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The Interlacing Log-concavity of the Boros-Moll Polynomials

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Abstract. We introduce the notion of interlacing log-concavity of a polynomial sequence $\{P_m(x)\}_{m\geq 0}$, where $P_m(x)$ is a polynomial of degree m with positive coefficients. This sequence is said to be interlacingly log-concave if the ratios of consecutive coefficients of $P_m(x)$ interlace the ratios of consecutive coefficients of $P_{m+1}(x)$ for any $m \geq 0$. The interlacing log-concavity of a sequence of polynomials is stronger than the log-concavity of the polynomials themselves. We show that the Boros-Moll polynomials are interlacingly log-concave. Furthermore, we give a sufficient condition for the interlacing log-concavity which implies that some classical combinatorial polynomials are interlacingly log-concave.

Keywords: interlacing log-concavity, log-concavity, the Boros-Moll polynomials

AMS Subject Classification: 05A20; 33F10

1 Introduction

In this paper, we introduce the notion of interlacing log-concavity of a polynomial sequence $\{P_m(x)\}$, which is stronger than the log-concavity of the polynomials $P_m(x)$ themselves. We show that the Boros-Moll polynomials are interlacingly log-concave.

Let $\{P_m(x)\}$ be a sequence of polynomials, where

$$P_m(x) = \sum_{i=0}^{m} a_i(m)x^m$$

is a polynomial of degree m. Let

$$r_i(m) = \frac{a_i(m)}{a_{i+1}(m)}.$$

We say that the polynomials $P_m(x)$ $(m \geq 0)$ are interlacingly log-concave if the ratios $r_i(m)$ interlace the ratios $r_i(m+1)$, that is,

$$r_0(m+1) \leq r_0(m) \leq r_1(m+1) \leq r_1(m) \leq \cdots \leq r_{m-1}(m+1) \leq r_{m-1}(m) \leq r_m(m+1).$$

Recall that a sequence $\{a_i\}_{0 \leq i \leq m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \cdots \leq \frac{a_{m-1}}{a_m}.$$

It is obvious that the interlacing log-concavity implies log-concavity.

The main objective of this paper is to prove the interlacing log-concavity of the Boros-Moll polynomials. For the background on these polynomials, see [2, 5–9, 14]. From now on, we shall use $P_m(a)$ to denote the Boros-Moll polynomial given by

$$P_m(x) = \sum_{j,k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(x+1)^j (x-1)^k}{2^{3(k+j)}}. \tag{1.1}$$

Boros and Moll [5] derived the following formula for the coefficient $d_i(m)$ of x^i in $P_m(x)$,

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \tag{1.2}$$

In [6], they showed that the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is unimodal and the maximum element appears in the middle. In other words,

$$d_0(m) < d_1(m) < \cdots < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]-1}(m) > \cdots > d_m(m). \tag{1.3}$$

They also established the unimodality by a different approach [1, 7]. Moll [14] conjectured that the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is log-concave. Kauers and Paule [12] proved this conjecture based on recurrence relations found by using a computer algebra approach. Chen and Xia [10] showed that the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ satisfies the ratio monotone property which implies the log-concavity and the spiral property. A combinatorial proof of the log-concavity of $P_m(a)$ has been found by Chen, Pang and Qu [11].

In addition to the Boros-Moll polynomials, we study polynomials whose coefficients satisfy a triangular recurrence relation. It is easy to show that the binomial coefficients, the Narayana numbers and the Bessel numbers are interlacingly log-concave. We also give a sufficient condition for the interlacing log-concavity of a sequence of polynomials and prove that the polynomials $x(x+1)\cdots(x+n-1)$, the Bell polynomials and the Whitney polynomials are interlacingly log-concave.

2 The interlacing log-concavity of $d_i(m)$

In this section, we show that for $m \geq 2$, the Boros-Moll polynomials $P_m(x)$ are interlacingly log-concave.

Theorem 2.1. *For $m \geq 2$ and $0 \leq i \leq m$, we have*

$$d_i(m)d_{i+1}(m+1) > d_{i+1}(m)d_i(m+1) \tag{2.1}$$

and

$$d_i(m)d_i(m+1) > d_{i-1}(m)d_{i+1}(m+1). \tag{2.2}$$

The proof relies on the following recurrence relations derived by Kauers and Paule [12]:

$$d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}d_i(m), \qquad 0 \leq i \leq m+1, \tag{2.3}$$

$$\begin{aligned} d_i(m+1) = & \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)}d_i(m) \\ & - \frac{i(i+1)}{(m+1)(m+1-i)}d_{i+1}(m), \qquad 0 \leq i \leq m, \end{aligned} \tag{2.4}$$

$$\begin{aligned} d_i(m+2) = & \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)}d_i(m+1) \\ & - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)}d_i(m), \qquad 0 \leq i \leq m+1, \end{aligned} \tag{2.5}$$

and for $0 \leq i \leq m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \tag{2.6}$$

Note that Moll [15] independently derived the recurrence relations (2.3) and (2.6) from which the other two relations can be easily deduced.

