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The Extended Zeilberger Algorithm with Parameters

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Dedicated to Professor Wen-Tsun Wu
on the occasion of his ninetieth birthday

Abstract. Two hypergeometric terms $f(k)$ and $g(k)$ are said to be similar if the ratio $f(k)/g(k)$ is a rational function of k. For similar hypergeometric terms $f_1(k), \ldots, f_m(k)$, we present an algorithm, called the extended Zeilberger algorithm, to derive a linear relation among the sums $F_i = \sum_k f_i(k)$ $(1 \leq i \leq m)$ with polynomial coefficients. When the summands $f_1(k), \ldots, f_m(k)$ contain a parameter x, we further impose the condition that the coefficients of F_i in the linear relation are x-free. Such linear relations with x-free coefficients can be used to determine the structure relations for orthogonal polynomials and to derive recurrence relations for the connection coefficients between two sequences of orthogonal polynomials. The extended Zeilberger algorithm can be easily adapted to basic hypergeometric terms. As examples, we use the algorithm or its q-analogue to establish linear relations among orthogonal polynomials and to derive recurrence relations with multiple parameters for hypergeometric sums and basic hypergeometric sums.

Keywords: Zeilberger's algorithm, the extended Zeilberger algorithm, the Gosper algorithm, hypergeometric series, orthogonal polynomials

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1. Introduction

Based on Gosper's algorithm, Zeilberger [15,16] developed a powerful method for proving identities on hypergeometric series and basic hypergeometric series. Let $F(n,k)$ be a double hypergeometric term, namely, $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are both rational functions of n and k. Zeilberger's algorithm is devised to find a double hypergeometric term $G(n,k)$ and polynomials $a_0(n), a_1(n), \ldots, a_m(n)$ which are independent of k such that

$$a_0(n)F(n,k) + \cdots + a_m(n)F(n+m,k) = G(n,k+1) - G(n,k). \tag{1.1}$$

Set

$$S(n) = \sum_{k=0}^{\infty} F(n,k).$$

Summing (1.1) over k, we deduce that

$$a_0(n)S(n) + \cdots + a_m(n)S(n+m) = G(n,\infty) - G(n,0). \tag{1.2}$$

Thus the identity

$$\sum_{k=0}^{\infty} F(n,k) = f(n) \tag{1.3}$$

can be justified by verifying that $f(n)$ also satisfies (1.2) and both sides of (1.3) share the same initial values.

The main idea of this paper is the observation that Zeilberger's approach can be extended to a more general telescoping problem. Let $f_1(k,a,b,\ldots,c)$, …, $f_m(k,a,b,\ldots,c)$ be m similar hypergeometric terms of k with parameters $a,b,\ldots,c$, namely, the ratios

$$\frac{f_i(k,a,b,\ldots,c)}{f_j(k,a,b,\ldots,c)} \quad \text{and} \quad \frac{f_i(k+1,a,b,\ldots,c)}{f_i(k,a,b,\ldots,c)}$$

are all rational functions of k and $a,b,\ldots,c$. Find a hypergeometric term $g(k,a,b,\ldots,c)$, that is, the ratio $g(k+1,a,b,\ldots,c)/g(k,a,b,\ldots,c)$ is a rational function of k and $a,b,\ldots,c$, and polynomial coefficients $a_1(a,b,\ldots,c)$, $a_2(a,b,\ldots,c)$, …, $a_m(a,b,\ldots,c)$ which are independent of k such that

$$a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) = g(k+1) - g(k). \tag{1.4}$$

For brevity, from now on we may omit the parameters $a,b,\ldots,c$ and write $f_i(k)$ for $f_i(k,a,b,\ldots,c)$, a_i for $a_i(a,b,\ldots,c)$, and $g(k)$ for $g(k,a,b,\ldots,c)$. Let

$$F_i = \sum_k f_i(k),$$

for $1 \leq i \leq m$. Once the telescoping relation (1.4) is established, summing over k often leads to a homogenous relation among the sums F_i,

$$a_1F_1 + a_2F_2 + \cdots + a_mF_m = 0.$$

We were informed by one of referees that Paule independently obtained a parametric variation of Zeilberger's algorithm in an unpublished manuscript [11]. Here is a description of Paule's algorithm. Given a hypergeometric term $f(k)$ and rational functions $r_1(k)$, $r_2(k)$, …, $r_m(k)$, find a hypergeometric term $g(k)$ and coefficients $a_1, a_2, \ldots, a_m$ which are independent of k such that

$$a_1r_1(k)f(k) + a_2r_2(k)f(k) + \cdots + a_mr_m(k)f(k) = g(k+1) - g(k).$$

Clearly, the above formulation of Paule is equivalent to the general telescoping problem (1.4) by setting $f_i(k) = r_i(k)f(k)$.

