The q-Log-convexity of the Narayana Polynomials of Type B

William Y. C. Chen[1], Robert L. Tang[2],
Larry X. W. Wang[3] and Arthur L. B. Yang[4]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

emails: [1]chen@nankai.edu.cn, [2]tangling@cfc.nankai.edu.cn,
[3]wxw@cfc.nankai.edu.cn, [4]yang@nankai.edu.cn

Abstract. We prove a conjecture of Liu and Wang on the q-log-convexity of the Narayana polynomials of type B. By using Pieri's rule and the Jacobi-Trudi identity for Schur functions, we obtain an expansion of a sum of products of elementary symmetric functions in terms of Schur functions with nonnegative coefficients. By the principal specialization this, leads to q-log-convexity. We also show that the linear transformation with respect to the triangular array of Narayana numbers of type B is log-convexity preserving.

Keywords: q-log-convexity, Schur positivity, Pieri's rule, the Jacobi-Trudi identity, principal specialization, Narayana numbers of type B.

AMS Classification: 05E05, 05E10

1 Introduction

The objective of this paper is to prove a conjecture of Liu and Wang [17] on the q-log-convexity of the polynomials

$$W_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k. \tag{1.1}$$

The polynomial $W_n(q)$ has appeared as the rank generating function of the lattice of noncrossing partitions of type B on $[n] = \{1, 2, \dots, n\}$, see Reiner [12] and Simion [14].

The coefficients $\binom{n}{k}^2$ have been called the Narayana numbers of type B by Fomin and Reading [9]. Consequently, we will call $W_n(q)$ the Narayana polynomials of type B. For the type A case, the rank generating function of the lattice of noncrossing partitions on $[n]$ is equal to the Narayana polynomial

$$N_n(q) = \sum_{k=0}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k+1}q^k. \tag{1.2}$$

Liu and Wang [17] also conjectured that $N_n(q)$ are q-log-convex. This conjecture has been proved by Chen, Wang and Yang [6].

The polynomials $W_n(q)$ also arise in the theory of growth series of the root lattice. Recall that the classical root lattice A_n is generated by $\mathcal{M} = \{\mathbf{e}_i - \mathbf{e}_j : 0 \le i, j \le n+1 \text{ with } i \ne j\}$. The growth series is defined to be the generating function

$$G(q) = \sum_{k \ge 0} S(k)q^k,$$

where $S(k)$ is the number of elements $\mathbf{u} \in A_n$ with length k. It is known that $G(q)$ is a rational function of the form

$$G(q) = \frac{h(q)}{(1-q)^d},$$

where d is the rank of A_n and $h(q)$ is a polynomial of degree less than or equal to d. The polynomial $h(q)$ is defined to be the coordinator polynomial of the growth series, see [2]. Recently, Ardila, Beck, Hosten, Pfeifle and K. Seashore [1] have shown that the above coordinator polynomial $h(q)$ of A_n equals the polynomial $W_n(q)$.

Recall that a sequence $\{a_k\}_{k \ge 0}$ of nonnegative numbers is said to be log-convex if $a_{k-1}a_{k+1} \ge a_k^2$ for any $k \ge 1$. The q-log-convexity is a property defined in [17] for sequences of polynomials over the field of real numbers, in contrast with the concept of q-log-concavity introduced by Stanley, see Brenti [3], and studied by Butler [4], Krattenthaler [10], Leroux [11] and Sagan [13]. Given a sequence $\{f_n(q)\}_{n \ge 0}$ of polynomials in q, we say that it is q-log-convex if for any $k \ge 1$ the difference

$$f_{k+1}(q)f_{k-1}(q) - f_k(q)^2$$

has nonnegative coefficients as a polynomial in q. It has been shown that many combinatorial polynomials are q-log-convex, such as the Bell polynomials, the Eulerian polynomials, the Bessel polynomials, the Ramanujan polynomials and the Dowling polynomials, see Liu and Wang [17], and Chen, Wang and Yang [7]. Meanwhile, it should be noticed that Butler and Flanigan [5] gave a different notion of q-log-convexity

in their study of q-Catalan numbers, namely, a sequence of polynomials $(f_k(q))_{k \geq 0}$ is called q-log-convex if

$$f_{m-1}(q)f_{n+1}(q) - q^{n-m+1}f_m(q)f_n(q)$$

has nonnegative coefficients for $n \geq m \geq 1$. However, the polynomials $W_n(q)$ do not form a q-log-convex sequence in the sense of Butler and Flanigan.

Clearly, if a sequence $\{f_n(q)\}_{n \geq 0}$ is q-log-convex, then for any fixed positive number q the sequence $\{f_n(q)\}_{n \geq 0}$ is log-convex. It is readily seen that the sequence of the central binomial coefficients $\{b_n\}_{n \geq 0}$ is log-convex, where

$$b_n = \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Došlić and Veljan [8] obtained the log-convexity of the sequence of the central Delannoy numbers $\{d_n\}_{n \geq 0}$, where

$$d_n = \sum_{k=0}^{n} \binom{n}{k}^2 2^k.$$

These two instance are supportive of the conjecture on the q-log-convexity of $W_n(q)$, see [17, Conjecture 5.3]. We will give a proof of this conjecture.

Theorem 1.1 *The polynomials $W_n(q)$ form a q-log-convex sequence.*

By using the principal specialization, the q-log-convexity of $W_n(q)$ follows from the Schur positivity of a sum of products of elementary symmetric functions. We will establish an identity on symmetric functions by showing that both sides satisfy the same recurrence relation. To derive the required recurrence relation, we employ the Jacobi-Trudi identity and Pieri's rule for Schur functions. We note that in general the polynomials

$$\sum_{k=0}^{n} \binom{n}{k}^m q^k, \quad n \geq 0$$

are not q-log-convex for $m \geq 3$.

The second result of this paper is concerned with a condition on linear transformations that preserve log-convexity. Let $\{a(n,k)\}_{0 \leq k \leq n}$ be a triangular array of numbers. The linear transformation on a sequence $\{x_k\}_{k \geq 0}$ with respect to a triangular array $\{a(n,k)\}_{0 \leq k \leq n}$ is defined by

$$y_n = \sum_{k=0}^{n} a(n,k)x_k.$$

Such a transformation is called *log-convexity preserving* if $\{y_n\}_{n \geq 0}$ is log-convex whenever $\{x_k\}_{k \geq 0}$ is log-convex. Liu and Wang [17] obtained a sufficient condition on a triangular array which ensures the corresponding transformation is log-convexity preserving.

Given a triangular array $\{a(n,k)\}_{0 \leq k \leq n}$, define $\alpha(n,r,k)$ by

$$\begin{aligned} \alpha(n,r,k) &= a(n+1,k)a(n-1,r-k) + a(n+1,r-k)a(n-1,k) \\ &\quad - 2a(n,r-k)a(n,k), \end{aligned} \tag{1.3}$$

where $n \geq 1$, $0 \leq r \leq 2n$ and $0 \leq k \leq \lfloor \frac{r}{2} \rfloor$. The sufficient condition of Liu and Wang is stated as follows.

Theorem 1.2 ([17, Theorem 4.8]) *Assume that the polynomials*

$$A_n(q) = \sum_{k=0}^{n} a(n,k)q^k$$

form a q-log-convex sequence. For any given n and r, if there exists an integer $k' = k'(n,r)$ such that $\alpha(n,r,k) \geq 0$ for $k \leq k'$ and $\alpha(n,r,k) \leq 0$ for $k > k'$, then the linear transformation with respect to the triangular array $\{a(n,k)\}_{0 \leq k \leq n}$ is log-convexity preserving.

Liu and Wang conjectured that the above sufficient condition holds for the coefficients of $W_n(q)$. We will show that this conjecture is true.

Theorem 1.3 *The linear transformation with respect to the triangular array $\{\binom{n}{k}^2\}_{0 \leq k \leq n}$ is log-convexity preserving.*

This paper is organized as follows. We recall some definitions and known results on symmetric functions in Section 2. In Section 3, we will give an inductive proof of the identity which implies the desired Schur positivity. In Section 4, we will present the proofs of Theorem 1.1 and Theorem 1.3.

2 Background on symmetric functions

Throughout this paper, we will adopt the notation and terminology on partitions and symmetric functions in Stanley [15]. Recall that a *partition* λ of a nonnegative integer n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots)$ of nonnegative integers satisfying $\sum_i \lambda_i =$

n, denoted $\lambda \vdash n$. We usually omit the parts $\lambda_i = 0$. We also denote a partition $\lambda \vdash n$ by $(n^{m_n}, \ldots, 2^{m_2}, 1^{m_1})$ if λ has m_i i's for $1 \le i \le n$. Let $\mathrm{Par}(n)$ denote the set of partitions of n.

If $\lambda \vdash n$, we draw a left-justified array of n squares with λ_i squares in the i-th row. This array is called the *Young diagram* of λ. By transposing the diagram of λ, we get the *conjugate partition* of λ, denoted λ'. We use $\mu \subseteq \lambda$ to denote that the Young diagram of μ is contained in the diagram of λ.