To prove (2.1), we need the following lemma.

Lemma 2.2. *Assume that $m \geq 2$. For $0 \leq i \leq m-2$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} < \frac{(4m+2i+3)d_{i+1}(m)}{(4m+2i+7)d_{i+2}(m)}. \tag{2.7}$$

Proof. We proceed by induction on m. When $m = 2$, it is easy to check that the result holds. Assume that the theorem is valid for n, namely,

$$\frac{d_i(n)}{d_{i+1}(n)} < \frac{(4n+2i+3)d_{i+1}(n)}{(4n+2i+7)d_{i+2}(n)}, \qquad 0 \leq i \leq n-2. \tag{2.8}$$

We aim to show that (2.7) holds for $n+1$, that is

$$\frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{(4n+2i+7)d_{i+1}(n+1)}{(4n+2i+11)d_{i+2}(n+1)}, \qquad 0 \leq i \leq n-1. \tag{2.9}$$

From the recurrence relation (2.3), it is easy to check that for $0 \leq i \leq n-1$,

$$\begin{aligned}
&(2i+4n+7)d_{i+1}^2(n+1) - (2i+4n+11)d_i(n+1)d_{i+2}(n+1) \\
&\quad = (2i+4n+7)\left(\frac{i+n+1}{n+1}d_i(n) + \frac{2i+4n+5}{2(n+1)}d_{i+1}(n)\right)^2 \\
&\qquad - (2i+4n+11)\left(\frac{i+n+2}{n+1}d_{i+1}(n) + \frac{2i+4n+7}{2(n+1)}d_{i+2}(n)\right) \\
&\qquad\quad \times \left(\frac{n+i}{n+1}d_{i-1}(n) + \frac{2i+4n+3}{2(n+1)}d_i(n)\right) \\
&\quad = \frac{A_1(n,i) + A_2(n,i) + A_3(n,i)}{4(n+1)^2},
\end{aligned}$$

where $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are given by

$$\begin{aligned}
A_1(n,i) &= 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\
&\quad - 4(n+i)(2i+4n+11)(i+n+2)d_{i+1}(n)d_{i-1}(n), \\
A_2(n,i) &= (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\
&\quad - (2i+4n+3)(2i+4n+11)(2i+4n+7)d_i(m)d_{i+2}(n), \\
A_3(n,i) &= (8i^3 + 40i^2 + 58i + 32n^3 + 42n + 80n^2 + 120ni + 40i^2n + 64n^2i + 8) \\
&\quad \cdot d_{i+1}(n)d_i(n) - 2(n+i)(2i+4n+11)(2i+4n+7)d_{i+2}(n)d_{i-1}(n).
\end{aligned}$$

We are going to show that $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are all positive for $0 \le i \le n-2$. By the induction hypothesis (2.8), we find that for $0 \le i \le n-2$,

$$\begin{aligned}
A_1(n,i) >& 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\
&- 4(n+i)(2i+4n+11)(i+n+2)\frac{(4n+2i+1)}{(4n+2i+5)} d_i^2(n) \\
=& 4\frac{35+96n+72i+64ni+40n^2+28i^2}{2i+4n+5} d_i^2(n),
\end{aligned}$$

which is positive. From (2.8) it follows that for $0 \le i \le n-2$,

$$\begin{aligned}
A_2(n,i) >& (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\
&- (2i+4n+3)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)}{(4n+2i+7)} d_{i+1}^2(n) \\
=& (40i+80n+76) d_{i+1}^2(n),
\end{aligned}$$

which is positive. By the induction hypothesis (2.8), we see that for $0 \le i \le n-2$,

$$d_i(n)d_{i+1}(n) > \frac{(2i+4n+5)(2i+4n+7)}{(2i+4n+3)(2i+4n+1)} d_{i-1}(n)d_{i+2}(n). \tag{2.10}$$

