Higher Order Log-Concavity in Euler’s Difference Table

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Abstract. For \(0 \leq k \leq n\), let \(e_k^n\) be the entries in Euler’s difference table and let \(d_k^n = e_k^n/k!\). Dumont and Randrianarivony showed \(e_k^n\) equals the number of permutations on \([n]\) whose fixed points are contained in \(\{1, 2, \ldots, k\}\). Rakotondrajao found a combinatorial interpretation of the number \(d_k^n\) in terms of \(k\)-fixed-points-permutations of \([n]\). We show that for any \(n \geq 1\), the sequence \(\{d_k^n\}_{0 \leq k \leq n}\) is both 2-log-concave and reverse ultra log-concave.

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1 Introduction

Euler’s difference table \((e_k^n)_{0 \leq k \leq n}\) is defined by \(e_n^n = n!\) and

\[ e_n^{k-1} = e_n^k - e_n^{k-1}, \]  

(1.1)

for \(1 \leq k \leq n\). Dumont and Randrianarivony [5] showed that \(e_n^k\) equals the number of permutations on \([n]\) whose fixed points are contained in \(\{1, 2, \ldots, k\}\). Clarke, Han and Zeng [4] gave a combinatorial interpretation of a \(q\)-analogue of Euler’s difference table. This combinatorial interpretation was further extended by Faliharimalala and Zeng [7, 8] to the wreath product \(C_\ell \wr S_n\) of the cyclic group and the symmetric group.

It is easily seen from the recurrence relation (1.1) that \(k!\) divides \(e_n^k\). Thus the number \(d_n^k = e_n^k/k!\) is always an integer. Rakotondrajao [13] has shown that \(d_n^k\) equals the number
of \( k \)-fixed-points-permutations of \([n]\), where a permutation \( \pi \in S_n \) is called a \( k \)-fixed-points-permutation if there are no fixed points in the last \( n - k \) positions and the first \( k \) elements are in different cycles. Based on this combinatorial explanation, Rakotondrajao [14] has found bijective proofs for the following recurrence relations

\[
d_k^n = (n - 1)d_{k-1}^{n-1} + (n - k - 1)d_{k-2}^{n-2}, \tag{1.2}
\]

and

\[
d_k^n = nd_{k-1}^{n-1} - d_{k-2}^{n-2}, \tag{1.3}
\]

where \( 0 \leq k \leq n - 1 \) and \( d_n^n = 1 \).

Recently, Eriksen, Freij and Wästlund [6] generalized the above recurrence relations to \( \lambda \)-colored permutations. By equating the right hand sides of (1.2) and (1.3), and changing the index from \( n - 1 \) to \( n \), we obtain the following relation for \( 1 \leq k \leq n - 1 \),

\[
d_k^n = d_{k-1}^{n-1} + (n - k)d_{k-1}^{n-1}. \tag{1.4}
\]

Applying the above relations (1.2) (1.3) and (1.4), we shall prove that for any \( n \geq 1 \), the sequence \( \{d_k^n\}_{0 \leq k \leq n} \) is 2-log-concave and reverse ultra log-concave.

## 2 The 2-log-concavity

In this section, we show that the sequence \( \{d_k^n\}_{0 \leq k \leq n} \) is 2-log-concave for any \( n \geq 1 \). Recall that a sequence \( \{a_k\}_{k \geq 0} \) of real numbers is said to be log-concave if \( a_k^2 \geq a_{k+1}a_{k-1} \) for all \( k \geq 1 \); see Stanley [15] and Brenti [2]. From the recurrence relation (1.4), it is easy to prove by induction that the sequence \( \{d_k^n\}_{0 \leq k \leq n} \) is log-concave.

