Adv. in Appl. Math.
49 (2012), no. 3-5, 271-284

A Franklin Type Involution for Squares

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Dedicated to Dennis Stanton on the Occasion of His Sixtieth Birthday

Abstract. We find an involution as a combinatorial proof of Ramanujan's partial theta identity. Based on this involution, we obtain a Franklin type involution on the set of partitions into distinct parts with the smallest part being odd. Compared with the involution of Bessenrodt and Pak, our involution possesses a weight preserving property that leads to a combinatorial proof of a weighted partition theorem derived by Alladi from Ramanujan's partial theta identity. This gives an indirect answer to a question of Berndt, Kim and Yee. Moreover, we obtain a partition theorem based on Andrews' identity and provide a combinatorial proof via certain weight assignment for our involution. A specialization of this partition theorem is related to an identity of Andrews concerning partitions into distinct nonnegative parts with the smallest part being even. Finally, we give an extension of our partition theorem which corresponds to a generalization of Andrews' identity.

Keywords: Franklin type involution, Ramanujan's partial theta identity, Andrews' identity.

AMS Subject Classifications: 05A17, 11P81

1 Introduction

The main result of this paper is a Franklin type involution for squares which is related to Ramanujan's partial theta identity and an identity of Andrews. As applications of this involution, we give an indirect solution to a problem proposed by Berndt, Kim and Yee [8] by providing a combinatorial interpretation of a partition theorem derived by Alladi [1] from Ramanujan's partial theta identity. Furthermore, we obtain a partition theorem based on Andrews' identity. A specialization of this theorem is related to an identity of Andrews on partitions into distinct nonnegative parts with the smallest part being even. Finally, we find a more general form of our partition theorem which corresponds to a generalization of Andrews' identity.

Recall that the celebrated involution of Franklin gives a combinatorial interpretation of Euler's pentagonal number theorem as stated below

$$(q;q)_\infty = 1 + \sum_{k=1}^{\infty} (-1)^k (q^{k(3k-1)/2} + q^{k(3k+1)/2}), \tag{1.1}$$

where the q-shifted factorial is defined by

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geq 1,$$

and

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n, \quad |q| < 1.$$

Let D denote the set of integer partitions into distinct parts, and let $D(n)$ denote the set of partitions of n into distinct parts. The relation (1.1) can be reinterpreted as the following number-theoretic identity

$$\sum_{\lambda \in D(n)} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^k, & \text{if } n = k(3k \pm 1)/2, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

where $\ell(\lambda)$ denotes the number of parts of λ.

Our Franklin type involution for squares will be concerned with the set of partitions of a non-negative integer into distinct parts with the smallest part being odd. Let us use P_{do} to denote the set of such partitions, and use $P_{do}(n)$ to denote the set of such partitions of n. To be more specific, we obtain the following number-theoretic identity which is analogous to (1.2),

$$\sum_{\lambda \in P_{do}(n)} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

It is clear that (1.3) implies a theorem of Fine [9] concerning the parity of the number of partitions in $P_{do}(n)$. Bessenrodt and Pak [7] constructed a involution for Fine's theorem, and Yee [14] gave an indirect bijective proof.

Moreover, for various weight assignments $\omega(\lambda)$ to partitions $\lambda \in P_{do}$, our involution turns out to be sign-reversing and weight-preserving. This property leads to several number-theoretic identities of the following form:

$$\sum_{\lambda \in P_{do}(n)} \omega(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

The first case is related to a problem proposed by Berndt, Kim and Yee [8] concerning a combinatorial interpretation of a number-theoretic identity derived by Alladi [1] from the following Ramanujan's partial theta identity from Ramanujan's lost notebook [12, p. 38],

$$1 + \sum_{k=1}^{\infty} \frac{(-q;q)_{k-1}(-a)^k q^{k(k+1)/2}}{(aq^2;q^2)_k} = \sum_{k=0}^{\infty} (-a)^k q^{k^2}. \tag{1.5}$$

Defining a weight function in terms of the gaps between the parts of partitions in P_{do}, Alladi [1] derived a partition theorem in the above form, see, Section 4. Though Berndt, Kim and Yee [8] have found a bijective proof Ramanujan's identity (1.5), it is not clear whether their involution can imply a combinatorial interpretation of Alladi's weighted partition theorem. As will be seen, our Franklin type involution gives a combinatorial proof of Alladi's weighted partition theorem, whereas the involution of Bessenrodt and Pak [7] is not weight preserving as far as Alladi's theorem is concerned.

The second case is concerned with a weighted partition theorem obtained by Alladi [2] from the following partial theta identity of Andrews [4, p. 157],

$$\sum_{n=0}^{\infty} q^{2n}(q^{2n+2};q^2)_\infty (aq^{2n+1};q^2)_\infty = \sum_{k=0}^{\infty} (-a)^k q^{k^2}. \tag{1.6}$$

By giving a weight function in terms of the odd parts of partitions in P_{do}, Alladi [2] derived a partition theorem in the form of (1.4), It turns out that our involution also applies to this partition theorem in terms of a different weight assignment.

As the third application of our involution, we give a combinatorial proof of a number-theoretic theorem on partitions into distinct parts with smallest part being even derived from Andrews' identity (1.6). Moreover, we note that a special case of this partition theorem is related to an identity of Andrews, first proposed as a problem in [3]. A generating function proof was given by Stenger [13].

