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The 2-log-convexity of the Apéry Numbers

William Y. C. Chen[a,b] and Ernest X. W. Xia[a]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

emails: chen@nankai.edu.cn, xxw@cfc.nankai.edu.cn

Abstract. We present an approach to proving the 2-log-convexity of sequences satisfying three-term recurrence relations. We show that the Apéry numbers, the Cohen-Rhin numbers, the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and 4 and the large Schröder numbers are all 2-log-convex. Numerical evidence suggests that all these sequences are k-log-convex for any $k \geq 1$ possibly except for a constant number of terms at the beginning.

1 Introduction

In his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, Apéry [2] introduced the following numbers A_n and B_n as given by

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \tag{1.1}$$

$$B_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}. \tag{1.2}$$

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The numbers A_n and B_n are often called the Apéry numbers. It has been shown by Apéry [2] that A_n and B_n satisfy the following three-term recurrence relations for $n \geq 2$,

$$A_n = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} A_{n-1} - \frac{(n-1)^3}{n^3} A_{n-2}, \tag{1.3}$$

$$B_n = \frac{11n^2 - 11n + 3}{n^2} B_{n-1} + \frac{(n-1)^2}{n^2} B_{n-2}, \tag{1.4}$$

where $A_0 = 1$, $A_1 = 5$, $B_0 = 1$, $B_1 = 3$; see also [10, 13]. Congruences of the Apéry numbers have been investigated by Ahlgren, Ekhad, Ono, and Zeilberger [1], Beukers [3, 4], Chowla and Clowes [5] and Gessel [9]. Note that the recurrence relations (1.3) and (1.4) can be derived by using Zeilberger's algorithm [14].

Cohen [6] and Rhin obtained the following recurrence relation of the numbers U_n in connection with the rational approximation of $\zeta(4)$, see also [11],

$$U_{n+1} = R(n)U_n + G(n)U_{n-1}, \qquad n \geq 1, \tag{1.5}$$

where $U_0 = 1$, $U_1 = 12$ and

$$R(n) = \frac{3(2n+1)(3n^2+3n+1)(15n^2+15n+4)}{(n+1)^5}, \quad G(n) = \frac{3n^3(3n-1)(3n+1)}{(n+1)^5}.$$

Expressions of U_n as double sums of products of binomial coefficients have been derived by Krattenthaler and Rivoal [11] and Zudilin [15, 16].

In this paper, we shall establish the 2-log-convexity of the sequences of the Apéry numbers A_n, B_n, the Cohen-Rhin numbers U_n and some other combinatorial sequences based on the three-term recurrence relations. Recall that an infinite positive sequence $\{a_n\}_{n=0}^{\infty}$ is said to be log-convex if for all $n \geq 1$,

$$a_n^2 \leq a_{n-1}a_{n+1}. \tag{1.6}$$

We say that $\{a_n\}_{n=0}^{\infty}$ is 2-log-convex if $\{a_n\}_{n=0}^{\infty}$ is log-convex and for all $n \geq 1$,

$$\left(a_n a_{n+2} - a_{n+1}^2\right)^2 \leq \left(a_{n-1}a_{n+1} - a_n^2\right)\left(a_{n+1}a_{n+3} - a_{n+2}^2\right). \tag{1.7}$$

Meanwhile, the sequence $\{a_n\}_{n=0}^{\infty}$ is called strictly log-convex (2-log-convex) if the inequality in (1.6) ((1.7)) is strict for all $n \geq 1$. Došlić [7] proved the log-convexity of A_n by induction. In fact, using similar arguments one can show that $\{B_n\}_{n=0}^{\infty}$ and $\{U_n\}_{n=0}^{\infty}$ are log-convex.