**Theorem 2.1** The sequence \( \{d_k^n\}_{0 \leq k \leq n} \) is log-concave.

The notion of high order log-concavity was introduced by Moll [12]; see also, [9]. Given a sequence \( \{a_k\}_{k \geq 0} \), define the operator \( \mathcal{L} \) as \( \mathcal{L}\{a_k\} = \{b_k\} \), where

\[
b_k = a_k^2 - a_{k-1}a_{k+1}.
\]

The log-concavity of \( \{a_k\} \) becomes non-negativity of \( \mathcal{L}\{a_k\} \). If the sequence \( \mathcal{L}\{a_k\} \) is not only nonnegative but also log-concave, then we say that \( \{a_k\} \) is 2-log-concave. In general, we say that \( \{a_k\} \) is \( l \)-log-concave if \( \mathcal{L}^l\{a_k\} \) is nonnegative, and that \( \{a_k\} \) is infinite log-concave if \( \mathcal{L}^l\{a_k\} \) is nonnegative for any \( l \geq 1 \). From numerical evidence, we conjecture that the sequence \( \{d_k^n\}_{0 \leq k \leq n} \) is infinitely log-concave.

Recently, Brändén [1] has proved that if a polynomial has only real and nonpositive zeros, then its coefficients form an infinite log-concave sequence. However, this is not the case for the polynomials \( \sum d_k^n x^k \), since not all polynomials \( \sum d_k^n x^k \) have only real zeros, for example, when \( n = 2 \), the polynomial \( x^2 + x + 1 \) does not have any real root. Nevertheless, we shall show that the sequence \( \{d_k^n\} \) is 2-log-concave in support of the general conjecture.
Theorem 2.2 The sequence \( \{d_n^k\}_{0 \leq k \leq n} \) is 2-log-concave. In other words, for \( n \geq 4 \) and \( 2 \leq k \leq n-2 \), we have
\[
\left( (d_n^k)^2 - d_n^{k-1}d_n^{k+1} \right)^2 - \left( (d_n^{k-1})^2 - d_n^{-2}d_n^k \right) \left( (d_n^{k+1})^2 - d_n^kd_n^{k+2} \right) \geq 0.
\] (2.1)

The idea to prove Theorem 2.2 can be outlined as follows.

1. As the first step, we reformulate the left hand side of inequality (2.1) as a cubic function \( f \) in \( d_n^{k+1}d_n^n \) by applying the recurrence relations (1.2), (1.3), (1.4) and a recurrence relation presented in Lemma 2.3.

2. We show that Theorem 2.2 follows from the assertion that \( f \geq 0 \) in the interval \( I = \left[ n + \frac{n-k}{n}, n + \frac{n-k}{n} + \frac{n-k}{n^2} \right] \), since it can be verified that for \( n \geq 4 \) and \( 2 \leq k \leq n-2 \),
\[
n + \frac{n-k}{n} \leq \frac{d_n^{k+1}}{d_n^k} \leq n + \frac{n-k}{n} + \frac{n-k}{n^2}.
\] (2.2)

3. In order to prove \( f > 0 \), we consider \( f \) as a continuous function in \( x \). It can be shown that \( f'(x) < 0 \) for \( x \in I \) and
\[
f \left( n + \frac{n-k}{n} + \frac{n-k}{n^2} \right) \geq 0.
\]
Hence we deduce that \( f \geq 0 \) in the interval \( I \). This proves Theorem 2.2.

Lemma 2.3 For \( 1 \leq k \leq n \), we have
\[
d_n^{k-1} = (k+1)(n-k)d_n^{k+1} - (n-2k+1)d_n^k.
\] (2.3)

Proof. First, from (1.1) it is easy to establish the following recurrence relation for \( 1 \leq k \leq n \),
\[
d_n^{k-1} = kd_n^k - d_n^{k-1}.
\] (2.4)
For \( 1 \leq k \leq n \), we find
\[
d_n^k = (k+1)d_n^{k+1} - d_n^{k-1}
\]
\[
= (k+1)d_n^{k+1} - \left( \frac{1}{n-k}d_n^k - \frac{1}{n-k}d_n^{k-1} \right) \quad \text{(by (1.4))}
\]
\[
= (k+1)d_n^{k+1} - \frac{1}{n-k}d_n^k + \frac{1}{n-k} (kd_n^k - d_n^{k-1}) \quad \text{(by (2.4))}
\]
\[
= (k+1)d_n^{k+1} + \frac{k-1}{n-k}d_n^k - \frac{1}{n-k}d_n^{k-1}.
\]
Consequently,
\[ d_{n-1}^k = (k + 1)(n - k)d_{n+1}^k - (n - 2k + 1)d_n^k, \]
as desired.

