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On Three and Four Vicious Walkers

William Y. C. Chen[1], Donna Q. J. Dou[2], Terence Y. J. Zhang[3]

[1,3]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

[2]School of Mathematics
Jilin University, Changchun 130012, P. R. China

emails: [1]chen@nankai.edu.cn, [2]qjdou@jlu.edu.cn,
[3]terry@mail.nankai.edu.cn

Abstract. We establish a reflection principle for three lattice walkers and use this principle to reduce the enumeration of configurations of three vicious walkers to the enumeration of configurations of two vicious walkers. More precisely, the reflection principle leads to a bijection between three walks (L_1, L_2, L_3) such that L_2 intersects both L_1 and L_3 and three walks (L_1, L_2, L_3) such that L_1 intersects L_3. Hence we find a combinatorial interpretation of the formula for the generating function for the number of configurations of three vicious walkers, originally derived by Bousquet-Mélou by using the kernel method, and independently by Gessel by using tableaux and symmetric functions. This answers a question posed by Gessel and Bousquet-Mélou. We also find a reflection principle for four vicious walks that leads to a combinatorial interpretation of a formula derived from Gessel's theorem.

Keywords: vicious walkers, watermelon, Catalan numbers, Ballot numbers, reflection principle.

AMS Classification Numbers: 05A15; 82B23

1 Introduction

The vicious walker model was introduced by Fisher [5] in 1984. A walker is said to be vicious if he does not like to meet any other walker at any point. Formally speaking, a configuration of r vicious walkers, also called an r-vicious walk, of length n, is an r-tuple of pairwise nonintersecting lattice walks of length n, consisting of up steps U (i.e., $(1,1)$) and down steps D (i.e., $(1,-1)$), starting from $(0,2i_1),(0,2i_2),\ldots,(0,2i_r)$ and ending at $(n,e_1),(n,e_2),\ldots,(n,e_r)$ where $i_r > \cdots > i_2 > i_1 = 0$ and $e_r > \cdots > e_2 > e_1$. More precisely, two lattice paths are said to be nonintersecting if they do not share any common points. In particular, an r-watermelon of length n is a configuration consisting of nonintersecting lattice paths of length n which start at the points $(0,0),(0,2),\ldots,(0,2r-2)$ and end at the points $(n,k),(n,k+2),\ldots,(n,k+2r-2)$ for some k. In other words, an r-watermelon is an r-vicious walk starting from adjacent points and ending at adjacent points. Note that two lattice points are said to be adjacent if they are on the same vertical line and their y-coordinates differ by 2. It is known that r-vicious walks can be represented by tableaux. So the theory of symmetric functions plays an important role in the study of r-vicious walks, see [10, 11, 12, 13, 15, 16].

The main objective of this paper is to present a combinatorial approach to the enumeration of 3-vicious walks and 4-vicious walks. Let us fix the starting points $(0,0)$, $(0,2i)$ and $(0,2i+2j)$. Let $V(i,j,n)$ be the set of 3-vicious walks (L_1,L_2,L_3) of length n, where L_1 is the path of the first walker starting from $(0,0)$, L_2 is the path of the second walker starting from $(0,2i)$, and L_3 is the path of the third walker starting from $(0,2i+2j)$. Define the generating function $V_{i,j}(t)$ to be

$$V_{i,j}(t) = \sum_{n=0}^{\infty} |V(i,j,n)| t^n, \tag{1.1}$$

where $|\cdot|$ denotes the cardinality of a set.

The enumeration of configurations of three vicious walkers has been solved independently by Bousquet-Mélou [1] by using the obstinate kernel method, and by Gessel [9] by using tableaux and symmetric functions. They obtained a formula for $V_{i,j}(t)$ in terms of the generating function of the Catalan numbers.

Let $C(t)$ be the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, that is,

$$C(t) = \sum_{n=0}^{\infty} C_n t^n.$$

Recall that $C(t)$ satisfies the recurrence relation

$$C(t) = 1 + tC^2(t). \tag{1.2}$$

Let

$$D(t) = tC^2(t) = C(t) - 1 = \sum_{n=0} C_{n+1} t^{n+1}. \tag{1.3}$$

The following elegant formula is due to Bousquet-Mélou [1] and Gessel [9].

Theorem 1.1 (Bousquet-Mélou [1] and Gessel [9])**.**

$$V_{i,j}(t) = \frac{1}{1-8t}(1-D^i(2t))(1-D^j(2t)). \tag{1.4}$$

In view of the relation (1.3) and the identity

$$\left(\frac{1+D(t)}{1-D(t)}\right)^2 = \frac{1}{1-4t}, \tag{1.5}$$

Gessel derived the following formula for $V_{i,j}(t)$.