A semistandard Young tableau of shape λ is an array $T = (T_{ij})$ of positive integers of shape λ such that it is weakly increasing in each row and strictly increasing in each column. The type of T is defined as the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T. Let $x = (x_1, x_2, \ldots)$ be a sequence of indeterminates. If $\mathrm{type}(T) = \alpha$, then we write
$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$
The *Schur function* $s_\lambda(x)$ is defined as the generating function

$$s_\lambda(x) = \sum_T x^T,$$

summed over semistandard Young tableaux T of shape λ. When $\lambda = \emptyset$, we set $s_\emptyset(x) = 1$.

It is well known that Schur functions $s_\lambda(x)$ form a basis for the ring of symmetric functions. A symmetric function $f(x)$ is called *Schur positive* if the coefficients a_λ are all nonnegative in the Schur expansion $f(x) = \sum_\lambda a_\lambda s_\lambda(x)$.

When $\lambda = (1^k)$ with $k \ge 1$, the Schur function $s_\lambda(x)$ becomes the k-th elementary symmetric function $e_k(x)$, i.e.,

$$s_{(1^k)}(x) = e_k(x) = \sum_{1 \le i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}. \tag{2.4}$$

The dual Jacobi-Trudi identity gives an expression of the Schur function $s_\lambda(x)$ in terms of the elementary symmetric functions.

Theorem 2.1 ([15, Corollary 7.16.2]) *Let λ be a partition with the largest part $\le n$ and λ' its conjugate. Then*

$$s_\lambda(x) = \det(e_{\lambda_i' - i + j}(x))_{i,j=1}^n,$$

where $e_0 = 1$ and $e_k = 0$ for $k < 0$.

Given any symmetric function $f(x)$, we may omit the variable set x if no confusion arises in the context. Now let us review the definition of the principal specialization ps_n^1 of a symmetric function. For any symmetric function f, the action of ps_n^1 is defined as

$$\mathrm{ps}_n^1(f) = f(\underbrace{1, \ldots, 1}_{n\ 1's}, 0, 0, \ldots).$$

In particular, by (2.4), we have $\mathrm{ps}_n^1(e_k) = \binom{n}{k}$ and

$$\mathrm{ps}_n^1(e_k) = \mathrm{ps}_{n-1}^1(e_k + e_{k-1}), \tag{2.5}$$

which is a restatement of the relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

We will also need the dual version of Pieri's rule which expresses the product of a Schur function s_μ and an elementary symmetric function e_k in terms of Schur functions.

Theorem 2.2 ([15]) *We have*

$$s_\mu e_k = \sum_\lambda s_\lambda,$$

summed over partitions λ such that $\mu \subseteq \lambda$ and the difference of the Young diagrams of λ and μ contains no two squares in the same row.

3 A Schur Positivity Identity

In this section, we aim to establish the following Schur positivity identity which will be used to prove the q-log-convexity of $W_n(q)$.

Theorem 3.1 *For any $r \geq 1$, we have*

$$\sum_{k=0}^r (e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_k e_{r-k}e_{r-k} - 2e_{k-1}e_k e_{r-k-1}e_{r-k}) = \sum_\lambda s_\lambda, \tag{3.6}$$

where λ sums over all partitions of $2r-2$ of the form $(4^{i_4}, 3^{2i_3}, 2^{2i_2}, 1^{2i_1})$ with i_1, i_2, i_3, i_4 being nonnegative integers.

Before proceeding to prove the above theorem, we will give the illustrations for $r = 3, 4, 5$. By using the Maple package ACE [18], or SF [16], we find that

$$\sum_{k=0}^3 (e_{k-1}e_{k-1}e_{3-k}e_{3-k} + e_{k-2}e_k e_{3-k}e_{3-k} - 2e_{k-1}e_k e_{3-k-1}e_{3-k})$$

6

$$= s_{(1^4)} + s_{(2^2)} + s_{(4)},$$

$$\sum_{k=0}^{4}(e_{k-1}e_{k-1}e_{4-k}e_{4-k} + e_{k-2}e_k e_{4-k}e_{4-k} - 2e_{k-1}e_k e_{4-k-1}e_{4-k})$$

$$= s_{(1^6)} + s_{(2^2, 1^2)} + s_{(4, 1^2)} + s_{(3^2)},$$

$$\sum_{k=0}^{5}(e_{k-1}e_{k-1}e_{5-k}e_{5-k} + e_{k-2}e_k e_{5-k}e_{5-k} - 2e_{k-1}e_k e_{5-k-1}e_{5-k})$$

$$= s_{(4, 2^2)} + s_{(4^2)} + s_{(1^8)} + s_{(2^2, 1^4)} + s_{(2^4)} + s_{(4, 1^4)} + s_{(3^2, 1^2)}.$$

Let $L(r)$ and $R(r)$ denote the left-hand side and the right-hand side of (3.6), respectively. We will proceed to prove Theorem 3.1 by showing that $L(r)$ and $R(r)$ satisfy the same recurrence relation. It is easy to find the recurrence relation of $R(r)$. To derive the recurrence relation of $L(r)$, we will need several lemmas. For the sake of presentation, for $t \geq 0$, we define

$$A_1(t, k, i, j) = e_k s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i})},$$

$$A_2(t, k, i, j) = e_k s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i})},$$

$$A_3(t, k, i, j) = e_{k-1} s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i+1})},$$

$$A_4(t, k, i, j) = e_k s_{(3^i, 2^{k-i+j-1}, 1^{4t-3k-2j-i+2})},$$

$$B_1(t, k, i, j) = e_k s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i-2})},$$

$$B_2(t, k, i, j) = e_k s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i-2})},$$

$$B_3(t, k, i, j) = e_{k-1} s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i-1})},$$

$$B_4(t, k, i, j) = e_k s_{(3^i, 2^{k-i+j-1}, 1^{4t-3k-2j-i})},$$

and

$$A_1(t) = \sum_{k=0}^{t}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k} A_1(t, k, i, j),$$

$$A_2(t) = \sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1} A_2(t, k, i, j),$$

$$A_3(t) = \sum_{k=1}^{t}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k} A_3(t, k, i, j),$$

$$A_4(t) = \sum_{k=1}^{t}\sum_{i=0}^{k-1}\sum_{j=0}^{2t-2k+1} A_4(t, k, i, j),$$

7

$$B_1(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} B_1(t,k,i,j),$$

$$B_2(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} B_2(t,k,i,j),$$

$$B_3(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} B_3(t,k,i,j),$$

$$B_4(t) = \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} B_4(t,k,i,j).$$

The following lemma gives explicit expressions for $L(r)$ according to the parity of r.

Lemma 3.2 For $t \geq 0$, we have

$$L(2t+1) = A_1(t) + A_2(t) - A_3(t) - A_4(t),$$
$$L(2t) = B_1(t) + B_2(t) - B_3(t) - B_4(t).$$

Proof. For any $r \geq 0$, we have

$$L(r) = \sum_{k=0}^{r} (e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_k e_{r-k}e_{r-k} - 2e_{k-1}e_k e_{r-k-1}e_{r-k})$$

$$= \sum_{k=0}^{r} (e_{k-1}e_{k-1}e_{r-k}e_{r-k} - e_{k-1}e_k e_{r-k-1}e_{r-k})$$

$$\qquad + \sum_{k=0}^{r} (e_{k-2}e_k e_{r-k}e_{r-k} - e_{k-1}e_k e_{r-k-1}e_{r-k})$$

$$= \sum_{k=0}^{r} e_{k-1}e_{r-k} \det \begin{pmatrix} e_{k-1} & e_k \\ e_{r-k-1} & e_{r-k} \end{pmatrix}$$

$$\qquad + \sum_{k=0}^{r} e_k e_{r-k} \det \begin{pmatrix} e_{k-2} & e_{k-1} \\ e_{r-k-1} & e_{r-k} \end{pmatrix}.$$

By the dual Jacobi-Trudi identity, we find that

$$L(r) = \sum_{\substack{k=0 \\ k-1 \geq r-k}}^{r} e_{k-1}e_{r-k}s_{(2^{r-k},\,1^{2k-r-1})} - \sum_{\substack{k=0 \\ k-1 < r-k-1}}^{r} e_{k-1}e_{r-k}s_{(2^k,\,1^{r-2k-1})}$$

8

$$+ \sum_{\substack{k=0 \\ k-2 \geq r-k}}^{r} e_k e_{r-k} s_{(2^{r-k},\, 1^{2k-r-2})} - \sum_{\substack{k=0 \\ k-2 < r-k-1}}^{r} e_k e_{r-k} s_{(2^{k-1},\, 1^{r-2k})}.$$

Applying the dual version of Pieri's rule to the products $e_{k-1} s_{(2^{r-k},\, 1^{2k-r-1})}$, $e_{r-k} s_{(2^k,\, 1^{r-2k-1})}$, $e_k s_{(2^{r-k},\, 1^{2k-r-}}$ and $e_{r-k} s_{(2^{k-1},\, 1^{r-2k})}$, we get

$$
\begin{aligned}
L(r) = {} & \sum_{\substack{k=0 \\ k-1 \geq r-k}}^{r} \sum_{i=0}^{r-k} \sum_{j=0}^{2k-r-1} e_{r-k} s_{(3^i,\, 2^{r-k-i+j},\, 1^{2k-r-2-j+k-i-j})} \\
& - \sum_{\substack{k=0 \\ k-1 < r-k-1}}^{r} \sum_{i=0}^{k} \sum_{j=0}^{r-2k-1} e_{k-1} s_{(3^i,\, 2^{k-i+j},\, 1^{r-2k-j+r-k-i-j-1})} \\
& + \sum_{\substack{k=0 \\ k-2 \geq r-k}}^{r} \sum_{i=0}^{r-k} \sum_{j=0}^{2k-r-2} e_{r-k} s_{(3^i,\, 2^{r-k-i+j},\, 1^{2k-r-2-j+k-i-j})} \\
& - \sum_{\substack{k=0 \\ k-2 < r-k-1}}^{r} \sum_{i=0}^{k-1} \sum_{j=0}^{r-2k} e_k s_{(3^i,\, 2^{k-i+j-1},\, 1^{r-2k-j+r-k-i-j})}.
\end{aligned}
$$

