Ramanujan J.
25 (2011), no. 1, 37-47

The Algorithm Z and Ramanujan's ${}_1\psi_1$ Summation

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Abstract. We use the Algortihm Z on partitions due to Zeilberger, in a variant form, to give a combinatorial proof of Ramanujan's ${}_1\psi_1$ summation formula.

Keywords: Algorithm Z, Ramanujan's ${}_1\psi_1$ summation, bijection, partition.

AMS Classification: 05A10

1 Introduction

Ramanujan's sum for ${}_1\psi_1$ has been extensively studied in the theory of q-series, which is usually stated in the following form:

$$ {}_1\psi_1(a;b;q,z)=\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty},\ |b/a|<|z|<1,\ |q|<1, \quad (1.1) $$

where the q-shifted factorial is defined by

$$ (a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n),\ (a;q)_n=(a;q)_\infty/(aq^n;q)_\infty. $$

The main result of this paper is a combinatorial proof of the above formula by using a variation of the Algorithm Z named after Zeilberger [7]. Since Hahn and Jackson published the first proofs in 1949 and 1950, many other proofs have been found, see, for example, Andrews [4], Andrews and Askey [2], Berndt [5], Fine [12], Ismail [15], Mimachi [17]. However, the combinatorial proofs have appeared only recently. Using the Frobenius notation for overpartitions, Corteel and Lovejoy [10] have found a bijective

proof of the constant term identity for the the following formulation of Ramanujan's ${}_1\psi_1$ summation:

$$\frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-a^{-1};q)_n(zqa)^n}{(-bq;q)_n}=\frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(bz^{-1};q)_\infty(azq;q)_\infty}. \tag{1.2}$$

Corteel [9] went on to find a bijection, by using particle seas, to show that the coefficients of z^N ($N\neq 0$) on both sides of (1.2) are equal as well, which leads to the completion of the combinatorial proof of (1.2). In the meantime, Yee [19] also found a combinatorial proof of (1.2) in the language of F-partitions defined as pairs of overpartitions with different sizes written in the two line notation.

In this paper, we shall present a new combinatorial proof of Ramanujan's ${}_1\psi_1$ sum based on a variation of the Algorithm Z. Conceptually, our bijection is rather simple despite that there are several steps which do not seem to be avoidable to accomplish the task of transformations of partitions. As will be seen, the Algorithm Z serves as the main ingredient of our combinatorial construction for Ramanujan's formula. To be precise, our bijection is devised for following restatement of Ramanujan's formula

$$\frac{(-q/a;q)_\infty(-b/az;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-a;q)_n}{(b;q)_n}z^n=\frac{(-b/a;q)_\infty(-az;q)_\infty(-q/az;q)_\infty}{(b;q)_\infty(z;q)_\infty}. \tag{1.3}$$

The Algorithm Z, as called by Andrews and Bressoud [3], was found by Zeilberger [7] (Proposition 3.1) as a combinatorial interpretation of the Gauss coefficient $\begin{bmatrix}n\\k\end{bmatrix}$ as defined by the following relation

$$\frac{1}{(q;q)_{i+j}}\begin{bmatrix}i+j\\i\end{bmatrix}=\frac{1}{(q;q)_i(q;q)_j}.$$

Using this algorithm, Andrews and Bressoud have found combinatorial proofs of some classical q-identities. The Algorithm Z has also been employed by Bessenrodt [6] to give a bijective proof of a theorem of Alladi and Gordon, and to give a combinatorial interpretation of the Lebesgue identity by Fu [13].

2 The Algorithm Z

In this section, we shall give an overview of the Algorithm Z and use it to give a combinatorial interpretation of q-binomial theorem, which is an important step of our combinatorial proof of Ramanujan's summation (1.3):

$$\sum_{n\geq 0}\frac{P_n(b,-a)}{(q;q)_n}z^n=\sum_{n\geq 0}\frac{(-a/b;q)_n}{(q;q)_n}(bz)^n=\frac{(-az;q)_\infty}{(bz;q)_\infty}. \tag{2.1}$$

where the polynomials

$$P_n(b,-a)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}a^kq^{\binom{k}{2}}b^{n-k}=\begin{cases}(b+a)(b+aq)\cdots(b+aq^{n-1}), & \text{if } n\geq 1;\\ 1, & \text{if } n=0,\end{cases}$$

are the Cauchy polynomials as called in [8].

