Noncrossing Linked Partitions and Large (3, 2)-Motzkin Paths

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Abstract. Noncrossing linked partitions arise in the study of certain transforms in free probability theory. We explore the connection between noncrossing linked partitions and (3, 2)-Motzkin paths, where a (3, 2)-Motzkin path can be viewed as a Motzkin path for which there are three types of horizontal steps and two types of down steps. A large (3, 2)-Motzkin path is a (3, 2)-Motzkin path for which there are only two types of horizontal steps on the x-axis. We establish a one-to-one correspondence between the set of noncrossing linked partitions of \{1, \ldots, n + 1\} and the set of large (3, 2)-Motzkin paths of length n, which leads to a simple explanation of the well-known relation between the large and the little Schröder numbers.

Keywords: Noncrossing linked partition, Schröder path, large (3, 2)-Motzkin path, Schröder number

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1 Introduction

The notion of noncrossing linked partitions was introduced by Dykema [5] in the study of the unsymmetrized T-transform in free probability theory. Let \([n]\) denote \(\{1, \ldots, n\}\). It has been shown that the generating function of the number of noncrossing linked partitions of \([n+1]\) is given by

\[
F(x) = \sum_{n=0}^{\infty} f_{n+1} x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.
\]  

(1.1)

This implies that the number of noncrossing linked partitions of \([n+1]\) is equal to the \(n\)-th large Schröder number \(S_n\), that is, the number of large Schröder paths of length \(2n\). To be more specific, a large Schröder path of length \(2n\) is a lattice path from \((0,0)\) to \((2n,0)\) consisting of up steps \((1,1)\), horizontal steps \((2,0)\) and down steps \((1,-1)\) that does not go below the \(x\)-axis. Notice that a large Schröder path is also called a Schröder path. The first few values of \(S_n\) are given below

\[1, 2, 6, 22, 90, 394, 1806, \ldots\]

The sequence of the large Schröder numbers is listed as entry A006318 in OEIS [8].

A bijection from the set of noncrossing linked partitions of \([n+1]\) to the set of large Schröder paths of length \(2n\) was established by Chen, Wu, and Yan [2].

In this paper, we aim to construct an explicit correspondence between noncrossing linked partitions and \((3,2)\)-Motzkin paths. Recall that a Motzkin path of length \(n\) is defined as a lattice path from \((0,0)\) to \((n,0)\) consisting of up steps \((1,1)\), horizontal steps \((1,0)\) and down steps \((1,-1)\) that does not go below the \(x\)-axis. A \((3,2)\)-Motzkin path is a Motzkin path for which each horizontal step colored by one of the three colors 1, 2, and 3, and each down step colored by one of the two colors 1 and 2.

It is known that the number of little Schröder paths of length \(2n\) equals the number of \((3,2)\)-Motzkin paths of length \(n - 1\), where a little Schröder path is defined as a large Schröder path such that there are no horizontal steps on the \(x\)-axis. Yan [10] found a bijective proof of this fact. The number of little Schröder paths of length \(2n\) is referred to as the little Schröder number \(s_n\). Since the large Schröder numbers and the little Schröder numbers are related by a factor of two, we see that the number of noncrossing linked partitions of \([n+1]\) is twice the number of \((3,2)\)-Motzkin paths of length \(n\).

In this paper, we introduce a class of Motzkin paths, called large \((3,2)\)-Motzkin paths, which are defined as \((3,2)\)-Motzkin paths such that each horizontal step at the \(x\)-axis is colored by one of the two colors 1 and 2. We shall show that noncrossing linked partitions of \([n+1]\) are in one-to-one correspondence with large \((3,2)\)-Motzkin paths.
paths of length $n$. By examining the connection between large $(3, 2)$-Motzkin paths and ordinary $(3, 2)$-Motzkin paths, we immediately get the relation between the large and the little Schröder numbers.

