Arithmetic Properties of Overpartition Pairs

William Y.C. Chen\textsuperscript{1} and Bernard L.S. Lin\textsuperscript{2}

\textsuperscript{1}Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
emails: chen@nankai.edu.cn, linlishuang@cfc.nankai.edu.cn

Abstract. Bringmann and Lovejoy introduced a rank for overpartition pairs and investigated its role in congruence properties of $pp(n)$, the number of overpartition pairs of $n$. In particular, they applied the theory of Klein forms to show that there exist many Ramanujan-type congruences for $pp(n)$. In this paper, we derive two Ramanujan-type identities and some explicit congruences for $pp(n)$. Moreover, we find three ranks as combinatorial interpretations of the fact that $pp(n)$ is divisible by three for any $n$. We also construct infinite families of congruences for $pp(n)$ modulo 3 and 5, and two congruence relations modulo 9.

Keywords: overpartition pairs, rank of overpartition pairs, congruence

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1 Introduction

A partition of a positive integer $n$ is a non-increasing sequence of positive integers whose sum is $n$. An overpartition $\lambda$ of $n$ is a partition of $n$ for which the first occurrence of a number may be overlined. Let $p(n)$ denote the number of overpartitions of $n$. Congruence properties for $p(n)$ have been extensively studied, see, for example, Fortin, Jacob and Mathieu [6], Hirschhorn and Sellers [8], Kim [11], Lovejoy and Osburn [13], and Mahlburg [14]. In this paper, we study arithmetic properties of the number of overpartition pairs of $n$. An overpartition pair $\pi$ of $n$ is a pair of overpartitions $(\lambda, \mu)$ such that the sum of all of the parts is $n$. Note that we allow $\lambda$ and $\mu$ to be an overpartition of zero. There is only one partition of zero, and there is only one overpartition of zero as well. Let $pp(n)$ denote the number of overpartition pairs of $n$. 
Then the generating function for $\overline{pp}(n)$ is

$$
\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \frac{(-q; q)^2_\infty}{(q; q^2)^\infty}.
$$

(1.1)

Throughout this paper, we assume that $|q| < 1$ and we adopt the following standard $q$-series notation

$$(a; q)_\infty = \prod_{k=1}^{\infty} (1 - aq^k).$$

Bringmann and Lovejoy [4] defined a rank for overpartition pairs to investigate congruence properties of $\overline{pp}(n)$. Let $\overline{NN}(m, n)$ denote the number of overpartition pairs of $n$ with rank $m$, and let $\overline{NN}(r, t, n)$ denote the number of overpartition pairs of $n$ with rank congruent to $r$ modulo $t$. They obtained a bivariate generating function for $\overline{NN}(m, n)$ from which they derived the following relation for $0 \leq r \leq 2$,

$$\overline{NN}(r, 3, 3n + 2) = \overline{pp}(3n + 2) / 3.
$$

This leads to the following Ramanujan-type congruence in the spirit of Ramanujan’s congruences on the partition function $p(n)$ modulo 5 and 7 (see, e.g., Berndt [3, Chapter 2]):

$$\overline{pp}(3n + 2) \equiv 0 \pmod{3}.
$$

(1.2)

Furthermore, by using the theory of Klein forms, Bringmann and Lovejoy [4] proved that there exist infinitely many Ramanujan-type congruences for $\overline{pp}(n)$. Let $l$ be an odd prime and let $t$ be an odd number which is a power of $l$ or relatively prime to $l$. Then for any positive integer $j$, there are infinitely many non-nested arithmetic progressions $An + B$ such that

$$\overline{NN}(r, t, An + B) \equiv 0 \pmod{l^j}
$$

(1.3)

for any $0 \leq r \leq t - 1$. Hence there are infinitely many non-nested arithmetic progressions $An + B$ satisfying

$$\overline{pp}(An + B) \equiv 0 \pmod{l^j}
$$

(1.4)

for any odd prime $l$ and any positive integer $j$. For the case $l = 2$, using the theory of modular forms, they have shown that (1.4) holds for any positive integer $j$.

However, the theory of Klein forms used to derive the congruence relation (1.4) is not constructive and it does not give explicit arithmetic progressions $An + B$ in the statement. So it is still desirable to find explicit congruences for $\overline{pp}(n)$. In this paper, we obtain some congruences for $\overline{pp}(n)$ modulo 3 and 5.