To prove (2.2), we need a lower bound on \( d_{n+1}^k / d_n^k \).

**Lemma 2.4** For \( n \geq 1 \) and \( 1 \leq k \leq n - 1 \), we have
\[ \frac{d_{n+1}^k}{d_n^k} \geq n + \frac{n - k}{n}. \]  

**Proof.** First we consider the case \( 1 \leq k \leq n - 2 \). We proceed by induction on \( n \). It is clear that (2.5) holds for \( n = 1 \) and \( n = 2 \). We now assume that (2.5) holds for \( n - 2 \), that is,
\[ \frac{d_{n-1}^k}{d_{n-2}^k} \geq n - 2 + \frac{n - k - 2}{n - 2}. \]  

By recurrence (1.2), we have
\[ \frac{d_{n+1}^k}{d_n^k} = \frac{n d_n^k + (n - k)d_{n-1}^k}{d_n^k} = n + (n - k) \frac{d_{n-1}^k}{d_n^k} = n + (n - k) \frac{d_{n-1}^k}{(n - 1)d_{n-1}^k + (n - k - 1)d_{n-2}^k}. \]  

Thus (2.5) can be recast as
\[ (n - 1) + (n - k - 1) \frac{d_{n-2}^k}{d_{n-1}^k} \leq n. \]  

So it suffices to check that
\[ \frac{d_{n-1}^k}{d_{n-2}^k} \geq n - k - 1. \]  

Since \( n \geq 3 \), by the induction hypothesis, we have
\[ \frac{d_{n-1}^k}{d_{n-2}^k} \geq n - 2 + \frac{n - 2 - k}{n - 2} \]
\[ = n - 1 - \frac{k}{n - 2} \]
\[ \geq n - k - 1. \]
as required.

We now turn to the case $k = n - 1$. By (1.3), we get

$$d_{n-1}^n = (n - 1)d_{n-1}^{n-1}.$$ 

By definition, we have $d_{n-1}^{n-1} = 1$. Moreover, it is easy to see that $d_{n}^{n-1} = n - 1$. Hence, by (1.4), we have

$$\frac{d_{n}^{n+1}}{d_{n}^{n}} = n + \frac{1}{n - 1} > n + \frac{1}{n}.$$ 

This completes this proof.

Next we give an upper bound on $d_{n+1}^{k}/d_{n}^{k}$.

**Lemma 2.5** For $n \geq 4$ and $2 \leq k \leq n - 2$, we have

$$\frac{d_{k}^{n+1}}{d_{k}^{n}} \leq n + \frac{n - k}{n} + \frac{n - k}{n^2}. \quad (2.7)$$

**Proof.** From (1.2) it follows that

$$\frac{d_{k}^{n+1}}{d_{k}^{n}} = n + (n - k)\frac{d_{k}^{n-1}}{d_{k}^{n}}$$

$$= n + (n - k)\frac{d_{k}^{n-1}}{(n - 1)d_{n-1}^{k} + (n - k - 1)d_{n-2}^{k}}.$$ 

Thus (2.7) can be rewritten as

$$(n - 1) + (n - k - 1)\frac{d_{n-2}^{k}}{d_{n-1}^{k-1}} \geq \frac{n^2}{n + 1},$$

that is,

$$\frac{d_{n-1}^{k-1}}{d_{n-2}^{k}} \leq (n + 1)(n - k - 1). \quad (2.8)$$

By recurrence (1.3) for $2 \leq k \leq n - 2$, we see that

$$\frac{d_{n-1}^{k-1}}{d_{n-2}^{k}} \leq n - 1,$$

which implies (2.8). This completes the proof.