Theorem 1.2 (Gessel [9])**.** *For any $i, j \geq 1$,*

$$V_{i,j}(t) = C^2(2t)\big(1 + D(2t) + \cdots + D^{i-1}(2t)\big)\big(1 + D(2t) + \cdots + D^{j-1}(2t)\big). \tag{1.6}$$

Both Bousquet-Mélou [1] and Gessel [9] proposed the problem of finding a combinatorial interpretation for the above formula for $V_{i,j}(t)$. The question of Bousquet-Mélou is concerned with the formula (1.4), while the question of Gessel is concerned with the formula in the form of (1.6). In this paper, we shall present a combinatorial interpretation of (1.4). As will be seen, the algebraic manipulations to transform the formula (1.4) to (1.6) can be explained combinatorially. So we have obtained combinatorial interpretations for both formulas (1.4) and (1.6).

In Section 3, we present an approach to the enumeration of 2-vicious walks. By reformulating the problem in terms of pairs of intersecting walks, we give a decomposition of a pair of converging walks, namely, two walks that do not intersect until they reach the same ending point, into two-chain watermelons, or 2-watermelons. Then we use Pólya's formula for the number of 2-watermelons of length n to derive the formula for the number of 2-vicious walks of length n. In Section 4, we find a connection between the Labelle merging algorithm, in the form presented by Chen, Pang, Qu and Stanley [3], and the classical ballot numbers. In the last section, we give a reflection principle for the enumeration of configurations of 4-vicious walks with prescribed starting points. More precisely, we give a combinatorial proof of a formula for the number of 4-vicious walks derived from Gessel's theorem [9].

2 The Reflection Principle

In this section, we give a reflection principle so that we can reduce the enumeration of 3-vicious walks to that of 2-vicious walks. This reduction leads to a combinatorial interpretation of the formula (1.4) for the generating function $V_{i,j}(t)$ of the number of 3-vicious walks as defined by (1.1).

Let us recall some basic definitions. Two walks L_1 and L_2 are said to be intersecting, denoted $L_1 \cap L_2 \neq \emptyset$, if L_1 and L_2 share a common point. Let $U(i, j, n)$ be the set of all 3-walks (L_1, L_2, L_3) of length n, where L_1, L_2 and L_3 start from $(0, 0)$, $(0, 2i)$ and $(0, 2i + 2j)$ respectively. Let

$$U_{i,j}(t) = \sum_{n=0}^{\infty} |U(i, j, n)| t^n.$$

It is obvious that

$$U_{i,j}(t) = \frac{1}{1 - 8t}. \tag{2.1}$$

We use $W_{12}(n)$, or W_{12} for short, to denote the set of 3-walks (L_1, L_2, L_3) in $U(i, j, n)$ such that L_1 and L_2 are nonintersecting. Similarly, we use $W_{23}(n)$, or W_{23} for short, to denote the set of 3-walks (L_1, L_2, L_3) in $U(i, j, n)$ such that L_2 and L_3 are nonintersecting. It is readily seen that the set $V(i, j, n)$ of three vicious walks of length n can be expressed as $W_{12} \cap W_{23}$. By the principle of inclusion and exclusion, we find that

$$|V(i, j, n)| = |W_{12} \cap W_{23}| = |W_{12}| + |W_{23}| - |W_{12} \cup W_{23}|. \tag{2.2}$$

In order to compute $|W_{12} \cup W_{23}|$, we let $M_{12,23}(n)$, or $M_{12,23}$ for short, denote the set of 3-walks (L_1, L_2, L_3) in $U(i, j, n)$ such that L_2 intersects both L_1 and L_3. Hence we have

$$|W_{12} \cup W_{23}| = |U(i, j, n)| - |M_{12,23}|. \tag{2.3}$$

We are now in a position to establish a reflection principle for the enumeration of the 3-walks in $M_{12,23}(n)$. Let $M_{13}(n)$, or M_{13} for short, denote the set of 3-walks (L_1, L_2, L_3) in $U(i, j, n)$ such that L_1 intersects L_3.

