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The ratio monotonicity of the q-derangement numbers

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Abstract. We introduce the notion of ratio monotonicity for polynomials with nonnegative coefficients, and we show that for $n \geq 6$, the q-derangement numbers $D_n(q)$ are strictly ratio monotone, except for the last term when n is even. This property implies the spiral property and log-concavity.

Keywords: q-derangement number, spiral property, log-concavity, ratio monotone property.

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1 Introduction

Let D_n be the set of derangements on $\{1, 2, \ldots, n\}$, and let $\mathrm{maj}(\pi)$ denote the major index of a permutation π. The q-derangement number $D_n(q)$ is defined as $\sum_{\pi \in D_n} q^{\mathrm{maj}(\pi)}$. The following formula is due to Gessel [5] (see also Gessel and Reutenauer [6]):

$$D_n(q) = [n]! \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \frac{1}{[k]!}, \tag{1.1}$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [1][2]\cdots[n]$. Combinatorial proofs of (1.1) have been found by Wachs [7], and Chen and Xu [3]. The polynomials $D_n(q)$ satisfy the following recurrence relation

$$D_n(q) = [n]D_{n-1}(q) + (-1)^n q^{\binom{n}{2}}, \qquad n \geq 2, \tag{1.2}$$

with $D_1(q) = 0$. Note that the q-derangement polynomials of type B have been introduced and studied independently by Chen, Tang and Zhao [2] and Chow [4].

Chen and Rota [1] showed that the q-derangement numbers are unimodal and conjectured the maximum coefficient appears in the middle. Zhang [8] confirmed this conjecture by showing that the q-derangement numbers satisfy the spiral property. For example, we have

$$\begin{aligned} D_6(q) =& q + 4q^2 + 9q^3 + 16q^4 + 24q^5 + 32q^6 + 37q^7 + 38q^8 + 35q^9 \\ & + 28q^{10} + 20q^{11} + 12q^{12} + 6q^{13} + 2q^{14} + q^{15}. \end{aligned}$$

Observe that $D_6(q)$ has the following spiral property

$$1 < 2 < 4 < 6 < 9 < 12 < 16 < 20 < 24 < 28 < 32 < 35 < 37 < 38,$$

where the last term is not taken into consideration. We say that a sequence $a(1), a(2), \ldots, a(n)$ of positive numbers is ratio monotone if

$$\frac{a(1)}{a(n)} \leq \frac{a(2)}{a(n-1)} \leq \cdots \leq \frac{a(\lfloor n/2 \rfloor)}{a(\lceil n/2 \rceil + 1)} \leq 1 \tag{1.3}$$

and

$$\frac{a(n)}{a(2)} \leq \frac{a(m-1)}{a(3)} \leq \cdots \leq \frac{a(\lfloor n/2 \rfloor + 2)}{a(\lceil n/2 \rceil)} \leq 1, \tag{1.4}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor function and the ceiling function. In the case that all the inequalities are strict, we say that the sequence is strictly ratio monotone. It can be easily seen that the ratio monotone property implies the spiral property and the log-concavity. For example, for the case of $D_6(q)$ without the last term, we see that

$$\frac{1}{2} < \frac{4}{6} < \frac{9}{12} < \frac{16}{20} < \frac{24}{28} < \frac{32}{35} < \frac{37}{38} < 1, \tag{1.5}$$

$$\frac{2}{4} < \frac{6}{9} < \frac{12}{16} < \frac{20}{24} < \frac{28}{32} < \frac{35}{37} < 1. \tag{1.6}$$

In the next section, we shall show that the q-derangement numbers $D_n(q)$ are strictly ratio monotone for $n \geq 6$ except for the last term when n is even.

2 The ratio monotonicity

In order to state the main result, we first introduce some notation. Set

$$B_n(q) = \begin{cases} D_n(q) - q^{\binom{n}{2}}, & \text{if } n \text{ is even}, \\ D_n(q), & \text{if } n \text{ is odd}. \end{cases} \tag{2.1}$$

Let $\delta_n = \binom{n}{2} - 1$ denote the degree of $B_n(q)$. Then (1.2) can be recast as

$$B_n(q) = \begin{cases} [n]B_{n-1}(q), & \text{if } n \text{ is even,} \\ [n]B_{n-1}(q) + [n-1]q^{\binom{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

Write

$$B_n(q) = b_n(1)q + b_n(2)q^2 + \cdots + b_n(\delta_n)q^{\delta_n}.$$

The ratio monotone property of $B_n(q)$ can be stated as follows.

Theorem 2.1. *For $n \geq 6$, we have*

$$\frac{b_n(1)}{b_n(\delta_n)} < \frac{b_n(2)}{b_n(\delta_n - 1)} < \cdots < \frac{b_n(\lceil \binom{n}{2}/2 \rceil - 1)}{b_n(\lfloor \binom{n}{2}/2 \rfloor + 1)} < 1, \tag{2.3}$$

$$\frac{b_n(\delta_n)}{b_n(2)} < \frac{b_n(\delta_n - 1)}{b_n(3)} < \cdots < \frac{b_n(\lceil \binom{n}{2}/2 \rceil + 1)}{b_n(\lfloor \binom{n}{2}/2 \rfloor)} < 1. \tag{2.4}$$

Theorem 2.1 implies the log-concavity of $B_n(q)$.