In view of (2.10), we deduce that

$$\begin{aligned}
A_3(n,i) >& (8i^3+40i^2+58i+32n^3+42n+80n^2+120ni+40i^2n+64n^2i+8)d_{i+1}(n)d_i(n) \\
&- 2(n+i)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)(4n+2i+1)}{(4n+2i+5)(4n+2i+7)} d_{i+1}(n)d_i(n) \\
=& 8\frac{5+22n+30i+44ni+24n^2+16i^2}{2i+4n+5} d_{i+1}(n)d_i(n),
\end{aligned}$$

which is positive for $0 \le i \le n-2$. Hence the inequality (2.9) holds for $0 \le i \le n-2$. It remains to show that (2.9) is true for $i = n-1$, that is,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} < \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}. \tag{2.11}$$

From (1.2) it follows that

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1}, \tag{2.12}$$

$$d_{n+1}(n+1) = \frac{1}{2^{n+1}}\binom{2n+2}{n+1}, \tag{2.13}$$

$$d_n(n+2) = \frac{(n+1)(4n^2+18n+21)}{2^{n+4}(2n+3)}\binom{2n+4}{n+2}. \tag{2.14}$$

Consequently,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} = \frac{n(4n^2+10n+7)}{2(2n+1)(2n+3)} < \frac{(2n+3)(6n+5)}{2(6n+9)} = \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}.$$

This completes the proof. ∎

We are in a position to prove (2.1). In fact we shall prove a stronger inequality.

Lemma 2.3. *Assume that $m \geq 2$. For $0 \leq i \leq m-1$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} > \frac{(2i+4m+5)d_i(m+1)}{(2i+4m+3)d_{i+1}(m+1)}. \tag{2.15}$$

Proof. By Lemma 2.2, we have for $0 \leq i \leq m-1$,

$$d_i^2(m) > \frac{2i+4m+5}{2i+4m+1}d_{i-1}(m)d_{i+1}(m). \tag{2.16}$$

From (2.16) and the recurrence relation (2.3), we find that for $0 \leq i \leq m-1$,

$$\begin{aligned}
d_{i+1}&(m+1)d_i(m) - \frac{2i+4m+5}{2i+4m+3}d_{i+1}(m)d_i(m+1) \\
&= \frac{2i+4m+5}{2(m+1)}d_{i+1}(m)d_i(m) + \frac{i+m+1}{m+1}d_i(m)^2 \\
&\qquad - \frac{2i+4m+5}{2i+4m+3}\left(\frac{2i+4m+3}{2(m+1)}d_i(m)d_{i+1}(m) + \frac{i+m}{m+1}d_{i-1}(m)d_{i+1}(m)\right) \\
&= \frac{i+m+1}{m+1}d_i^2(m) - \frac{(4m+2i+5)(m+i)}{(4m+2i+3)(m+1)}d_{i-1}(m)d_{i+1}(m) \\
&> \left(\frac{m+1+i}{m+1} - \frac{(4m+2i+1)(m+i)}{(4m+2i+3)(m+1)}\right)d_i^2(m) \\
&= \frac{6m+4i+3}{(4m+2i+3)(m+1)}d_i^2(m) > 0,
\end{aligned}$$

which yields (2.15). This completes the proof of the lemma. ∎

We now turn to the proof of (2.2).

Lemma 2.4. *Assume that $m \geq 2$. For $0 \leq i \leq m-1$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} < \frac{d_{i+1}(m+1)}{d_{i+2}(m+1)}. \tag{2.17}$$

Proof. We proceed by induction on m. It is easily seen that the theorem holds for $m = 2$. We assume that the lemma is true for $n \geq 2$, i.e.,

$$\frac{d_i(n)}{d_{i+1}(n)} < \frac{d_{i+1}(n+1)}{d_{i+2}(n+1)}, \qquad 0 \leq i \leq n-1. \tag{2.18}$$

It will be shown that the theorem holds for $n+1$, that is,

$$\frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{d_{i+1}(n+2)}{d_{i+2}(n+2)}, \qquad 0 \leq i \leq n. \tag{2.19}$$

Recall that the sequence $\{d_i(n+1)\}_{0 \leq i \leq n+1}$ is unimodal. Furthermore, from (1.3) or the ratio monotone property [10], we see that the maximum element appears in the middle, namely, $d_i(n+1) < d_{i+1}(n+1)$ when $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$ and $d_i(n+1) > d_{i+1}(n+1)$ when $\left[\frac{n+1}{2}\right] \leq i \leq n$. We shall consider three cases. The first case is $d_i(n+1) < d_{i+1}(n+1)$, namely, $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$. From the recurrence relation (2.3), we find that for $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$,

$$\begin{aligned}
&d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) \\
&\quad = \frac{2i+4n+9}{2(n+2)} d_{i+1}^2(n+1) + \frac{i+n+2}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\qquad - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1) - \frac{i+n+3}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\quad = \frac{2i+4n+9}{2(n+2)} d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1) \\
&\qquad - \frac{1}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\quad > \frac{2i+4n+7}{2(n+2)} d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1),
\end{aligned}$$

which is positive by Lemma 2.2. It follows that for $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$,

$$d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) > 0. \tag{2.20}$$

Hence this completes the proof of the first case.