Theorem 2.1. *For $n \geq 1$, there exists a bijection between $M_{12,23}(n)$ and $M_{13}(n)$.*

Proof. We proceed to construct a map Φ from $M_{12,23}(n)$ to $M_{13}(n)$. Let (L_1, L_2, L_3) be a 3-walk in $M_{12,23}(n)$. We consider the following two cases. If $L_1 \cap L_3 \neq \emptyset$, then $(L_1, L_2, L_3) \in M_{13}(n)$. In this case, we define $\Phi((L_1, L_2, L_3)) = (L_1, L_2, L_3)$.

Now we may assume that $L_1 \cap L_3 = \emptyset$. We first consider the case that L_2 meets L_1 before it meets L_3. Suppose that P is the first intersection point of L_2 and L_1. Apply the usual reflection operation on L_1 and L_2, and denote the resulting paths by L_1' and L_2', namely, L_1' consists of the first segment of L_1 up to the point P followed by the last segment of L_2 starting from the point P, and L_2' consists of the first segment of L_2

up to the point P followed by the last segment of L_1 starting from the point P. Figure 2.1 is an illustration of the reflection.

Let $L_3' = L_3$ and let $\Phi((L_1, L_2, L_3)) = (L_1', L_2', L_3')$. It can be seen that L_1' intersects L_3'. Thus we have $(L_1', L_2', L_3') \in M_{13}(n)$.

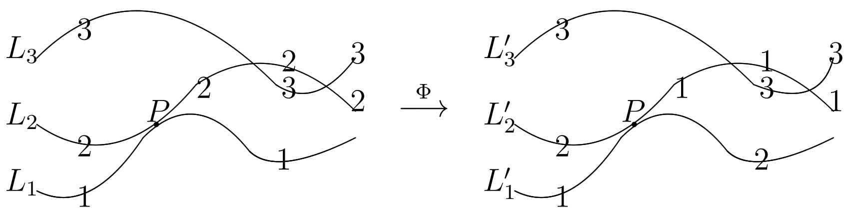

Figure 2.1: The reflection principle.

It is not difficult to see that the above procedure is reversible. We are still left with the case when L_2 intersects L_3 before meeting L_1. This case can be dealt with in the same manner. Thus we have reached the conclusion that Φ is a bijection. ∎

Combining (2.2), (2.3) and Theorem 2.1, we obtain the following relation

$$|V(i,j,n)| = |W_{12}| + |W_{23}| + |M_{13}| - |U(i,j,n)|. \tag{2.4}$$

Let W_{13} be the set of three walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 never meets L_3, and define the generating functions for $|W_{12}|$, $|W_{23}|$ and $|W_{13}|$ by $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$ respectively. From (2.4) it follows that

$$|V(i,j,n)| = |W_{12}| + |W_{23}| - |W_{13}|. \tag{2.5}$$

Proposition 2.2.

$$V_{i,j}(t) = W_{12}(t) + W_{23}(t) - W_{13}(t). \tag{2.6}$$

The above formula can be viewed as a reduction of the enumeration of 3-vicious walks to the enumeration of 2-vicious walks. Let $N(i,n)$ be the set of 2-vicious walks (L_1, L_2) of length n starting at $(0,0)$ and $(0,2i)$ respectively, and denote the corresponding generating function by

$$N_i(t) = \sum_{n=0}^{\infty} |N(i,n)| t^n.$$

Bousquet-Mélou [1] and Gessel [9] derived the following formula

$$N_i(t) = \frac{1}{1-4t}(1 - D^i(t)). \tag{2.7}$$

As pointed out by Gessel [9], the above expression for $N_i(2t)$ can be deduced from the formula (1.6) for $V_{i,j}(t)$ by taking the limit $j \to \infty$, and by using the identity (1.5).

Using the above expression for $N_i(t)$, we can derive the following formulas for the generating functions $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$:

$$W_{12}(t) = \frac{1 - D^i(2t)}{1 - 8t}, \; W_{23}(t) = \frac{1 - D^j(2t)}{1 - 8t}, \; W_{13}(t) = \frac{1 - D^{i+j}(2t)}{1 - 8t}. \tag{2.8}$$

Now we see that the formula (1.4) in Theorem 1.1 follows from the above formulas and the relation (2.6).

We note that Gessel [9] obtained the following identity

$$V_{i,j}(t) = N_i(2t) + N_j(2t) - N_{i+j}(2t), \tag{2.9}$$

in agreement with the combinatorial statement (2.6) derived from the reflection principle.