This paper is organized as follows. In Section 2, we give a general framework to prove the 2-log-convexity of a sequence $\{S_n\}_{n=0}^{\infty}$ based on a lower bound f_n and an upper bound g_n for the ratio S_n/S_{n-1}, where the numbers S_n satisfy a three-term recurrence relation. Section 3 demonstrates how to find the bounds f_n and g_n. Section 4 is devoted to the computations of the upper bounds for the ratios A_n/A_{n-1}, B_n/B_{n-1} and U_n/U_{n-1}. In Section 5, we show that the sequences of A_n, B_n, U_n, the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and 4 and the large Schröder numbers are all 2-log-convex. We conclude this paper with a conjecture on the infinite log-convexity in the spririt of the infinite log-concavity introduced by Moll [12].

2 A criterion

In this section, we present a criterion for the 2-log-convexity of a sequence $\{S_n\}_{n=0}^{\infty}$ satisfying a three-term recurrence relation. We need the assumption that the ratio S_n/S_{n-1} has a lower bound f_n and an upper bound g_n.

Theorem 2.1. *Suppose $\{S_n\}_{n=0}^{\infty}$ is a positive log-convex sequence that satisfies the recurrence relation*

$$S_n = b(n)S_{n-1} + c(n)S_{n-2} \tag{2.1}$$

for $n \geq 2$. Let

$$\begin{aligned}
a_3(n) =&2b(n+2)b^2(n+1) + 2b(n+1)c(n+2) - b^3(n+1)\\
&- b(n+1)b(n+2)b(n+3) - b(n+3)c(n+2) - c(n+3)b(n+1),\\
a_2(n) =&4b(n+1)b(n+2)c(n+1) + 2c(n+1)c(n+2) + b^2(n+1)b(n+2)b(n+3)\\
&+ b(n+1)b(n+3)c(n+2) + b^2(n+1)c(n+3) - 3c(n+1)b^2(n+1)\\
&- b(n+3)b(n+2)c(n+1) - c(n+3)c(n+1) - b^2(n+2)b^2(n+1)\\
&- 2b(n+2)b(n+1)c(n+2) - c^2(n+2),\\
a_1(n) =&- c(n+1)\big(2b(n+2)c(n+2) - 2b(n+2)c(n+1)\\
&- 2b(n+3)b(n+2)b(n+1) - b(n+3)c(n+2) - 2c(n+3)b(n+1)\\
&+ 3c(n+1)b(n+1) + 2b^2(n+2)b(n+1)\big),\\
a_0(n) =&- c^2(n+1)\left(c(n+1) - b(n+2)b(n+3) - c(n+3) + b^2(n+2)\right)
\end{aligned}$$

and

$$\Delta(n) = 4a_2^2(n) - 12a_1(n)a_3(n).$$

Assume that $a_3(n) < 0$ and $\Delta(n) > 0$ for all $n \geq N$, where N is a positive integer. If there exist f_n and g_n such that for all $n \geq N$,

(C_1) $f_n \leq \frac{S_n}{S_{n-1}} < g_n$;

(C_2) $f_n \geq \frac{-2a_2(n)-\sqrt{\Delta(n)}}{6a_3(n)}$;

(C_3) $a_3(n)g_n^3 + a_2(n)g_n^2 + a_1(n)g_n + a_0(n) > 0$,

then $\{S_n\}_{n=N}^{\infty}$ is strictly 2-log-convex, that is, for $n \geq N$,

$$\left(S_{n-1}S_{n+1} - S_n^2\right)\left(S_{n+1}S_{n+3} - S_{n+2}^2\right) > \left(S_nS_{n+2} - S_{n+1}^2\right)^2. \tag{2.2}$$

Proof. By the recurrence relation (2.1), we have

$$\begin{aligned}
&\left(S_{n-1}S_{n+1} - S_n^2\right)\left(S_{n+1}S_{n+3} - S_{n+2}^2\right) - \left(S_nS_{n+2} - S_{n+1}^2\right)^2 \\
&\quad = S_{n+1}\left(2S_nS_{n+1}S_{n+2} + S_{n-1}S_{n+1}S_{n+3} - S_{n+1}^3 - S_n^2S_{n+3} - S_{n-1}S_{n+2}^2\right) \\
&\quad = S_{n+1}\left(a_3(n)S_n^3 + a_2(n)S_n^2S_{n-1} + a_1(n)S_nS_{n-1}^2 + a_0(n)S_{n-1}^3\right).
\end{aligned}$$