Let us give a brief review of some terminology. Let $m_n$ denote the $n$-th $(3, 2)$-Motzkin number, that is, the number of $(3, 2)$-Motzkin paths with $n$ steps. An irreducible large $(3, 2)$-Motzkin path is defined as a large $(3, 2)$-Motzkin path that does not touch the $x$-axis except for the origin and the destination. Bear in mind that a horizontal step on the $x$-axis is considered as an irreducible large $(3, 2)$-Motzkin path. The length of a path is defined to be the number of steps in the path. Denote the set of large $(3, 2)$-Motzkin paths by $L$ and the set of large $(3, 2)$-Motzkin paths of length $n$ by $L_n$. Let $l_n$ be the number of paths in $L_n$.

By the decomposition of a large $(3, 2)$-Motzkin path into irreducible segments, we see that the generating function $L(x) = \sum_{n=0}^{\infty} l_n x^n$ satisfies the functional equation

$$L(x) = 1 + 2xL(x) + 2x^2M(x)L(x), \quad (1.2)$$

where

$$M(x) = \sum_{n=0}^{\infty} m_n x^n = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x^2} \quad (1.3)$$

is the generating function of the $(3, 2)$-Motzkin numbers. A similar decomposition has been used by Cheon, Lee, and Shapiro [3] to derive generating function identities for the Catalan numbers and the Fine numbers. From (1.2) and (1.3) it follows that $L(x) = F(x)$. This yields

$$l_n = f_{n+1}. \quad (1.4)$$

Using the connection between the large $(3, 2)$-Motzkin paths and ordinary $(3, 2)$-Motzkin paths, we are led to a simple explanation of the following relation:

$$l_n = 2m_{n-1}. \quad (1.5)$$

Since the little Schröder number $s_n$ is equal to the $(3, 2)$-Motzkin number $m_{n-1}$ (Chen, Li, Shapiro, and Yan [1] and Yan [10]), we find that relation (1.5) is equivalent to the well-known relation

$$S_n = 2s_n. \quad (1.6)$$

Combinatorial interpretations of (1.6) have been given by Shapiro and Sulanke [9], Deutsch [4], Gu, Li, and Mansour [6], and Huq [7].
2 Noncrossing Linked Partitions

In this section, we give a bijection from the set of large $(3, 2)$-Motzkin paths of length $n$ to the set of noncrossing linked partitions of $[n + 1]$.

A linked partition of $[n]$ is a collection of nonempty subsets $B_1, \ldots, B_k$ of $[n]$, called blocks, such that the union of $B_1, \ldots, B_k$ is $[n]$ and any two distinct blocks are nearly disjoint. Two blocks $B_i$ and $B_j$ are said to be nearly disjoint if for any $k \in B_i \cap B_j$, one of the following conditions holds:

(a) $k = \min(B_i), |B_i| > 1$ and $k \neq \min(B_j)$, or
(b) $k = \min(B_j), |B_j| > 1$ and $k \neq \min(B_i)$.

We say that $\pi = \{B_1, \ldots, B_k\}$ is a noncrossing linked partition if in addition, for any two distinct blocks $A$ and $B$ in $\pi$, there does not exist $a, b \in A$ and $c, d \in B$ such that $a < c < b < d$. Let $NCL(n)$ denote the set of noncrossing linked partitions of $[n]$.

In this paper, we adopt the linear representation of linked partitions, introduced by Chen, Wu, and Yan [2]. For a linked partition $\pi$ of $[n]$, first we draw $n$ vertices $1, \ldots, n$ on a horizontal line in increasing order. For each block $B = \{i_1, \ldots, i_k\}$, we write the elements $i_1, \ldots, i_k$ in increasing order, and we use $\min(B)$ to denote the minimum element $i_1$ of $B$. If $k \geq 2$, then we draw an arc joining $i_1$ and any other vertex in $B$. We shall use a pair $(i, j)$ to denote an arc between $i$ and $j$, where we assume that $i < j$.

It can be seen that a linked partition is noncrossing if and only if it does not contain any crossing arcs in its linear representation. For example, the linear representation of a noncrossing linked partition $\pi = \{1, 4, 9\}\{2, 3\}\{5, 6\}\{6, 7\}\{8\}$ is illustrated in Figure 2.1, where 6 belongs to both blocks $\{5, 6\}$ and $\{6, 7\}$.