As to the question of finding a combinatorial interpretation of the generating function formula (1.4), the reflection principle (Theorem 2.1) along with the combinatorial interpretations of the formulas for $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$ can be considered as an answer. Moreover, it is easy to deduce (1.6) from (1.4) by utilizing the identity (1.5), which can be explained combinatorially in two steps. The first step is to use the identity

$$4^n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k}, \tag{2.10}$$

which is equivalent to the identity

$$\sum_{n=0}^{\infty} \binom{2n}{n} t^n = \frac{1}{\sqrt{1-4t}}. \tag{2.11}$$

There are several combinatorial proofs of (2.10), see, for example, Kleitman [14] and Sved [22]. The second step is to show that

$$\frac{1 + D(t)}{1 - D(t)} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n. \tag{2.12}$$

Note that $\frac{1+D(t)}{1-D(t)}$ can be written as $\frac{C(t)}{1-tC^2(t)}$. A combinatorial interpretation of the identity

$$\frac{C(t)}{1 - tC^2(t)} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n$$

is given by Chen, Li and Shapiro [2] in terms of doubly rooted plane trees and the butterfly decomposition.

3 Converging Walks and 2-Watermelons

In this section, we present an approach to computing the number of the 2-vicious walks by counting pairs of converging walks. A pair of walks is said to be converging if they never meet until they reach a common ending point. We shall show that pairs of converging walks can be counted by applying Pólya's formula for two-chain watermelons, or 2-watermelons [19]. More specifically, we shall give a decomposition of a pair of converging walks into 2-watermelons.

Like the definition of $M_{13}(n)$ given in the previous section, we let $M_{12}(n)$, or M_{12} for short, be the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 intersects L_2. Similarly, we can define $M_{23}(n)$, or M_{23} for short. Then we have the following relations

$$|M_{12}| = |U(i,j,n)| - |W_{12}|, \quad |M_{23}| = |U(i,j,n)| - |W_{23}|.$$

From (2.4) it follows that

$$|V(i,j,n)| = |U(i,j,n)| + |M_{13}| - |M_{12}| - |M_{23}|.$$

Let $M_{12}(t)$, $M_{23}(t)$ and $M_{13}(t)$ denote the generating functions for $|M_{12}(n)|$, $|M_{23}(n)|$ and $|M_{13}(n)|$, respectively.

Proposition 3.1. *We have*

$$V_{i,j}(t) = U_{i,j}(t) - M_{12}(t) - M_{23}(t) + M_{13}(t). \tag{3.1}$$

It will be shown that $M_{12}(t)$, $M_{13}(t)$ and $M_{23}(t)$ can be computed by using the following formula for the number of 2-watermelons as derived by Levine [18] and Pólya [19], see also, Fürlinger and Hofbauer [6], Gessel [7], and Shapiro [21].

Proposition 3.2. *The number of 2-watermelons with each walk having n steps is C_{n+1}.*

Using the above formula, one sees that the generating function of the number of 2-watermelons equals $C^2(t)$. Note that 2-watermelons of length n correspond to pairs of converging walks of length $n+1$ with adjacent starting points. In general, let $T(i,n)$ be the set of pairs of converging walks (L_1, L_2) of length n, where L_1 starts from $(0,0)$ and L_2 starts from $(0,2i)$. Define

$$T_i(t) = \sum_{n\geq 0} |T(i,n)| t^n.$$

Proposition 3.3. *For any $i \geq 1$, $T_i(t) = D^i(t)$.*

Proof. Let $L_1 = A_0A_1 \ldots A_n$ and $L_2 = B_0B_1 \ldots B_n$, where a walk is represented by a sequence of points. For $0 \le k \le i$, let j_k be the minimum index such that the difference of the y-coordinates of (A_{j_k}, B_{j_k}) equals to $2i - 2k$. It is clear that $j_0 = 0$ and $j_i = n$. We now decompose (L_1, L_2) into i 2-walks: $(L_1^{(1)}, L_2^{(1)}), \ldots, (L_1^{(i)}, L_2^{(i)})$, where $L_1^{(k)} = A_{j_{k-1}}A_{j_{k-1}+1} \ldots A_{j_k}$ and $L_2^{(k)} = B_{j_{k-1}}B_{j_{k-1}+1} \ldots B_{j_k}$. Figure 3.1 is an illustration of the decomposition.