Setting $r = 2t$ or $r = 2t + 1$, we obtain the required relations. ∎

To find a recurrence relation for $L(r)$, we need the operator Δ^μ associated with a partition μ, which acts on symmetric functions. This operator was introduced by Chen, Wang and Yang [6]. Given two partitions λ and μ, let $\lambda \cup \mu$ be the partition whose parts are obtained by taking the union of the parts of λ and μ. For a symmetric function f with the expansion

$$f = \sum_\lambda a_\lambda s_\lambda,$$

the action of Δ^μ on f is defined by

$$\Delta^\mu(f) = \sum_\lambda a_\lambda s_{\lambda \cup \mu}.$$

In order to compute the difference $L(2t+1) - \Delta^{(1,1)} L(2t)$, we will evaluate $A_m(t, k, i, j) - \Delta^{(1,1)}(B_m(t, k, i, j))$ for $1 \leq m \leq 4$. In fact, we will be able to express these differences as double sums of Schur functions. For $t \geq 0$, let

$$T_1(t, i, j, k) = \sum_{a=0}^{\beta_1 + 1} \sum_{b=0}^{\min(\beta_2, \beta_3 + 1)} P(t, k+1, i, j-1, a, b),$$

$$T_2(t, i, j, k) = \sum_{a=0}^{\beta_1} \sum_{b=0}^{\min(\beta_2, \beta_3)} P(t, k, i, j, a, b),$$

9

$$T_3(t,i,j,k) = \sum_{a=0}^{\beta_1+1} \sum_{b=0}^{\min(\beta_2,\beta_3+1)} P(t,k+1,i,j-1,a,b),$$

$$T_4(t,i,j,k) = \sum_{a=0}^{\beta_1} \sum_{b=0}^{\min(\beta_2,\beta_3)} P(t,k,i,j,a,b),$$

$$T_5(t,i,j,k) = \sum_{a=0}^{\beta_1-1} \sum_{b=0}^{\min(\beta_2,\beta_3-1)} P(t,k,i,j,a,b),$$

$$T_6(t,i,j,k) = \sum_{a=0}^{\beta_1-2} \sum_{b=0}^{\min(\beta_2,\beta_3-2)} P(t,k-1,i,j+1,a,b),$$

$$T_7(t,i,j,k) = \sum_{a=0}^{\beta_1-1} \sum_{b=0}^{\min(\beta_2-1,\beta_3-1)} P(t,k,i,j,a,b),$$

$$T_8(t,i,j,k) = \sum_{a=0}^{\beta_1-2} \sum_{b=0}^{\min(\beta_2-1,\beta_3-2)} P(t,k-1,i,j+1,a,b),$$

where

$$\begin{aligned}
P(t,k,i,j,a,b) &= s_{(4^a,\, 3^{i-a+b},\, 2^{4t-2k-2i-j-b},\, 1^{4k+i+2j-a-b-4t})}, \\
\beta_1 &= 4k+i+2j-4t, \\
\beta_2 &= k-i+j, \\
\beta_3 &= 4k+i+2j-a-4t.
\end{aligned}$$

We now have the following relations.

Lemma 3.3 *Let $t \geq 0$.*

(i) *If $0 \leq k \leq t-1, 0 \leq i \leq k, 0 \leq j \leq 2t-2k-1$, then*

$$A_1(t,k,i,j) - \Delta^{(1,1)}(B_1(t,k,i,j)) = T_1(t,i,j,k) + T_2(t,i,j,k).$$

(ii) *If $0 \leq k \leq t-1, 0 \leq i \leq k, 0 \leq j \leq 2t-2k-2$, then*

$$A_2(t,k,i,j) - \Delta^{(1,1)}(B_2(t,k,i,j)) = T_3(t,i,j,k) + T_4(t,i,j,k).$$

(iii) *If $1 \leq k \leq t-1, 0 \leq i \leq k, 0 \leq j \leq 2t-2k-1$, then*

$$A_3(t,k,i,j) - \Delta^{(1,1)}(B_3(t,k,i,j)) = T_5(t,i,j,k) + T_6(t,i,j,k).$$

10

(iv) *If* $1 \le k \le t, 0 \le i \le k-1, 0 \le j \le 2t - 2k$, *then*

$$A_4(t,k,i,j) - \Delta^{(1,1)}(B_4(t,k,i,j)) = T_7(t,i,j,k) + T_8(t,i,j,k).$$

Proof. We only give the proof of (i). The proofs of (ii), (iii) and (iv) are analogous to that of (i). Recall that

$$A_1(t,k,i,j) = e_k s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i})}, \quad B_1(t,k,i,j) = e_k s_{(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i-2})}.$$

Applying the dual version of Pieri's rule, we see that

$$A_1(t,k,i,j) = \sum_{(a,b,c,d)} s_{(4^a, 3^{i-a+b}, 2^{k-i+j-b+c}, 1^{4t-3k-2j-i-c+d})},$$

where the sum ranges over nonnegative integer sequences (a,b,c,d) satisfying

$$a + b + c + d = k, \quad a \le i, \ b \le k - i + j, \quad c \le 4t - 3k - 2j - i.$$

Note that the shape $(4^a, 3^{i-a+b}, 2^{k-i+j-b+c}, 1^{4t-3k-2j-i-c+d})$ is obtained from the Young diagram of $(3^i, 2^{k-i+j}, 1^{4t-3k-2j-i})$ by adding a squares in the fourth column, b squares in the third column, c squares in the second column, and d squares in the first column. Similarly, we see that

$$B_1(t,k,i,j) = \sum_{(a,b,c,d)} s_{(4^a, 3^{i-a+b}, 2^{k-i+j-b+c}, 1^{4t-3k-2j-i-2-c+d})},$$

summed over nonnegative integer sequences (a,b,c,d) satisfying

$$a + b + c + d = k, \ a \le i, \ b \le k - i + j, \ c \le 4t - 3k - 2j - i - 2.$$

Therefore,

$$A_1(t,k,i,j) - \Delta^{(1,1)}(B_1(t,k,i,j))$$

$$= \sum_{\substack{(a,b,c,d) \\ c=4t-3k-2j-i}} s_{(4^a, 3^{i-a+b}, 2^{k-i+j-b+c}, 1^{4t-3k-2j-i-c+d})}$$

$$+ \sum_{\substack{(a,b,c,d) \\ c=4t-3k-2j-i-1}} s_{(4^a, 3^{i-a+b}, 2^{k-i+j-b+c}, 1^{4t-3k-2j-i-c+d})},$$

where both sums range over nonnegative integers a, b, d satisfying

$$a \le i, \ b \le k - i + j, \ a + b + c + d = k.$$

11

Since $0 \le j \le 2t - 2k - 1$, we have

$$k - (4t - 3k - 2j - i) = 4k + 2j + i - 4t$$
$$\le 4k + 2(2t - 2k - 1) + i - 4t \le i,$$

and

$$k - (4t - 3k - 2j - i - 1) = 4k + 2j + i - 4t + 1$$
$$\le 4k + 2(2t - 2k - 1) + i - 4t + 1 \le i.$$

Hence
$$A_1(t, k, i, j) - \Delta^{(1,1)}(B_1(t, k, i, j)) = T_1(t, i, j, k) + T_2(t, i, j, k).$$

This completes the proof of (i). ∎

In light of the above lemma, we will show that $L(2t + 1) - \Delta^{(1,1)}L(2t)$ can be expressed in terms of ten sums $T_1(t), T_2(t), \ldots, T_{10}(t)$, as defined below,

$$T_1(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} T_1(t, i, j, k), \quad T_2(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} T_2(t, i, j, k),$$

$$T_3(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} T_3(t, i, j, k), \quad T_4(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} T_4(t, i, j, k),$$

$$T_5(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} T_5(t, i, j, k), \quad T_6(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} T_6(t, i, j, k),$$

$$T_7(t) = \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} T_7(t, i, j, k), \qquad T_8(t) = \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} T_8(t, i, j, k),$$

$$T_9(t) = \sum_{k=0}^{t} e_k s_{(3^k, 2^{2t-2k})}, \qquad\qquad T_{10}(t) = \sum_{k=0}^{t-1} e_k s_{(3^{k+1}, 2^{2t-2k-2}, 1)}.$$

Lemma 3.4 *For $t \ge 0$, we have*

$$L(2t + 1) - \Delta^{(1,1)}L(2t) = (T_1(t) + T_2(t) + T_3(t) + T_4(t))$$
$$- (T_5(t) + T_6(t) + T_7(t) + T_8(t)) + (T_9(t) - T_{10}(t)).$$