To conclude this paper, we extend our involution to derive a more general identity

$$\begin{aligned} &\sum_{n=0}^{\infty} q^{2mn}(q^{2mn+2m};q^{2m})_\infty (aq^{2mn+1};q^2)_\infty \\ &= 1 + \sum_{k=1}^{\infty} (-a)^k q^{k^2} \prod_{j=1}^{k} (1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}), \end{aligned} \tag{1.7}$$

which reduces to the following identity of Andrews [4, p. 157] when setting $a = -1$,

$$\begin{aligned} &\sum_{n=0}^{\infty} q^{2mn}(q^{2mn+2m};q^{2m})_\infty(-q^{2mn+1};q^2)_\infty \\ &= 1+\sum_{k=1}^{\infty} q^{k^2}\prod_{j=1}^{k}(1+q^{2j}+q^{4j}+\cdots+q^{2(m-1)j}). \end{aligned} \tag{1.8}$$

Notice that (1.8) is a generalization of (1.6).

2 An involution for Ramanujan's identity

In this section, we shall construct an involution which leads to a combinatorial proof of Ramanujan's partial theta identity as stated in the previous section:

$$1+\sum_{k=1}^{\infty}\frac{(-q;q)_{k-1}(-a)^k q^{k(k+1)/2}}{(aq^2;q^2)_k}=\sum_{k=0}^{\infty}(-a)^k q^{k^2}. \tag{2.1}$$

This involution plays a key role in the Franklin type involution presented in the next section which can be viewed as a bijective proof of Alladi's partition theorem derived from (2.1) with respect to certain weight assignment.

Berndt, Kim and Yee [8] provided a bijective proof of (2.1) based on the interpretation of the numerator $(-q;q)_{k-1}q^{k(k+1)/2}$ in terms of parity sequences. Yee [15] made an attempt to give another bijective proof of (2.1). However, the bijection presented in the current version of [15] is not complete. It only deals with the Part B case of our construction.

Let D_k be the set of partitions π into k distinct parts with the smallest part being 1 such that $\pi_i - \pi_{i+1} \leq 2$, and let E_k denote the set of partitions σ with even parts not exceeding $2k$, that is, each σ_i is even and $\sigma_1 \leq 2k$. We are going to establish an involution on $D_k \times E_k$. Throughout this paper, T_k standards for the triangular partition $(2k-1, 2k-3, \ldots, 3, 1)$.

Theorem 2.1 *There exists an involution on the set $D_k \times E_k$ under which the pair of partitions $(T_k, \emptyset)$ remains invariant.*

To construct the desired involution on $D_k \times E_k$, we introduce a statistic called the modular leg hook length of a partition in D_k. Adopting the notation in [11], we use $[\lambda]_2$ to denote the 2-modular diagram of a partition λ defined to be a Young diagram filled with 1 or 2 such that the last cell of row i is filled with 1 if and only if λ_i is odd. Given a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k) \in D_k$, let us consider the 2-modular diagram. Suppose that π_i is an even part other than the largest part, we can associate it with a modular leg hook H_i which consists of the squares in the i-th row in the 2-modular diagram and the squares in first column above the i-th row. For a modular leg hook H_i, the length of this hook, denoted by $|H_i|$, is defined to be the sum of the numbers filled in the hook, and its height is referred to the number of squares in the first column.

We are now ready to describe the construction of the involution on $D_k \times E_k$. Let us denote this involution by φ.

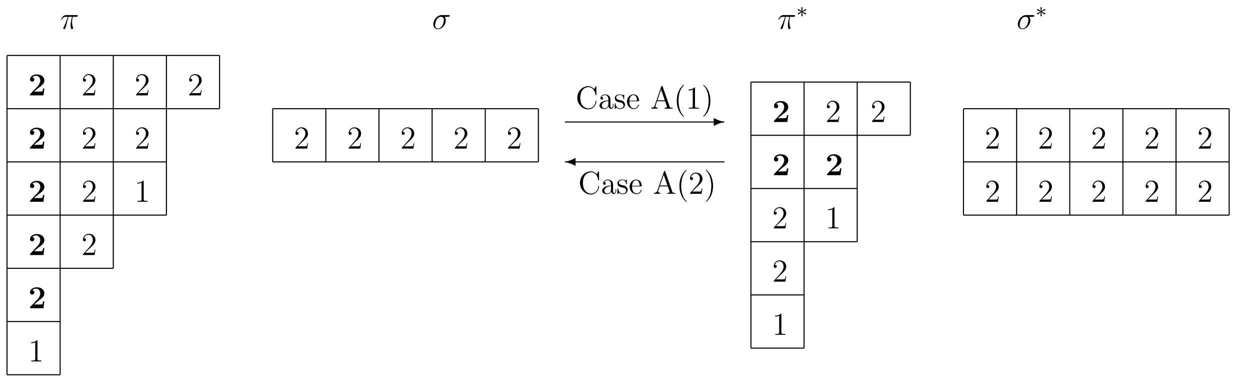
The involution φ on $D_k \times E_k$: Given a pair of partitions $(\pi, \sigma) \in D_k \times E_k$, represent π and σ by their 2-modular diagrams, respectively. In fact, the desired involution consists of two involutions.

Part A: We have the following two cases.