We are now ready to give the proof of Theorem 2.2.

**Proof of Theorem 2.2.** It is easy to check that the theorem holds for $n = 4, 5, 6$. So we may assume that $n \geq 7$. 

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We now consider the function $f \left( \frac{d_{n+1}^k}{d_n^k} \right)$. By the recurrences (1.2), (1.3), (1.4) and (2.3), we can derive the following relations,

\[
\begin{align*}
    d_n^{k-2} &= (n-k+1)(n-k+3)d_n^k - (n-2k+3)d_{n+1}^k, \\
    d_n^{k-1} &= d_{n+1}^k - (n-k+1)d_n^k, \\
    d_n^{k+1} &= \frac{1}{(k+1)(n-k)} \left( d_{n+1}^k - kd_n^k \right), \\
    d_n^{k+2} &= \frac{1}{(k+1)(k+2)(n-k-1)(n-k)} \left( (n-2k-1)d_{n+1}^k + (n+k^2)d_n^k \right).
\end{align*}
\]

It follows that (2.1) can be rewritten as

\[
A \left( C_3(n,k) \left( \frac{d_{n+1}^k}{d_n^k} \right)^3 + C_2(n,k) \left( \frac{d_{n+1}^k}{d_n^k} \right)^2 \left( \frac{d_n^k}{d_{n+1}^k} \right) + C_1(n,k) \left( \frac{d_{n+1}^k}{d_n^k} \right)^2 + C_0(n,k) \left( \frac{d_n^k}{d_{n+1}^k} \right)^3 \right) \geq 0,
\]

where

\[
A = \frac{d_n^k}{(k+1)^2(n-k)^2(k+2)(n-k-1)},
\]

\[
C_3(n,k) = -n^2 - 5n + 6k + 6, \\
C_2(n,k) = n^3 + n^2k + 5n^2 + 3nk - 10k^2 + n - 16k - 6, \\
C_1(n,k) = n^2 - 2n + 14k + 14k^2 + n^3 + 10nk^2 - 10n^2k - n^3k - 3nk, \\
C_0(n,k) = -4n^2 - 12k^2 - 12k^3 + 10nk + 18nk^2 - 9n^2k + n^2k^2 - n^3k.
\]

Since $d_n^k$ are positive, it suffices to show that

\[
C_3(n,k) \left( \frac{d_{n+1}^k}{d_n^k} \right)^3 + C_2(n,k) \left( \frac{d_{n+1}^k}{d_n^k} \right)^2 + C_1(n,k) \left( \frac{d_{n+1}^k}{d_n^k} \right) + C_0(n,k) \geq 0. \tag{2.9}
\]

We now consider the function

\[
f(x) = C_3(n,k)x^3 + C_2(n,k)x^2 + C_1(n,k)x + C_0(n,k),
\]

with

\[
f'(x) = 3C_3(n,k)x^2 + 2C_2(n,k)x + C_1(n,k). \tag{2.10}
\]

We aim to show that $f'(x) < 0$, for $2 \leq k \leq n-1$ and $x \in I$.

It can be shown that $f'(-1) < 0$, $f'(k) > 0$, $f'(n) > 0$ and $C_3(n,k) < 0$. The proofs will be given later. Using the facts $f'(-1) < 0$, $f'(k) > 0$ and $f'(n) > 0$, we deduce that $f'(x)$ has a zero in the interval $[-1,k]$ and a zero in the interval $[k,n]$. This implies that $f'(x)$ has no zeros in the interval $I$ since $f'(x)$ is a quadratic function. Since $f'(n) > 0$
and $C_3(n, k) < 0$, we see that $f'(x) < 0$ in the interval $I$. In other words, $f(x)$ is strictly decreasing in the interval $I$.

It will be also shown that

$$f \left( n + \frac{n-k}{n} + \frac{n-k}{n^2} \right) > 0. \quad (2.11)$$

Combining with the fact that $f(x)$ is strictly decreasing in $I$, we obtain that $f(x) > 0$ in $I$, as desired.