A partition λ of a nonnegative integer with r parts is denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$. The number of parts, called the length of λ, is denoted by $l(\lambda)$, and the sum of parts, called the weight of λ, is denoted by $|\lambda|$. The conjugate of λ is denoted by $\lambda' = (\lambda'_1, \ldots)$, where λ'_i is the number of positive parts of λ that are greater than or equal to i. The following bijection is call the Algorithm Z.

Theorem 2.1 *There is a bijection between the set of pairs of partitions (α, β) where α has $s-r$ parts and β has r parts, and the set of pairs of partitions (μ, ν), where μ has s parts and ν has r parts with each part not exceeding $s-r$. We call μ the insertion partition and call ν the record partition.*

Proof. Given a partition α with $s-r$ parts, denoted by $(\alpha_1, \ldots, \alpha_{s-r})$, and a partition β with r parts, denoted by $(\beta_1, \ldots, \beta_r)$, we may insert β into α to create a pair of partitions μ and ν. The insertion algorithm can be described as the following recursive procedure.

- If $\beta_1 \leq \alpha_{s-r}$, we insert β_1 into α so that we get a new partition $(\alpha_1, \alpha_2, \ldots, \alpha_{s-r+1})$, where $\alpha_{s-r+1} = \beta_1$. Moreover, we use a zero part as a record of the insertion position.
- If $\beta_1 > \alpha_{s-r}$, we recursively insert $\beta_1 - 1$ into the partition $(\alpha_1, \alpha_2, \ldots, \alpha_{s-r-1})$. Suppose that the recursive procedure ends up with $\beta_1 - \nu_1$ being inserted, we use a part ν_1 to record the position of $\beta_1 - \nu_1$. Obviously, we have $0 \leq \nu_1 \leq s-r$.

Conversely, given a partition $(\alpha_1, \ldots, \alpha_{s-r+1})$ and a number ν_1 with $0 \leq \nu_1 \leq s-r$, we may extract the part β_1 from the given partition. It is easy to see that above procedure is reversible.

After the part β_1 has been inserted to α, we may iterate the above procedure to insert remaining parts of β. Eventually, we obtain a pair of partitions (μ, ν). This completes the proof. ∎

As an example, taking $\alpha = (5, 3, 2, 1)$, $\beta = (4, 3, 0)$ with $s = 7, r = 3$, we have

$$\mu = (5, 3, 2, 2, 2, 1, 0), \quad \nu = (2, 1, 0).$$

Below is the illustration of the insertion procedure

5	3		2		1	
5	3	2	2	2	1	0
		2		1		0
		4		3		0

Corollary 2.2 *There is a bijection ϕ between the set of pairs of partitions (α, β) and the set of pairs of partitions (μ, ν) satisfying the following conditions*

- *α has i distinct parts, β has j parts;*
- *μ has $i+j$ parts, ν has i distinct parts with each part $\leq i+j-1$;*
- $|\alpha|+|\beta|=|\mu|+|\nu|$.

Proof. Given a pair of partitions (α,β), where α has i distinct parts and β has j parts, we denote by $\overline{\alpha}$ the partition $(\alpha_1-i+1,\alpha_2-i+2,\ldots,\alpha_i-0)$. Applying the Algorithm Z to $(\beta,\,\bar{\alpha})$ yields the desired partition μ into exactly $i+j$ parts and a partition $\bar{\nu}$ into exactly i parts with each part $\leq j$. Set $\nu=(\bar{\nu}_1+i-1,\ldots,\bar{\nu}_i+0)$. It is clear that $|\mu|+|\nu|=|\alpha|+|\beta|$. Hence the pair of partitions (μ,ν) satisfy the conditions in the corollary. Since each step is reversible, we have established a bijection. This completes the proof. ∎

It is clear that Corollary 2.2 leads to a combinatorial proof of the q-binomial theorem. The first partition-theoretic proof of (2.1) is due to Andrews [1]. There are other proofs of this classical identity, for example, by overpartitions [11] and by MacMahon diagrams [16, 18].

3 A Variation of the Algorithm Z

In this section, we give a variation of the Algorithm Z. This algorithm plays a key role in our combinatorial proof of Ramanujan's summation formula.