For the case of modulo 3, we obtain a Ramanujan-type identity

$$
\sum_{n=0}^{\infty} \overline{pp}(3n + 2) q^n = 12 \frac{(q^2; q^6)^6 \left( q^3; q^3 \right)^6}{(q; q^6)^{14}},
$$

(1.5)
which is analogous to Ramanujan’s identity (see, e.g., Berndt [3, Theorem 2.3.4])

\[
\sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \frac{(q^5; q^5)^5_{\infty}}{(q; q)^6_{\infty}}.
\] (1.6)

Furthermore, we show that there are infinite families of congruences modulo 3 satisfied by \( \overline{pp}(n) \). For example, for any \( \alpha \geq 1 \) and \( n \geq 0 \),

\[
\overline{pp}(9^\alpha(3n + 1)) \equiv \overline{pp}(9^\alpha(3n + 2)) \equiv 0 \pmod{3}.
\] (1.7)

For the case of modulo 5, we obtain three Ramanujan-type congruences

\[
\overline{pp}(20n + 11) \equiv \overline{pp}(20n + 15) \equiv \overline{pp}(20n + 19) \equiv 0 \pmod{5},
\] (1.8)

for any \( n \geq 0 \). We also find infinite families of congruences modulo 5. For example, for any \( \alpha \geq 1 \) and \( n \geq 0 \),

\[
\overline{pp}(5^\alpha(5n + 2)) \equiv \overline{pp}(5^\alpha(5n + 3)) \equiv 0 \pmod{5}.
\] (1.9)

Motivated by the work of Paule and Radu [17] on some strange congruences, we obtain similar congruences for \( \overline{pp}(n) \). For example, for any \( k \geq 0 \),

\[
\overline{pp}(5 \cdot 29^k) \equiv 3(k + 1) \pmod{5}
\] (1.10)

and

\[
\overline{pp}(2 \cdot 13^k) \equiv 3(k + 1) \pmod{9}.
\] (1.11)

In order to give combinatorial interpretations of the fact that \( \overline{pp}(3n + 2) \) is divisible by 3, we find three ranks of overpartition pairs that serve this purpose.

This paper is organized as follows. In Section 2, we obtain two Ramanujan-type identities and some Ramanujan-type congruences modulo 5 and 64. In Section 3, we give three combinatorial interpretations for the congruence (1.2). Section 4 gives infinite families of congruences modulo 3 and 5. In Section 5, we obtain congruences modulo 9 which are similar to the congruences of Paule and Radu for the number of broken 2-diamond partitions.

## 2 Ramanujan-type identities and congruences

In this section, we establish two Ramanujan-type identities and derive some congruence relations modulo 5 and 64.
Theorem 2.1. We have
\[
\sum_{n=0}^{\infty} p^p(3n+2)q^n = 12 \frac{(q^2; q^2)_\infty^6 (q^3; q^3)_\infty^6}{(q; q)_\infty^{14}}, \quad (2.1)
\]
\[
\sum_{n=0}^{\infty} p^p(4n+3)q^n = 32 \frac{(q^2; q^2)_\infty^{20}}{(q; q)_\infty^{22}}. \quad (2.2)
\]

To prove the above identities, we recall two Ramanujan’s theta functions \( \varphi(q) \) and \( \psi(q) \), namely,
\[
\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.
\]
The following two identities are due to Gauss (see, e.g., Berndt [3, p.11]):
\[
\varphi(-q) = \frac{(q; q^2)_\infty^2}{(q^2; q^2)_\infty}, \quad \psi(q) = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty^2}.
\]
As shown by Hirschhorn and Sellers [7], the generating function of \( p(n) \) is
\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\varphi(-q)}. \quad (2.3)
\]
This implies that the generating function of \( p^p(n) \) equals
\[
\sum_{n=0}^{\infty} p^p(n)q^n = \frac{1}{\varphi(-q)^2}. \quad (2.3)
\]
The following dissection formula of Hirschhorn and Sellers [7] plays a key role in the proof of Theorem 2.1.

Lemma 2.1. Let
\[
A(q) = \frac{(q; q)_\infty (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty (q^3; q^3)_\infty}.
\]
Then we have
\[
\frac{1}{\varphi(-q)} = \frac{\varphi(-q^9)}{\varphi(-q^3)^4} \left( \varphi(-q^9)^2 + 2q\varphi(-q^9)A(q^3) + 4q^2A(q^3)^2 \right)
\]
\[
= \frac{1}{\varphi(-q^4)^4} \left( \varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)^2\psi(q^8)^2 + 8q^3\psi(q^8)^3 \right). \quad (2.4)
\]