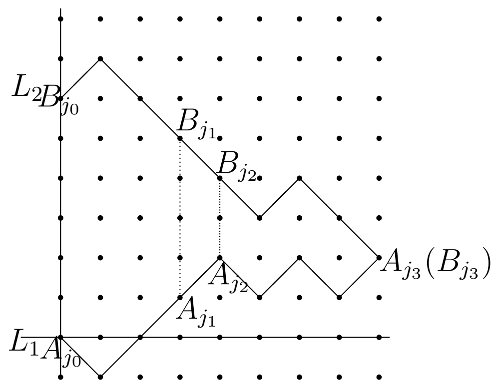

Figure 3.1: The decomposition of a pair of converging walks.

Observe that by the choice of j_k, the rightmost pair of steps in $(L_1^{(k)}, L_2^{(k)})$ must be (U, D). Moreover, if we delete this pair of steps, the resulting upper walk can be lowered by $2i - 2k$ units without intersecting the lower walk to form a 2-watermelon. An example is given in Figure 3.2.

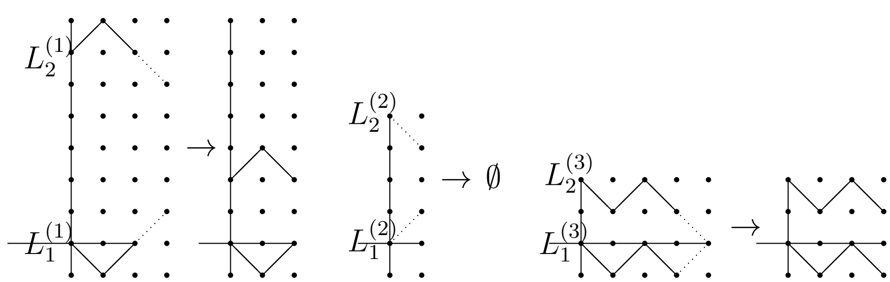

Figure 3.2: From 2-walks to 2-watermelons

By Proposition 3.2, the generating function for the number of 2-walks $(L_1^{(k)}, L_2^{(k)})$ equals $D(t) = t \cdot C^2(t)$. This completes the proof. ∎

Let $M(i, n)$ be the set of intersecting 2-walks (L_1, L_2) of length n, where L_1 and L_2

start from $(0,0), (0,2i)$ respectively. Define

$$M_i(t) = \sum_{n\geq 0} |M(i,n)| t^n.$$

Observe that every pair of intersecting paths (L_1, L_2) can be decomposed into a pair of converging paths and a pair of arbitrary paths starting from the same point. Thus we have the following formula.

Corollary 3.4. *For any $i \geq 1$,*

$$M_i(t) = \frac{D^i(t)}{1-4t}. \tag{3.2}$$

Notice that

$$M_i(t) + N_i(t) = \frac{1}{1-4t}. \tag{3.3}$$

So the formula (2.7) for $N_i(t)$ can be deduced from the above relation (3.3). Moreover, it is easy to see that $M_{12}(t)$, $M_{23}(t)$ and $M_{13}(t)$ can be computed by using the above formula (3.2) for $M_i(t)$. So we get

$$M_{12}(t) = \frac{D^i(2t)}{1-8t}, \quad M_{23}(t) = \frac{D^j(2t)}{1-8t}, \quad M_{13}(t) = \frac{D^{i+j}(2t)}{1-8t}, \tag{3.4}$$

in agreement with (2.8). Substituting (3.4) into (3.1), we obtain Theorem 1.1.

We note that by Proposition 3.3 and the Lagrange inversion formula, we can deduce that

$$|T(i,n)| = \frac{i}{n}\binom{2n}{n-i}. \tag{3.5}$$

So the formula (2.7) implies an explicit expression for $|N(i,n)|$:

$$|N(i,n)| = 4^n - \sum_{k=i}^{n} \frac{i}{k}\binom{2k}{k-i} 4^{n-k}. \tag{3.6}$$

As will be seen in the next section, the set $T(i,n)$ of converging paths are in one-to-one correspondence with partial Dyck paths which are counted by the ballot numbers.

4 Connection to the Ballot Numbers

In this section, we put the Labelle merging algorithm in a more general setting, and show that the direct correspondence formulated by Chen, Pang, Qu and Stanley [3] leads to a connection between pairs of converging walks and the classical ballot numbers.

We shall represent a walk as a sequence of steps rather than points. Let (L_1, L_2) be a 2-watermelon of length n, and let $L_1 = p_1p_2\cdots p_n$ and $L_2 = q_1q_2\cdots q_n$, where $p_i, q_i = U$ or D. Set $U' = D$ and $D' = U$. Using the direct correspondence in [3], the watermelon (L_1, L_2) can be represented by a Dyck path of length $2n+2$:

$$Uq_1p_1'q_2p_2'\cdots q_np_n'D.$$

It is not difficult to see that the above map is a bijection. Figure 4.1 gives an illustration.

