{1}{2}\right)_j \left(j + \frac{1}{2}\right)_k \left(\frac{a+1}{2}\right)^j \left(\frac{-a+1}{2}\right)^k. \tag{2.2}$$

Applying the bijection described in Proposition 2.3 between Meixner bi-endofunctions on $[m]$ and 3-colored permutations on $[m]$, we find that (2.2) can be rewritten as the summation of weights of 3-colored permutations on $[m]$ with the following weight assignments. A white, black, or red element is given the weight

$$(-a+1)/2, \quad (a+1)/2, \quad 1.$$

A cycle containing only white elements is given the weight $1/2$, a cycle containing at least one black element is given the weight $-m - 1/2$, and a cycle consisting of only red elements is given the weight $-m$. Now, the weight of a 3-colored permutation is defined as the product of the weights of the elements and the weights of the cycles.

On the other hand, the total weight of 3-colored permutations on $[m]$ can be computed based on the cycle decompositions of permutations on $[m]$. Given a permutation π on $[m]$ and a cycle D in π with r elements, if D is a cycle consisting of white elements, then the weight contribution is

$$\frac{1}{2} \left(\frac{-a+1}{2}\right)^r. \tag{2.3}$$

If D is used to form a cycle containing at least one black element, the total weight contribution equals

$$\left(-m - \frac{1}{2}\right) \sum_{i=1}^{r} \binom{r}{i} \left(\frac{a+1}{2}\right)^i \left(\frac{-a+1}{2}\right)^{r-i}. \tag{2.4}$$

If D is used to a cycle containing only red elements, the total weight contribution equals $-m$. Combining the above three cases, we get the total weight contribution of the cycle D to the summation of weights of 3-colored permutations

$$-m + \frac{1}{2} \left(\frac{-a+1}{2}\right)^r + \left(-m - \frac{1}{2}\right) \sum_{i=1}^{r} \binom{r}{i} \left(\frac{a+1}{2}\right)^i \left(\frac{-a+1}{2}\right)^{r-i},$$

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which simplifies to

$$-2m - \frac{1}{2} + (m+1)\left(\frac{-a+1}{2}\right)^r. \tag{2.5}$$

Note that we can easily give a combinatorial argument for the above computation.

We continue to show that the right hand side of (2.1) can also be expressed as a summation over permutations on $[m]$ with each cycle having the above weight (2.5). By the definition of the weight of a Meixner endofunction, it is easily seen that the sum of weights over Meixner endofunctions on $[m]$ equals the the summation on the right hand side of (2.1):

$$\sum_{i+j=m} \binom{m}{i,j}\left(\frac{1}{2} - j\right)_j (m+1)_i \left(\frac{-a-1}{2}\right)^i. \tag{2.6}$$

Applying the bijection in Proposition 2.2 between Meixner endofunctions on $[m]$ and bi-colored permutations on $[m]$, (2.6) can be expressed as a summation of weights of bi-colored permutations on $[m]$ with the following weight assignments. The weight of a white element is given by 1, the weight of a black element is given by $(-a-1)/2$, the weight of a cycle consisting of only white elements is given by $1/2 - m$ and the weight of a cycle containing at least one black element is given by $1 + m$.

Analogously, the total weight of the bi-colored permutations on $[m]$ can be computed based on the cycle decompositions of permutations on $[m]$. Given a permutation π on $[m]$ and a cycle D in π with r elements, if D is a cycle consisting of white elements, the weight contribution is $1/2 - m$. If D is used to form a cycle containing at least one black element, the total weight contribution equals

$$(1+m)\sum_{i=1}^{r} \binom{r}{i}\left(\frac{-a-1}{2}\right)^i 1^{r-i} = (1+m)\left[\left(\frac{-a+1}{2}\right)^r - 1\right]. \tag{2.7}$$

Summing up the above two cases, we get the total weight contribution of the cycle D to the summation of weights of bi-colored permutations on $[m]$:

$$\frac{1}{2} - m + (1+m)\left[\left(\frac{-a+1}{2}\right)^r - 1\right] = -2m - \frac{1}{2} + (m+1)\left(\frac{-a+1}{2}\right)^r. \tag{2.8}$$

Comparing (2.5) and (2.8), we see that the weight assignment to 3-colored permutations is equivalent to the weight assignment to bi-colored permutations. This completes the combinatorial proof of the identity (2.1). ∎

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