Corollary 2.2. *For $n \geq 6$, the polynomials $B_n(q)$ are log-concave, that is,*

$$\frac{b_n(1)}{b_n(2)} < \frac{b_n(2)}{b_n(3)} < \cdots < \frac{b_n(\delta_n - 2)}{b_n(\delta_n - 1)} < \frac{b_n(\delta_n - 1)}{b_n(\delta_n)}. \tag{2.5}$$

To prove Theorem 2.1 we need the following lemmas.

Lemma 2.3. *For positive numbers $c_1, c_2, \ldots, c_{k+1}, d_1, d_2, \ldots, d_{k+1}$ satisfying*

$$\frac{d_1}{c_1} < \frac{d_2}{c_2} < \cdots < \frac{d_k}{c_k} < \frac{d_{k+1}}{c_{k+1}},$$

we have

$$\frac{d_1 + d_2 + \cdots + d_k}{c_1 + c_2 + \cdots + c_k} < \frac{d_1 + d_2 + \cdots + d_k + d_{k+1}}{c_1 + c_2 + \cdots + c_k + c_{k+1}}, \tag{2.6}$$

$$\frac{d_1 + d_2 + \cdots + d_k}{c_1 + c_2 + \cdots + c_k} < \frac{d_2 + \cdots + d_k + d_{k+1}}{c_2 + \cdots + c_k + c_{k+1}}, \tag{2.7}$$

$$\frac{d_1 + d_2 + \cdots + d_k + d_{k+1}}{c_1 + c_2 + \cdots + c_k + c_{k+1}} < \frac{d_2 + \cdots + d_k + d_{k+1}}{c_2 + \cdots + c_k + c_{k+1}}. \tag{2.8}$$

The proof of Lemma 2.3 is straightforward, and the details are omitted. Using recurrence relation (2.2), it is easy to verify the following lemma.

Lemma 2.4. *If $n \geq 3$, we have $b_n(1) = 1$, $b_n(2) = n-2$, and*

$$\begin{aligned}
b_n(3) &= \frac{n(n-3)}{2}, \\
b_n(4) &= \frac{(n^2-4)(n-3)}{6}, \\
b_n(\delta_n) &= \lceil n/2 \rceil - 1, \\
b_n(\delta_n - 1) &= \lceil n^2/4 - n/2 \rceil, \\
b_n(\delta_n - 2) &= \lceil n^3/12 - n^2/8 - n/12 \rceil - 1.
\end{aligned}$$

Corollary 2.5. *For $n \geq 3$,*

$$\frac{b_n(1)}{b_n(\delta_n)+1} < \frac{b_n(2)}{b_n(\delta_n - 1)}. \tag{2.9}$$

For $n \geq 4$ and n even,

$$\frac{b_n(\delta_n)+1}{b_n(2)} < \frac{b_n(\delta_n - 1)}{b_n(3)}. \tag{2.10}$$

For $n \geq 6$,

$$\frac{b_n(\delta_n) + b_n(\delta_n - 1) + 1}{b_n(2) + b_n(3)} < \frac{b_n(\delta_n - 2)}{b_n(4)}. \tag{2.11}$$

We are now ready to present the proof of Theorem 2.1.

Proof of Theorem 2.1. We claim that for $n \geq 6$ the coefficients of $B_n(q)$ satisfy the following relations

$$\frac{b_n(1)}{b_n(\delta_n)} < \frac{b_n(2)}{b_n(\delta_n - 1)} < \cdots < \frac{b_n(\delta_n - 1)}{b_n(2)} < \frac{b_n(\delta_n)}{b_n(1)}, \tag{2.12}$$

$$\frac{b_n(\delta_n)}{b_n(2)} < \frac{b_n(\delta_n - 1)}{b_n(3)} < \cdots < \frac{b_n(3)}{b_n(\delta_n - 1)} < \frac{b_n(2)}{b_n(\delta_n)}. \tag{2.13}$$

To prove the above assertion we use induction on n. For $n = 6, 7$, it is easy to check that the claim is valid. Suppose that the claim holds for n. We now proceed to show that the it holds for $n+1$, that is,

$$\frac{b_{n+1}(i)}{b_{n+1}(\delta_{n+1} - i + 1)} < \frac{b_{n+1}(i+1)}{b_{n+1}(\delta_{n+1} - i)}, \qquad 1 \leq i \leq \delta_{n+1} - 1, \tag{2.14}$$

$$\frac{b_{n+1}(\delta_{n+1} - i + 1)}{b_{n+1}(i+1)} < \frac{b_{n+1}(\delta_{n+1} - i)}{b_{n+1}(i+2)}, \qquad 1 \leq i \leq \delta_{n+1} - 2. \tag{2.15}$$

We only consider the case when n is even, since the case when n is odd can be dealt with by using the same argument. Assume that $n \geq 6$ and n is even. From (2.2) we get the following recurrence relation for $b_n(k)$,

$$b_{n+1}(k) = \begin{cases} \sum_{i=k-n}^{k} b_n(i), & 1 \leq k < \binom{n}{2}, \\ 1 + \sum_{i=k-n}^{k} b_n(i), & \binom{n}{2} \leq k < \binom{n+1}{2}, \end{cases} \tag{2.16}$$

with the convention that $b_n(i) = 0$ if $i < 1$ or $i > \delta_n$.