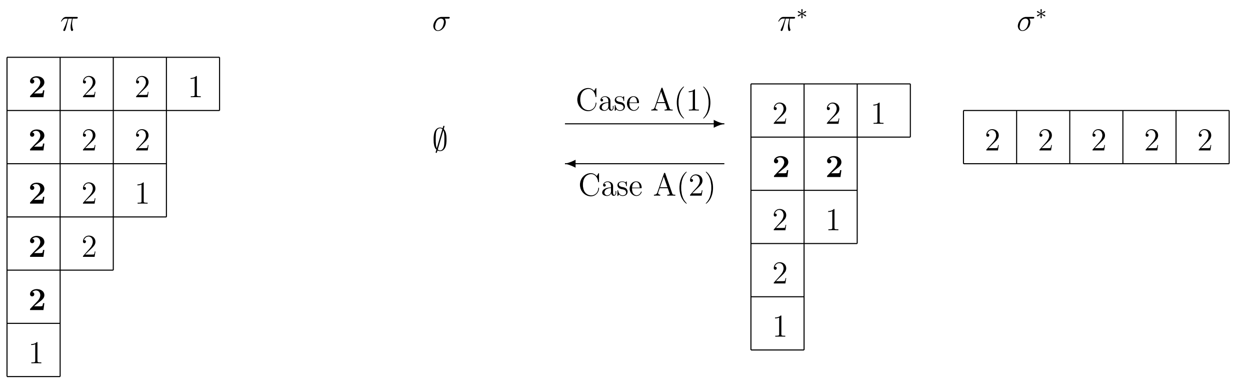
(1) Suppose that there exists a modular leg hook in π such that after the deletion of this hook the resulting partition is in D_{k-1}, then we choose such a hook with maximum height and denote it by $H_r(\pi)$. If $|H_r(\pi)| \geq \sigma_1$. Then delete $H_r(\pi)$ from π and add it to σ as a new part. We denote the resulting partitions by π^* and σ^*, respectively. Since $|H_r(\pi)| \leq 2k - 2$, we have $(\pi^*, \sigma^*) \in D_{k-1} \times E_{k-1}$.

(2) Suppose that either there is the modular leg hook $H_r(\pi)$ in π and $|H_r(\pi)| < \sigma_1$ or there does not exist the modular leg hook $H_r(\pi)$ in π and $\pi_1 + 2 < \sigma_1$. Then insert σ_1 into π as a modular leg hook in the 2-modular diagram of π. To be precise, this operation can be described as follows. Let i be the largest positive integer such that $\sigma_1 - 2i > \pi_{i+1}$, that is, for $j > i$ we have $\sigma_1 - 2j < \pi_{j+1}$. Then we add 2 to each of the first i parts $\pi_1, \pi_2, \ldots, \pi_i$, and insert $\sigma_1 - 2i$ as a new part before the part π_{i+1}. Since $\sigma_1 \leq 2k - 2$ and any two consecutive parts of π differ by at most 2, the resulting pair of partitions, denoted by (π^*, σ^*), belongs to $D_{k+1} \times E_k$. Furthermore, there exists the modular leg hook $H_r(\pi^*)$ in π^* and $|H_r(\pi^*)| \geq \sigma_1^*$.

Below is an example.



For the case when there does not exist a modular leg hook, we have the following example.

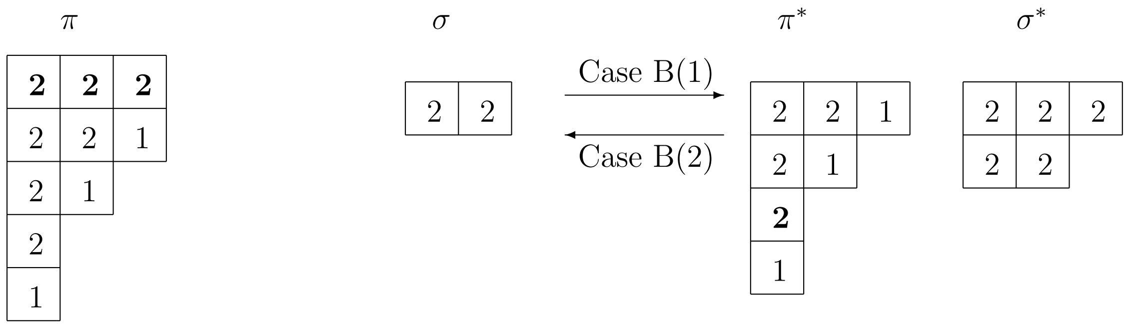


Part B: Suppose that there does not exist the modular leg hook $H_r(\pi)$ in π and $\sigma_1 \leq \pi_1 + 2$. If π has even parts, then we choose the largest even part of π, and denote it by π_r. We consider the following two cases.

(1) If $\pi_r \geq \sigma_1$, then remove the part π_r in π and add it to σ. We denote the resulting partitions by π^* and σ^*, respectively. Since $\pi_r \leq 2k-2$, we see that $(\pi^*, \sigma^*) \in D_{k-1} \times E_{k-1}$.

(2) If $\pi_r < \sigma_1$, then remove the part σ_1 in σ and add it to π. Denote the resulting partitions by π^* and σ^*, respectively. Since $\pi_i - \pi_{i+1} \leq 2$ for each i, σ_1 can be inserted either between two odd parts of π or at the top of π. Therefore, $(\pi^*, \sigma^*) \in D_{k+1} \times E_k$.

Here is an example.



In an extreme case, for $\pi = (7, 6, 5, 4, 3, 2, 1)$ and $\sigma = \emptyset$, we have $\pi^* = (7, 5, 4, 3, 2, 1)$ and $\sigma^* = (6)$ under the involution φ.

Finally, we are left with the case when π has no even parts and σ is the empty partition. In this situation, there is only one pair of partitions $(T_k, \emptyset)$, which is defined as the fixed point of the involution.

It is straightforward to check that the above correspondence is an involution. Except for the fixed point, the mapping changes the number of even parts of π by 1 and preserves the number of odd parts at the same time. Indeed, the above involution serves as a combinatorial proof of Ramanujan's partial theta identity (2.1).