We now finish the proofs of the above claims. First, we show that $f'(-1) < 0$. Clearly, we have

$$f'(-1) = -(k + 1)(n^3 + 12n^2 - 10nk + 19n - 34k - 30).$$

For $n \geq 7$ and $2 \leq k \leq n - 2$, we find

$$n^3 + 12n^2 - 10nk + 19n - 34k - 30 \geq n^3 + 12n(k + 2) + 19n - 30 - 10nk - 34k \geq (n^3 - 30) + 2nk + (43n - 34k) > 0.$$ 

This implies that $f'(-1) < 0$.

Next we shall verify that $f'(k) > 0$ and $f'(n) > 0$. For $x = k$, we have

$$f'(k) = (k + 1)(n - k)(n^2 + n + 2k - 2).$$

Since $n > k$ and $k > 1$, we see that $f'(k) > 0$.

For $x = n$, we have

$$f'(n) = -(n - k)(n^3 + 4n^2 - 10nk + 14k - 21n + 14). \quad (2.12)$$

To prove $f'(n) < 0$, it suffices to show that for $2 \leq k \leq n - 2$,

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 > 0.$$ 

We consider two cases. For $2 \leq k \leq n - 3$, we have

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 = n \left( (n - 3)^2 + 10(n - k - 3) \right) + 14k + 14 > 0,$$

On the other hand, for $k = n - 2$, we have

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 = n(n - 3)^2 + 4n - 14 > 0.$$ 

Thus $f'(n) < 0$ holds for $2 \leq k \leq n - 2$. 

To prove \( f'(x) > 0 \), we need to verify that \( C_3(n, k) < 0 \). Since \( n \geq k + 2 \), it is easily seen that

\[
C_3(n, k) = -(n^2 + 5n - 6k - 6) \\
\leq - ((k + 2)^2 + 5(k + 2) - 6k - 6) \\
\leq -(k^2 + 3k + 8) < 0.
\]

Till now, we have proved the facts \( f'(-1) < 0 \), \( f'(k) > 0 \), \( f'(n) > 0 \) and \( C_3(n, k) < 0 \). Finally, we finish the proof of (2.11). It is easily checked that

\[
f \left( n + \frac{n - k}{n} + \frac{n - k}{n^2} \right) = \frac{h(k)(n-k)^2}{n^6},
\]

where

\[
h(k) = (-10n^4 - 26n^3 - 28n^2 - 18n - 6)k^2 + (-n^6 + 20n^5 + 27n^4 + 19n^3 - 7n - 6)k \\
+ (n^7 - 10n^6 - 4n^5 - 4n^4 + 9n^3 + 7n^2 + 6n).
\]

We continue to show that \( h(k) \geq 0 \) for \( n \geq 7 \) and \( 2 \leq k \leq n - 2 \). We now consider \( h(x) \) as a continuous function in \( x \), that is,

\[
h(x) = (-10n^4 - 26n^3 - 28n^2 - 18n - 6)x^2 + (-n^6 + 20n^5 + 27n^4 + 19n^3 - 7n - 6)x \\
+ (n^7 - 10n^6 - 4n^5 - 4n^4 + 9n^3 + 7n^2 + 6n).
\]

Since the leading coefficient of \( h(x) \) is negative, we only need to prove that \( h(2) > 0 \) and \( h(n - 1) > 0 \). For \( n \geq 7 \), we have

\[
h(n - 1) = n(n^5 - 3n^4 + 2n^3 + 2n^2 + 2n + 1) \\
= n(n^3(n - 1)(n - 2) + 2n^2 + 2n + 1) > 0,
\]

and

\[
h(2) = n^7 - 12n^6 + 36n^5 + 10n^4 - 57n^3 - 105n^2 - 80n - 36 \\
= n^5(n - 5)(n - 7) + n^4(n - 6) + 16n^3(n - 7) + 55n^2(n - 7) \\
+ 80n(n - 1) + 200n^2 - 36 > 0.
\]