![Figure 2.1: The linear representation of $\pi = \{1, 4, 9\}\{2, 3\}\{5, 6\}\{6, 7\}\{8\}$.](image)

Below is the main result of this paper.

**Theorem 2.1** There is a bijection from the set of large $(3, 2)$-Motzkin paths of length $n$ to the set of noncrossing linked partitions of $[n + 1]$.

**Proof.** To establish the correspondence, we define a map $\varphi$ from $L_n$ to $NCL(n + 1)$ in terms of a recursive procedure. Let $P$ be a large $(3, 2)$-Motzkin path in $L_n$, which is
represented as a sequence on \( \{u, d_1, d_2, h_1, h_2, h_3\} \), where \( u \) is an up step, \( d_i \) is an down step with color \( i \) for \( i = 1, 2 \), and \( h_j \) is a horizontal step with color \( j \) for \( j = 1, 2, 3 \). We proceed to construct a noncrossing linked partition \( \pi = \varphi(P) \).

If \( P = \emptyset \), then set \( \varphi(P) = \{1\} \). If \( P \) is nonempty, then it can be decomposed into a sequence of irreducible large \((3,2)\)-Motzkin paths, say, \( P = P_1 P_2 \cdots P_k \). Note that a horizontal step on the \( x \)-axis is an irreducible large \((3,2)\)-Motzkin path. For each segment \( P_i \), let \( p_i \) denote the length of \( P_i \). We wish to construct a noncrossing linked partition \( \varphi(P_i) \) on the set \( \{1, \ldots, p_i + 1\} \). We can then recover a noncrossing linked partition \( \pi \) by piecing together the noncrossing linked partitions \( \varphi(P_1), \varphi(P_2), \ldots, \varphi(P_k) \) and relabeling the elements from left to right with \( 1, \ldots, n + 1 \).

**Case 1:** \( P_i \) contains only one step. If \( P_i = h_1 \), then set \( \varphi(P_i) = \{1, 2\} \); if \( P_i = h_2 \), then set \( \varphi(P_i) = \{1\} \{2\} \). Figure 2.2 is an illustration of this case.

**Case 2:** \( P_i \) contains at least two steps. In this case, we may write \( P_i \) in the form \( u Q_1 h_3 Q_2 h_3 \cdots h_3 Q_r d \), where \( r \geq 1 \), \( d = d_1 \) or \( d_2 \), and \( Q_j \in L \) is a large \((3,2)\)-Motzkin path that is allowed to be empty. Then \( \varphi(P_i) \) can be generated by the following operations on the linear representations of \( \varphi(Q_1), \varphi(Q_2), \ldots, \varphi(Q_r) \).

For the case \( d = d_1 \), arrange the linear representations of \( \varphi(Q_1), \varphi(Q_2), \ldots, \varphi(Q_r) \) from left to right, and relabel the vertices also from left to right by \( 1, \ldots, p_i - 1 \). For \( j = 1, \ldots, r - 1 \), add an arc connecting the minimal vertex of \( \varphi(Q_j) \) and the minimal vertex of \( \varphi(Q_{j+1}) \). Then add two vertices \( p_i \) and \( p_i + 1 \) to the right of \( \varphi(Q_r) \). Finally, add an arc connecting the minimal vertex of \( \varphi(Q_r) \) and the vertex \( p_i \) and add an arc connecting 1 and the vertex \( p_i + 1 \). See Figure 2.3.

For the case \( d = d_2 \), the construction of \( \varphi(P_i) \) is similar to the case \( d = d_1 \), except that we do not add the arc connecting the vertex 1 and the minimal vertex of \( \varphi(Q_2) \). See Figure 2.4. If \( r = 1 \), namely \( P_i = u Q_1 d_2 \), then \( p_i \) is an isolated vertex in \( \varphi(P_i) \).