Theorem 3.1 *Let s,t,k,m be nonnegative integers. There is a bijection φ between the set of pairs of partitions (α,β) and the set of pairs of partitions (μ,ν) satisfying the conditions*

- *α has s distinct parts with each part $\geq m$ and β has t parts with each part $\geq k+s+t-1$;*
- *If $s,t>0$, then μ has $s+t$ distinct nonnegative parts with $\mu_s-\mu_{s+1}\geq m+1$ and ν has t distinct parts with $k\leq\nu_i\leq k+s+t-1$ for each $1\leq i\leq t$;*

  If $s>0$ and $t=0$, then $\mu=\alpha$ and ν is an empty partition;

  If $s=0$, $t>0$, then $\mu=(\beta_1-k,\beta_2-k-1,\ldots,\beta_t-k-t+1)$ and $\nu=(k+t-1,k+t-2,\ldots,k)$;

  If $s=t=0$, then both μ and ν are empty partitions.
- $|\alpha|+|\beta|=|\mu|+|\nu|$.

Proof. Given two partitions $\alpha=(\alpha_1,\ldots,\alpha_s)$ and $\beta=(\beta_1,\ldots,\beta_t)$ satisfying above conditions. We shall only consider the case when $s,t>0$ because the other three cases are trivial.

Set $\bar{\alpha}_i=\alpha_i-m+t$ and $\bar{\beta}_j=\beta_j-k-s-j+1$. It is easy to chcek that $\bar{\alpha}=(\bar{\alpha}_1,\bar{\alpha}_2,\ldots,\bar{\alpha}_s)$ and $\bar{\beta}=(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_t)$ form two partitions with $\bar{\alpha}_s\geq t$ and $\bar{\beta}_t\geq 0$. So we may insert $\bar{\beta}$ into $\bar{\alpha}$ to create a pair of partitions (μ,ν) via the following procedure.

- If $\bar{\beta}_1 \geq \bar{\alpha}_1$, we insert $\bar{\beta}_1$ into $\bar{\alpha}$ to form a new partition $\delta = (\delta_1, \delta_2, \ldots, \delta_{s+1}) = (\bar{\beta}_1, \bar{\alpha}_1 - 1, \bar{\alpha}_2 - 1, \ldots, \bar{\alpha}_s - 1)$. Moreover, we set $\nu_1 = k + s + t - 1$ to record the insertion position.

- Otherwise, we assume that j_1 is the largest integer such that $\bar{\alpha}_{j_1} > \bar{\beta}_1$. Then we insert $\bar{\beta}_1$ into $\bar{\alpha}$ to form a new partition $\delta = (\delta_1, \delta_2, \ldots, \delta_{s+1}) = (\bar{\alpha}_1, \ldots, \bar{\alpha}_{j_1}, \bar{\beta}_1, \bar{\alpha}_{j_1+1} - 1, \ldots, \bar{\alpha}_s - 1)$. In this case, we use $\nu_1 = k + s + t - j_1 - 1$ to record the insertion position of $\bar{\beta}_1$. Obviously, $k + t - 1 \leq \nu_1 \leq k + s + t - 2$.

Conversely, given a partition $\delta = (\delta_1, \delta_2, \ldots, \delta_{s+1})$ and a number ν_1 with $k + t - 1 \leq \nu_1 \leq k+s+t-1$, we may extract the part $\bar{\beta}_1$ from δ. It is clear that the above procedure is reversible.

Similarly, we can insert $\bar{\beta}_2$ into the partition $\delta = (\delta_1, \delta_2, \ldots, \delta_{s+1})$. Applying the insertion algorithm repeatedly to $\bar{\beta}_2, \ldots, \bar{\beta}_t$, we come to a partition $\bar{\mu}$ with $s + t$ parts and the desired partition ν, where $\nu = (\nu_1, \ldots, \nu_t)$ with $\nu_i = k + s + t - j_i - 1$ for each $1 \leq i \leq t$. Furthermore, one sees that $k + t - i \leq \nu_i \leq k + s + t - i$. On the other hand, we get the desired partition μ by setting $\mu = \{\bar{\mu}_1 + m, \ldots, \bar{\mu}_s + m, \bar{\mu}_{s+1}, \ldots \bar{\mu}_{s+t}\}$. This completes the proof. ∎