Proof of (2.1): Note that the generating function for partitions $\pi \in D_k$ equals

$$(-q; q)_{k-1} q^{k(k+1)/2},$$

and the generating function for partitions $\sigma \in E_k$ equals

$$\frac{1}{(q^2; q^2)_k}.$$

Thus the left hand side of (2.1) corresponds to partitions $(\pi, \sigma) \in \bigcup_{k=0}^{\infty} D_k \times E_k$ with weight $(-1)^{\ell(\pi)} a^{\ell(\pi)+\ell(\sigma)}$. Notice that the involution φ changes the parity of $\ell(\pi)$ and preserves the quantity $\ell(\pi) + \ell(\sigma)$. The fixed point $(T_k, \emptyset)$ corresponds to the right hand side of (2.1). In view of the involution φ, we obtain the identity (2.1). ∎

3 A Franklin type involution for squares

In this section, we shall construct a Franklin type involution on $P_{do}(n)$, namely, the set of partitions of n into distinct parts with the smallest part being odd, where the involution φ on $D_k \times E_k$ given in the previous section serves as the main ingredient.

This involution will be used to give a combinatorial proof of a partition theorem derived by Alladi from Ramanujan's partial theta identity.

It should be noted that Bessenrodt and Pak [7] have established a different involution on $P_{do}(n)$ by using Vahlen's involution and Sylvester's transformation, which leads to the following theorem in the spirit of Euler's pentagonal number theorem.

Theorem 3.1 *For any positive integer n, we have*

$$\sum_{\lambda \in P_{do}(n)} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.1 implies the following theorem of Fine [9], see also [7, 10].

Theorem 3.2 *The number of partitions of n into distinct parts with the smallest being odd is odd if and only if n is a square.*

An indirect bijective proof of Fine's theorem has been given by Yee [14]. Compared with the involution of Bessenrodt and Pak, our involution also leads to the above theorem, and it further possesses a weight preserving property for the purpose of giving a combinatorial proof of Alladi's theorem. Meanwhile, both the involution of Bessenrodt and Pak and our involution have weighted versions for the another partition theorem of Alladi and for the partition theorems derived from identities of Andrews that will be considered later. Our involution, denoted by Ψ, can be described as follows.

Step 1. Extraction of parts from λ: For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_{do}(n)$, represent it by the 2-modular diagram $[\lambda]_2$, from which we can construct a pair of partitions $(\pi, \sigma) \in D_k \times E_k$. Initially, set $\pi = \lambda$, $\sigma = \emptyset$ and $t = k$. Then iterate the following procedure until $t = 1$:

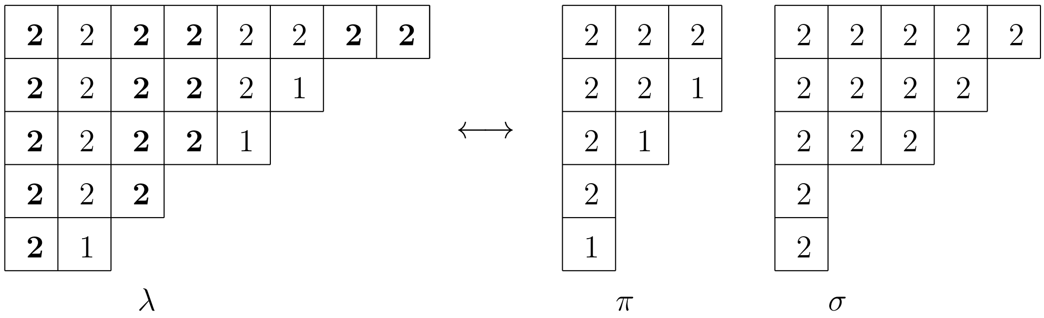
- Suppose that there exists i such that $\pi_t - \pi_{t+1} = 2i + r_t$, where $1 \leq r_t \leq 2$ and π_{k+1} is defined to be 0.
- Subtract $2i$ from each of the parts $\pi_1, \pi_2, \ldots, \pi_t$;
- Rearrange the parts to form a new partition π and add i parts of size $2t$ to σ. Replace t by $t-1$.

When $t = 1$, we get a pair of partitions $(\pi, \sigma) \in D_k \times E_k$. It is clear that

$$\ell(\lambda) = \ell(\pi), \quad \ell_e(\lambda) = \ell_e(\pi), \quad \ell_o(\lambda) = \ell_o(\pi),$$

where $\ell_e(\lambda)$ (resp. $\ell_o(\lambda)$) denotes the number of even (resp. odd) parts of λ.

Here is an example.



Step 2. Apply the involution φ on $D_k \times E_k$: For a pair of partitions $(\pi, \sigma) \in D_k \times E_k$, use the involution φ to generate a pair of partitions $(\pi^*, \sigma^*) \in D_{k+1} \times E_k$ or $D_{k-1} \times E_{k-1}$.

Step 3. Insertion of parts of σ^* to π^*: For a pair of partitions $(\pi^*, \sigma^*) \in D_{k+1} \times E_k$ or $D_{k-1} \times E_{k-1}$, consider their 2-modular diagrams. Let $\lambda^* = \pi^* + c_2(\sigma^*)$, where $c_2(\sigma^*)$ denotes the 2-modular conjugate partition obtained from $[\sigma^*]_2$, and for partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, $\lambda + \mu$ is defined to be the partition $(\lambda_1+\mu_1, \lambda_2+\mu_2, \ldots)$. Clearly, we have $\lambda^* \in P_{do}(n)$. It is obvious that

$$\ell(\lambda^*) = \ell(\pi^*), \quad \ell_e(\lambda^*) = \ell_e(\pi^*), \quad \ell_o(\lambda^*) = \ell_o(\pi^*).$$

Based on the above procedure, we can see that the mapping Ψ is a bijection. Moreover, it is easily seen that

$$\ell(\lambda^*) = \ell(\lambda) \pm 1, \quad \ell_e(\lambda^*) = \ell_e(\lambda) \pm 1, \quad \ell_o(\lambda^*) = \ell_o(\lambda), \tag{3.1}$$

where the $\pm$ sign means either plus or minus. In other words, Ψ changes the parity of the number of parts. It is easy to check that only when n is a square, say, $n = k^2$, there is exactly one partition which is undefined for Ψ, that is, $\lambda = (2k-1, 2k-3, \ldots, 3, 1)$. Therefore, the involution Ψ gives a combinatorial proof of theorem 3.1.