It should be mentioned that for the purpose of computing the structure relations for orthogonal polynomials, our algorithm has an additional feature that it requires the x-free condition on the coefficients $a_1, \ldots, a_m$.

We remark that equation (1.4) can be solved in more general contexts. Let $(\mathbb{F}, \sigma)$ be a difference field, i.e., a field $\mathbb{F}$ with an automorphism $\sigma\colon \mathbb{F} \to \mathbb{F}$. Let $\mathbb{K} = \{a \in \mathbb{F}\colon \sigma(a) = a\}$ be its constant field. When the field $\mathbb{F}$ is a $\Pi\Sigma$-extension of $\mathbb{K}$, Karr [8] gave an algorithm to solve the following equation for $g \in \mathbb{F}$ and $a_1, \ldots, a_m \in \mathbb{K}$,

$$a_1f_1 + \cdots + a_mf_m = \sigma(g) - g, \tag{1.5}$$

where $f_1, \ldots, f_m$ are given elements in $\mathbb{F}$. Schneider [14] considered more general parameterized linear difference equations and provided a simplified version of Karr's algorithm.

We also note that Chyzak [5] extended Zeilberger's algorithm to general holonomic functions. Let $\mathbb{O}$ be an Ore algebra acting on f and $\partial_1, \ldots, \partial_m, \partial \in \mathbb{O}$. Chyzak provided algorithms to find an element $Q \in \mathbb{O}$ and $\boldsymbol{x}$-free coefficients η_i such that

$$\eta_1\partial_1 f + \cdots + \eta_m\partial_m f = \partial(Qf).$$

Here the $\boldsymbol{x}$-free condition corresponds to the usual k-free condition on the coefficients $\eta_1, \ldots, \eta_m$.

For the purpose of this paper, we assume that $f_i(k)$ are similar hypergeometric terms. We show that under this assumption Equaiton (1.4) can be solved by using the same technique as in Zeilberger's algorithm. Notice that the algorithms of Schneider and Chyzak rely on Abramov's algorithm. Our algorithm is based on Gosper's algorithm.

As an application of our algorithm, we compute the coefficients of the structure relations of orthogonal polynomials. For instance, let

$$P_n(x) = \sum_k P_{n,k}(x)$$

be the hypergeometric representation of the Jacobi polynomials as given in (3.3). Set

$$f_1(k) = P_{n,k}(x),\ f_2(k) = P'_{n+1,k}(x),\ f_3(k) = P'_{n,k}(x),\ f_4(k) = P'_{n-1,k}(x),$$

where $P'_{n,k}(x)$ denotes the derivative of $P_{n,k}(x)$ with respect to x. The extended Zeilberger algorithm enables us to give the structure relation for $P_n(x)$,

$$P_n(x) = \tilde{a}_n P'_{n+1}(x) + \tilde{b}_n P'_n(x) + \tilde{c}_n P'_{n-1}(x). \tag{1.6}$$

Sometimes it is necessary to impose an additional condition that the coefficients $a_1, \ldots, a_m$ in (1.4) are not only independent of k but also independent of some other parameters such as the variable x. For example, $\tilde{a}_n, \tilde{b}_n$ and $\tilde{c}_n$ in (1.6) are required to be independent of the variable x. Based on this parameter-free property, Chen and Sun [4] have developed a computer algebra approach to proving identities on Bernoulli polynomials and Euler polynomials.

It should be mentioned that Koepf and Schmersau [10] have presented two algorithms for deriving the structure relations for orthogonal polynomials by utilizing variations of Zeilberger's algorithm. Our extended Zeilberger algorithm serves as a unification of their algorithms and applies to more general cases. For instance, we are able to derive relations involving orthogonal polynomials with different parameters. Moreover, our algorithm can be used to derive recurrence relations for the connection coefficients between two sequences of orthogonal polynomials.