Since $\{S_n\}_{n=0}^{\infty}$ is a positive sequence, in order to prove (2.2), it suffices to show that for all $n \geq N$,

$$a_3(n)\left(\frac{S_n}{S_{n-1}}\right)^3 + a_2(n)\left(\frac{S_n}{S_{n-1}}\right)^2 + a_1(n)\frac{S_n}{S_{n-1}} + a_0(n) > 0. \tag{2.3}$$

Consider the polynomial $f(x) = a_3(n)x^3 + a_2(n)x^2 + a_1(n)x + a_0(n)$. Note that

$$f'(x) = 3a_3(n)x^2 + 2a_2(n)x + a_1(n).$$

Since $a_3(n) < 0$ and $\Delta(n) > 0$ for all $n \geq N$, we see that the quadratic function $f'(x)$ is negative for $x > \frac{-2a_2(n)-\sqrt{\Delta(n)}}{6a_3(n)}$. Thus, $f(x)$ is strictly decreasing on the interval $[\frac{-2a_2(n)-\sqrt{\Delta(n)}}{6a_3(n)}, +\infty)$. From the assumption $g_n > f_n \geq \frac{-2a_2(n)-\sqrt{\Delta(n)}}{6a_3(n)}$, it follows that $f(x)$ is strictly decreasing on the interval $[f_n, g_n]$. Since $\frac{S_n}{S_{n-1}} \in [f_n, g_n]$, it remains to show that $f(g_n) > 0$ for any $n \geq N$, which is equivalent to condition (C_3), that is,

$$a_3(n)g_n^3 + a_2(n)g_n^2 + a_1(n)g_n + a_0(n) > 0$$

for any $n \geq N$. This completes the proof. □

3 A heuristic approach to computing the bounds

In this section, we present a procedure to derive a lower bound f_n and an upper bound g_n for the ratio S_n/S_{n-1} based on a three-term recurrence relation of S_n. We first describe how to obtain an upper bound g_n as required in Theorem 2.1. As will be seen, this procedure is not guaranteed to give an upper bound g_n, but it is practically valid for many cases.

Assume that $\lim_{n\to\infty} b(n) = b$ and $\lim_{n\to\infty} c(n) = c$, where b and c are two constants and $b^2 + 4c > 0$. All sequences considered in this paper satisfy this condition. Let

$$x_0 = \frac{b + \sqrt{b^2 + 4c}}{2}. \tag{3.1}$$

We begin with the case $c(n) < 0$, and we shall try to construct g_n which satisfies the condition (C_3) together with the following inequality:

$$g_{n+1} - \left(b(n+1) + \frac{c(n+1)}{g_n} \right) > 0. \tag{3.2}$$

In fact, the condition (3.2) is essential to find an upper bound g_n for S_n/S_{n-1}. As will be seen in the following lemma, if we find a function g_n satisfying (3.2) and $S_n/S_{n-1} < g_n$ for small n, then we can deduce that g_n is an upper bound for S_n/S_{n-1} for any n.

Lemma 3.1. *Let S_n be the sequence defined by the recurrence relation* (2.1)*. Assume that N is a positive integer such that $c(n) < 0$ for $n \geq N$. If $\frac{S_N}{S_{N-1}} \leq g_N$ and the condition* (3.2) *holds for $n \geq N$, then we have for $n \geq N$,*

$$\frac{S_n}{S_{n-1}} \leq g_n. \tag{3.3}$$

Proof. We use induction on n. Obviously, the lemma holds for $n = N$. We assume that it is true for $n = m \geq N$, that is, $\frac{S_m}{S_{m-1}} < g_m$. Since $c(m) < 0$ for $m \geq N$, we see that

$$c(m+1)\frac{S_{m-1}}{S_m} < \frac{c(m+1)}{g_m}. \tag{3.4}$$

We now consider the case $n = m + 1$. From (2.1) and (3.4) it follows that

$$\frac{S_{m+1}}{S_m} = b(m+1) + c(m+1)\frac{S_{m-1}}{S_m} \leq b(m+1) + \frac{c(m+1)}{g_m}. \tag{3.5}$$