Proof of Theorem 2.1. Substituting the 3-dissection formula (2.4) into (2.3), we see that
\[
\sum_{n=0}^{\infty} p^p(n)q^n = \frac{\varphi(-q^9)^2}{\varphi(-q^3)^8} \left( \varphi(-q^9)^2 + 2q\varphi(-q^9)A(q^3) + 4q^2A(q^3)^2 \right)^2. \quad (2.6)
\]
Choosing those terms on each side for which the powers of \( q \) are of the form \( 3n + 2 \), we find that
\[
\sum_{n=0}^{\infty} \overline{pp}(3n + 2)q^{3n + 2} = \frac{\varphi(-q^9)^2}{\varphi(-q^3)^8} \left( 8q^2\varphi(-q^9)^2A(q^3)^2 + 4q^2\varphi(-q^9)^2A(q^3)^2 \right)
\]
\[
= 12q^2A(q^3)^2\varphi(-q^9)^4
\]
Dividing both sides by \( q^2 \) and replacing \( q^3 \) by \( q \), we obtain that
\[
\sum_{n=0}^{\infty} \overline{pp}(3n + 2)q^n = 12A(q^2)\varphi(-q^3)^4
\]
This yields (2.1). Similarly,
\[
\sum_{n=0}^{\infty} \overline{pp}(n)q^n = \frac{1}{\varphi(-q^4)^8} \left( \varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3 \right)^2.
\]
Choosing the terms for which the powers of \( q \) are of the form \( 4n + 3 \), we find that
\[
\sum_{n=0}^{\infty} \overline{pp}(4n + 3)q^{4n+3} = \frac{1}{\varphi(-q^4)^8} \left( 16q^3\varphi(q^4)^3\psi(q^8)^3 + 16q^3\varphi(q^4)^3\psi(q^8)^3 \right)
\]
\[
= 32q^3\varphi(q^4)^3\psi(q^8)^3
\]
Dividing both sides by \( q^3 \) and replacing \( q^4 \) by \( q \), we deduce that
\[
\sum_{n=0}^{\infty} \overline{pp}(4n + 3)q^n = 32\varphi(q)^3\psi(q^2)^3
\]
which is equivalent to (2.2). This completes the proof.

In view of Theorem 2.1, it can be seen that \( \overline{pp}(3n + 2) \) and \( \overline{pp}(4n + 3) \) are divisible by 4. In fact, for all \( n \geq 1 \), \( \overline{pp}(n) \) is divisible by 4, since
\[
\sum_{n=0}^{\infty} \overline{pp}(n)q^n \equiv \left( 1 + 2\sum_{n=0}^{\infty}(-q)^n \right)^2 \sum_{n=0}^{\infty} \overline{pp}(n)q^n \text{ (mod 4)}
\]
\[
= \varphi(-q)^2 \frac{1}{\varphi(-q)^2} = 1.
\]
In fact, Keister, Sellers and Vary [10] have shown that for \( n \geq 1 \),
\[
\overline{pp}(n) \equiv \begin{cases} 4 \text{ (mod 8),} & \text{if } n \text{ is a square or twice a square,} \\ 0 \text{ (mod 8),} & \text{otherwise.} \end{cases}
\]
Recently, Kim [12] gave a combinatorial proof of the above fact and studied arithmetic properties of $\overline{pp}(n)$ modulo powers of 2.

With the aid of (2.2) and the following relation for any prime $p$,

$$(q; q)_\infty^p \equiv (q^p; q^p)_\infty (\mod p),$$

(2.8)

we are led to the following congruence relations modulo 5 and 64.

**Corollary 2.1.** For any nonnegative integer $n$,

$$\overline{pp}(8n + 7) \equiv 0 \pmod{64},$$

(2.9)

$$\overline{pp}(20n + 11) \equiv 0 \pmod{5},$$

(2.10)

$$\overline{pp}(20n + 15) \equiv 0 \pmod{5},$$

(2.11)

$$\overline{pp}(20n + 19) \equiv 0 \pmod{5}. $$

(2.12)

**Proof.** From (2.2) and (2.8) with $p = 2$, we have

$$\sum_{n=0}^{\infty} \overline{pp}(4n + 3)q^n \equiv \frac{(q^2; q^2)^2_\infty}{(q^2; q^2)_\infty^2} \equiv (q^2; q^2)_\infty^9 \pmod{2}. $$

This yields congruence (2.9) by equating the coefficients of $q^{2n+1}$ for $n \geq 0$. Again by (2.2) and (2.8) with $p = 5$, we see that

$$\sum_{n=0}^{\infty} \overline{pp}(4n + 3)q^n \equiv 2\frac{(q_{10}; q_{10})^4_\infty}{(q^5; q^5)_\infty^4} \cdot \frac{1}{(q; q)_\infty^2} \pmod{5}. $$

(2.13)

Let $p_2(n)$ be defined by

$$\sum_{n=0}^{\infty} p_2(n)q^n = \frac{1}{(q; q)_\infty^2}. $$

It has been shown by Ramanathan [18] that for $n \geq 0$,

$$p_2(5n + 2) \equiv p_2(5n + 3) \equiv p_2(5n + 4) \equiv 0 \pmod{5}. $$

Combining (2.13) and the above three congruences, we deduce the congruence relations (2.10), (2.11) and (2.12). This completes the proof.