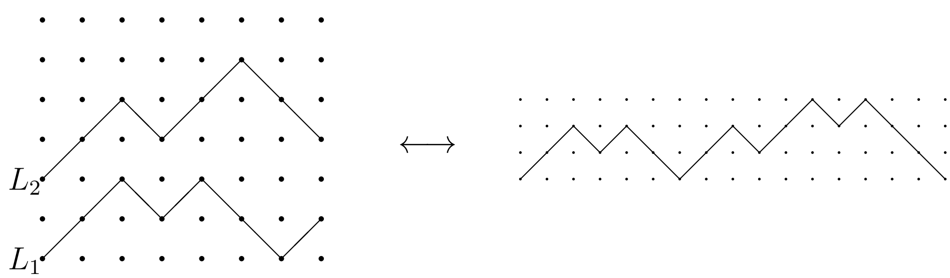

Figure 4.1: From a 2-watermelon to a Dyck path.

Using the above merging algorithm, we may encode a pair of converging walks (L_1, L_2) in $T(i,n)$ by a partial Dyck path P in the sense that P does not have to start from the origin but it ends up with a point on the x-axis. We should note that a partial Dyck path is usually defined as a lattice path starting from the origin $(0,0)$ with up and down steps that does not go below the x-axis. Define $P(i,n)$ to be the set of partial Dyck paths of length $2n$ which start from $(0,2i)$ and never return to the x-axis except for the final destination. It is well known that the number of partial Dyck paths in $P(i,n)$ is given by the classical ballot number. The following proposition establishes a connection between converging walks and partial Dyck paths.

Proposition 4.1. *For $n \geq 1$, there exists a bijection between $T(i,n)$ and $P(i,n)$.*

Proof. Given a pair of converging walks (L_1, L_2) in $T(i,n)$, let $L_1 = p_1p_2\cdots p_n$ and $L_2 = q_1q_2\cdots q_n$, where $p_i, q_i = U$ or D. Then (L_1, L_2) can be represented by a partial Dyck path P of length $2n$ starting from $(0,2i)$:

$$P = q_1p_1'q_2p_2'\cdots q_np_n'.$$

Clearly, P returns to the x-axis at the ending point and never touches the x-axis before the ending point, that is, $P \in P(i,n)$. It is easy to check that the above correspondence is a bijection. Figure 4.2 is an illustration. ∎

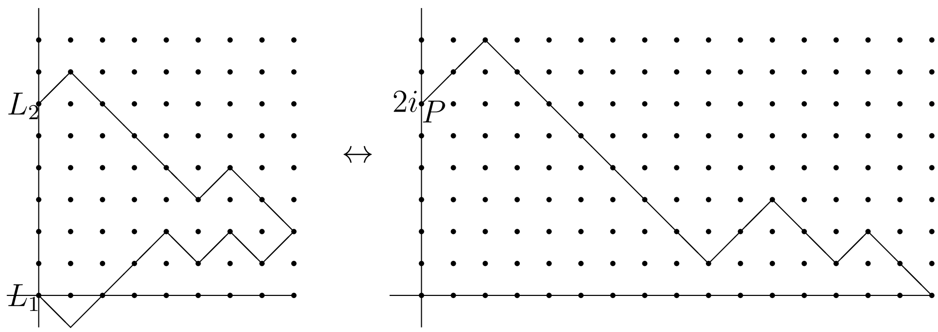

Figure 4.2: From a pair of converging walks to a partial Dyck path

It is easy to see that there is a one-to-one correspondence between $P(i,n)$ and the set of Dyck paths of length $2n+2i$ with $2i$ returns to the x-axis, which implies the same formula for $|P(i,n)|$ as the formula (3.5) for $|T(i,n)|$, see Deutsch [4]. On the other hand, $|P(i,n)|$ can be expressed as the classical ballot number $b(n+i-1,n-i)$, where

$$b(n,i)=\binom{n+i}{i}-\binom{n+i}{i-1}=\frac{n+1-i}{n+1+i}\binom{n+i+1}{i},$$

see, for example, Riordan [20].