From (3.2) and (3.5) we deduce that for $m \geq N$,

$$g_{m+1} - \frac{S_{m+1}}{S_m} \geq g_{m+1} - \left(b(m+1) + \frac{c(m+1)}{g_m} \right) > 0,$$

which is the statement of the lemma for $n = m + 1$. This completes the proof. □

Now we present a heuristic procedure to find the desired upper bound g_n. Let $g_n = x_0$ as given by (3.1). If g_n satisfies the conditions (C_3) and (3.2), then g_n is the desired choice. Otherwise, let $g_n = x_0 + \frac{x}{n}$. Substitute g_n into (3.2) and let $Y(n)$ denote the numerator of the left hand side of (3.2), which is often a polynomial in n and x. Setting the coefficient of the highest degree in n of $Y(n)$ to be 0, we obtain an equation in x. If x_1 is the unique solution of this equation, then we set $g_n = x_0 + \frac{x_1}{n}$. If $g_n = x_0 + \frac{x_1}{n}$ satisfies the conditions (C_3) and (3.2), then g_n is the desired choice. Otherwise, set $g_n = x_0 + \frac{x_1}{n} + \frac{x}{n^2}$ and repeat the above process to find a solution x_2 of the equation. By iteration, we may find $x_0, x_1, \ldots, x_i$ such that $g_n = x_0 + \frac{x_1}{n} + \frac{x_2}{n^2} + \cdots + \frac{x_i}{n^i}$ satisfies the conditions (C_3) and (3.2).

For example, let $S_n = A_n$, where A_n is Apéry number defined by (1.1). Since $\lim\limits_{n\to\infty} b(n) = 34$ and $\lim\limits_{n\to\infty} c(n) = -1$, by the definition of A_n, we have $x_0 = 17 + 12\sqrt{2}$. Since $g_n = 17 + 12\sqrt{2}$ does not satisfy the condition (C_3) in Theorem 2.1, we further consider $g_n = 17 + 12\sqrt{2} + \frac{x}{n}$. Let $Y(n)$ denote the numerator of the left hand side of (3.2). It is easy to see that $Y(n)$ is a cubic polynomial in n with the leading coefficient equal to

$$E_1 = -(17\sqrt{2} - 24)(48x + 864\sqrt{2} + 1224).$$

Setting $E_1 = 0$ gives $x_1 = -\frac{51}{2} - 18\sqrt{2}$. Again, $g_n = x_0 + \frac{x_1}{n}$ does not satisfy (3.2). So we continue to consider $g_n = x_0 + \frac{x_1}{n} + \frac{x}{n^2}$ and we find that $x_2 = \frac{609}{64}\sqrt{2} + \frac{27}{2}$. Now, $g_n = x_0 + \frac{x_1}{n} + \frac{x_2}{n^2}$ does not satisfy the condition (C_3). Repeating the above procedure, we find that $x_3 = -\frac{225}{128}\sqrt{2} - \frac{645}{256}$ and $g_n = x_0 + \frac{x_1}{n} + \frac{x_2}{n^2} + \frac{x_3}{n^3}$ satisfies (3.2) and the condition (C_3).

For the case $c(n) > 0$, we aim to construct an upper bound g_n which satisfies condition (C_3) and the following inequality

$$g_n - \left(b(n) + \frac{c(n)}{b(n-1) + \frac{c(n-1)}{g_{n-2}}} \right) > 0. \tag{3.6}$$

Similarly, if we find a function g_n satisfying (3.6) and $S_n/S_{n-1} < g_n$ for certain n, then we can deduce that g_n is an upper bound for any n. To be precise, we have the following lemma.