By (2.12), (2.13) and Corollary 2.5, we obtain that

$$\frac{b_n(1)}{b_n(\delta_n)+1} < \frac{b_n(2)}{b_n(\delta_n-1)} < \cdots < \frac{b_n(\delta_n-1)}{b_n(2)} < \frac{b_n(\delta_n)+1}{b_n(1)}, \tag{2.17}$$

$$\frac{b_n(\delta_n)+1}{b_n(2)+1} < \frac{b_n(\delta_n-1)}{b_n(3)} < \cdots < \frac{b_n(3)}{b_n(\delta_n-1)} < \frac{b_n(2)+1}{b_n(\delta_n)+1}. \tag{2.18}$$

Thus (2.14) can be deduced from (2.17) and (2.16) by using Lemma 2.3. For $1 \leq i \leq n$, use (2.6). For $n < i \leq \delta_n - 1$, use (2.7). For $\delta_n - 1 < i \leq \delta_{n+1} - 1$, use (2.8). Again, using Lemma 2.3 and Corollary 2.5, (2.15) can be deduced from (2.18) and (2.16). For $1 \leq i < n$, use (2.6). For $n < i < \delta_n - 1$, use (2.7). For $\delta_n - 1 < i \leq \delta_{n+1} - 2$, use (2.8).

Now special attention should be paid to the case $i = n$. It follows from (2.11) and (2.13) that

$$\frac{b_n(\delta_n) + b_n(\delta_n-1) + 1}{b_n(2) + b_n(3)} < \frac{b_n(\delta_n - n)}{b_n(n+2)}. \tag{2.19}$$

By (2.13) we find that

$$\frac{\sum_{i=2}^{n-1} b_n(\delta_n - i)}{\sum_{i=4}^{n+1} b_n(i)} < \frac{b_n(\delta_n - n)}{b_n(n+2)}. \tag{2.20}$$

In view of (2.12) and (2.7), we obtain

$$\frac{\sum_{i=2}^{n+1} b_n(i)}{\sum_{i=1}^{n} b_n(\delta_n - i)} < \frac{\sum_{i=1}^{n} b_n(\delta_n - i)}{\sum_{i=2}^{n+1} b_n(i)},$$

which yields that

$$\sum_{i=2}^{n+1} b_n(i) < \sum_{i=1}^{n} b_n(\delta_n - i). \tag{2.21}$$

From (2.19), (2.20) and (2.21) we conclude that

$$\frac{1 + \sum_{i=0}^{n-1} b_n(\delta_n - i)}{1 + \sum_{i=2}^{n+1} b_n(i)} < \frac{\sum_{i=0}^{n} b_n(\delta_n - i)}{\sum_{i=2}^{n+2} b_n(i)},$$

which can be restated as the case $i = n$ of (2.15) based on (2.16).

The case $i = \delta_n - 1$ can be verified by the same argument, and hence the above claim is confirmed.

In order to prove (2.3) and (2.4), it remains to verify that $\frac{b_n(r)}{b_n(s)} < 1$ and $\frac{b_n(u)}{b_n(v)} < 1$, where

$$r = \lceil n(n-1)/4 \rceil - 1, \quad s = \lfloor n(n-1)/4 \rfloor + 1, \quad u = \lceil n(n-1)/4 \rceil + 1, \quad v = \lfloor n(n-1)/4 \rfloor .$$

If $n \equiv 0, 1 \pmod 4$, then $r + 1 = s - 1$. By (2.12), we see that $\frac{b_n(r)}{b_n(s)} < \frac{b_n(r+1)}{b_n(s-1)} = 1$. If $n \equiv 2, 3 \pmod 4$, then $r = s - 1$. By (2.12), we get $\frac{b_n(r)}{b_n(s)} < \frac{b_n(r+1)}{b_n(s-1)} = \frac{b_n(s)}{b_n(r)}$. Hence, in either case, we have $\frac{b_n(r)}{b_n(s)} < 1$, and so (2.3) holds.

If $n \equiv 0, 1 \pmod 4$, then $u = v + 1$. By (2.13), we see that $\frac{b_n(u)}{b_n(v)} < \frac{b_n(u-1)}{b_n(v+1)} = \frac{b_n(v)}{b_n(u)}$. If $n \equiv 2, 3 \pmod 4$, then $u - 1 = v + 1$. By (2.13), we find $\frac{b_n(u)}{b_n(v)} < \frac{b_n(u-1)}{b_n(v+1)} = 1$. Consequently, we have $\frac{b_n(u)}{b_n(v)} < 1$ in either case, and so (2.4) holds. This completes the proof. ∎

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