In the above correspondence, the partition μ is also called the insertion partition and ν is called the record partition. As an example, let $k = 3, m = 2, s = 4, t = 3$ and $\alpha = (8, 7, 5, 3)$, $\beta = (12, 11, 9)$. Then we have $\bar{\alpha} = (9, 8, 6, 4)$ and $\bar{\beta} = (5, 3, 0)$, and

$$\bar{\mu} = (9, 8, 6, 5, 3, 2, 0), \quad \nu = (6, 5, 3), \quad \mu = (11, 10, 8, 7, 3, 2, 0).$$

The above correspondence is illustrated as follows

9	8	6			4	
9	8	6	5	3	2	0
			6	5		3
			5	3		0

It is worth mentioning that the conditions $\alpha_s > m$ and $\beta_t > k + s + t - 1$ can be recast in terms of the single statement that $\mu_{s+t} > 0$. This observation will be useful in the proof of Theorem 4.1.

We now turn our attention to the minor difference between the Algorithm Z and the above variation. Given two partitions α and β, we may apply the Algorithm Z to a pair of partitions $(\bar{\alpha}, \bar{\beta})$, where $\bar{\alpha} = (\alpha_1 - s - m + 1, \alpha_2 - s - m + 2, \ldots, \alpha_s - m)$ and $\bar{\beta} = (\beta_1 - s - t - k + 1, \ldots, \beta_t - s - t - k + 1)$. It can be seen that the record partition of (α, β) and the record partition of $(\bar{\alpha}, \bar{\beta})$ differ only by a staircase partition $(k + t - 1, k + t - 2, \ldots, k)$. For the above example, one has $\bar{\alpha} = (3, 3, 2, 1)$, $\bar{\beta} = (3, 2, 0)$. Inserting $\bar{\beta}$ into $\bar{\alpha}$ via the Algorithm Z gives $\bar{\mu} = (3, 3, 2, 2, 1, 1, 0)$, $\bar{\nu} = (1, 1, 0)$ as illustrated below

3	3	2			1	
3	3	2	2	1	1	0
			1	1		0
			3	2		0

It is not hard to see that our combinatorial proof of Ramanujan's formula can be restated in terms of the original Algorithm Z. Nevertheless, the variation seems to be more convenient for the sake of presentation.

Corollary 3.2 *There is a bijection between the set of pairs of partitions* (α, β) *and the set of triples of partitions* $(n; \mu, \nu, \gamma)$ *satisfying the conditions*

- α *has distinct nonnegative parts and* β *has nonnegative parts;*
- μ *has* n *distinct nonnegative parts,* ν *has either distinct nonnegative parts with each part* $\leq n-1$ *(corresponding to* $\beta_1 \geq l(\alpha)$*) or is an empty partition (corresponding to* $\beta_1 < l(\alpha)$*), and* γ *has nonnegative parts with each part* $\leq n-1$*;*
- $|\alpha| + |\beta| = |\mu| + |\nu| + |\gamma|$.

Proof. Assume that n is the largest number satisfying $\beta_{n-l(\alpha)} \geq n-1$. If such an n exists, then set $\gamma = (\beta_{n-l(\alpha)+1}, \ldots, \beta_{l(\beta)})$, which is a partition with each part $\leq n-1$. Denote by $\bar{\beta}$ the partition $(\beta_1, \ldots, \beta_{n-l(\alpha)})$and apply the bijection φ in Theorem 3.1 to $(\alpha, \bar{\beta})$ for $m = 0$ and $k = 0$, we get a partition μ having n distinct nonnegative parts and a partition ν having distinct parts with $0 \leq \nu_i \leq n-1$. If there does not exist such an n, namely $\beta_1 \leq l(\alpha) - 1$, then we set $n = l(\alpha)$, $\mu = \alpha$, $\gamma = \beta$ and set ν to be the empty partition. This completes the proof. ∎

The above corollary can be regarded as a combinatorial interpretation of the following identity [14, Exercise 1.6 (ii)]:

$$\frac{(-a;q)_\infty}{(b;q)_\infty} = \sum_{n=0}^{\infty} \frac{P_n(a,-b)q^{\binom{n}{2}}}{(q;q)_n(b;q)_n}. \tag{3.1}$$