Lemma 3.2. *Let S_n be defined by (2.1). If there exists a positive integer N such that the inequality (3.6) holds, $\frac{S_N}{S_{N-1}} \le g_N$, $\frac{S_{N+1}}{S_N} \le g_{N+1}$ and $c(n) > 0$ for $n \ge N$, then we have for $n \ge N$,*

$$\frac{S_n}{S_{n-1}} \le g_n. \tag{3.7}$$

Proof. We conduct induction on n. Clearly, the lemma holds for $n = N$ and $n = N+1$. Assume that it is true for $n = m - 2 \geq N$, that is,

$$\frac{S_{m-2}}{S_{m-3}} \leq g_{m-2}. \tag{3.8}$$

We shall show that the lemma is true for $n = m$, that is,

$$\frac{S_m}{S_{m-1}} \leq g_m. \tag{3.9}$$

Since $c(n) > 0$ for $n \geq N$, from (2.1) and (3.8) it follows that

$$\begin{aligned} \frac{S_m}{S_{m-1}} &= b(m) + c(m)\frac{S_{m-2}}{S_{m-1}} = b(m) + \frac{c(m)}{b(m-1) + c(m-1)\frac{S_{m-3}}{S_{m-2}}} \\ &\leq b(m) + \frac{c(m)}{b(m-1) + \frac{c(m-1)}{g_{m-2}}}. \end{aligned} \tag{3.10}$$

In view of (3.6) and (3.10), we find that

$$g_m - \frac{S_m}{S_{m-1}} \geq g_m - \left(b(m) + \frac{c(m)}{b(m-1) + \frac{c(m-1)}{g_{m-2}}} \right) > 0,$$

which yields (3.9). This completes the proof. $\square$

Now we can use the same approach as in the case $c(n) < 0$ to find an upper bound g_n. Moreover, if we have obtain an approximation g_n that does not simultaneously satisfy (3.2) ((3.6)) and the condition (C_3), instead of going further to update the estimation of g_n, we may try to adjust some coefficients to find a desired bound. For example, let $S_n = B_n$, where B_n is defined by (1.2). At some point, we get

$$\begin{aligned} g_n =& \frac{11}{2} + \frac{5\sqrt{5}}{2} - \left(\frac{11}{2} + \frac{5\sqrt{5}}{2} \right) \frac{1}{n} \\ &+ \left(\frac{7}{10}\sqrt{5} + \frac{3}{2} \right) \frac{1}{n^2} + \frac{1}{25n^3} + \left(\frac{1}{50} + \frac{23\sqrt{5}}{1250} \right) \frac{1}{n^4}. \end{aligned} \tag{3.11}$$

Here g_n satisfies the condition (C_3) in Theorem 2.1, but it fails to satisfy (3.6). If we replace the coefficient $\frac{1}{50}$ in (3.11) by $\frac{1}{25}$, then the adjusted bound g_n' satisfies both conditions (C_3) and (3.6).

To conclude this section, we need to mention that it is much easier to find a lower bound f_n for the ratio S_n/S_{n-1}. In many cases, we have $f(n) = b(n)$ when $b(n)$ and $c(n)$ are positive for $n \geq N$ and $f_n = b(n) + c(n)$ when $c(n)$ is negative and $S_n \geq S_{n-1}$ for $n \geq N$.

4 Upper bounds for A_n/A_{n-1}, B_n/B_{n-1} and U_n/U_{n-1}

In this section, we shall use the heuristic approach described in the previous section to find upper bounds for the ratios A_n/A_{n-1}, B_n/B_{n-1} and U_n/U_{n-1}.