There are two examples that are worth mentioning. One is an identity due to Andrews [2] which is used in the evaluation of the Mills-Robbins-Rumsey determinant, and the other is Jackson's terminating q-analogue of Dixon's sum ${}_3\phi_2$. While in theory one can use Zeilberger's algorithm and q-Zeilberger's algorithm to derive recurrence relations, it is practically difficult to accomplish these tasks. Using the extended algorithm and its q-analogue, we easily find two simple recurrence relations with multiple parameters.

Let us give a brief review of some notation and terminology. A function $t(k)$ is called a hypergeometric term if $t(k+1)/t(k)$ is a rational function of k. A hypergeometric series is defined by

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the raising factorial. The q-shifted factorial is defined by

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

and we write

$$(a_1,\ldots,a_m;q)_k = (a_1;q)_k\cdots(a_m;q)_k.$$

A basic hypergeometric series is defined by

$${}_r\phi_s\left[\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array}\middle| q;z\right] = \sum_{k=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_k}{(b_1,\cdots b_s;q)_k}\frac{z^k}{(q;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r+1}.$$

2. The extended Zeilberger algorithm

Let $f_1(k), f_2(k), \ldots, f_m(k)$ be similar hypergeometric terms with parameters $a, b, \ldots, c$. Recall that two hypergeometric terms $f(k)$ and $g(k)$ are said to be similar if the ratio $f(k)/g(k)$ is a rational function of k and the parameters. We assume that

$$\frac{f_1(k+1)}{f_1(k)} = \frac{u(k)}{v(k)}, \tag{2.1}$$

and for $i = 1, 2, \ldots, m$,

$$\frac{f_i(k)}{f_1(k)} = \frac{p_i(k)}{Q(k)}, \tag{2.2}$$

where $u(k), v(k), p_i(k), Q(k)$ are polynomials in k and the parameters $a, b, \ldots, c$. Then we have

$$\frac{f_i(k+1)}{f_i(k)} = \frac{f_i(k+1)/f_1(k+1)}{f_i(k)/f_1(k)}\frac{f_1(k+1)}{f_1(k)} = \frac{p_i(k+1)Q(k)u(k)}{p_i(k)Q(k+1)v(k)}$$

and

$$\frac{f_i(k)}{f_j(k)} = \frac{f_i(k)/f_1(k)}{f_j(k)/f_1(k)} = \frac{p_i(k)}{p_j(k)}$$

are rational functions of k and $a, b, \ldots, c$. Thus (2.1) and (2.2) are equivalent to the statement that $f_1(k), f_2(k), \ldots, f_m(k)$ are similar hypergeometric terms.

Our aim is to find k-free polynomials $a_1, \ldots, a_m$ in the parameters $a, b, \ldots, c$ and a hypergeometric term $g(k)$ with parameters $a, b, \ldots, c$ such that

$$a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) = g(k+1) - g(k). \tag{2.3}$$

Since $f_1(k), \ldots, f_m(k)$ are similar hypergeometric terms, the sum

$$t_k = a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) \tag{2.4}$$

is a also hypergeometric term of k with parameters $a, b, \ldots, c$. Like Zeilberger's algorithm, we can apply Gosper's algorithm [7] to find k-free polynomials $a_1, \ldots, a_m$ and a hypergeometric term $g(k)$ such that (2.3) holds. It follows from (2.4) that

$$\begin{aligned} \frac{t_{k+1}}{t_k} &= \frac{f_1(k+1)}{f_1(k)} \frac{\sum_{i=1}^m a_i f_i(k+1)/f_1(k+1)}{\sum_{i=1}^m a_i f_i(k)/f_1(k)} \\ &= \frac{u(k)Q(k)}{v(k)Q(k+1)} \frac{\sum_{i=1}^m a_i p_i(k+1)}{\sum_{i=1}^m a_i p_i(k)}. \end{aligned}$$