Proof. By Lemma 3.2, $L(2t + 1) - \Delta^{(1,1)}L(2t)$ equals

$$(A_1(t) - \Delta^{(1,1)}B_1(t)) + (A_2(t) - \Delta^{(1,1)}B_2(t))$$
$$- (A_3(t) - \Delta^{(1,1)}B_3(t)) - (A_4(t) - \Delta^{(1,1)}B_4(t))$$

12

$$= \left(\sum_{k=0}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k} A_1(t,k,i,j) - \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \Delta^{(1,1)}(B_1(t,k,i,j)) \right)$$

$$+ \left(\sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} A_2(t,k,i,j) - \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \Delta^{(1,1)}(B_2(t,k,i,j)) \right)$$

$$- \left(\sum_{k=1}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k} A_3(t,k,i,j) - \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \Delta^{(1,1)}(B_3(t,k,i,j)) \right)$$

$$- \left(\sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k+1} A_4(t,k,i,j) - \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} \Delta^{(1,1)}(B_4(t,k,i,j)) \right)$$

$$= \sum_{k=0}^{t} \sum_{i=0}^{k} A_1(t,k,i,2t-2k) + \sum_{k=0}^{t-1} \sum_{i=0}^{k} A_2(t,k,i,2t-2k-1)$$

$$+ \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \left(A_1(t,k,i,j) - \Delta^{(1,1)}(B_1(t,k,i,j)) \right)$$

$$+ \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \left(A_2(t,k,i,j) - \Delta^{(1,1)}(B_2(t,k,i,j)) \right)$$

$$- \sum_{k=1}^{t} \sum_{i=0}^{k} A_3(t,k,i,2t-2k) - \sum_{k=1}^{t} \sum_{i=0}^{k-1} A_4(t,k,i,2t-2k+1)$$

$$- \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \left(A_3(t,k,i,j) - \Delta^{(1,1)}(B_3(t,k,i,j)) \right)$$

$$- \sum_{k=1}^{t} \sum_{i=0}^{k-1} \sum_{j=0}^{2t-2k} \left(A_4(t,k,i,j) - \Delta^{(1,1)}(B_4(t,k,i,j)) \right).$$

Applying Lemma 3.3, we get the desired relation. ∎

In order to derive a recurrence relation of $L(r)$, we still need to compute

$$(L(2(t+1)+1) - \Delta^{(1,1)} L(2(t+1))) - \Delta^{(2,2)}(L(2t+1) - \Delta^{(1,1)} L(2t)).$$

By Lemma 3.4, the above expression equals

$$\sum_{i=1}^{4} (T_i(t+1) - \Delta^{(2,2)} T_i(t)) - \sum_{i=5}^{8} (T_i(t+1) - \Delta^{(2,2)} T_i(t))$$

$$+ (T_9(t+1) - \Delta^{(2,2)} T_9(t)) + (T_{10}(t+1) - \Delta^{(2,2)} T_{10}(t)). \tag{3.7}$$

13

Set

$$Q_1(t,k,i,j,a,b) = s_{(4^a,\, 3^{k-a+b+1},\, 2^{4t-4k-j-b-1},\, 1^{5k+2j-a-b-4t+3})},\tag{3.8}$$

$$Q_2(t,k,i,j,a,b) = s_{(4^a,\, 3^{k-a+j+1},\, 2^{4t-3k-i-2j},\, 1^{3k+2i+j-a-4t+1})},\tag{3.9}$$

$$\gamma_1 = 5k + 2j + 2 - 4t,\tag{3.10}$$

$$\gamma_2 = 5k + 2j - a + 2 - 4t,\tag{3.11}$$

$$\gamma_3 = 3k + 2i + j - 4t,\tag{3.12}$$

and let

$$T_{11}(t) = \sum_{k=0}^{t-1}\ \sum_{j=0}^{2t-2k-1}\ \sum_{a=0}^{\gamma_1}\ \sum_{b=0}^{\min(j,\gamma_2)} Q_1(t,k,i,j,a,b),$$

$$T_{12}(t) = \sum_{k=0}^{t}\sum_{i=0}^{k}\ \sum_{j=0}^{2t-2k-1}\ \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{21}(t) = \sum_{k=0}^{t-1}\ \sum_{j=-1}^{2t-2k-2}\ \sum_{a=0}^{\gamma_1+1}\ \sum_{b=0}^{\min(j+1,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{22}(t) = \sum_{k=0}^{t-1}\sum_{i=-1}^{k-1}\ \sum_{j=0}^{2t-2k-1}\ \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b),$$

$$T_{31}(t) = \sum_{k=0}^{t}\ \sum_{j=0}^{2t-2k-2}\ \sum_{a=0}^{\gamma_1}\ \sum_{b=0}^{\min(j,\gamma_2)} Q_1(t,k,i,j,a,b),$$

$$T_{32}(t) = \sum_{k=0}^{t}\sum_{i=0}^{k}\ \sum_{j=0}^{2t-2k-2}\ \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{41}(t) = \sum_{k=0}^{t-1}\ \sum_{j=-1}^{2t-2k-3}\ \sum_{a=0}^{\gamma_1+1}\ \sum_{b=0}^{\min(j+1,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{42}(t) = \sum_{k=0}^{t-1}\sum_{i=-1}^{k-1}\ \sum_{j=0}^{2t-2k-2}\ \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b),$$

$$T_{51}(t) = \sum_{k=0}^{t-1}\ \sum_{j=-1}^{2t-2k-2}\ \sum_{a=0}^{\gamma_1}\ \sum_{b=0}^{\min(j+1,\gamma_2)} Q_1(t,k,i,j,a,b),$$

14

$$T_{52}(t) = \sum_{k=0}^{t-1} \sum_{i=-1}^{k-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{61}(t) = \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+2,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{62}(t) = \sum_{k=-2}^{t-3} \sum_{i=0}^{k+2} \sum_{j=2}^{2t-2k-3} \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b),$$

$$T_{71}(t) = \sum_{k=0}^{t} \sum_{j=-2}^{2t-2k-2} \sum_{a=0}^{\gamma_1+1} \sum_{b=-1}^{\min(j+1,\gamma_2)} Q_1(t,k,i,j,a,b),$$

$$T_{72}(t) = \sum_{k=0}^{t} \sum_{i=0}^{k-1} \sum_{j=-1}^{2t-2k-1} \sum_{a=0}^{\gamma_3} Q_2(t,k,i,j,a,b),$$

$$T_{81}(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j,\gamma_2+1)} Q_1(t,k,i,j,a,b),$$

$$T_{82}(t) = \sum_{k=-1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{\gamma_3+1} Q_2(t,k,i,j,a,b).$$

The following lemma shows that each $T_i(t+1) - \Delta^{(2,2)}T_i(t)$ can be expressed in terms of two sums for $1 \le i \le 8$, where each sum is a quadruple sum of Schur functions.

Lemma 3.5 *For $t \ge 0$ and $1 \le i \le 8$, we have*

$$T_i(t+1) - \Delta^{(2,2)}T_i(t) = T_{i1}(t) + T_{i2}(t).$$

Proof. We will give only the proof of the identity for $T_1(t)$, since the other cases can be verified by the same argument. Observe that

$$T_1(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{4k+i+2j-4t+1} \sum_{b=0}^{\min(k-i+j,4k+i+2j-a-4t+1)}$$

$$\left(s_{(4^a, 3^{i-a+b}, 2^{4t-2k-2i-j-b-1}, 1^{4k+i+2j-a-b-4t+2})} \right).$$

We have

$$T_1(t+1) = \sum_{k=0}^{t} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k+1} \sum_{a=0}^{4k+i+2j-4t-3} \sum_{b=0}^{\min(k-i+j,4k+i+2j-a-4t-3)}$$

15

$$\left(s_{\left(4^a,\,3^{i-a+b},\,2^{4t-2k-2i-j-b+3},\,1^{4k+i+2j-a-b-4t-2}\right)}\right)$$

$$=\sum_{k=-1}^{t-1}\sum_{i=0}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,\,4k+i+2j-a-4t+1)}$$

$$\left(s_{\left(4^a,\,3^{i-a+b},\,2^{4t-2k-2i-j-b+1},\,1^{4k+i+2j-a-b-4t+2}\right)}\right)$$

$$=\sum_{k=0}^{t-1}\sum_{i=0}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,\,4k+i+2j-a-4t+1)}$$

$$\left(s_{\left(4^a,\,3^{i-a+b},\,2^{4t-2k-2i-j-b+1},\,1^{4k+i+2j-a-b-4t+2}\right)}\right),$$

where the last equality holds because the upper bound $4k+i+2j-4t+1$ of a is negative for $k=-1$. Hence we deduce that

$$T_1(t+1)-\Delta^{(2,2)}T_1(t)$$

$$=\sum_{k=0}^{t-1}\sum_{i=k+1}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,\,4k+i+2j-a-4t+1)}$$

$$\left(s_{\left(4^a,\,3^{i-a+b},\,2^{4t-2k-2i-j-b+1},\,1^{4k+i+2j-a-b-4t+2}\right)}\right)$$

$$+\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,\,4k+i+2j-a-4t+1)}$$

$$\left(s_{\left(4^a,\,3^{i-a+b},\,2^{4t-2k-2i-j-b+1},\,1^{4k+i+2j-a-b-4t+2}\right)}\right)$$

$$-\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j,\,4k+i+2j-a-4t+1)}$$

$$\left(s_{\left(4^a,\,3^{i-a+b},\,2^{4t-2k-2i-j-b+1},\,1^{4k+i+2j-a-b-4t+2}\right)}\right)$$

$$=\sum_{k=0}^{t-1}\sum_{i=k+1}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1,\,4k+i+2j-a-4t+1)}$$

$$\left(s_{\left(4^a,\,3^{i-a+b},\,2^{4t-2k-2i-j-b+1},\,1^{4k+i+2j-a-b-4t+2}\right)}\right)$$

$$+\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{\substack{a=0\\k-i+j+1\le 4k+i+2j-a-4t+1}}^{4k+i+2j-4t+1}\sum_{b=k-i+j+1}^{k-i+j+1}$$