For example, when $n = 10$, there are six partitions in $P_{do}(10)$, namely,

$$9+1, \quad 7+3, \quad 4+3+2+1,$$

$$7+2+1, \quad 6+3+1, \quad 5+4+1.$$

The involution Ψ gives the following correspondence

$$9+1 \leftrightarrow 7+2+1, \quad 7+3 \leftrightarrow 6+3+1, \quad 4+3+2+1 \leftrightarrow 5+4+1.$$

While under the involution of Bessenrodt and Pak [7], the corresponding relations between them are given by:

$$9+1 \leftrightarrow 6+3+1, \quad 7+3 \leftrightarrow 7+2+1, \quad 4+3+2+1 \leftrightarrow 5+4+1.$$

4 Alladi's partition theorems

In this section, we apply the involution Ψ presented in the previous section to give a combinatorial interpretation of a weighted partition theorem derived by Alladi [1] from Ramanujan's partial theta identity (1.5). While Berndt, Kim and Yee [8] constructed an involution for the identity (1.5), they raised the question of how to translate their involution into a combinatorial proof of Alladi's weighted partition theorem. Even though our involution is not a direct answer to their question, it is likely that there is no easy way to make the translation. If so, our combinatorial interpretation can be considered as an indirect answer to the question of Berndt, Kim and Yee. The theorem of Alladi is stated as follows.

Theorem 4.1 *For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in P_{do}$, define δ_i to be the least integer $\geq (\lambda_i - \lambda_{i+1})/2$, where λ_{l+1} is defined to be 0. Define the weight of λ by*

$$\omega_g(\lambda) = (-1)^l \prod_{i=1}^{l} a^{\delta_i}. \tag{4.1}$$

Then we have

$$\sum_{\lambda \in P_{do}(n)} \omega_g(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Proof. For $\lambda \in P_{do}(n)$, let (π, σ) be the pair of partitions obtained from λ in Step 1 of the Franklin type involution Ψ. It can be seen that the exponent of a in $\omega_g(\lambda)$ equals $\ell(\pi) + \ell(\lambda)$. It is also clear that the quantity $\ell(\pi) + \ell(\lambda)$ remains unchanged in Step 2, that is

$$\ell(\pi) + \ell(\lambda) = \ell(\pi^*) + \ell(\lambda^*).$$

Let $\lambda^* = \pi^* + c_2(\sigma^*)$ in Step 3, then the exponent of a in $\omega_g(\lambda^*)$ equals $\ell(\pi^*) + \ell(\lambda^*)$. Thus the involution Ψ preserves the exponent of a in $\omega_g(\lambda)$. In view of the property (3.1), we see that $\omega_g(\lambda)$ and $\omega_g(\lambda^*)$ have opposite signs. Therefore, the partitions λ in $P_{do}(n)$ cancel each except for the the partition $\lambda = (2k-1, 2k-3, \ldots, 3, 1)$ which has weight $(-a)^k$ for $n = k^2$. This completes the proof. ∎

For example, when $n = 9$, there are five partitions in $P_{do}(9)$, that is,

$$8+1, \quad 6+3,$$

$$9, \quad 6+2+1, \quad 5+3+1.$$

Under the involution Ψ, the partitions are paired as follows

$$8+1 \leftrightarrow 9, \quad 6+3 \leftrightarrow 6+2+1,$$

while the triangular partition $5+3+1$ remains fixed. Meanwhile, the weights of the partitions are given by

$$\omega_g(8+1) = a^5, \quad \omega_g(6+3) = a^4,$$

and

$$\omega_g(9) = -a^5, \quad \omega_g(6+2+1) = -a^4, \quad \omega_g(5+3+1) = -a^3.$$

From (3.1), it can be seen that the Franklin type involution Ψ preserves the number of odd parts of $\lambda \in P_{do}(n)$. Thus, the involution Ψ can be used to give a combinatorial interpretation of another weight partition theorem derived by Alladi [2] from Andrews' identity (1.6).

Theorem 4.2 *For $\lambda \in P_{do}$, define the weight of λ by*

$$\omega_o(\lambda) = (-1)^l a^{\ell_o(\lambda)}. \tag{4.3}$$

Then we have

$$\sum_{\lambda \in P_{do}(n)} \omega_o(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof. Let $\lambda \in P_{do}(n)$. From (3.1), it is easily seen that the involution Ψ changes the number of even parts of λ by 1 and preserves the number of odd parts. Consequently, the involution Ψ preserves the exponent of a given in the weight $\omega_o(\lambda)$ and reverses the sign of $\omega_o(\lambda)$. When n is a square, say, $n = k^2$, there exists exactly one partition which is undefined for Ψ, that is $\lambda = (2k-1, 2k-3, \ldots, 3, 1)$ whose weight equals $(-a)^k$. This completes the proof. ∎

We should note that the involution of Bessenrodt and Pak [7] also preserves the number of odd parts of $\lambda \in P_{do}(n)$. Thus it implies a combinatorial proof of Theorem 4.2 as well.

We note that Theorem 4.2 can be translated back to the following identity:

$$\sum_{n=1}^{\infty} -aq^{2n-1}(q^{2n};q^2)_\infty(aq^{2n+1};q^2)_\infty = \sum_{n=1}^{\infty}(-a)^n q^{n^2}, \tag{4.5}$$

which takes a different form compared with the identity (1.6). Nevertheless, as shown by Alladi [2], (4.5) can be deduced from (1.6). In the next section, we shall give a combinatorial proof of (1.6).