Suppose that

$$\frac{u(k)Q(k)}{v(k)Q(k+1)} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}$$

is a Gosper representation, i.e., $a(k), b(k), c(k)$ are polynomials such that $\gcd(a(k), b(k+h)) = 1$ for all non-negative integers h. Then a Gosper representation of t_{k+1}/t_k is given by

$$\frac{t_{k+1}}{t_k} = \frac{a(k)}{b(k)} \frac{c(k+1)P(k+1)}{c(k)P(k)},$$

where

$$P(k) = \sum_{i=1}^m a_i p_i(k). \tag{2.5}$$

Gosper's algorithm states that $g(k)$ exists if and only if there exists a polynomial $x(k)$ such that

$$a(k)x(k+1) - b(k-1)x(k) = c(k)P(k). \tag{2.6}$$

Moreover, the degree bound d for $x(k)$ can be estimated by $a(k)$ and $b(k)$. Suppose that

$$x(k) = \sum_{i=0}^d c_i k^i.$$

By equating coefficients of k^i, we obtain a system of linear equations in $a_1, \ldots, a_m$ and $c_0, c_1, \ldots, c_d$. Solving this system of linear equations, we find the coefficients $a_1, \ldots, a_m$ and

$$g(k) = \frac{b(k-1)x(k)}{c(k)Q(k)} f_1(k).$$

In summary, the extended Zeilberger algorithm can be described by the following steps. Given m similar hypergeometric terms $f_1(k), \ldots, f_m(k)$, we wish to find k-free coefficients $a_1, a_2, \ldots, a_m$ and a hypergeometric term $g(k)$ satisfying (2.3).

Step 1. Compute the rational functions

$$r_i(k) = \frac{f_i(k)}{f_1(k)} \quad \text{and} \quad r(k) = \frac{f_1(k+1)}{f_1(k)}.$$

Set $Q(k)$ to be the common denominator of $r_1(k), \ldots, r_m(k)$, set

$$p_i(k) = r_i(k)Q(k),$$

and let $P(k)$ be given by (2.5).

Step 2. Compute a Gosper representation of

$$r(k)\frac{Q(k)}{Q(k+1)} = \frac{a(k)}{b(k)}\frac{c(k+1)}{c(k)}.$$

Step 3. Compute the degree bound d for $x(k)$ and solve Equation (2.6) by the method of undetermined coefficients to obtain the k-free coefficients $a_1, \ldots, a_m$ and the polynomial $x(k)$.

Step 4. The hypergeometric term $g(k)$ is then given by

$$g(k) = \frac{b(k-1)x(k)}{c(k)Q(k)}f_1(k).$$

We remark that when $d < 0$ in Step 3, we should set $x(k) = 0$ because we need the solution $g(k) = 0$ in this case.

Suppose that $F(n, k)$ is a double hypergeometric term. Clearly, the extended Zeilberger algorithm reduces to Zeilberger's algorithm by taking $f_i(k) = F(n + i - 1, k)$.

As will be seen, in some applications it is necessary to require that the coefficients $a_1, \ldots, a_m$ be independent of some parameters, say, the parameter a. For this purpose, we should express the solutions $(a_1, \ldots, a_m, g(k))$ of equation (2.3) in the following form,

$$\begin{aligned} &a_1 = v_1, \ldots, a_r = v_r, \\ &a_{r+1} = h_{r+1}(v_1, \ldots, v_r), \ldots, a_m = h_m(v_1, \ldots, v_r), \\ &g(k) = h(k, v_1, \ldots, v_r)f_1(k), \end{aligned} \tag{2.7}$$

where $v_1, \ldots, v_r$ are variables and $h_{r+1}, \ldots, h_m, h$ are linear combinations of $v_1, \ldots, v_r$ with coefficients being rational functions of the parameters $a, b, \ldots, c$. Thus, for $r+1 \leq i \leq m$, the functions h_i can be written as

$$h_i = p_i(v_1, \ldots, v_r, a, b, \ldots, c)/q_i(a, b, \ldots, c),$$

where p_i, q_i are relatively prime polynomials in $a, b, \ldots, c$. Now consider the additional requirement that $a_1, a_2, \ldots, a_m$ are independent of the parameter a. This means that all the coefficients of

$$p_i(v_1, \ldots, v_r, a, b, \ldots, c) - a_i q_i(a, b, \ldots, c)$$

in a must be zero except for the constant term. This condition gives rise to a system of linear equations in $a_1, \ldots, a_m$ and $v_1, \ldots, v_r$. Upon solving these equations, we eventually find $a_1, a_2, \ldots, a_m$ which are independent of k and the parameter a. The above version of the extended Zeilberger algorithm is still called the extended Zeilberger algorithm.