5 4-Vicious Walkers

In this section, we present a reflection principle for 4-vicious walks that leads to a reduction from 4-vicious walks to 2-vicious walks. We first give some definitions. Let $U(i,j,k,n)$ be the set of 4-walks (L_1,L_2,L_3,L_4) of length n, where L_1, L_2, L_3 and L_4 start from $(0,0)$, $(0,2i)$, $(0,2i+2j)$ and $(0,2i+2j+2k)$ respectively. Let $V(i,j,k,n)$ be the set of 4-vicious walks (L_1,L_2,L_3,L_4) in $U(i,j,k,n)$. Define the generating function $V_{i,j,k}(t)$ by

$$V_{i,j,k}(t)=\sum_{n\geq 0}|V(i,j,k,n)|t^n.$$

The following formula for $|V(i,j,k,n)|$ is a consequence of Gessel's theorem [9]. Let $v(i,n)$ denote the number of 2-vicious walks in $N(i,n)$ as given by the generating function (2.7). Recall that an explicit formula for $v(i,n)$ is given by (3.6).

Theorem 5.1. *For any $i,j,k\geq 1$, we have*

$$|V(i,j,k,n)|=v(i,n)v(k,n)-v(i+j,n)v(j+k,n)+v(i+j+k,n)v(j,n). \tag{5.1}$$

In order to give a combinatorial interpretation of the above formula (5.1), we shall establish a reflection principle for certain classes of 4-vicious walks. For $1 \leq r < s \leq 4$, we use $W_{rs}(n)$, or W_{rs} for short, to denote the set of 4-walks (L_1, L_2, L_3, L_4) in $U(i, j, k, n)$ such that L_r and L_s are nonintersecting. Similarly, for $1 \leq r < s \leq 4$, we use $M_{rs}(n)$, or M_{rs} for short, to denote the set of 4-walks (L_1, L_2, L_3, L_4) in $U(i, j, k, n)$ such that L_r and L_s are intersecting. Clearly, the set $V(i, j, k, n)$ of 4-vicious walks of length n can be expressed as $W_{12} \cap W_{23} \cap W_{34}$. It is also clear that

$$|W_{12} \cap W_{23} \cap W_{34}| = |W_{12} \cap W_{34}| - |W_{12} \cap M_{23} \cap W_{34}|. \tag{5.2}$$

Note that it is easy to find a formula for $|W_{12} \cap W_{34}|$ based on the number of 2-vicious walks. To compute $|W_{12} \cap M_{23} \cap W_{34}|$, we may use the following reflection principle.

Theorem 5.2. *There exists a bijection between the set $W_{13} \cap W_{24}$ and the set $(W_{14} \cap W_{23}) \cup (W_{12} \cap M_{23} \cap W_{34})$.*

Proof. We proceed to construct a map ψ from $W_{13} \cap W_{24}$ to $(W_{14} \cap W_{23}) \cup (W_{12} \cap M_{23} \cap W_{34})$. Let (L_1, L_2, L_3, L_4) be a 4-walk in $W_{13} \cap W_{24}$. We have the following situations.

(1) $L_1 \cap L_2 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$ and $L_3 \cap L_4 = \emptyset$. Evidently, $(L_1, L_2, L_3, L_4) \in W_{12} \cap M_{23} \cap W_{34}$. In this case, no operation is needed for ψ, that is, $\psi((L_1, L_2, L_3, L_4)) = (L_1, L_2, L_3, L_4)$.

(2) $L_1 \cap L_4 = \emptyset$ and $L_2 \cap L_3 = \emptyset$. So we have $(L_1, L_2, L_3, L_4) \in W_{14} \cap W_{23}$. In this case, we simply do nothing as in Case (1).

(3) $L_2 \cap L_3 \neq \emptyset$, either $L_1 \cap L_2 \neq \emptyset$ or $L_3 \cap L_4 \neq \emptyset$, L_2 meets L_3 before it meets L_1 (when $L_1 \cap L_2 = \emptyset$, we naturally assume that L_2 meets L_3 before it meets L_1), L_3 meets L_2 before it meets L_4 (when $L_3 \cap L_4 = \emptyset$, we naturally assume that L_3 meets L_2 before it meets L_4). In this case, we apply the usual reflection operation on L_2 and L_3, and denote the resulting paths by L_2' and L_3', respectively. Let $L_1' = L_1$, $L_4' = L_4$ and let $\psi((L_1, L_2, L_3, L_4)) = (L_1', L_2', L_3', L_4')$. It is easy to see that $(L_1', L_2', L_3', L_4') \in W_{12} \cap M_{23} \cap W_{34}$.