5 A partition theorem derived from Andrews' identity

As we have seen in the previous section, Theorem 4.2 is a direct translation of the identity (4.5) rather than Andrews' identity (1.6). We first derive a partition theorem from (1.6). Then we give a combinatorial proof by applying the involution Ψ. Recall that Q denotes the set of partitions into distinct non-negative parts with the smallest part being even. Let $Q(n)$ denote such partitions of n in Q. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in Q$, define the weight of λ by

$$\omega_e(\lambda) = (-1)^{l-1} a^{\ell_o(\lambda)}. \tag{5.1}$$

Then we have the following partition identity.

Theorem 5.1 *We have*

$$\sum_{\lambda \in Q(n)} \omega_e(\lambda) = \begin{cases} (-a)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Proof. Let λ be a partition in $Q(n)$. Let $s(\lambda)$ denote the smallest part of the partition λ, and let $ss(\lambda)$ denote the second small part of λ. Define an involution ψ by the following procedure. Three cases are considered.

(i) Assume that $s(\lambda) = 0$ and $ss(\lambda)$ is even. Delete the part $s(\lambda)$ in λ and denote the resulting partition by λ^*. It can be seen that $\lambda^* \in Q(n)$.

(ii) Assume that $s(\lambda) \neq 0$. Add 0 to λ as a new part. Denote the resulting partition by λ^*. Then we have $\lambda^* \in Q(n)$.

(iii) Assume that $s(\lambda) = 0$ and $ss(\lambda)$ is odd. In this case, λ can be considered as a partition in $P_{do}(n)$ by disregarding the zero part $s(\lambda)$ so that we can apply Ψ to λ.

According to the above construction, ψ is a sign-reversing and weight-preserving involution for which the partition $\lambda = (2k-1, 2k-3, \ldots, 3, 1, 0) \in Q(n)$ is defined as the fixed point for $n = k^2$. This completes the proof. ∎

For example, $n = 10$, there are fourteen partitions in $Q(10)$:

$$10, \quad 8+2, \quad 6+4, \quad 5+3+2$$

$$10+0, \quad 8+2+0, \quad 6+4+0, \quad 5+3+2+0$$

$$9+1+0, \quad 7+3+0, \quad 4+3+2+1+0,$$

$$7+2+1+0, \quad 6+3+1+0, \quad 5+4+1+0.$$

In this example, the involution ψ gives the following correspondence

$$10 \leftrightarrow 10+0, \quad 8+2 \leftrightarrow 8+2+0, \quad 6+4 \leftrightarrow 6+4+0, \quad 5+3+2 \leftrightarrow 5+3+2+0$$

$$9+1+0 \leftrightarrow 7+2+1+0,\ 7+3+0 \leftrightarrow 6+3+1+0,\ 4+3+2+1+0 \leftrightarrow 5+4+1+0.$$

The weights of partitions in $Q(10)$ are listed below, and it can be seen that ψ is indeed weight-preserving and sign-reversing,

$$\omega_e(10) = 1, \quad \omega_e(8+2) = -1, \quad \omega_e(6+4) = -1, \quad \omega_e(5+3+2) = a$$

$$\omega_e(9+1+0) = a^2, \quad \omega_e(7+3+0) = a^2, \quad \omega_e(4+3+2+1+0) = a^2,$$

$$\omega_e(10+0) = -1, \quad \omega_e(8+2+0) = 1, \quad \omega_e(6+4+0) = 1, \quad \omega_e(5+3+2+0) = -a$$

$$\omega_e(7+2+1+0) = -a^2, \quad \omega_e(6+3+1+0) = -a^2, \quad \omega_e(5+4+1+0) = -a^2.$$

We remark that the involution of Bessenrodt and Pak can be modified to prove Theorem (5.1).

6 Connection to another identity of Andrews

In this section, we consider the special case of Theorem 5.1 when setting $a = -1$, that is,

$$\sum_{n=0}^{\infty} q^{2n}(q^{2n+2}; q^2)_\infty(-q^{2n+1}; q^2)_\infty = \sum_{k=0}^{\infty} q^{k^2}, \tag{6.1}$$

which turns out to be related to a problem proposed by Andrews [3] in 1972, see also, Andrews [4, pp. 156-157]. The original problem of Andrews is stated below.

A Problem of Andrews. Let $q_e(n)$ (resp. $q_o(n)$) denote the number of partitions in $Q(n)$ that have an even number (resp. odd number) of even parts. Prove that

$$q_o(n) - q_e(n) = \begin{cases} 1, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{6.2}$$

Clearly, the left hand side of (6.1) counts the number of partitions λ in Q with the sign $(-1)^{\ell_e(\lambda)-1}$ attached to λ. The sign $(-1)^{\ell_e(\lambda)-1}$ equals the weight of λ by setting $a = -1$ in (5.1), namely,

$$\omega_e(\lambda) = (-1)^{l-1} a^{\ell_o(\lambda)}.$$

Thus we can apply the above involution ψ defined in the previous section give a combinatorial interpretation of the identity (6.2).

When $a = -1$, the identity (5.2) can be rewritten as

$$\begin{aligned} \sum_{\lambda \in Q(n)} \omega_e(\lambda) &= \sum_{\lambda \in Q(n)} (-1)^{l-1} (-1)^{\ell_o(\lambda)} = \sum_{\lambda \in Q(n)} (-1)^{\ell_e(\lambda)-1} \\ &= q_o(n) - q_e(n) = \begin{cases} 1, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{6.3}$$

It is clear from (3.1) that the involution ψ only changes the number of even parts of $\lambda \in Q(n)$ by 1. Thus the identity (6.2) follows from the involution ψ.