For example, the above algorithm can be used to derive linear relations on Chebyshev polynomials of the first kind and their derivatives.

Example 2.1 The Chebyshev polynomials of the first kind $T_n(x)$ are given by

$$T_n(x) = {}_2F_1\left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix} \,\middle|\, \frac{1-x}{2} \right), \tag{2.8}$$

see [9, Section 1.8.2]. We aim to find a structure relation of the form

$$T_n(x) = \alpha_n T'_{n+1}(x) + \beta_n T'_n(x) + \gamma_n T'_{n-1}(x),$$

where the coefficients $\alpha_n, \beta_n, \gamma_n$ do not depend on x. Let

$$T_{n,k}(x) = \frac{(-n)_k (n)_k}{(1/2)_k k!} \left(\frac{1-x}{2} \right)^k$$

be the k-th summand in (2.8). We first ignore the x-freeness requirement and apply the extended Zeilberger algorithm to the four similar hypergeometric terms with parameters n and x

$$f_1(k) = T_{n,k}(x),\ f_2(k) = T'_{n+1,k}(x),\ f_3(k) = T'_{n,k}(x),\ f_4(k) = T'_{n-1,k}(x).$$

We find that

$$a_1 = v_1,\ a_2 = v_2,\ a_3 = -\frac{x(v_1 + 2v_2(n+1))}{n},\ a_4 = \frac{v_1 + v_2(n+1)}{n-1}, \tag{2.9}$$

and

$$g(k) = \frac{k(2k-1)((v_1 + 2v_2(n+1))k - (v_1 + 2v_2)(n+1))}{(1-x)(n+1-k)(n-1+k)n} T_{n,k}(x). \tag{2.10}$$

Now we impose the x-freeness condition to get an additional equation

$$v_1 + 2v_2(n+1) = 0,$$

which yields

$$a_1 = v_1,\ a_2 = -\frac{v_1}{2(n+1)},\ a_3 = 0,\ a_4 = \frac{v_1}{2(n-1)},$$

and

$$g(k) = \frac{k(2k-1)v_1}{(x-1)(n+1-k)(n-1+k)} T_{n,k}(x).$$

It follows that

$$v_1 T_{n,k}(x) - \frac{v_1}{2(n+1)} T'_{n+1,k}(x) + \frac{v_1}{2(n-1)} T'_{n-1,k}(x) = g(k+1) - g(k).$$

Summing over k, we deduce that

$$T_n(x) = \frac{1}{2(n+1)} T'_{n+1}(x) - \frac{1}{2(n-1)} T'_{n-1}(x). \tag{2.11}$$

3. Hypergeometric series

It is easy to see that the method to derive the relation in Example 2.1 is valid in the general case. So we can use the extended Zeilberger algorithm to express the derivatives of orthogonal polynomials in terms of the polynomials themselves, and vice versa.

Let $P_n(x)$ be a sequence of continuous orthogonal polynomials. Let $P_{n,k}(x)$ be the k-th summand in the hypergeometric representation of $P_n(x)$ and $P'_{n,k}(x)$ be the derivative of $P_{n,k}(x)$. It is easily seen that $P'_{n,k}(x)$ is similar to $P_{n,k}(x)$. This enables us to derive the structure relations for $P_n(x)$ as given below

$$\sigma(x) P'_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \tag{3.1}$$

and

$$P_n(x) = \bar{a}_n P'_{n+1}(x) + \bar{b}_n P'_n(x) + \bar{c}_n P'_{n-1}(x), \tag{3.2}$$

where $\sigma(x)$ is a polynomials in x of degree less than or equal to 2 and $a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n$ are constants not depending on x. To derive (3.1), we set

$$f_1(k) = \sigma(x) P'_{n,k}(x),\ \ f_2(k) = P_{n+1,k}(x),\ \ f_3(k) = P_{n,k}(x),\ \ f_4(k) = P_{n-1,k}(x).$$

To establish (3.2), we set

$$f_1(k) = P_{n,k}(x),\ \ f_2(k) = P'_{n+1,k}(x),\ \ f_3(k) = P'_{n,k}(x),\ \ f_4(k) = P'_{n-1,k}(x).$$