**3 Three ranks for overpartition pairs**

In this section, we give three combinatorial interpretations for the fact that $\overline{pp}(3n + 2)$ is divisible by 3.
The first rank of an overpartition pair \( \pi = (\lambda, \mu) \), denoted \( r_1(\pi) \), is defined to be \( n_1(\lambda) - n_1(\mu) \), where \( n_1(\lambda) \) denotes the number of parts of an overpartition \( \lambda \). As usual, let \( R_1(m,n) \) denote the number of overpartition pairs of \( n \) with \( r_1(\pi) = m \) and let \( R_1(s,t,n) \) denote the number of overpartition pairs of \( n \) with \( r_1(\pi) \equiv s \pmod{t} \).

By symmetry, we see that \( R_1(m,n) = R_1(-m,n) \), and so \( R_1(s,t,n) = R_1(t - s, t, n) \).

It is easy to derive the bivariate generating function for \( R_1(m,n) \), that is,

\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} R_1(m,n) z^m q^n = \frac{(-q z; q)_\infty}{(q z; q)_\infty} \cdot \frac{(-q/z; q)_\infty}{(q/z; q)_\infty}.
\] (3.1)

Here we adopt the convention that the empty overpartition pair of 0 has rank zero. Similarly, this convention is valid for the other two ranks that will be introduced in this section. The following theorem shows that the rank \( r_1(\pi) \) leads to a classification of overpartition pairs of \( 3n + 2 \) into three equinumerous sets.

**Theorem 3.1.** For \( 0 \leq s \leq 2 \), we have

\[
R_1(s,3,3n+2) = \frac{\mp(3n+2)}{3}.
\] (3.2)

**Proof.** Substituting \( z = \xi = e^{2\pi i/3} \) into (3.1) and using the symmetry relation \( R_1(1,3,n) = R_1(2,3,n) \), we find that

\[
\sum_{n=0}^{\infty} (R_1(0,3,n) - R_1(1,3,n)) q^n = \frac{(-q \xi; q)_\infty}{(q \xi; q)_\infty} \frac{(-q \xi^2; q)_\infty}{(q \xi^2; q)_\infty}
= \frac{(-q^3; q^3)_\infty}{(q^3; q^3)_\infty} \frac{(q; q)_\infty}{(-q; q)_\infty}
= \frac{(-q^3; q^3)_\infty}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} (-1)_n q^{n^2}.
\] (3.3)

Here the second equality follows from identity

\((1 - x^3) = (1 - x)(1 - x\xi)(1 - x\xi^2)\).

Equating the coefficients of \( q^{3n+2} \) on both sides of (3.3), and observing that there are no squares congruent to 2 modulo 3, we conclude that

\( R_1(0,3,3n+2) = R_1(1,3,3n+2) \),

and so

\( R_1(0,3,3n+2) = R_1(1,3,3n+2) = R_1(2,3,3n+2) = \frac{\mp(3n+2)}{3} \).

This completes the proof. \( \blacksquare \)
We now introduce the second rank $r_2$. Let $\pi = (\lambda, \mu)$ be an overpartition pair. Define

$$r_2(\pi) = n_2(\lambda) - n_2(\mu),$$

(3.4)

where $n_2(\lambda)$ denotes the number of overlined parts of an overpartition $\lambda$. Similarly, let $R_2(m, n)$ denote the number of overpartition pairs of $n$ with $r_2(\pi) = m$ and let $R_2(s, t, n)$ denote the number of overpartition pairs of $n$ with $r_2(\pi) \equiv s \pmod{t}$. Then we have the following relation.

**Theorem 3.2.** For $n \geq 0$, we have

$$R_2(0, 3, 3n + 2) \equiv R_2(1, 3, 3n + 2) \equiv R_2(2, 3, 3n + 2) \pmod{3}.$$  (3.5)

*Proof.* It is routine to check that

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} R_2(m, n) z^m q^n = \frac{(-q^3; q^3)_{\infty}}{(q^3; q^3)_{\infty}} \cdot \frac{(-q/3; q)_{\infty}}{(q; q)_{\infty}}.$$  (3.6)

Using the fact that $R_2(1, 3, n) = R_2(2, 3, n)$ and setting $z = \xi = e^{2\pi i/3}$ in (3.6), we find

$$\sum_{n=0}^{\infty} (R_2(0, 3, n) - R_2(1, 3, n)) q^n = \frac{(-q^3; q^3)_{\infty}}{(q^3; q^3)_{\infty}} \cdot \frac{(-q^3; q^3)_{\infty}}{(q; q)_{\infty}} \cdot \frac{(-q^3; q^3)_{\infty}}{(q^3; q^3)_{\infty}} \sum_{n=-\infty}^{\infty} (-q)^n.$$  (3.7)

$$\equiv \frac{(-q^3; q^3)_{\infty}}{(q^3; q^3)_{\infty}} \sum_{n=-\infty}^{\infty} (-q)^{n^2} \pmod{3}.$$

Since there are no squares congruent to 2 modulo 3, we see that

$$R_2(0, 3, 3n + 2) - R_2(1, 3, 3n + 2) \equiv 0 \pmod{3},$$

and hence the proof is complete.  