For example, when $n = 9$, the five partitions enumerated by $q_e(9)$ are

$$8+1+0, \quad 7+2+0, \quad 6+3+0, \quad 5+4+0, \quad 4+3+2,$$

and the six partitions enumerated by $q_o(9)$ are

$$9+0, \quad 7+2, \quad 6+2+1+0, \quad 5+4, \quad 5+3+1+0, \quad 4+3+2+0.$$

Under the involution ψ, the partitions are paired as follows

$$8+1+0 \leftrightarrow 9+0, \quad 7+2+0 \leftrightarrow 7+2, \quad 6+3+0 \leftrightarrow 6+2+1+0,$$

$$5+4+0 \leftrightarrow 5+4, \quad 4+3+2 \leftrightarrow 4+3+2+0.$$

The partition $5+3+1+0$ is the fixed point.

7 A more general partition theorem

In this section, we present the following weighted form of Andrews' identity (1.8):

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{2mn}(q^{2mn+2m};q^{2m})_\infty(aq^{2mn+1};q^2)_\infty \\ &= 1+\sum_{k=1}^{\infty}(-a)^k q^{k^2}\prod_{j=1}^{k}(1+q^{2j}+q^{4j}+\cdots+q^{2(m-1)j}),\end{aligned} \tag{7.1}$$

which reduces to (1.8) by setting $a=-1$ and reduces to (1.6) by setting $m=1$. By extending the involution Ψ, we obtain a combinatorial interpretation of the above generalization. Notice that one can also extend the involution of Bessenrodt and Pak to give a combinatorial proof of (7.1).

Let us introduce some notation. For a positive integer m, let $A_{k,m}$ denote the set of partitions into k distinct nonnegative parts such that all the even parts are multiples of $2m$ and the smallest part is even. Let $A_m=\bigcup_{k=0}^{\infty} A_{k,m}$ and let $A_m(n)$ be the set of partitions of n in A_m. In this notation, the generating function for partitions $\lambda \in A_m$ equals

$$\sum_{n=0}^{\infty} q^{2mn}(-q^{2mn+2m};q^{2m})_\infty(-q^{2mn+1};q^2)_\infty. \tag{7.2}$$

To give a combinatorial interpretation of the right hand side of (7.1), let $H_{k,m}$ denote the set of partitions $\lambda_{k,m}$ such that each part of $\lambda_{k,m}$ is less than or equal to k and the multiplicity of each part is an even number less than $2m$. Then the generating function for partitions $\lambda_{k,m}$ in $H_{k,m}$ equals

$$\prod_{j=1}^{k}(1+q^{2j}+q^{4j}+\cdots+q^{2(m-1)j}).$$

The factor q^{k^2} equals the generating function of the triangular partition

$$T_k=(2k-1,2k-3,\ldots,3,1).$$

In order to give a combinatorial explanation of the identity (7.1), we shall give another interpretation of the right hand side of (7.1). To this end, let $B_{k,m}$ denote the set of partitions $\pi=(\pi_1,\pi_2,\ldots,\pi_k)$ into distinct odd parts such that the difference of consecutive parts is less than or equal to $2m$, namely, $\pi_i-\pi_{i+1}\le 2m$ for $1\le i\le k$ with the convention that $\pi_{k+1}=0$. Set

$$B_m=\bigcup_{k=0}^{\infty} B_{k,m},$$

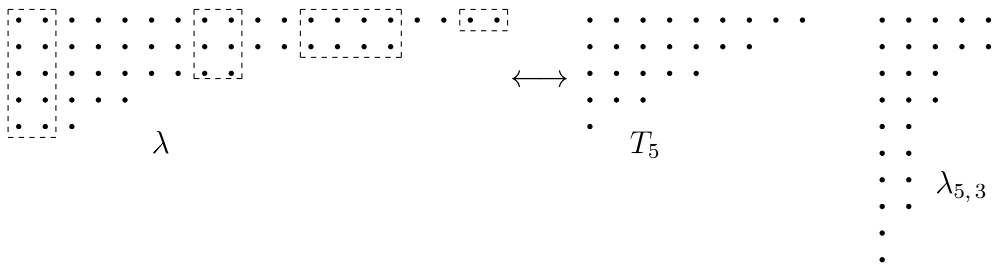
and let $B_m(n)$ be the set of partitions of n in B_m. Then we have the following correspondence.

Theorem 7.1 *There exists a bijection between the set $B_{k,m}$ and the set $\{T_k\} \times H_{k,m}$.*

Proof. We proceed to construct a bijection from $B_{k,m}$ to $\{T_k\} \times H_{k,m}$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in B_{k,m}$, using the Ferrers diagram, we can generate the triangular partition T_k and a partition $\lambda_{k,m} \in H_{k,m}$ by the following procedure. Let i be the largest integer such that $\lambda_i - \lambda_{i+1} = 2j > 2$ and $j \leq m$ with the convention that $\lambda_{k+1} = 0$. Then we remove $2(j-1)$ columns of length i from λ and add them to $\lambda_{k,m}$ as rows. Repeating this procedure until there does not exist such i. Finally, the remaining partition is the triangular partition T_k. It can be seen that $\lambda_{k,m} \in H_{k,m}$.

The above construction is reversible. Given the triangular T_k and a partition $\lambda_{k,m} \in H_{k,m}$, let $\lambda = T_k + \lambda'_{k,m}$. Then, we have $\lambda \in B_{k,m}$. This completes the proof. ∎

Below is an example when $\lambda = (19, 15, 9, 5, 3) \in B_{5,3}$.