Example 3.1 The monic Jacobi polynomials are given by

$$P_n(x) = \frac{(a+1)_n 2^n}{(n+a+b+1)_n}\ {}_2F_1\left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right), \tag{3.3}$$

see [9, Section 1.8]. Let $P_{n,k}(x)$ denote the k-th summand. Its derivative with respect to x equals

$$P'_{n,k}(x) = -\frac{(a+1)_n 2^n}{(n+a+b+1)_n} \frac{(-n)_k (n+a+b+1)_k}{2(a+1)_k (k-1)!} \left(\frac{1-x}{2}\right)^{k-1}.$$

Consider the similar terms

$$f_1(k) = (1-x^2)P'_{n,k}(x),\ f_2(k) = P_{n+1,k}(x),\ f_3(k) = P_{n,k}(x),\ f_4(k) = P_{n-1,k}(x),$$

and

$$f_1(k) = P_{n,k}(x),\ f_2(k) = P'_{n+1,k}(x),\ f_3(k) = P'_{n,k}(x),\ f_4(k) = P'_{n-1,k}(x),$$

respectively. By the extended Zeilberger algorithm with parameters n and x, we find that

$$(1-x^2)P'_n(x) = -nP_{n+1}(x) + \frac{2n(a-b)(n+a+b+1)}{(2n+2+a+b)(2n+a+b)} P_n(x) + \frac{4(n+b)(a+n)(n+a+b+1)(n+a+b)n}{(2n+a+b+1)(2n+a+b-1)(2n+a+b)^2} P_{n-1}(x).$$

and

$$P_n(x) = \frac{1}{n+1} P'_{n+1}(x) + \frac{2(a-b)}{(2n+2+a+b)(2n+a+b)} P'_n(x) - \frac{4(n+b)(a+n)n}{(2n+a+b+1)(2n+a+b-1)(2n+a+b)^2} P'_{n-1}(x).$$

The following example is concerned with expressing orthogonal polynomials with shifted parameters in terms of the original polynomials and their derivatives.

Example 3.2 Let

$$P_n^{(a,b)}(x) = \frac{(a+1)_n 2^n}{(n+a+b+1)_n} {}_2F_1\left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right)$$

be the Jacobi polynomials as in Example 3.1. By applying the extended Zeilberger algorithm to $f_1(k) = P_{n,k}^{(a+1,b)}(x)$ ($f_1(k) = P_{n,k}^{(a,b+1)}(x)$, respectively) and

$$f_2(k) = P_{n+1,k}^{(a,b)}{}'(x),\ f_3(k) = P_{n,k}^{(a,b)}{}'(x),\ f_4(k) = P_{n-1,k}^{(a,b)}{}'(x),$$

we are led to the known relations

$$P_n^{(a+1,b)}(x) = \frac{1}{n+1} P_{n+1}^{(a,b)}{}'(x) + \frac{2(a+1+n)}{(2n+2+a+b)(2n+a+b+1)} P_n^{(a,b)}{}'(x)$$

and

$$P_n^{(a,b+1)}(x) = \frac{1}{n+1} P_{n+1}^{(a,b)}{}'(x) - \frac{2(b+1+n)}{(2n+2+a+b)(2n+a+b+1)} P_n^{(a,b)}{}'(x),$$

which are due to Koepf and Schmersau [10]. Moreover, we can deduce the following relations which seem to be new,

$$\begin{aligned} P_n^{(a+1,b-1)}(x) = & \frac{1}{n+1} P_{n+1}^{(a,b)}{}'(x) + \frac{4(a+1+n)}{(2n+2+a+b)(2n+a+b)} P_n^{(a,b)}{}'(x) \\ & + \frac{4(a+1+n)(a+n)n}{(2n+a+b-1)(2n+a+b+1)(2n+a+b)^2} P_{n-1}^{(a,b)}{}'(x), \end{aligned}$$

and

$$\begin{aligned} P_n^{(a-1,b+1)}(x) = & \frac{1}{n+1} P_{n+1}^{(a,b)}{}'(x) - \frac{4(b+1+n)}{(2n+2+a+b)(2n+a+b)} P_n^{(a,b)}{}'(x) \\ & + \frac{4(b+1+n)(b+n)n}{(2n+a+b-1)(2n+a+b+1)(2n+a+b)^2} P_{n-1}^{(a,b)}{}'(x). \end{aligned}$$

The extended Zeilberger algorithm can also be employed to compute the connection coefficients of two sequences of orthogonal polynomials. Ronveaux [13] developed an approach to the computation of recurrence relations for the connection coefficients by utilizing structure relations of orthogonal polynomials. The extended Zeilberger algorithm can be directly used to serve this purpose. As an example, let us consider the connection coefficients of two sequences of Meixner polynomials with different parameters.