16

$$\left(s_{(4^a, 3^{i-a+b}, 2^{4t-2k-2i-j-b+1}, 1^{4k+i+2j-a-b-4t+2})}\right)$$

$$-\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{\substack{a=0 \\ k-i+j+1\leq 4k+i+2j-a-4t+1}}^{4k+i+2j-4t+1}\sum_{b=0}^{k-i+j}$$

$$\left(s_{(4^a, 3^{i-a+b}, 2^{4t-2k-2i-j-b+1}, 1^{4k+i+2j-a-b-4t+2})}\right)$$

$$=\sum_{k=0}^{t-1}\sum_{i=k+1}^{k+1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{4k+i+2j-4t+1}\sum_{b=0}^{\min(k-i+j+1, 4k+i+2j-a-4t+1)}$$

$$\left(s_{(4^a, 3^{i-a+b}, 2^{4t-2k-2i-j-b+1}, 1^{4k+i+2j-a-b-4t+2})}\right)$$

$$+\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{\substack{a=0 \\ k-i+j+1\leq 4k+i+2j-a-4t+1}}^{4k+i+2j-4t+1}\sum_{b=k-i+j+1}^{k-i+j+1}$$

$$\left(s_{(4^a, 3^{i-a+b}, 2^{4t-2k-2i-j-b+1}, 1^{4k+i+2j-a-b-4t+2})}\right)$$

$$=\sum_{k=0}^{t-1}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{5k+2j+2-4t}\sum_{b=0}^{\min(j, 5k+2j-a+2-4t)}$$

$$\left(s_{(4^a, 3^{k-a+b+1}, 2^{4t-4k-j-b-1}, 1^{5k+2j-a-b-4t+3})}\right)$$

$$+\sum_{k=0}^{t-1}\sum_{i=0}^{k}\sum_{j=0}^{2t-2k-1}\sum_{a=0}^{3k+2i+j-4t}$$

$$\left(s_{(4^a, 3^{k-a+j+1}, 2^{4t-3k-i-2j}, 1^{3k+2i+j-a-4t+1})}\right),$$

as desired. This completes the proof. ∎

In order to compute (3.7), it is necessary to simplify

$$\sum_{i=1}^{4}(T_i(t+1) - \Delta^{(2,2)}T_i(t)) - \sum_{i=5}^{8}(T_i(t+1) - \Delta^{(2,2)}T_i(t)). \tag{3.13}$$

By the above lemma, (3.13) equals

$$\sum_{i=1}^{4}(T_{i1}(t) + T_{i2}(t)) - \sum_{i=5}^{8}(T_{i1}(t) + T_{i2}(t)). \tag{3.14}$$

Moreover, we need to rearrange the terms into groups in order to reduce the relevant quadruple sums to triple sums, and further to double sums. For $t \geq 0$, let

$$N_1(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} S_{(4^a,\, 3^{k-a+j+1},\, 2^{4t-4k-2j+1},\, 1^{5k+j-1-4t-a})},$$

$$N_2(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j-4t+2} S_{(4^a,\, 3^{k-a+j+1},\, 2^{4t-4k-2j-1},\, 1^{5k+j+3-4t-a})},$$

$$N_3(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{k} \sum_{b=0}^{\min(2t-2k-2,\, k-a)} S_{(4^a,\, 3^{k-a+b+1},\, 2^{2t-2k-b},\, 1^{k-a-b+1})},$$

$$N_4(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{k} \sum_{b=0}^{\min(2t-2k-1,\, k-a)} S_{(4^a,\, 3^{k+1-a+b},\, 2^{2t-2k-b},\, 1^{k+1-a-b})},$$

$$N_5(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,5k+j-4t+3)}^{5k+2j-4t+3} S_{(4^a,\, 3^{6k-4t+2j-2a+4},\, 2^{8t-9k-3j+a-4})},$$

$$N_6(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=\max(0,5k+j-4t+1)}^{5k+2j-4t+1} S_{(4^a,\, 3^{6k-4t+2j-2a+2},\, 2^{8t-9k-3j+a-1})},$$

$$M_1(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j+2-4t} S_{(4^a,\, 3^{k+1-a+j},\, 2^{4t-2j-1-4k},\, 1^{5k+j-4t+3-a})},$$

$$M_2(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j-4t-1} S_{(4^a,\, 3^{k+1-a+j},\, 2^{4t+1-4k-2j},\, 1^{5k+j-4t-1-a})},$$

$$M_3(t) = \sum_{k=0}^{t-1} \sum_{i=1}^{k} \sum_{a=0}^{k-2t+2i-4} S_{(4^a,\, 3^{2t-k-a},\, 2^{k+4-i},\, 1^{k+2i-2t-4-a})},$$

$$M_4(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{a=0}^{k+2i-2t-1} S_{(4^a,\, 3^{2t-a-k},\, 2^{k-i+2},\, 1^{k+2i-2t-a})},$$

$$M_5(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,3k+j+2i-4t+1)}^{3k+j+2i-4t+1} S_{(4^a,\, 3^{k+1-a+j},\, 2^{4t-2j-i-3k})},$$

$$M_6(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k} \sum_{j=0}^{2t-2k-1} \sum_{a=\max(0,3k+j+2i-4t-1)}^{3k+j+2i-4t-1} S_{(4^a,\, 3^{k+1-a+j},\, 2^{4t-2j-i-3k+1})}.$$

18

The following lemma shows that one can group the terms $T_{i1}(t)$ and $T_{i2}(t)$ to reduce (3.14) to triple sums.

Lemma 3.6 *For $t \geq 0$, we have*

$$T_{41}(t) - T_{61}(t) = - N_1(t),$$

$$T_{31}(t) - T_{71}(t) = N_2(t) - N_3(t),$$

$$T_{11}(t) - T_{81}(t) = N_4(t) - N_5(t),$$

$$T_{21}(t) - T_{51}(t) = N_6(t),$$

$$T_{32}(t) - T_{72}(t) = - M_1(t)$$

$$T_{42}(t) - T_{62}(t) = M_2(t) - M_3(t),$$

$$T_{12}(t) - T_{82}(t) = M_4(t) - M_5(t),$$

$$T_{22}(t) - T_{52}(t) = M_6(t).$$

Proof. We will only give the derivation of the first identity, since the other relations can be verified by the same argument. Using the notation $Q_1(t, k, i, j, a, b)$, γ_1 and γ_2, as given in (3.8)–(3.11), we find

$$T_{41}(t) - T_{61}(t) = \sum_{k=0}^{t-1} \sum_{j=-1}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+1, \gamma_2+1)} Q_1(t, k, i, j, a, b)$$

$$- \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+2, \gamma_2+1)} Q_1(t, k, i, j, a, b)$$

$$= \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+1, \gamma_2+1)} Q_1(t, k, i, j, a, b)$$

$$- \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{\gamma_1+1} \sum_{b=0}^{\min(j+2, \gamma_2+1)} Q_1(t, k, i, j, a, b)$$

$$= - \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{\substack{a=0 \\ j+2 \leq \gamma_2+1}}^{\gamma_1+1} \sum_{b=j+2}^{j+2} Q_1(t, k, i, j, a, b)$$

$$= - \sum_{k=0}^{t-1} \sum_{j=-2}^{2t-2k-3} \sum_{a=0}^{5k+j-4t+1} Q_1(t, k, i, j, a, j+2).$$

19

Substituting j with $j-2$ in the last summation, we are led to the required relation. This completes the proof. ∎