From Theorem 7.1, we conclude that the generating function for partitions $\lambda \in B_m$ equals

$$1 + \sum_{k=1}^{\infty} q^{k^2} \prod_{j=1}^{k} (1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}). \tag{7.3}$$

Using the identities (7.2) and (7.3), we obtain the the following number-theoretic interpretation of the identity (7.1) in terms of weighted partitions.

Theorem 7.2 *Assume that $m \geq 1$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in A_m$, define the weight of λ by*

$$\omega_1(\lambda) = (-1)^{l-1} a^{\ell_o(\lambda)}. \tag{7.4}$$

On the other hand, for $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \in B_m$, define the weight of μ by

$$\omega_2(\mu) = (-a)^l. \tag{7.5}$$

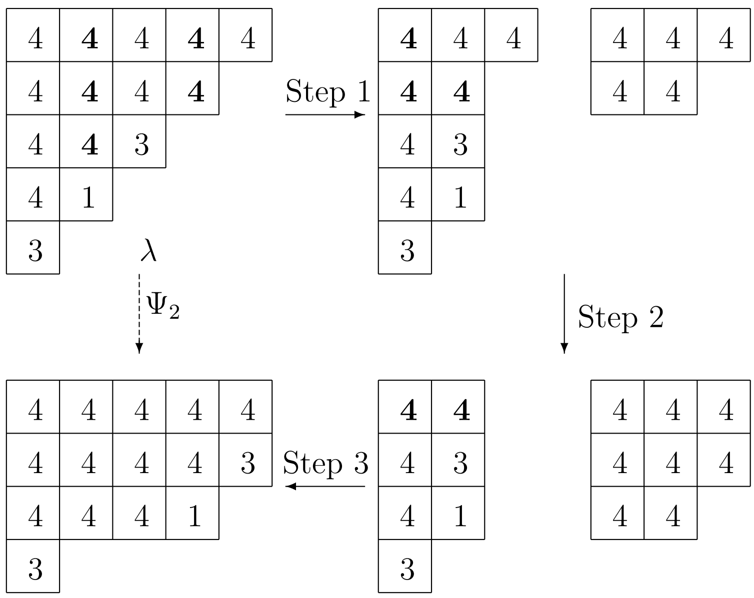
Then the following relation holds

$$\sum_{\lambda \in A_m(n)} \omega_1(\lambda) = \sum_{\mu \in B_m(n)} \omega_2(\mu). \tag{7.6}$$

Since $A_1 = Q$ and B_1 consists of only triangular partitions $T_k = (2k-1, 2k-3, \ldots, 3, 1)$, Theorem 7.2 reduces to Theorem 5.1 when setting $m = 1$.

The proof of Theorem 7.2 relies on the notion of $2m$-modular diagrams, see [11]. Recall that the $2m$-modular diagram of a partition λ is defined to be Young diagram by placing the integer $2m$ in the squares of each row possibly except for the last square, and the last square of each row may be filled with an integer not exceeding $2m$.

Let $P_{do}^m(n)$ denote the set of partitions of n into distinct parts such that all the even parts are multiples of $2m$ and the smallest part is odd. Using the $2m$-modular diagrams of partitions, we can extend the Franklin type involution Ψ on $P_{do}(n)$ to $P_{do}^m(n)$, and we denote it by Ψ_m. The explicit construction of Ψ_m is analogous to the three steps of the involution Ψ in Section 3, and hence it is omitted. Furthermore, we can extend the involution ψ on $Q(n)$ to $A_m(n)$ with the aid of Ψ_m to give a combinatorial proof of Theorem 7.2. Since the proof of Theorem 7.2 is similar to that of Theorem 5.1, it is also omitted. Here is an example of the involution ψ_m for $m = 2$. For $\lambda = (20, 16, 11, 5, 3, 0) \in A_2(55)$, we have $\psi_2(\lambda) = (20, 19, 13, 3, 0) \in A_2(55)$. The following figure is an illustration of the procedure to construct $\psi_2(\lambda)$.



Acknowledgments. We are grateful to Krishnaswami Alladi, Bruce C. Berndt, George E. Andrews, Mourad E. H. Ismail, Byungchan Kim, Igor Pak and Ae Ja Yee for helpful comments and suggestions. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] K. Alladi, A partial theta identity of Ramanujan and its number theoretic interpretation, Ramanujan J., to appear.

[2] K. Alladi, A combinatorial study and comparison of partial theta identities of Andrews and Ramanujan, Ramanujan J., to appear.

[3] G.E. Andrews, Problem 5865, Amer. Math. Monthly, 79 (1972) 668.

[4] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1998.

[5] G.E. Andrews and B.C. Berndt, Ramanujans Lost Notebook, Part II, Springer, New York, 2009.

[6] G.E. Andrews, Parity in partition identities, Ramanujan J., to appear.

[7] C. Bessenrodt and I. Pak, Partition congruences by involutions, European J. Combin., 25 (2004) 1139-1149.

[8] B.C. Berndt, B. Kim and A.J. Yee, Ramanujan's Lost Notebook: Combinatorial proofs of identities associated with Heine's transformation or partial theta functions, J. Combin. Theory Ser. A., to appear.

[9] N. J. Fine, Some new results on partitions, Proc. Nat. Acad. Sci. USA 34 (1948), 616-618.

[10] I. Pak, On Fines partition theorems, Dyson, Andrews, and missed opportunities, Math. Intelligencer 25 (2003) 10-16.

[11] I. Pak, Partition bijections, a survey, Ramanujan J., 12 (2006) 5-75.

[12] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.

[13] A. Stenger, Solution to Problem 5865, Amer. Math. Monthly 80 (1973) 1147-1148.

[14] A.J. Yee, Bijective proofs of a theorem of Fine and related partition identities, Internat. J. Number Theory, to appear.

[15] A.J. Yee, Ramanujans partial theta series and parity in partitions, Ramanujan J., to appear.