Example 3.3 Let $M_n^{(\gamma,\mu)}(x)$ be the monic Meixner polynomials defined by

$$M_n^{(\gamma,\mu)}(x) = (\gamma)_n \left(\frac{\mu}{\mu-1}\right)^n {}_2F_1\left(\begin{matrix} -n, -x \\ \gamma \end{matrix} \,\middle|\, 1 - \frac{1}{\mu} \right),$$

see [9, p. 45]. We wish to find a recurrence relation for the connection coefficients $C_m(n)$ defined by

$$M_n^{(\gamma,\mu)}(x) = \sum_{m=0}^{n} C_m(n) M_m^{(\delta,\nu)}(x). \tag{3.4}$$

To this end, we first find a difference operator which eliminates $M_n^{(\gamma,\mu)}(x)$. This task can be accomplished by applying the extended Zeilberger algorithm to the similar terms

$$f_1(k) = M_{n,k}^{(\gamma,\mu)}(x),\ f_2(k) = M_{n,k}^{(\gamma,\mu)}(x+1),\ \text{and}\ f_3(k) = M_{n,k}^{(\gamma,\mu)}(x-1),$$

where

$$M_{n,k}^{(\gamma,\mu)}(x) = (\gamma)_n \left(\frac{\mu}{\mu-1}\right)^n \frac{(-n)_k(-x)_k}{(\gamma)_k k!} \left(1-\frac{1}{\mu}\right)^k.$$

From the telescoping relation generated by the extended Zeilberger algorithm, we deduce that

$$(x\mu + \mu\gamma + x - n + n\mu)M_n^{(\gamma,\mu)}(x) - \mu(\gamma + x)M_n^{(\gamma,\mu)}(x+1) - xM_n^{(\gamma,\mu)}(x-1) = 0.$$

Let

$$S_m(x) = (x\mu + \mu\gamma + x - n + n\mu)M_m^{(\delta,\nu)}(x) - \mu(\gamma + x)M_m^{(\delta,\nu)}(x+1) - xM_m^{(\delta,\nu)}(x-1).$$

It follows from (3.4) that

$$\sum_{m=0}^{n} C_m(n)S_m(x) = 0. \tag{3.5}$$

Suppose that we can express $S_m(x)$ in terms of a suitable basis $\{B_m(x)\}$, namely,

$$S_m(x) = a_m B_{m+1}(x) + b_m B_m(x) + c_m B_{m-1}(x), \tag{3.6}$$

where a_m, b_m and c_m are independent of x. Substituting (3.6) into (3.5), by the linear independence of $B_m(x)$, we see that the coefficients of $B_m(x)$ are all zeros. This implies that

$$a_{m-1}C_{m-1}(n) + b_m C_m(n) + c_{m+1}C_{m+1}(n) = 0. \tag{3.7}$$

It remains to find the polynomials $B_m(x)$ in order to determine the coefficients a_m, b_m and c_m. In view of relation (3.6), we assume that $B_m(x)$ is a hypergeometric term that is similar to $S_m(x)$ so that we can solve the equation

$$S_m(x) - a_m B_{m+1}(x) - b_m B_m(x) - c_m B_{m-1}(x) = 0$$

by using the extended Zeilberger algorithm. In fact, we may choose

$$B_m(x) = \Delta(M_m^{(\delta,\nu)}(x)) = M_m^{(\delta,\nu)}(x+1) - M_m^{(\delta,\nu)}(x).$$

It is easily checked that $B_m(x)$ satisfies (3.6) and the corresponding coefficients are given by

$$a_m = \frac{(\mu-1)(n-m)}{m+1}, \quad c_m = \frac{(\nu-\mu)(\delta+m-1)m\nu}{(1-\nu)^2},$$

$$b_m = \frac{-\nu\mu m - m\mu + 2m\nu + \nu\mu\gamma - \nu n + \nu\delta - \nu + \mu - \nu\mu\delta - \mu\gamma + \nu n\mu}{1-\nu}.$$

So we obtain a recurrence relation (3.7) for the connection coefficients $C_m(n)$.