Lemma 4.1. *Let*

$$\begin{aligned} P(n) =& 17+12\sqrt{2}-\left(\frac{51}{2}+18\sqrt{2}\right)\frac{1}{n} \\ & +\left(\frac{27}{2}+\frac{609}{64}\sqrt{2}\right)\frac{1}{n^2}-\left(\frac{645}{256}+\frac{225\sqrt{2}}{128}\right)\frac{1}{n^3}. \end{aligned} \tag{4.1}$$

For $n \geq 2$, we have $\frac{A_n}{A_{n-1}} < P(n)$.

Proof. For the Apéry numbers A_n, we use Lemma 3.1 by setting $N = 2$ and $g_n = P(n)$. Evidently, $\frac{A_2}{A_1} < P(2)$. Also, it is easily checked that

$$\begin{aligned} P(n+1) &- \left(\frac{(2n+1)(17n^2+17n+5)}{(n+1)^3}-\frac{n^3}{(n+1)^3P(n)}\right) \\ &= \frac{9(17-12\sqrt{2})(5664n^2-3560\sqrt{2}n+1225)}{256(256n^3-384n^2-60\sqrt{2}n+288n+90\sqrt{2}-165)(n+1)^3}, \end{aligned}$$

which is positive for $n \geq 2$. By lemma 3.1, we see that $P(n)$ is an upper bound for A_n/A_{n-1} when $n \geq 2$. This completes the proof. □

Lemma 4.2. *Let*

$$\begin{aligned} T(n) =& \frac{11}{2}+\frac{5\sqrt{5}}{2}-\left(\frac{11}{2}+\frac{5\sqrt{5}}{2}\right)\frac{1}{n} \\ & +\left(\frac{7}{10}\sqrt{5}+\frac{3}{2}\right)\frac{1}{n^2}+\frac{1}{25n^3}+\left(\frac{1}{25}+\frac{23\sqrt{5}}{1250}\right)\frac{1}{n^4}. \end{aligned} \tag{4.2}$$

For $n \geq 20$, we have $\frac{B_n}{B_{n-1}} < T(n)$.

Proof. Set $N = 20$ and $g_n = T(n)$ in Lemma 3.2. It is easy to check that $\frac{B_{20}}{B_{19}} < T(20)$ and $\frac{B_{21}}{B_{20}} < T(21)$. Moreover, it is not difficult to verify that

$$T(n) - \left(\frac{11n^2-11n+3}{n^2}+\frac{(n-1)^2}{n^2\left(\frac{11n^2-33n+25}{(n-1)^2}+\frac{(n-2)^2}{(n-1)^2}\frac{1}{T(n-2)}\right)}\right)$$

$$= \frac{(123\sqrt{5}-275)J(n)}{1250n^4K(n)},$$

where $J(n)$ and $K(n)$ are given by

$$\begin{aligned}J(n) =&1718750n^6 - 4656250\sqrt{5}n^5 - 18026250n^5 + 98010000n^4\\&+ 38885750\sqrt{5}n^4 - 136205250\sqrt{5}n^3 - 310595950n^3 + 248642319\sqrt{5}n^2\\&+ 557184100n^2 - 233557457\sqrt{5}n - 522290000n + 199152500 + 89063225\sqrt{5},\\K(n) =&2500n^6 - 30000n^5 + 150000n^4 - 500\sqrt{5}n^4 - 401100n^3 + 4500\sqrt{5}n^3\\&+ 642325n^2 - 30881\sqrt{5}n^2 - 619575n + 78143\sqrt{5}n - 60525\sqrt{5} + 278125.\end{aligned}$$

It follows that $J(n)$ and $K(n)$ are positive for $n \geq 20$. Hence we have

$$\frac{11n^2-11n+3}{n^2} + \frac{(n-1)^2}{n^2\left(\frac{11n^2-33n+25}{(n-1)^2} + \frac{(n-2)^2}{(n-1)^2}\frac{1}{T(n-2)}\right)} < T(n). \tag{4.3}$$

In view of Lemma 3.2, we deduce that $T(n)$ is an upper bound for B_n/B_{n-1} when $n \geq 20$. ☐

Using the same procedure, we find the following upper bound for U_n/U_{n-1}. The proof is omitted.