4 The Combinatorial Proof

In this section, we aim to give a combinatorial proof of Ramanujan's ${}_1\psi_1$ summation formula (1.3). When $N \geq 0$, the coefficient of z^N on the left-hand side equals the generating function for the quintuples $(n; \alpha, \beta, \gamma, \lambda, \mu)$ subject to the following conditions:

- α has distinct and positive parts,
- β has positive parts,
- γ has distinct nonnegative parts,

- λ has distinct nonnegative parts with each part $\leq n-1$,
- μ has nonnegative parts with each part $\leq n-1$,

where the exponents of a and b are used to keep track of $l(\lambda)-l(\alpha)-l(\gamma)$ and $l(\gamma)+l(\mu)$ respectively, and N records $n-l(\gamma)$. The coefficient of z^N on the right-hand side is the generating function for the quintuples (A,B,C,D,E) of partitions with the following restrictions:

- Both A and C have distinct nonnegative parts,
- Both B and D have nonnegative parts,
- E has distinct and positive parts,

where the exponents of a and b are used to keep track of $l(C)-l(A)-l(E)$ and $l(A)+l(B)$ respectively, and N records the number $l(C)+l(D)-l(E)$. Let $\mathscr{A}$ and $\mathscr{B}$ be the sets of the quintuples $(n;\alpha,\beta,\gamma,\lambda,\mu)$ and (A,B,C,D,E), as defined above.

Theorem 4.1 *There is a bijection between $\mathscr{A}$ and $\mathscr{B}$.*

Proof. Given a quintuple $(n;\alpha,\beta,\gamma,\lambda,\mu)$ with $N=n-l(\gamma)$. As an example, for $N=4$, $n=9$, let

$$\alpha=(10,9,5,3,2),\quad \beta=(13,11,10,9,9,5,4,4,2),\quad \gamma=(9,6,4,2,1),$$

$$\lambda=(7,6,5,3,1),\quad \mu=(5,4,4,1).$$

We shall use this example to illustrate the operations at every step.

Step 1. Find the largest number p such that $\lambda_p \geq N$. Then $\bar{\lambda}=(\lambda_1,\ldots,\lambda_p)$ is a partition into distinct parts with $N\leq\lambda_i\leq n-1$ and $1\leq i\leq p$. Let $F=(\lambda_{p+1},\ldots,\lambda_{l(\lambda)})$, where p is the largest number such that $\lambda_{p+1}\leq N-1$. Clearly, $l(\bar{\lambda})\leq n-N=l(\gamma)$.

Applying the bijection φ^{-1} in Theorem 3.1 to the pair $(\gamma,\bar{\lambda})$ with $m=0$ and $k=N$, we obtain a partition A with distinct nonnegative parts, and a partition $\bar{B}$ with every part $\geq n-1$. Now we can put the parts of $\bar{B}$ and μ together to form the desired partition B. Note that if such an integer p does not exist, that is, $\lambda_1\leq N-1$, then we have $F=\lambda$, $A=\gamma$ and $B=\mu$.

For the above example, we have

$$\bar{\lambda}=(7,6,5),\quad F=(3,1),\quad A=(6,1),\quad B=(12,11,10,5,4,4,1).$$

Step 2. Find the largest number l such that $\beta_{l-l(\alpha)}\geq N+l$, and set $\bar{\beta}=(\beta_1,\ldots,\beta_{l-l(\alpha)})$. Add enough zero parts if necessary to the conjugate of the partition $(\beta_{l-l(\alpha)+1},\ldots,\beta_{l(\beta)})$ to obtain a partition $\bar{D}$ with $N+l$ parts.

Now we can apply the mapping φ to $(\alpha,\bar{\beta})$ with $m=0$ and $k=N$ to generate a partition E with l distinct positive parts, since $\alpha_{l(\alpha)}>0$ and $\beta_{l-l(\alpha)}\geq N+l$. Meanwhile,

we also obtain a partition $\bar{F}$ into distinct parts with each part $\geq N \leq N+l-1$. So we can put F and $\bar{F}$ together to create a partition $\bar{C}$ into distinct nonnegative parts with each part $\leq N+l-1$. Note that if such an integer l does not exist, that is, $\beta_1 \leq N+l(\alpha)$, we may set $E=\alpha$ and $\bar{C}=F$. In this case, $\bar{D}$ is a partition with $N+l(\alpha)$ parts, which can be obtained from the conjugate of β with some zero parts added if needed.