For $t \geq 0$, let

$$C_1(t) = \sum_{k=1}^{t} \sum_{a=0}^{5k-4t-3} S_{(4^a,\, 3^{k-a},\, 2^{4t-4k+3},\, 1^{5k-4t-a-2})},$$

$$C_2(t) = \sum_{k=1}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=\max(0,5k+j-4t-1)}^{5k+j-4t-1} S_{(4^a,\, 3^{k-a+j+1},\, 2^{4t-4k-2j+1})},$$

$$C_3(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t+1} S_{(4^a,\, 3^{2t-k-a},\, 2,\, 1^{3k-2t-a+2})},$$

$$C_4(t) = \sum_{k=0}^{t-1} \sum_{a=\max(0,5k-4t+1)}^{5k-4t+1} S_{(4^a,\, 3^{k+1-a},\, 2^{4t-4k})},$$

$$C_5(t) = \sum_{k=1}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,5k+j-4t+2)}^{5k+j-4t+2} S_{(4^a,\, 3^{k+j-a+2},\, 2^{4t-4k-2j-2})},$$

$$D_1(t) = \sum_{k=0}^{t-2} \sum_{j=1}^{2t-2k-3} \sum_{a=\max(0,5k+j-4t+3)}^{5k+j-4t+3} S_{(4^a,\, 3^{k+1-a+j},\, 2^{4t-2j-4k-1})},$$

$$D_2(t) = \sum_{a=0}^{t-3} S_{(4^a,\, 3^{t-a},\, 2^3,\, 1^{t-a-2})},$$

$$D_3(t) = \sum_{k=0}^{t-2} \sum_{a=0}^{5k-4t+2} S_{(4^a,\, 3^{k+1-a},\, 2^{4t-4k-1},\, 1^{5k-4t-a+3})},$$

$$D_4(t) = \sum_{k=0}^{t-2} \sum_{a=0}^{3k-2t} S_{(4^a,\, 3^{2t-k-a-1},\, 2^3,\, 1^{3k-2t-a+1})},$$

$$D_5(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t-1} S_{(4^a,\, 3^{2t-a-k},\, 2^2,\, 1^{3k-2t-a})},$$

$$D_6(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t-3} S_{(4^a,\, 3^{2t-a-k},\, 2^3,\, 1^{3k-2t-a-2})},$$

$$D_7(t) = \sum_{k=0}^{t-1} \sum_{i=0}^{k-2} \sum_{a=\max(0,k+2i-2t)}^{k+2i-2t} S_{(4^a,\,3^{2t-k-a},\,2^{k-i+2})},$$

$$D_8(t) = \sum_{k=1}^{t-1} \sum_{i=0}^{k-1} \sum_{a=\max(0,k-2t+2i)}^{k-2t+2i} S_{(4^a,\,3^{2t-a-k},\,2^{k-i+2})},$$

$$D_9(t) = \sum_{k=1}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=\max(0,5k+j-4t+1)}^{5k+j-4t+1} S_{(4^a,\,3^{k+1-a+j},\,2^{4t-4k-2j})}.$$

The following lemma enables us to express (3.14) in terms of the double sums C_i and D_i.

Lemma 3.7 *For any $t \geq 0$, we have*

$$N_2(t) - N_1(t) = C_1(t) - C_2(t),$$

$$N_4(t) - N_3(t) = C_3(t),$$

$$N_6(t) - N_5(t) = C_4(t) + C_5(t),$$

$$M_2(t) - M_1(t) = D_1(t) - D_2(t) - D_3(t) - D_4(t),$$

$$M_4(t) - M_3(t) = D_5(t) + D_6(t) - D_7(t),$$

$$M_6(t) - M_5(t) = D_8(t) - D_9(t).$$

Proof. We have

$$N_2(t) - N_1(t) = \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-2} \sum_{a=0}^{5k+j-4t+2} S_{(4^a,\,3^{k-a+j+1},\,2^{4t-4k-2j-1},\,1^{5k+j+3-4t-a})}$$

$$- \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} S_{(4^a,\,3^{k-a+j+1},\,2^{4t-4k-2j+1},\,1^{5k+j-1-4t-a})}$$

$$= \sum_{k=1}^{t} \sum_{j=0}^{2t-2k} \sum_{a=0}^{5k+j-4t-3} S_{(4^a,\,3^{k-a+j},\,2^{4t-4k-2j+3},\,1^{5k+j-2-4t-a})}$$

$$- \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} S_{(4^a,\,3^{k-a+j+1},\,2^{4t-4k-2j+1},\,1^{5k+j-1-4t-a})}$$

21

$$= \sum_{k=1}^{t} \sum_{j=-1}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-2} s_{(4^a, 3^{k-a+j+1}, 2^{4t-4k-2j+1}, 1^{5k+j-1-4t-a})}$$

$$- \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} s_{(4^a, 3^{k-a+j+1}, 2^{4t-4k-2j+1}, 1^{5k+j-1-4t-a})}$$

$$= \sum_{k=1}^{t} \sum_{j=-1}^{-1} \sum_{a=0}^{5k+j-4t-2} s_{(4^a, 3^{k-a+j+1}, 2^{4t-4k-2j+1}, 1^{5k+j-1-4t-a})}$$

$$+ \sum_{k=1}^{t} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-2} s_{(4^a, 3^{k-a+j+1}, 2^{4t-4k-2j+1}, 1^{5k+j-1-4t-a})}$$

$$- \sum_{k=0}^{t-1} \sum_{j=0}^{2t-2k-1} \sum_{a=0}^{5k+j-4t-1} s_{(4^a, 3^{k-a+j+1}, 2^{4t-4k-2j+1}, 1^{5k+j-1-4t-a})}.$$

It follows that

$$N_2(t) - N_1(t) = C_1(t) - C_2(t).$$

Similarly, one can verify the other identities. This completes the proof. ∎

Let us now turn to the Schur expansions of $T_9(t+1) - \Delta^{(2,2)}T_9(t)$ and $T_{10}(t+1) - \Delta^{(2,2)}T_{10}(t)$. Let

$$E_1(t) = \sum_{a=0}^{t+1} s_{(4^a, 3^{t+1-a}, 1^{t+1-a})},$$

$$E_2(t) = \sum_{k=0}^{t} \sum_{a=0}^{3k-2t-1} s_{(4^a, 3^{2t-k-a+1}, 2, 1^{3k-2t-1-a})},$$

$$E_3(t) = \sum_{k=0}^{t} \sum_{a=0}^{3k-2t-2} s_{(4^a, 3^{2t-k-a+2}, 1^{3k-2t-2-a})},$$

$$E_4(t) = \sum_{a=0}^{t} s_{(4^a, 3^{t+1-a}, 1^{t+1-a})},$$

$$E_5(t) = \sum_{a=0}^{t-1} s_{(4^a, 3^{t+1-a}, 2, 1^{t-1-a})},$$

$$E_6(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t+1} s_{(4^a, 3^{2t-k-a}, 2, 1^{3k+2-2t-a})},$$

22

$$E_7(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t} s_{(4^a,\, 3^{2t+1-k-a},\, 1^{3k+1-2t-a})},$$

$$E_8(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t} s_{(4^a,\, 3^{2t-k-a},\, 2,\, 2,\, 1^{3k-2t-a})}$$

$$E_9(t) = \sum_{k=0}^{t-1} \sum_{a=0}^{3k-2t-1} s_{(4^a,\, 3^{2t+1-k-a},\, 2,\, 1^{3k-1-2t-a})}.$$

Lemma 3.8 *For any $t \geq 0$, we have*

$$T_9(t+1) - \Delta^{(2,2)} T_9(t) = E_1(t) + E_2(t) + E_3(t), \tag{3.15}$$

$$T_{10}(t+1) - \Delta^{(2,2)} T_{10}(t) = E_4(t) + E_5(t) + E_6(t) + E_7(t) + E_8(t) + E_9(t). \tag{3.16}$$

Proof. We will only present the proof of the identity (3.16), because it is easier to prove (3.15) by means of Pieri's rule. Recall that

$$T_{10}(t) = \sum_{k=0}^{t-1} e_k s_{(3^{k+1},\, 2^{2t-2k-2},\, 1)}.$$

Thus,

$$T_{10}(t+1) - \Delta^{(2,2)} T_{10}(t) = \sum_{k=0}^{t} e_k s_{(3^{k+1},\, 2^{2t-2k},\, 1)} - \Delta^{(2,2)} \left(\sum_{k=0}^{t-1} e_k s_{(3^{k+1},\, 2^{2t-2k-2},\, 1)} \right)$$

$$= e_t s_{(3^{t+1},\, 1)} + \sum_{k=0}^{t-1} e_k s_{(3^{k+1},\, 2^{2t-2k},\, 1)}$$

$$- \sum_{k=0}^{t-1} \Delta^{(2,2)} \left(e_k s_{(3^{k+1},\, 2^{2t-2k-2},\, 1)} \right).$$

Applying the dual version of Pieri's rule, we deduce that

$$e_t s_{(3^{t+1},\, 1)} = \sum_{a=0}^{t} s_{(4^a,\, 3^{t+1-a},\, 1^{t+1-a})} + \sum_{a=0}^{t-1} s_{(4^a,\, 3^{t+1-a},\, 2,\, 1^{t-1-a})},$$

and, for $0 \leq k \leq t-1$,

$$e_k s_{(3^{k+1},\, 2^{2t-2k-2},\, 1)} = \sum_{a,b,c,d} s_{(4^a,\, 3^{k+1-a+b},\, 2^{2t-2k-2-b+c},\, 1^{1-c+d})},$$

23

where

$$0 \le a \le k + 1,\ 0 \le b \le 2t - 2k - 2,\ 0 \le c \le 1,\ d \ge 0,\ a + b + c + d = k.$$

Therefore,

$$e_k s_{(3^{k+1},\, 2^{2t-2k},\, 1)} - \Delta^{(2,2)}\big(e_k s_{(3^{k+1},\, 2^{2t-2k-2},\, 1)}\big)$$

$$= \sum_{a,b,c,d} s_{(4^a,\, 3^{k+1-a+b},\, 2^{2t-2k-b+c},\, 1^{1-c+d})},$$

where

$$0 \le a \le k + 1,\ 2t - 2k - 1 \le b \le 2t - 2k,\ 0 \le c \le 1,\ d \ge 0,\ a + b + c + d = k.$$