It is worth mentioning that Andrews, Lewis and Lovejoy [1] investigated the arithmetic properties of the number $PD(n)$ of partitions of $n$ with designated summands, whose generating function is given by (3.7), that is,

$$\sum_{n=0}^{\infty} PD(n) q^n = \frac{(q^6; q^6)_{\infty}}{(q; q)_{\infty} (q^2; q^2)_{\infty} (q^3; q^3)_{\infty}}.$$  For example, it has been shown that $PD(3n + 2)$ is divisible by three. It should also be mentioned that Chan [5] studied the number $a(n)$ given by

$$\sum_{n=0}^{\infty} a(n) q^n = \frac{1}{(q; q)_{\infty} (q^2; q^2)_{\infty}}.$$  

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and derived a Ramanujan-type identity for $a(3n + 2)$, that is,

$$\sum_{n=0}^{\infty} a(3n + 2)q^n = 3\frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q; q)_\infty^4 (q^2; q^2)_\infty^4}. \quad (3.8)$$

From (3.7) and (3.8), we get the following formula.

**Corollary 3.1.** We have

$$\sum_{n=0}^{\infty} (R_2(0, 3, 3n + 2) - R_2(1, 3, 3n + 2)) q^n = 3\frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q; q)_\infty^5 (q^2; q^2)_\infty^3}. \quad (3.9)$$

Finally, we turn to the third rank $r_3$ of an overpartition pair $\pi = (\lambda, \mu)$, which is defined by

$$r_3(\pi) = n_3(\lambda) - n_3(\mu), \quad (3.10)$$

where $n_3(\lambda)$ denotes the number of non-overlined parts of an overpartition $\lambda$. Similarly, let $R_3(m, n)$ denote the number of overpartition pairs of $n$ with $r_3(\pi) = m$ and let $R_3(s, t, n)$ denote the number of overpartition pairs of $n$ with $r_3(\pi) \equiv s \pmod{t}$. Then we have the following relation.

**Theorem 3.3.** For $0 \leq s \leq 2$, we have

$$R_3(s, 3, 3n + 2) = \frac{pp(3n + 2)}{3}. \quad (3.11)$$

**Proof.** It is easy to derive that

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} R_3(m, n) z^m q^n = \frac{(-q; q)_\infty^2}{(q z; q)_\infty (q/ z; q)_\infty}. \quad (3.12)$$

Using the fact that $R_3(1, 3, n) = R_3(2, 3, n)$ and setting $z = \xi = e^{2\pi i/3}$ in (3.12), we find that

$$\sum_{n=0}^{\infty} (R_3(0, 3, n) - R_3(1, 3, n)) q^n = \frac{(-q; q)_\infty^2 (q/ \xi; q)_\infty}{(q^3; q^3)_\infty}$$

$$= \frac{(-q; q)_\infty^2 (q; q)_\infty}{(q^3; q^3)_\infty}$$

$$= \frac{1}{(q^3; q^3)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$ 

Note that there are no triangular numbers that are congruent to 2 modulo 3. It follows that

$$R_3(0, 3, 3n + 2) = R_3(1, 3, 3n + 2).$$
Since $R_3(1,3,3n+2) = R_3(2,3,3n+2)$, the proof is complete.

To conclude this section, we remark that the rank $r_3$ can be used to give combinatorial explanations of many Ramanujan-type congruences for $\overline{pp}(n)$ which plays an analogous role to the rank introduced by Bringmann and Lovejoy [4] for congruences also for overpartition pairs. To be specific, we have the following theorem. The proof is similar to the proof of Bringmann and Lovejoy. But the rank $r_3$ seems to simpler.

**Theorem 3.4.** Let $l$ be an odd prime, and let $t$ be an odd number which is a power of $l$ or relatively prime to $l$. Then for any positive integer $j$, there are infinitely many non-nested arithmetic progressions $An+B$ such that

$$R_3(r,t,An+B) \equiv 0 \pmod{l^j} \quad (3.13)$$

for any $0 \leq r \leq t-1$.