For the above example, we have

$$\bar{\beta}=(13,11), \quad \bar{D}=(7,7,6,6,4,3,3,3,3,1,0),$$

$$E=(12,11,7,5,4,3,1), \quad \bar{C}=(6,4,3,1).$$

Applying the bijection ϕ^{-1} in Corollary 2.2 to ($\bar{D}$,$\bar{C}$), we obtain the partition C into distinct nonnegative parts and the partition D into nonnegative parts. For the above example, we have

$$C=(10,7,6,2), \quad D=(7,7,6,6,3,3,0).$$

Whence we have constructed a quintuple (A,B,C,D,E) for which

$$|A|+|B|+|C|+|D|+|E|=|\alpha|+|\beta|+|\gamma|+|\lambda|+|\mu|.$$

Notice that the exponents of a and b remain unchanged during the above procedure. Since each step is reversible, we have established a bijection between $\mathscr{A}$ and $\mathscr{B}$. This completes the proof. ∎

When $N=-m<0$, by multiplying both sides of (1.3) by $\frac{(b;q)_{-m}}{(-a;q)_{-m}}$, we get

$$\frac{(-q/a;q)_\infty(-b/az;q)_\infty}{(q;q)_\infty}\sum_{l=0}^{\infty}\frac{(-aq^{-m};q)_l}{(bq^{-m};q)_l}z^{l-m} = \frac{(-b/a;q)_\infty(-az;q)_\infty(-q/az;q)_\infty(-aq^{-m};q)_m}{(bq^{-m};q)_\infty(z;q)_\infty}. \quad (4.1)$$

Substituting b by bq^m and using Euler's identity,

$$(-bq^m/az;q)_\infty=\sum_{n=0}^{\infty}\frac{(b/az)^n q^{mn+\binom{n}{2}}}{(q;q)_n},$$

the coefficients of z^N on both sides can be written as

$$\frac{a^{-m}q^{\binom{m+1}{2}}(-q^{m+1}/a;q)_\infty}{(q;q)_\infty}\sum_{l=0}^{\infty}\frac{P_l(bq^m/a,-b)q^{\binom{l}{2}}}{(b;q)_l(q;q)_l} = \left[z^{-m}\right]\frac{(-bq^m/a;q)_\infty(-az;q)_\infty(-q/az;q)_\infty}{(b;q)_\infty(z;q)_\infty}, \quad (4.2)$$

where $[x^n]F(x)$ denotes the coefficient of x^n in $F(x)$.

Each term on the left-hand side of (4.2) can be interpreted as the generating function for the quintuples $(l;\alpha,\beta,\gamma,\lambda,\mu)$ defined as follows

- α has distinct and positive parts with $\alpha_{l(\alpha)-m+1} = m$,
- β has positive parts,
- λ has distinct nonnegative parts with each part $\leq l-1$,
- γ has distinct l positive parts. Let $s = l - l(\lambda)$. Then $\gamma_s - \gamma_{s+1} \geq m+1$ if $0 < l(\lambda) < l$ and $\gamma_s \geq m$ if $l(\lambda) = 0$.
- μ has nonnegative parts with each part $\leq l-1$,

where the exponent of a records $l(\lambda) - l(\alpha) - l(\gamma)$ and the exponent of b keeps track of $l(\gamma) + l(\mu)$.

Clearly, the right-hand side of (4.2) is the generating function for the quintuples (A, B, C, D, E) defined as follows

- A has distinct parts with each part $\geq m$,
- Both B and D have nonnegative parts,
- C have distinct nonnegative parts,
- E has distinct and positive parts,

where the exponent of a records $l(C) - l(A) - l(E)$, the exponent of b keeps track of $l(A) + l(B)$ and $l(C) + l(D) - l(E) = -m$.

Let $\mathscr{C}$ and $\mathscr{D}$ be the sets of quintuples $(l; \alpha, \beta, \gamma, \lambda, \mu)$ and (A, B, C, D, E) as given before.