In view of the ranges of b and c, the above sum can be divided into four sums

$$\sum_{a=0}^{3k-2t+1} s_{(4^a,\, 3^{2t-k-a},\, 2,\, 1^{3k+2-2t-a})} + \sum_{a=0}^{3k-2t} s_{(4^a,\, 3^{2t+1-k-a},\, 1^{3k+1-2t-a})}$$

$$+ \sum_{a=0}^{3k-2t} s_{(4^a,\, 3^{2t-k-a},\, 2,\, 2,\, 1^{3k-2t-a})} + \sum_{a=0}^{3k-2t-1} s_{(4^a,\, 3^{2t+1-k-a},\, 2,\, 1^{3k-1-2t-a})}.$$

This completes the proof. ∎

We are now ready to give a recurrence relation for $L(r)$ by using the above lemmas. Note that it is easy to establish a recurrence relation for $R(r)$. For $t \ge 0$, let

$$\begin{aligned}
R_{o,1}(t) &= R(2t+1) - \Delta^{(1,1)} R(2t), \\
R_{e,1}(t) &= R(2t+2) - \Delta^{(1,1)} R(2t+1), \\
L_{o,1}(t) &= L(2t+1) - \Delta^{(1,1)} L(2t), \\
L_{e,1}(t) &= L(2t+2) - \Delta^{(1,1)} L(2t+1),
\end{aligned}$$

and

$$\begin{aligned}
R_{o,2}(t) &= R_{o,1}(t+1) - \Delta^{(2,2)} R_{o,1}(t), \\
R_{e,2}(t) &= R_{e,1}(t+1) - \Delta^{(2,2)} R_{e,1}(t), \\
L_{o,2}(t) &= L_{o,1}(t+1) - \Delta^{(2,2)} L_{o,1}(t), \\
L_{e,2}(t) &= L_{e,1}(t+1) - \Delta^{(2,2)} L_{e,1}(t).
\end{aligned}$$

The following lemma gives the recurrence relations of $L(r)$ and $R(r)$.

Lemma 3.9 *Let* $\mathrm{Par}_{\{3,4\}}(n)$ *denote the set of partitions of n with parts 3 and 4. Then for any $t \ge 0$, we have*

$$R_{o,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+4)} s_\lambda, \quad R_{e,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+6)} s_\lambda, \tag{3.17}$$

$$L_{o,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+4)} s_\lambda, \quad L_{e,2}(t) = \sum_{\lambda \in \mathrm{Par}_{\{3,4\}}(4t+6)} s_\lambda. \tag{3.18}$$

24

Proof. The two identities in (3.17) are easy to check. We will focus on the two identities in (3.18). Here we will consider only the identity concerning $L_{o,2}(t)$, since the identity for $L_{e,2}(t)$ can be verified in the same way.

By Lemma 3.4, we obtain

$$L_{o,2}(t) = \sum_{m=1}^{4}(T_m(t+1) - \Delta^{(2,2)}T_m(t)) - \sum_{m=5}^{8}(T_m(t+1) - \Delta^{(2,2)}T_m(t))$$
$$+ (T_9(t+1) - \Delta^{(2,2)}T_9(t)) - (T_{10}(t+1) - \Delta^{(2,2)}T_{10}(t)).$$

With the aid of Lemma 3.5, we find that

$$\begin{aligned}
L_{o,2}(t) =&(T_{41}(t) - T_{61}(t)) + (T_{31}(t) - T_{71}(t)) + (T_{11}(t) - T_{81}(t))\\
&+ (T_{21}(t) - T_{51}(t)) + (T_{32} - T_{72}(t)) + (T_{42}(t) - T_{62}(t))\\
&+ (T_{12}(t) - T_{82}(t)) + (T_{22}(t) - T_{52}(t))\\
&+ (T_9(t+1) - \Delta^{(2,2)}T_9(t)) - (T_{10}(t+1) - \Delta^{(2,2)}T_{10}(t)).
\end{aligned}$$

Applying Lemmas 3.6–3.8, we obtain

$$\begin{aligned}
L_{o,2}(t) =&(N_2(t) - N_1(t)) + (N_4(t) - N_3(t)) + (N_6(t) - N_5(t))\\
&+ (M_2(t) - M_1(t)) + (M_4(t) - M_3(t)) + (M_6(t) - M_5(t))\\
&+ (E_1(t) + E_2(t) + E_3(t))\\
&- (E_4(t) + E_5(t) + E_6(t) + E_7(t) + E_8(t) + E_9(t))\\
=&(C_1(t) - C_2(t) + C_3(t) + C_4(t) + C_5(t))\\
&+ (D_1(t) - D_2(t) - D_3(t) - D_4(t) + D_5(t)\\
&+ D_6(t) - D_7(t) + D_8(t) - D_9(t))\\
&+ (E_1(t) + E_2(t) + E_3(t) - E_4(t) - E_5(t)\\
&- E_6(t) - E_7(t) - E_8(t) - E_9(t))\\
=&[(C_3(t) - E_6(t)) + (E_2(t) - E_9(t)) - E_5(t)]\\
&+ [(C_5(t) - D_9(t)) + (D_5(t) - E_8(t)) + C_4(t)]\\
&+ [(D_6(t) - D_4(t)) + (C_1(t) - D_3(t)) - D_2(t)]\\
&+ [(D_8(t) - D_7(t)) + (D_1(t) - C_2(t))]\\
&+ [(E_1(t) - E_4(t)) + (E_3(t) - E_7(t))]
\end{aligned}$$

25

$$=0+0+0+0+\sum_{k=1}^{t+1} s_{(4^{3k-2t-2}, 3^{4t-4k+4})},$$

where we have invoked the following relations:

$$C_3(t) - E_6(t) = 0,$$

$$E_2(t) - E_9(t) = E_5(t),$$

$$D_6(t) - D_4(t) = 0,$$

$$E_1(t) - E_4(t) = s_{(4^{t+1})},$$

$$C_5(t) - D_9(t) = \sum_{k=1}^{t-1} \sum_{a=\max(0,3k-2t)}^{3k-2t} s_{(4^a, 3^{2t-k-a}, 2^2)}$$

$$- \sum_{k=1}^{t-1} \sum_{a=\max(0,5k-4t+1)}^{5k-4t+1} s_{(4^a, 3^{k+1-a}, 2^{4t-4k})},$$

$$C_1(t) - D_3(t) = \sum_{a=0}^{t-3} s_{(4^a, 3^{t-a}, 2^3, 1^{t-a-2})},$$

$$D_5(t) - E_8(t) = -\sum_{k=0}^{t-1} \sum_{a=\max(0,3k-2t)}^{3k-2t} s_{(4^a, 3^{2t-k-a}, 2^2)},$$

$$D_8(t) - D_7(t) = \sum_{k=1}^{t-1} \sum_{a=\max(0,3k-2t-2)}^{3k-2t-2} s_{(4^a, 3^{2t-a-k}, 2^3)},$$

$$D_1(t) - C_2(t) = -\sum_{k=0}^{t-2} \sum_{a=\max(0,3k-2t+1)}^{3k-2t+1} s_{(4^a, 3^{2t-a-k-1}, 2^3)},$$

$$E_3(t) - E_7(t) = \sum_{k=1}^{t} \sum_{a=\max(0,3k-2t-2)}^{3k-2t-2} s_{(4^a, 3^{2t-k-a+2})}.$$

So we have arrived at the desired Schur expansion of $L_{o,2}(t)$. This completes the proof. ∎

Based on the above lemma, we obtain the following identities.

Lemma 3.10 *For $t \geq 0$, we have*

$$L_{o,1}(t) = R_{o,1}(t), \quad L_{e,1}(t) = R_{e,1}(t).$$

Proof. We conduct induction on t. We first consider that relation $L_{o,1}(t) = R_{o,1}(t)$. Clearly, the equality holds for $t = 0$ and $t = 1$. Assume that $L_{o,1}(t-1) = R_{o,1}(t-1)$. Observe that

$$L_{o,1}(t) = L_{o,2}(t-1) + \Delta^{(2,2)} L_{o,1}(t-1),$$
$$R_{o,1}(t) = R_{o,2}(t-1) + \Delta^{(2,2)} R_{o,1}(t-1).$$

It follows from Lemma 3.9 that $L_{o,2}(t-1) = R_{o,2}(t-1)$. Hence by induction we have $L_{o,1}(t) = R_{o,1}(t)$. Similarly, it can be shown that $L_{e,1}(t) = R_{e,1}(t)$. ∎

We have established the recurrence relations for $L(r)$ and $R(r)$. We now give the proof of Theorem 3.1.

Proof of Theorem 3.1. We conduct induction on r. It is easy to verify that the identity holds for $r = 1$. Assume that $L(r) = R(r)$. We wish to prove that $L(r+1) = R(r+1)$. If $r = 2t$, we have

$$L(r+1) = L(2t+1) = \Delta^{(1,1)} L(2t) + L_{o,1}(t),$$
$$R(r+1) = R(2t+1) = \Delta^{(1,1)} R(2t) + R_{o,1}(t).$$

From Lemma 3.10 it follows that $L(r+1) = R(r+1)$. If $r = 2t+1$, then

$$L(r+1) = L(2t+2) = \Delta^{(1,1)} L(2t+1) + L_{e,1}(t),$$
$$R(r+1) = R(2t+2) = \Delta^{(1,1)} R(2t+1) + R_{e,1}(t).$$

By the inductive hypothesis and Lemma 3.10, we arrive at the conclusion that $L(r+1) = R(r+1)$. This completes the proof. ∎

4　The q-log-convexity

In this section, we aim to prove Theorem 1.1 and Theorem 1.3. The proof of Theorem 1.1 is based on the identity (3.6).