**Proof.** Note that the generating function for $R_3(s,t,n)$ can be decomposed into a linear combination of certain modular forms similar to the case for $\overline{NN}(r,t,n)$. Suppose that $t$ is an odd integer and $0 \leq s < t$. Let $\zeta_t = e^{2\pi i t}$ and define the rank of the overpartition pair of 0 to be 0. Then we have

$$\sum_{n=0}^{\infty} R_3(s,t,n)q^n = \frac{1}{t} \sum_{k=0}^{t-1} \zeta_t^{-ks} R_3(\zeta_t^k; q),$$

where

$$R_3(z; q) = \frac{(-q; q)_\infty^2}{(qz; q)_\infty(q/z; q)_\infty}.$$

Observe that $R_3(\zeta_t^k; q)$ differs from $R(\zeta_t^k; q)$ (see Bringmann and Lovejoy [4, Proposition 2.4]) only by a factor $\frac{4}{(1+\zeta_t^k)(1+\zeta_t^{-k})}$. Hence the argument of Bringmann and Lovejoy for (1.3) can be carried over to deduce relation (3.13). This completes the proof.

4 Infinite families of congruences modulo 3 and 5

In this section, we obtain a formula for $\overline{pp}(3n)$ modulo 3 based on the number of representations of $n$ as a sum of two squares. We further derive a formula for $\overline{pp}(5n)$ modulo 5 in connection with the number of representations of $n$ in the form $x^2 + 5y^2$. As consequences, we give infinite families of congruences modulo 3 and 5.

**Theorem 4.1.** If the prime factorization of $n$ is given by

$$n = 2^a \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

(4.1)
where \( p_i \equiv 1 \pmod{4} \) and \( q_j \equiv 3 \pmod{4} \). Then
\[
\overline{p^p}(3n) \equiv (-1)^n \prod_{i=1}^{r}(1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2} \pmod{3}.
\] (4.2)

Proof. First, it is easy to see that
\[
\varphi(-q)^3 \equiv \varphi(-q^3) \pmod{3}
\]
and
\[
\varphi(-q) = \varphi(-q^9) + qB(q^3),
\]
where \( B(q) \) is an infinite series in \( q \) with integer coefficients. Hence,
\[
\sum_{n=0}^{\infty} \overline{p^p}(n)q^n = \frac{\varphi(-q)}{\varphi(-q^3)} \equiv \frac{\varphi(-q)}{\varphi(-q^3)} \pmod{3}
\]
\[
= \frac{\varphi(-q^9) + qB(q^3)}{\varphi(-q^3)}.
\]
Extracting the terms \( q^{3n} \) for \( n \geq 0 \), and replacing \( q^3 \) by \( q \), we find that
\[
\sum_{n=0}^{\infty} \overline{p^p}(3n)q^n = \frac{\varphi(-q^3)}{\varphi(-q)} \equiv \varphi(-q)^2 \pmod{3}.
\] (4.3)

Let \( r_2(n) \) denote the number of representations of \( n \) as a sum of two squares. So we have
\[
\varphi(-q)^2 = \sum_{n=0}^{\infty} (-1)^n r_2(n)q^n.
\] (4.4)

From (4.3) and (4.4) it follows that
\[
\overline{p^p}(3n) \equiv (-1)^n r_2(n) \pmod{3}.
\] (4.5)

Given the prime factorization of \( n \) in the form of (4.1), it is well known that (see, e.g., Berndt [3] or Grosswald [9])
\[
r_2(n) = 4 \prod_{i=1}^{r}(1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2}.
\] (4.6)

Combining (4.5) and (4.6), we get the desired formula (4.2).

Theorem 4.2. Assume that \( p \) is prime with \( p \equiv 3 \pmod{4} \), and \( s \) is an integer with \( 1 \leq s < p \). Then for any \( \alpha \geq 0 \) and \( n \geq 0 \), we have
\[
\overline{p^p}(3p^{2\alpha+1}(pn + s)) \equiv 0 \pmod{3}.
\] (4.7)

In particular, setting \( p = 3 \), we have for any \( \alpha \geq 1 \) and \( n \geq 0 \),
\[
\overline{p^p}(9^\alpha(3n + 1)) \equiv 0 \pmod{3}
\] (4.8)

and
\[
\overline{p^p}(9^\alpha(3n + 2)) \equiv 0 \pmod{3}.
\] (4.9)
Proof. Recall that \( r_2(n) = 0 \) if and only if there exists a prime congruent to 3 modulo 4 that has an odd exponent in the canonical factorization of \( n \). It can be seen that

\[
r_2(p^{2n+1}(pn+s)) = 0,
\]

since \( p \) is not a factor of \( pn+s \). By (4.5) we obtain the congruence relation (4.7). This completes the proof.