(4) $L_1 \cap L_4 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$, $L_1 \cap L_2 \neq \emptyset$, and L_2 meets L_1 before it meets L_3.

(5) $L_1 \cap L_4 \neq \emptyset$, $L_2 \cap L_3 \neq \emptyset$, L_2 meets L_1 before it meets L_3, and L_3 meets L_2 before it meets L_4.

(6) $L_1 \cap L_4 \neq \emptyset$, $L_2 \cap L_3 = \emptyset$.

In Cases (4), (5) and (6), we apply the usual reflection operation on L_1 and L_2, and denote the resulting paths by L_1' and L_2', respectively. Let $L_3' = L_3$, $L_4' = L_4$ and

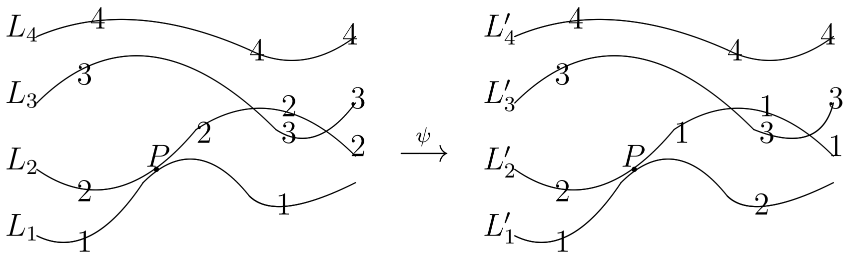

Figure 5.1: The action of ψ on a 4-walk in Case (4).

let $\psi((L_1, L_2, L_3, L_4)) = (L_1', L_2', L_3', L_4')$. Then we have $(L_1', L_2', L_3', L_4') \in W_{14} \cap W_{23}$. Figure 5.1 is an illustration of the reflection operation on (L_1, L_2, L_3, L_4) in Case (4).

(7) $L_1 \cap L_2 = \emptyset$, $L_1 \cap L_4 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$, $L_3 \cap L_4 \neq \emptyset$, and L_3 meets L_4 before it meets L_2.

(8) $L_1 \cap L_2 \neq \emptyset$, $L_1 \cap L_4 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$, $L_3 \cap L_4 \neq \emptyset$, L_3 meets L_4 before it meets L_2, and L_2 meets L_3 before it meets L_1.

(9) $L_1 \cap L_4 \neq \emptyset$, $L_2 \cap L_3 \neq \emptyset$, L_3 meets L_4 before it meets L_2.

In Cases (7), (8) and (9), we use the usual reflection operation on L_3 and L_4, and denote the resulting paths by L_3' and L_4', respectively. Let $L_1' = L_1$, $L_2' = L_2$ and let $\psi((L_1, L_2, L_3, L_4)) = (L_1', L_2', L_3', L_4')$. It is easy to check that $(L_1', L_2', L_3', L_4') \in W_{14} \cap W_{23}$.

It is not difficult to see that the above procedure is reversible. Thus we have reached the assertion that ψ is a bijection. ∎

Using the above reflection principle, we can give a combinatorial proof of Theorem 5.1. In view of the bijection in Theorem 5.2, we find that

$$|W_{12} \cap M_{23} \cap W_{34}| = |W_{13} \cap W_{24}| - |W_{14} \cap W_{23}|. \tag{5.3}$$

Substituting (5.3) into (5.2), we deduce that

$$|W_{12} \cap W_{23} \cap W_{34}| = |W_{12} \cap W_{34}| - |W_{13} \cap W_{24}| + |W_{14} \cap W_{23}|. \tag{5.4}$$

It is evident that a 4-walk in $W_{12} \cap W_{34}$ corresponds to a pair of 2-vicious walks (V_1, V_2), where V_1 is a 2-vicious walk (L_1, L_2) of length n, with L_1 and L_2 starting from $(0, 0)$ and $(0, 2i)$ respectively, and V_2 is a 2-vicious walk (L_3, L_4) of length n, with L_3 and L_4 starting from $(0, 2i + 2j)$ and $(0, 2i + 2j + 2k)$ respectively. Consequently,

$$|W_{12} \cap W_{34}| = v(i, n)v(k, n).$$

Similarly, we have

$$|W_{13} \cap W_{24}| = v(i + j, n)v(j + k, n),$$

and

$$|W_{14} \cap W_{23}| = v(i+j+k, n) \cdot v(j, n).$$

Hence we arrive at (5.1). This completes the proof. ∎

It would be interesting to find a general reflection principle for r-vicious walks for $r > 4$.

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