Lemma 4.3. *Let*

$$\begin{aligned}Q(n) =&135 + 78\sqrt{3} - \left(\frac{675}{2} + 195\sqrt{3}\right)\frac{1}{n} + \left(\frac{9737}{48}\sqrt{3} + 351\right)\frac{1}{n^2}\\&- \left(\frac{3497}{32}\sqrt{3} + \frac{6045}{32}\right)\frac{1}{n^3} + \left(\frac{841763}{27648}\sqrt{3} + \frac{2701}{32}\right)\frac{1}{n^4}.\end{aligned} \tag{4.4}$$

For $n \geq 100$, we have $\frac{U_n}{U_{n-1}} < Q(n)$.

5 The 2-log-convexity

Based on the criterion given in Theorem 2.1 and the upper bounds obtained in the previous section, we shall give the proofs of the 2-log-convexity of the sequences of Apéry numbers and other aforementioned combinatorial numbers.

Theorem 5.1. *The sequence $\{A_n\}_{n=0}^{\infty}$ is strictly 2-log-convex.*

Proof. We first consider the case $n \geq 2$. To apply Theorem 2.1, let

$$b(n) = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} \quad \text{and} \quad c(n) = -\frac{(n-1)^3}{n^3}.$$

It is straightforward to check that $a_3(n) < 0$ and $\Delta(n) > 0$ for $n \geq 2$. Since

$$\binom{n-1}{k}^2 \binom{n-1+k}{k}^2 \geq \binom{n-2}{k}^2 \binom{n-2+k}{k}^2,$$

we have $A_{n-1} \geq A_{n-2}$. Let

$$f_n = \frac{33n^3 - 48n^2 + 24n - 4}{n^3}.$$

Thus, by the recurrence relation (1.3), we see that

$$\begin{aligned} \frac{A_n}{A_{n-1}} =& \frac{34n^3 - 51n^2 + 27n - 5}{n^3} - \frac{(n-1)^3}{n^3} \frac{A_{n-2}}{A_{n-1}} \\ \geq& \frac{34n^3 - 51n^2 + 27n - 5 - (n-1)^3}{n^3} = f_n. \end{aligned} \tag{5.1}$$

Set $g_n = P(n)$, where $P(n)$ is given by (4.1). We proceed to verify the conditions (C_1), (C_2) and (C_3) in Theorem 2.1. By (5.1) and Lemma 4.1, we find that $f_n \leq \frac{A_n}{A_{n-1}} < g_n$, which is the condition (C_1). Define $R_1(n) = 6a_3(n)f_n + 2a_2(n)$. It is easily checked that $R_1(n) = -4\frac{H_1(n)}{L_1(n)}$, where $H_1(n)$ and $L_1(n)$ are polynomials in n and the leading coefficients of $H_1(n)$ and $L_1(n)$ are positive. Hence we deduce that $R_1(n) < 0$ for $n \geq 2$. Similarly, define $R_2(n) = \Delta(n) - R_1^2(n)$, which can be rewritten as $-96\frac{H_2(n)}{L_2(n)}$ where $H_2(n)$ and $L_2(n)$ are polynomials in n and the leading coefficients of $H_2(n)$ and $L_2(n)$ are positive. Consequently, we deduce $R_2(n) < 0$ for $n \geq 2$. It follows that for $n \geq 2$,

$$6a_3(n)f_n + 2a_2(n) < -\sqrt{\Delta(n)},$$

which is equivalent to the following inequality for $n \geq 2$:

$$f_n > \frac{-2a_2(n) - \sqrt{\Delta(n)}}{6a_3(n)}.$$

This is exactly the condition (C_2). Finally, it remains to verify the condition (C_3). To this end, we find that