**Theorem 4.3.** Let \( R(n,x^2+5y^2) \) denote the number of representations of \( n \) by the quadratic form \( x^2+5y^2 \). Then for any \( n \geq 0 \), we have

\[
\overline{pp}(5n) \equiv (-1)^n R(n,x^2+5y^2) \pmod{5}.
\]

**Proof.** It is easy to see that \( \varphi(-q)^8 \) is a modular form of weight 4 on \( \Gamma_0(2) \), where

\[
\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{2} \right\}.
\]

For the background on modular forms, see Ono [16]. Now \( \varphi(-q)^8|T_5 \) is also a modular form of weight 4 on \( \Gamma_0(2) \). Here \( T_5 \) is the Hecke operator which acts on

\[
\varphi(-q)^8 := \sum_{n=0}^{s} r(n)q^n
\]

defined by

\[
\varphi(-q)^8|T_5 = \sum_{n=0}^{\infty} r(5n)q^n + \sum_{n=0}^{\infty} 125r(n)q^{5n}.
\]

By Sturm’s theorem (see [16, p.40]), we have

\[
\varphi(-q)^8|T_5 \equiv \varphi(-q)^8 \pmod{5},
\]

and so

\[
\sum_{n=0}^{\infty} r(5n)q^n \equiv \varphi(-q)^8 \pmod{5}. \tag{4.11}
\]

On the other hand,

\[
\varphi(-q)^8 = \varphi(-q)^{10} \cdot \frac{1}{\varphi(-q)^2} \equiv \varphi(-q)^{5} \sum_{n=0}^{\infty} \overline{pp}(n)q^n \pmod{5}.
\]

Considering the terms for which the powers of \( q \) are multiples of 5, and replacing \( q^5 \) by \( q \), we deduce that

\[
\sum_{n=0}^{\infty} r(5n)q^n \equiv \varphi(-q)^2 \sum_{n=0}^{\infty} \overline{pp}(5n)q^n \pmod{5}. \tag{4.12}
\]
Combining (4.11) and (4.12), we deduce that

$$
\sum_{n=0}^{\infty} \overline{pp}(5n) q^n \equiv \varphi(-q)^6 \equiv \varphi(-q) \varphi(-q^5) \pmod{5}.
$$

This completes the proof.

The formula for $R(n, x^2 + 5y^2)$ due to Berkovich and Yesilyurt [2] leads to the following formula for $\overline{pp}(5n)$ modulo 5.

**Theorem 4.4.** If the prime factorization of $n$ is given by

$$
n = 2^a 5^b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},
$$

where $p_i \equiv 1, 3, 7, \text{ or } 9 \pmod{20}$ and $q_j \equiv 11, 13, 17, \text{ or } 19 \pmod{20}$. Then we have

$$
\overline{pp}(5n) \equiv (-1)^n \left(1 + (-1)^{a+t}\right) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2} \pmod{5},
$$

where $t$ is the number of prime factors of $n$, counting multiplicity, that are congruent to 3 or 7 modulo 20.

**Proof.** Given the prime factorization of $n$ in the form of (4.13), it is known that (see Berkovich and Yesilyurt [2, Corollary 3.3])

$$
R(n, x^2 + 5y^2) = \left(1 + (-1)^{a+t}\right) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2}.
$$

Combining (4.10) and (4.15), we get (4.14). This completes the proof.

As consequences of the above theorem, we have the following congruences.

**Corollary 4.1.** Let $p$ be a prime with $p \equiv 11, 13, 17, \text{ or } 19 \pmod{20}$. Then for any odd positive integer $t$ and any positive integer $n$ that is not divisible by $p$, we have

$$
\overline{pp}(5p^t n) \equiv 0 \pmod{5}.
$$

**Corollary 4.2.** Let $p$ be a prime with $p \equiv 1 \pmod{20} \text{ or } p \equiv 9 \pmod{20}$. Then for any positive integer $k$, we have

$$
\overline{pp}(5p^k) \equiv 3(k+1) \pmod{5}.
$$

Based on Theorem 4.4, we get two infinite families of congruences modulo 5.
Theorem 4.5. For any $\alpha \geq 1$ and $n \geq 0$, we have

\[ \overline{pp}(5^\alpha(5n + 2)) \equiv 0 \pmod{5} \]  \tag{4.18}

and

\[ \overline{pp}(5^\alpha(5n + 3)) \equiv 0 \pmod{5}. \]  \tag{4.19}

Proof. Considering the possible residues of $x^2 + 5y^2$ modulo 5, we find that

\[ R(5n + 2, x^2 + 5y^2) = R(5n + 3, x^2 + 5y^2) = 0. \]