We now come to the second case $\left[\frac{n+1}{2}\right] \leq i \leq n-1$. From the recurrence relations (2.3) and (2.4), it follows that for $\left[\frac{n+1}{2}\right] \leq i \leq n-1$,

$$\begin{aligned}
&d_{i+1}(n+2)d_{i+1}(n+1) - d_{i+2}(n+2)d_i(n+1) \\
&\quad = \left(\frac{(4n-2i+5)(n+i+3)}{2(n+2)(n+1-i)} d_{i+1}(n+1) - \frac{(i+1)(i+2)}{(n+2)(n+1-i)} d_{i+2}(n+1) \right)
\end{aligned}$$

$$\times \left(\frac{n+1+i}{n+1}d_i(n) + \frac{4n+2i+5}{2(n+1)}d_{i+1}(n)\right)$$

$$- \left(\frac{n+3+i}{n+2}d_{i+1}(n+1) + \frac{4n+2i+11}{2(n+2)}d_{i+2}(n+1)\right)$$

$$\times \left(\frac{(4n-2i+3)(n+i+1)}{2(n+1)(n+1-i)}d_i(n) - \frac{i(i+1)}{(n+1)(n+1-i)}d_{i+1}(n)\right)$$

$$= B_1(n,i)d_{i+1}(n+1)d_i(n) + B_2(n,i)d_{i+1}(n+1)d_{i+1}(n)$$

$$+ B_3(n,i)d_{i+2}(n+1)d_i(n) + B_4(n,i)d_{i+2}(n+1)d_{i+1}(n),$$

where $B_1(n,i)$, $B_2(n,i)$, $B_3(n,i)$ and $B_4(n,i)$ are given by

$$B_1(n,i) = \frac{(n+i+3)(n+1+i)}{(n+2)(n+1-i)(n+1)}, \tag{2.21}$$

$$B_2(n,i) = \frac{(n+i+3)(16n^2+40n+25+4i)}{4(n+2)(n+1-i)(n+1)}, \tag{2.22}$$

$$B_3(n,i) = -\frac{(n+1+i)(41+16n^2+56n-4i)}{4(n+2)(n+1-i)(n+1)}, \tag{2.23}$$

$$B_4(n,i) = -\frac{(i+1)(4n+5-i)}{(n+2)(n+1-i)(n+1)}. \tag{2.24}$$

Since $\left[\frac{n+1}{2}\right] \le i \le n-1$, it follows from (1.3) that $d_{i+1}(n+1) > d_{i+2}(n+1)$ and $d_i(n) > d_{i+1}(n)$. Thus we get

$$d_{i+1}(n+1)d_i(n) > d_{i+1}(n+1)d_{i+1}(n), \tag{2.25}$$

$$d_{i+1}(n+1)d_{i+1}(n) > d_{i+2}(n+1)d_{i+1}(n). \tag{2.26}$$

Observe that $B_1(n,i)$ and $B_2(n,i)$ are positive, and $B_3(n,i)$ and $B_4(n,i)$ are negative. By the induction hypothesis (2.18), (2.25) and (2.26), we find that for $\left[\frac{n+1}{2}\right] \le i \le n-1$,

$$d_{i+1}(n+2)d_{i+1}(n+1) - d_{i+2}(n+2)d_i(n+1)$$

$$> \left(B_1(n,i) + B_2(n,i) + B_3(n,i) + B_4(n,i)\right) d_{i+1}(n+1)d_{i+1}(n)$$

$$= \frac{24n+10n^2-8ni+8i^2+13}{2(n+2)(n+1-i)(n+1)}d_{i+1}(n+1)d_{i+1}(n) > 0. \tag{2.27}$$

From the inequalities (2.20) and (2.27), it follows that (2.19) holds for $0 \le i \le n-1$. It is still necessary to show that (2.19) is true for $i = n$, that is,

$$\frac{d_n(n+1)}{d_{n+1}(n+1)} < \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)}. \tag{2.28}$$

For the recurrence relation (2.6), setting $i = n+2$, we find that

$$\frac{d_n(n+1)}{d_{n+1}(n+1)} = \frac{2n+3}{2} < \frac{2n+5}{2} = \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)},$$

as desired. Hence the proof is complete by induction. ∎

Therefore, from Lemmas 2.3 and 2.4 it immediately follows the interlacing log-concavity of the Boros-Moll polynomials.