$$\begin{aligned} &a_3(n)g_n^3 + a_2(n)g_n^2 + a_1(n)g_n + a_0(n) \\ &= 9\left(30733178557 + 21731638968\sqrt{2}\right)\frac{H_3(n)}{L_3(n)}, \end{aligned} \tag{5.2}$$

where $H_3(n)$ and $L_3(n)$ are polynomials in n. Observe that the leading coefficients of $H_3(n)$ and $L_3(n)$ are both positive. This implies that the right hand side of (5.2) is positive for $n \geq 2$. Now we are left with the case $n = 1$, that is

$$(A_0A_2 - A_1^2)(A_2A_4 - A_3^2) > (A_1A_3 - A_2^2)^2,$$

which can be easily checked. This completes the proof. □

Theorem 5.2. *The sequence* $\{B_n\}_{n=0}^{\infty}$ *is strictly* 2*-log-convex.*

Proof. For $n \geq 20$, apply Theorem 2.1 with

$$f_n = \frac{11n^2 - 11n + 3}{n^2},$$

and $g_n = T(n)$, where $T(n)$ is given by (4.2). Using the argument in the proof of Theorem 5.1, we find that f_n and g_n satisfy all the conditions in Theorem 2.1. Finally, it is easy to verify that for $1 \leq n \leq 19$,

$$\left(B_{n-1}B_{n+1} - B_n^2\right)\left(B_{n+1}B_{n+3} - B_{n+2}^2\right) > \left(B_nB_{n+2} - B_{n+1}^2\right)^2.$$

This completes the proof. □

Theorem 5.3. *The sequence* $\{U_n\}_{n=0}^{\infty}$ *is strictly* 2*-log-convex.*

The above theorem follows from Theorem 2.1 by setting

$$f_n = \frac{3(2n-1)(3n^2 - 3n + 1)(15n^2 - 15n + 4)}{n^5}$$

and setting $g_n = Q(n)$, where $Q(n)$ is given by (4.4). The proof is similar to that of Theorem 5.1, and it is omitted.

Došlić [7,8] has proved the log-convexity of several well-known sequences of combinatorial numbers such as the Motzkin numbers M_n, the Fine numbers F_n, the Franel numbers $F_n^{(3)}$ and $F_n^{(4)}$ of order 3 and 4, and the large Schröder numbers s_n. Based on the recurrence relations satisfied by these numbers, we utilize Theorem 2.1 to deduce that these sequences are all strictly 2-log-convex possibly except for a fixed number of terms at the beginning.

We conclude this paper with a conjecture concerning the infinite log-convexity of the Aéry numbers. The notion of infinite log-convexity is analogous to that of infinite log-concavity introduced by Moll [12]. Given a sequence $A = \{a_i\}_{0 \leq i \leq \infty}$, define the operator $\mathcal{L}$ by

$$\mathcal{L}(A) = \{b_i\}_{0 \leq i \leq \infty},$$

where $b_i = a_{i-1}a_{i+1} - a_i^2$ for $i \geq 1$. We say that $\{a_i\}_{0 \leq i \leq \infty}$ is k-log-convex if $\mathcal{L}^j\left(\{a_i\}_{0 \leq i \leq \infty}\right)$ is log-convex for $j = 0, 1, \ldots, k-1$, and that $\{a_i\}_{0 \leq i \leq \infty}$ is ∞-log-convex if $\mathcal{L}^k\left(\{a_i\}_{0 \leq i \leq \infty}\right)$ is log-convex for any $k \geq 0$.

Conjecture 5.4. The sequences $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, $\{U_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ are infinitely log-convex. The sequences $\{M_n\}_{n=0}^{\infty}$, $\{F_n\}_{n=0}^{\infty}$, $\{F_n^{(3)}\}_{n=0}^{\infty}$ and $\{F_n^{(4)}\}_{n=0}^{\infty}$ are k-log-convex for any $k \geq 1$ except for a constant number (depending on k) of terms at the beginning.

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