Theorem 4.2 *There is a bijection between $\mathscr{C}$ and $\mathscr{D}$.*

Proof. Let (A, B, C, D, E) be a quintuple with $N = l(C) + l(D) - l(E)$. As an example, for $N = -2$, assume that

$$A = (12, 11, 7, 5, 4), \quad B = (15, 13, 12, 11, 11, 7, 6, 6, 4, 2, 1, 1), \quad C = (7, 6, 4, 1, 0),$$

$$D = (9, 8, 5, 5, 4, 1), \quad E = (22, 19, 18, 17, 15, 12, 11, 10, 8, 7, 6, 3, 1).$$

We shall use this example to illustration the operation at each step.

Step 1. If $B_1 \leq l(A) - 1$, we set $l = l(A)$, $\mu = B$, $\gamma = A$ and set λ to be the empty partition. Otherwise, we find the largest number l such that $B_{l-l(A)} \geq l - 1$, and set $\bar{B} = (B_1, \ldots, B_{l-l(A)})$. Now $(B_{l-l(A)+1}, \ldots, B_{l(B)})$ is the desired partition μ. Apply φ in Theorem 3.1 to $(A, \bar{B})$ with $k = 0$. In the case $s = l(A) > 0$, we get a partition γ into l distinct nonnegative parts with $\gamma_{l(A)} - \gamma_{l(A)+1} \geq m+1$ and a partition λ into distinct nonnegative parts with each part $\leq l-1$. When $s = 0$, we get a partition $\gamma = (B_1, B_2 - 1, \ldots, B_l - l + 1)$ and a partition $\lambda = (l-1, l-2, \ldots, 0)$.

For the above example, we have

$$\bar{B} = (15, 13, 12, 11, 11), \mu = (7, 6, 6, 4, 2, 1, 1),$$

$$\gamma = (17, 16, 12, 11, 9, 6, 5, 4, 3, 2), \lambda = (7, 5, 3, 1, 0).$$

Step 2. Applying the bijection ϕ in Corollary 2.2 to (C, D) yields a partition $\bar{C}$ into distinct nonnegative parts with each part $\leq n-1$ and a partition $\bar{D}$ into n nonnegative parts. Evidently, we have $l(C) + l(D) = n$, and hence $l(E) = n + m$.

For the above example, we find $\bar{C} = (5, 4, 3, 1, 0)$, $\bar{D} = (9, 8, 5, 5, 4, 2, 2, 1, 1, 0, 0)$.

Step 3. After removing a staircase partition $(n+m, n+m-1, \ldots, 1)$ from E, we are left with a partition with $l(E)$ nonnegative parts, whose conjugate is denoted by $\bar{E}$. Add $m+1$ to each part of $\bar{C}$ to obtain a partition $\tilde{C}$. Now we may construct a partition $\tilde{D}$ by adding a staircase $(n-1, n-2, \ldots, 0)$ to $\bar{D}$, then adding $m+1$ to the first $l(D)$ parts.

Applying φ^{-1} in Theorem 3.1 to $(\tilde{D}, \tilde{C})$ with k replaced by $m+1$ and m replaced by $m+1$ yields a partition $\bar{\alpha}$ into $l(D)$ distinct parts with each part $\geq m+1$ and a partition $\bar{\beta}$ into $l(C)$ parts with each part $\geq n+m = l(E)$. Combining $\bar{\alpha}$ with a staircase partition $(m, m-1, \ldots, 1)$ gives the partition α, and combining $\bar{\beta}$ with $\bar{E}$ gives the required partition β.

For the above example, we get

$$\bar{E} = (12, 11, 11, 8, 5, 5, 4, 1, 1), \quad \tilde{C} = (8, 7, 6, 4, 3), \quad \tilde{D} = (22, 20, 16, 15, 13, 10, 6, 4, 3, 1, 0),$$

$$\beta = (16, 16, 15, 13, 13, 12, 11, 11, 8, 5, 5, 4, 1, 1), \quad \alpha = (17, 15, 11, 10, 8, 4, 2, 1).$$

Thus we have constructed a quintuple $(l; \alpha, \beta, \gamma, \lambda, \mu)$ such that

$$|\alpha| + |\beta| + |\gamma| + |\lambda| + |\mu| = |A| + |B| + |C| + |D| + |E|.$$

Moreover, the exponents of a and b are preserved at every step. It should be mentioned that each step of the above procedure is reversible. This completes the proof. ∎

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

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