3 Polynomials with triangular relations on coefficients

Many combinatorial polynomials admit triangular relations on the coefficients. The log-concavity of polynomials of this kind of polynomials have been extensively studied. We show that many classical polynomials are interlacingly log-concave. First, it is easy to check that the binomial coefficients, the Narayana numbers

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1},$$

and the Bessel numbers

$$B(n,k) = \frac{(2n-k-1)!}{2^k(n-k)!(k-1)!}$$

are interlacingly log-concave.

Moreover, we give a criterion that applies to many combinatorial sequences such as the Stirling numbers of the first kind without signs, the Stirling numbers of the second kind, and the Whitney numbers.

Theorem 3.1. *Suppose that for any $n \geq 0$,*

$$G_n(x) = \sum_{k=0}^{n} T(n,k)x^k$$

is a polynomial of degree n which has only real zeros, and suppose that the coefficients $T(n,k)$ satisfy a recurrence relation of the following triangular form

$$T(n,k) = f(n,k)T(n-1,k) + g(n,k)T(n-1,k-1).$$

If

$$\frac{(n-k)k}{(n-k+1)(k+1)} f(n+1,k+1) \leq f(n+1,k) \leq f(n+1,k+1) \tag{3.1}$$

and

$$g(n+1,k+1) \le g(n+1,k) \le \frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1), \qquad (3.2)$$

then the polynomials $G_n(x)$ are interlacingly log-concave.

Proof. Since the polynomial $G_n(x)$ has only real zeros, by Newton's inequality, we have

$$k(n-k)T(n,k)^2 \ge (k+1)(n-k+1)T(n,k-1)T(n,k+1).$$

Hence

$$\begin{aligned}
&T(n,k)T(n+1,k+1) - T(n+1,k)T(n,k+1)\\
&\quad = f(n+1,k+1)T(n,k)T(n,k+1) + g(n+1,k+1)T(n,k)^2\\
&\qquad - f(n+1,k)T(n,k)T(n,k+1) - g(n+1,k)T(n,k-1)T(n,k+1)\\
&\quad \ge (f(n+1,k+1) - f(n+1,k))\, T(n,k)T(n,k+1)\\
&\qquad + \left(\frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1) - g(n+1,k) \right) T(n,k-1)T(n,k+1),
\end{aligned}$$

which is positive by (3.1) and (3.2). It follows that

$$\frac{T(n,k)}{T(n,k+1)} \ge \frac{T(n+1,k)}{T(n+1,k+1)}. \qquad (3.3)$$

On the other hand, we have

$$\begin{aligned}
&T(n,k+1)T(n+1,k+1) - T(n,k)T(n+1,k+2)\\
&\quad = f(n+1,k+1)T(n,k+1)^2 + g(n+1,k+1)T(n,k)T(n,k+1)\\
&\qquad - f(n+1,k+2)T(n,k)T(n,k+2) - g(n+1,k+2)T(n,k+1)T(n,k)\\
&\quad \ge \left(f(n+1,k+1) - \frac{(n-k-1)(k+1)}{(n-k)(k+2)} f(n+1,k+2) \right) T(n,k+1)^2\\
&\qquad + (g(n+1,k+1) - g(n+1,k+2))T(n,k+1)T(n,k).
\end{aligned}$$

It follows from (3.1) that

$$\frac{T(n,k)}{T(n,k+1)} \le \frac{T(n+1,k+1)}{T(n+1,k+2)}. \qquad (3.4)$$

This completes the proof. ∎

Employing Theorem 3.1, we show that many combinatorial polynomials which have only real zeros are interlacingly log-concave. For example,

(1) The polynomials

$$x(x+1)(x+2)\cdots(x+n-1),$$

whose coefficients are the Stirling numbers of the first kind without signs, which satisfy the recurrence relation

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1);$$

(2) The Bell polynomials whose coefficients are the Stirling numbers of the second kind $S(n,k)$, which satisfy the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k);$$

(3) The Whitney polynomials

$$W_n(x) = \sum_{k=0}^{n} W_m(n,k)x^k,$$

which have only real zeros, see Benoumhani [3, 4]. The coefficients $W_m(n,k)$ satisfy the recurrence relation

$$W_m(n,k) = (1+mk)W_m(n-1,k) + W_m(n-1,k-1).$$

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