Proof of Theorem 1.1. For any $n \geq 1$, we have $\mathrm{ps}_n^1(e_k) = \binom{n}{k}$. So we have

$$W_n(q) = \sum_{k=0}^n \binom{n}{k}^2 q^k = \sum_{k=0}^n (\mathrm{ps}_n^1(e_k))^2 q^k.$$

Thus, for any $r \geq 0$, the coefficient of q^r in $W_{n-1}(q)W_{n+1}(q) - (W_n(q))^2$ is given by

$$\sum_{k=0}^r \mathrm{ps}_{n-1}^1(e_k)^2 \mathrm{ps}_{n+1}^1(e_{r-k})^2 - \mathrm{ps}_n^1(e_k)^2 \mathrm{ps}_n^1(e_{r-k})^2.$$

By (2.5), the above sum can be written as

$$\mathrm{ps}_{n-1}^1 \left(\sum_{k=0}^{r} e_k{}^2 (e_{r-k} + 2e_{r-k-1} + e_{r-k-2})^2 - (e_k + e_{k-1})^2 (e_{r-k} + e_{r-k-1})^2 \right).$$

To evaluate the above sum, we first expand the squares, and then apply the following relations

$$\sum_{k=0}^{r} e_k^2 e_{r-k-2}^2 = \sum_{k=0}^{r} e_{k-1}^2 e_{r-k-1}^2,$$

$$\sum_{k=0}^{r} e_{k-1}^2 e_{r-k}^2 = \sum_{k=0}^{r} e_k^2 e_{r-k-1}^2,$$

$$\sum_{k=0}^{r} e_k^2 e_{r-k} e_{r-k-1} = \sum_{k=0}^{r} e_{r-k}^2 e_k e_{k-1},$$

$$\sum_{k=0}^{r} e_k^2 e_{r-k} e_{r-k-2} = \sum_{k=0}^{r} e_{r-k}^2 e_k e_{k-2},$$

$$\sum_{k=0}^{r} e_{k-1}^2 e_{r-k} e_{r-k-1} = \sum_{k=0}^{r} e_{r-k-1}^2 e_k e_{k-1} = \sum_{k=0}^{r} e_k^2 e_{r-k-1} e_{r-k-2}.$$

After simplification we reach the following expression

$$2\,\mathrm{ps}_{n-1}^1 \left(\sum_{k=0}^{r} e_{k-1}^2 e_{r-k}^2 + e_{k-2} e_k e_{r-k}^2 - 2e_{k-1} e_k e_{r-k-1} e_{r-k} \right).$$

By (3.6), we see that the sum in the above expression is Schur positive. This implies that the polynomials $W_n(q)$ form a q-log-convex sequence, completing the proof. ∎

To prove Theorem 1.3, we introduce the following polynomials. For any $n \geq 1$ and $0 \leq r \leq 2n$, define

$$\begin{aligned}
f_1(x) &= (n+1)^2 (n - x + 1)^2 (n - x)^2, \\
f_2(x) &= (n+1)^2 (n - (r - x) + 1)^2 (n - (r - x))^2, \\
f_3(x) &= n^2 (n - x + 1)^2 (n - (r - x) + 1)^2.
\end{aligned}$$

Set

$$f(x) = f_1(x) + f_2(x) - 2f_3(x).$$

Proof of Theorem 1.3. It suffices to show that the polynomials $W_n(q)$ satisfy the condition in Theorem 1.2. First, it is easy to see that $\alpha(n, r, k)$, as defined by the

recurrence relation (1.3), can be rewritten as

$$\alpha(n,r,k) = \frac{1}{n^2(n-k+1)^2(n-r+k+1)^2} \binom{n}{k}^2 \binom{n}{r-k}^2 f(k), \qquad (4.19)$$

where $n \geq 1$, $0 \leq r \leq 2n$ and $0 \leq k \leq \lfloor \frac{r}{2} \rfloor$. Clearly, for any n and r, if $k \leq r - n - 1$, then $n \leq (r-k) - 1$ and $\alpha(n,r,k) = 0$. To ensure that the condition in Theorem 1.2 is satisfied, we need to show that $\alpha(n,r,k) \leq 0$ for $r - n - 1 < k \leq \lfloor \frac{r}{2} \rfloor$. Let us consider the value of $f(k)$ for a given r. Differentiating $f(x)$ with respect to x, we obtain

$$f'(x) = 2(2x - r)g(x),$$

where

$$g(x) = 2x^2 + 4nx^2 - 4nrx - 2rx + 4nr^2 - 17n^2r + 2n^2r^2 - 12nr$$
$$- 8n^3r + 1 + 8n + 21n^2 + 8n^4 + 22n^3 + 2r^2 - 3r.$$

Differentiating $g(x)$ with respect to x, we get

$$g'(x) = 2(2x - r)(1 + 2n).$$

This shows that $g'(x) \leq 0$ for $x \leq \frac{r}{2}$. Hence $g(x)$ is decreasing on the interval $(-\infty, \frac{r}{2}]$. Since $g(x) \to +\infty$ when $x \to -\infty$, the polynomial $g(x)$ has at most one real root on the interval $(-\infty, \frac{r}{2}]$, say x_0, if it exists. Consequently, $f'(x)$ either has a unique root $\frac{r}{2}$ on the interval $(-\infty, \frac{r}{2}]$, or has two real roots x_0 and $\frac{r}{2}$. In the former case, $f(x)$ is decreasing on the interval $(-\infty, \frac{r}{2}]$. In the latter case, $f(x)$ is decreasing on the interval $(-\infty, x_0]$ and increasing on the interval $[x_0, \frac{r}{2}]$. Combining the above two cases, it suffices to show that $f(r/2) \leq 0$. Observe that

$$f(r/2) = 2(n+1)^2 \left(n - \frac{r}{2} + 1\right)^2 \left(n - \frac{r}{2}\right)^2 - 2n^2 \left(n - \frac{r}{2} + 1\right)^4$$
$$= -\frac{r(2n(2n-r) + (2n-r) + 2n)(2 + 2n - r)^2}{8},$$

which is nonpositive for $0 \leq r \leq 2n$. By (4.19), there exists an integer $k' = k'(n,r)$ such that $\alpha(n,r,k) \geq 0$ for $k \leq k'$ and $\alpha(n,r,k) \leq 0$ for $k > k'$. Therefore, by Theorem 1.1 and Theorem 1.2, we conclude that the linear transformation defined by the triangular array $\{\binom{n}{k}^2\}_{0 \leq k \leq n}$ preserves log-convexity. This completes the proof. ∎

References

[1] F. Ardila, M. Beck, S. Hosten, J. Pfeifle and K. Seashore, Root polytopes and growth series of root lattices, arXiv:Math.CO/0809.5123v1.

[2] M. Benson, Growth series of finite extensions of $\mathbf{Z}^n$ are rational, Invent. Math. 73 (1983), 251–269.

[3] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: An update, Contemp. Math. 178 (1994), 71–89.

[4] L.M. Butler, The q-log-concavity of q-binomial coefficients, J. Combin. Theory Ser. A 54 (1990), 54–63.

[5] L.M. Butler and W.P. Flanigan, A note on log-convexity of q-Catalan numbers, Ann. Combin. 11 (2007), 369–373.

[6] W.Y.C. Chen, L.X.W. Wang and A.L.B. Yang, Schur positivity and the q-log-convexity of the Narayana polynomials, arXiv:math.CO/0806.1561.

[7] W.Y.C. Chen, L.X.W. Wang and A.L.B. Yang, Recurrence relations for strongly q-log-convex polynomials, Canad. Math. Bull., to appear.

[8] T. Došlić and D. Veljan, Calculus proofs of some combinatorial inequalities, Math. Inequal. Appl. 6 (2003), 197–209.

[9] S. Fomin and N. Reading, Root systems and generalized associahedra, Geometric Combinatorics, 63–131, IAS/Park City Math. Ser. 13, Amer. Math. Soc., Providence, RI, 2007.

[10] C. Krattenthaler, On the q-log-concavity of Gaussian binomial coefficients, Monatsh. Math. 107 (1989), 333–339.

[11] P. Leroux, Reduced matrices and q-log-concavity properties of q-Stirling numbers, J. Combin. Theory Ser. A 54 (1990), 64–84.

[12] V. Reiner, Noncrossing partitions for classical reflection groups, Discrete Math. 177 (1992), 195–222.

[13] B. E. Sagan, Inductive proofs of q-log concavity, Discrete Math. 99 (1992), 298–306.

[14] R. Simion, Combinatorial statistics on type-B analogues of noncrossing partitions and restricted permutations, Electron. J. Combin. 7 (2000), #R9.

[15] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, New York/Cambridge, 1999.

[16] J. R. Stembridge, A Maple package for symmetric functions, J. Symbolic Comput. 20 (1995), 755–768.

[17] L.L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, Adv. Appl. Math. 39 (2007), 453–476.

[18] S. Veigneau, ACE, an Algebraic Combinatorics Environment for the computer algebra system MAPLE, User's Reference Manual, Version 3.0, IGM 98–11, Université de Marne-la-Vallée, 1998.