Thus we reach the conclusion that \( h(k) > 0 \) for \( n \geq 7 \) and \( 2 \leq k \leq n - 2 \). This completes the proof.
3 The reverse ultra log-concavity

In this section, we show that for any $n \geq 1$, the sequence $\{d^k_n\}_{0 \leq k \leq n}$ is reverse ultra log-concave. Recall that a sequence $\{a_k\}_{0 \leq k \leq n}$ is called ultra log-concave if $\{a_k/\binom{n}{k}\}$ is log-concave. This condition can be restated as

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \quad (3.1)$$

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. If a sequence $\{a_k\}_{0 \leq k \leq n}$ is ultra log-concave, then the sequence $\{k!a_k\}_{0 \leq k \leq n}$ is log-concave, see Liggett [11].

In comparison with ultra log-concavity, a sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (3.1), that is,

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0. \quad (3.2)$$

Chen and Gu [3] have shown the Boros-Moll polynomials are reverse ultra log-concave. The following theorem states that the sequence $\{d^k_n\}_{0 \leq k \leq n}$ is reverse ultra log-concave.

**Theorem 3.1** For $n \geq 1$ and $1 \leq k \leq n - 1$, we have

$$\frac{d^{k-1}_n}{\binom{n}{k-1}} \cdot \frac{d^{k+1}_n}{\binom{n}{k+1}} \geq \left( \frac{d^k_n}{\binom{n}{k}} \right)^2,$$

or equivalently,

$$(n-k+1)(k+1)d^{k-1}_n d^{k+1}_n \geq k(n-k) (d^k_n)^2. \quad (3.3)$$

**Proof.** According to the recurrence relations (1.4) and (2.3), we find that (3.3) can be reformulated as

$$(n-k+1) \left( \frac{d^{k+1}_{n+1}}{d^k_n} \right)^2 - (n-k+1)(n+1) \left( \frac{d^{k+1}_{n+1}}{d^k_n} \right) + k(2n-2k+1) \geq 0. \quad (3.4)$$

The discriminant of the quadratic polynomial in $d^{k+1}_{n+1}/d^k_n$ on the left hand side of (3.4) equals

$$\Delta = ((n-k+1)(n+1))^2 - 4k(n-k+1)(2n-2k+1).$$

We aim to show that $\Delta > 0$ for $1 \leq k \leq n - 1$. We can rewrite $\Delta$ as follows

$$\Delta = (n-k+1)[(n-k-1)((n+1)^2 - 8k) + 2((n+1)^2 - 6k)].$$

Since $(n+1)^2 - 6k \geq (n+1)^2 - 8k = (n-3)^2 \geq 0$, it follows that $\Delta > 0$ for $1 \leq k \leq n - 1$, as desired.
Therefore, the above quadratic function has two distinct real zeros. If we can prove that for \(1 \leq k \leq n - 1\), \(\frac{d_{n+1}^k}{d_n^k}\) is larger than the large zero, then (3.4) holds since \(n - k + 1 > 0\). Thus we still have to show that

\[
\frac{d_{n+1}^k}{d_n^k} > \frac{(n-k+1)(n+1) + \sqrt{\Delta}}{2(n-k+1)} = \frac{n+1}{2} + \frac{\sqrt{\Delta}}{2(n-k+1)}
\]  \tag{3.5}

In view of (2.5), we see that (3.5) can be deduced from the following inequality

\[
n + \frac{n-k}{n} \geq \frac{n+1}{2} + \frac{\sqrt{\Delta}}{2(n-k+1)},
\]

which is equivalent to

\[
(n-k+1)(n^2 + n - 2k) \geq n\sqrt{\Delta}.
\]

Evidently,

\[
\left( (n-k+1)(n^2 + n - 2k) \right)^2 - n^2 \Delta
\]

\[
= 4k(n-k+1)(n-k)(n^2 - n + k - 1),
\]

which is nonnegative for \(1 \leq k \leq n - 1\). This completes the proof.

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