Finally, we join the last vertex of \( \varphi(P_i) \) and the first vertex of \( \varphi(P_{i+1}) \), for \( i = 1, \ldots, k - 1 \). Now \( \pi = \varphi(P) \) can be obtained by relabeling the vertices from left to right with \( \{1, \ldots, n + 1\} \). It can be seen that \( \pi \) is a noncrossing linked partition of
Figure 2.3: The case for \( d = d_1 \).

Figure 2.4: The case for \( d = d_2 \).

Figure 2.5 is an illustration of the operation of piecing together noncrossing linked partitions that correspond to irreducible large \((3, 2)\)-Motzkin paths, where we use a dotted arc to represent a boundary arc. More precisely, a boundary arc of a partition is an arc that is not covered by any other arc. To show that \( \varphi \) is a bijection, we aim to construct the inverse map \( \varphi^{-1} \) from noncrossing linked partitions in \( NCL(n+1) \) to large \((3, 2)\)-Motzkin paths in \( L_n \). Let \( \pi \) be a noncrossing linked partition in \( NCL(n+1) \). As the inverse step of decomposing a large \((3, 2)\)-Motzkin path into irreducible segments, we can decompose a noncrossing linked partition also into irreducible segments. We say that a noncrossing linked partition \( \pi \) of \( [n+1] \) is irreducible if it has a boundary arc or it is \( \{1\} \{2\} \) for \( n = 1 \). It is easy to decompose \( \pi \) into irreducible segments. In the linear representation of \( \pi \), if there is a boundary arc from 1 to \( j \), for \( j \geq 2 \), then the partition of [\( j \)] consisting of the arcs of the linear representation of \( \pi \) forms an irreducible noncrossing linked partition.
Removing the vertices $1, \ldots, j-1$, we obtain a noncrossing linked partition. If $1$ is an isolated vertex, then we may form an irreducible partition $\{1\}\{2\}$. Removing the vertex $1$, we obtain a noncrossing linked partition. In either case, we can iterate this process to decompose $\pi$ into irreducible segments.

It is routine to verify that for any irreducible noncrossing linked partition, one can reverse every step of the map $\varphi$ to obtain an irreducible large $(3,2)$-Motzkin path. Thus the map $\varphi$ is a bijection. This completes the proof.

For example, the decomposition of $\pi = \{1,3,5\}\{2\}\{4\}\{5,6\}\{7\}\{8\} \in NCL(8)$ is shown in Figure 2.6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{decomposition.png}
\caption{The decomposition of $\pi = \{1,3,5\}\{2\}\{4\}\{5,6\}\{7\}\{8\}$.}
\end{figure}

An example of the above bijection is given in Figure 2.7.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{bijection.png}
\caption{Bijection $\varphi: L_{12} \rightarrow NCL(13)$.}
\end{figure}

The above bijection implies that the large Schröder number $S_n$ equals the number $l_n$ of large $(3,2)$-Motzkin paths of length $n$. On the other hand, there is a one-to-one correspondence between $(3,2)$-Motzkin paths of length $n - 1$ and little Schröder paths.
of length $2n$. Therefore, the relation $S_n = 2s_n$ can be rewritten as

$$l_n = 2m_{n-1},$$

(2.7)

that is, the number of large $(3, 2)$-Motzkin paths of length $n$ is twice the number of ordinary $(3, 2)$-Motzkin paths of length $n - 1$. Here we give a combinatorial interpretation of this fact. Let $P$ be a $(3, 2)$-Motzkin path of length $n - 1$. If $P$ does not have any horizontal step $h_3$ on the $x$-axis, then we can get two large $(3, 2)$-Motzkin paths by adding a horizontal step $h_1$ or $h_2$ at the end of $P$. Otherwise, we remove the first horizontal step $h_3$ on the $x$-axis in $P$, and elevate the path after this $h_3$ horizontal step by adding an up step at the beginning and a down step at the end so that the resulting path is a large $(3, 2)$-Motzkin path of length $n$. In this case, there are also two choices for the last down step. It is easy to see that the above construction is reversible. Hence we obtain (2.7).

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References