In light of (4.10), we deduce that

\[ \overline{pp}(25n + 10) \equiv (-1)^{5n+2}R(5n + 2, x^2 + 5y^2) \equiv 0 \pmod{5} \]  \tag{4.20}

and

\[ \overline{pp}(25n + 15) \equiv (-1)^{5n+3}R(5n + 3, x^2 + 5y^2) \equiv 0 \pmod{5}. \]  \tag{4.21}

Observe that formula (4.14) for $\overline{pp}(5n)$ modulo 5 is independent of the exponent of 5 in the factorization of $n$. This means that for $\alpha \geq 1$,

\[ \overline{pp}(5n) \equiv \overline{pp}(5^\alpha n) \pmod{5}. \]  \tag{4.22}

Combining (4.20), (4.21) and (4.22), we obtain the desired congruence relations (4.18) and (4.19). This completes the proof. \qed

5 Further congruences for overpartition pairs

In this section, we find some congruences for $\overline{pp}(n)$ modulo 9 which are similar to the congruences for the number of broken 2-diamond partitions obtained by Paule and Radu [17]. Let us begin with the congruences modulo 9.

Theorem 5.1. For any prime with $p \equiv 1 \pmod{12}$, we have

\[ \overline{pp}(3n + 2)p \equiv \frac{\overline{pp}(2p)}{3}\overline{pp}(3n + 2) \pmod{9}, \]  \tag{5.1}

for all positive integers $n$ such that $3n + 2 \not\equiv 0 \pmod{p}$.

To prove the above theorem, we need the following lemma which is a special case of Newman [15, Theorem 3].

Lemma 5.1. For each prime $p$ with $p \equiv 1 \pmod{12}$ and for all positive integers $n$,

\[ b\left(np + \frac{2p-2}{3}\right) + p^4b\left(\frac{n}{p} - 2\frac{p-1}{3p}\right) = b\left(\frac{2p-2}{3}\right)b(n), \]  \tag{5.2}

where $b(n)$ is defined by

\[ \sum_{n=0}^{\infty} b(n)q^n = (q; q)_{\infty}^4(q^2; q^2)_{\infty}^6. \]
Since the equality is derived by equating coefficients of series in $q$, it is safe to assume that $b(t) = 0$ if $t$ is not a nonnegative integer. We are now ready to give a proof of Theorem 5.1.

**Proof of Theorem 5.1.** By the generating function of $\overline{pp}(3n + 2)$ as given in (2.1), we see that
\[
\sum_{n=0}^{\infty} \frac{\overline{pp}(3n + 2)}{3} q^n \equiv (q;q)_\infty^4 (q^2;q^2)_\infty^6 \pmod{3}.
\]

From the definition of $b(n)$, we deduce that for $n \geq 0$,
\[
\frac{\overline{pp}(3n + 2)}{3} \equiv b(n) \pmod{3}. \quad (5.3)
\]

On the other hand, for those prime $p$ with $p \equiv 1 \pmod{12}$ and those $n$ such that $3n + 2$ is not a multiple of $p$, it follows that $b \left( \frac{n}{p} - 2 \frac{p-1}{3p} \right) = 0$. Thus, by Lemma 5.1 we obtain
\[
b \left( np + \frac{2p - 2}{3} \right) = b \left( \frac{2p - 2}{3} \right) b(n). \quad (5.4)
\]

Substituting (5.3) into (5.4), we get
\[
\frac{1}{3} \overline{pp}(3np + 2p) \equiv \frac{1}{9} \overline{pp}(2p) \overline{pp}(3n + 2) \pmod{3},
\]
as required.

Next, we use Lemma 5.1 to obtain the following congruence in the spirit of Paule and Radu [17].

**Theorem 5.2.** For any $k \geq 0$, we have
\[
\overline{pp}(2 \cdot 13^k) \equiv 3(k + 1) \pmod{9}. \quad (5.5)
\]

**Proof.** Let $p$ be a prime with $p \equiv 1 \pmod{12}$. Then setting $n = 2(p^{k+1} - 1)/3$ in (5.2) and using (5.3), we get
\[
\frac{1}{3} \overline{pp}(2p^{k+2}) + \frac{1}{3} \overline{pp}(2p^k) \equiv \frac{1}{9} \overline{pp}(2p) \overline{pp}(2p^{k+1}) \pmod{3}.
\]

When $p = 13$, since $\overline{pp}(26) \equiv 6 \pmod{9}$, we deduce that
\[
\overline{pp}(2 \cdot 13^{k+2}) + \overline{pp}(2 \cdot 13^k) \equiv 2 \overline{pp}(2 \cdot 13^{k+1}) \pmod{9}. \quad (5.6)
\]

Given the initial conditions $\overline{pp}(2) \equiv 3 \pmod{9}$, $\overline{pp}(26) \equiv 6 \pmod{9}$, by iteration of (5.6), we reach the conclusion (5.5). This completes the proof.

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References