In general, we can use the following procedure to derive a recurrence relation for the connection coefficients of two sequences of orthogonal polynomials $P_n(x)$ and $Q_n(x)$. We first derive a linear differential or difference operator L_n which annihilates $P_n(x)$, i.e., $L_nP_n(x) = 0$. Then we try to find a suitable basis $B_m(x)$ for the space of polynomials such that $L_nQ_m(x)$ can be expressed as a linear combination of $B_{m-r}(x), B_{m-r+1}(x), \ldots, B_{m+r}(x)$ for a fixed integer r, say,

$$L_nQ_m(x) = \sum_{i=-r}^{r} a_i B_{m-r}(x).$$

We may consider $Q_m(x)$, $Q'_m(x)$, and $\Delta Q_m(x)$ as a choice of $B_m(x)$. Let $C_m(n)$ be the connection coefficients between $P_n(x)$ and $Q_n(x)$ defined by

$$P_n(x) = \sum_{m=0}^{n} C_m(n)Q_m(x).$$

Then we are led to a recurrence relation for $C_m(n)$ as given by

$$\sum_{i=-r}^{r} a_{m-i}C_{m-i}(n) = 0.$$

Let $F(n, k)$ be a hypergeometric term with parameters $a, b, \ldots, c$. In Zeilberger's algorithm, we only shift the variable n. While in the extended Zeilberger algorithm, the shifts of other parameters are allowed so that one may expect simpler recurrence relations. For example, we consider the following identity due to Andrews [1,2], which was used in the evaluation of the Mills-Robbins-Rumsey determinant.

Example 3.4 For $n \geq 0$, we have

$$ {}_5F_4\left(\begin{matrix} -2n-1, x+2n+2, x-z+\frac{1}{2}, x+n+1, z+n+1 \\ \frac{x+1}{2}, \frac{x+2}{2}, 2z+2n+2, 2x-2z+1 \end{matrix} \middle| 1 \right) = 0. \tag{3.8}$$

Let $f(n, x, z)$ be the sum on the left hand side. The extended Zeilberger algorithm gives the following recurrence relation

$$f(n, x, z) = c_1 f(n-1, x, z) + c_2 f(n-1, x+1, z+1),$$

where

$$c_1 = \frac{n(2z+2n+1)(2n+2z-x)(n+3z-2x)}{(x+1+2n)(2n+1+z)(2x+1-2z+2n)(-x+6z+8n)},$$

$$c_2 = \frac{(2z-x)nP(x,z,n)}{(x+1)(x+1+2n)(2n+1+z)(2x+1-2z+2n)(-x+6z+8n)},$$

and

$$P(x,z,n) = 2x^3 - 8nx^2 - 12zx^2 - x^2 + zx - 4n^2x - 3nx + 14nzx - x \\ + 18z^2x + 8n + 49nz + 98n^2z + 6z + 12z^2 + 43n^2 + 60n^3 + 42nz^2.$$

Hence (3.8) is valid since it holds for $n = 0$.

4. q-Hypergeometric series

The extended Zeilberger algorithm can be readily adapted to basic hypergeometric terms t_k with parameters $a, b, \ldots, c$, that is, the ratio t_{k+1}/t_k is a rational function of q^k and the parameters. Let $f_1(k)$, $f_2(k), \ldots, f_m(k)$ be similar q-hypergeometric terms, namely, $f_i(k)/f_j(k)$ and $f_i(k+1)/f_i(k)$ are rational functions of q^k and the parameters for $1 \leq i, j \leq m$. The objective of the q-analogue of the extended Zeilberger algorithm, called the extended q-Zeilberger algorithm, is to find a q-hypergeometric term $g(k)$ and k-free polynomial coefficients a_1, $a_2, \ldots, a_m$ such that

$$a_1 f_1(k) + a_2 f_2(k) + \cdots + a_m f_m(k) = g(k+1) - g(k). \tag{4.1}$$

The detailed description of the extended q-Zeilberger algorithm is similar to that of the ordinary case, hence it is omitted. We shall give two examples to demonstrate how to use the extended q-Zeilberger algorithm to compute the structure relations for q-orthogonal polynomials.

Example 4.1 The discrete q-Hermite polynomials are given by

$$H_n(x) = q^{\binom{n}{2}} \, {}_2\phi_1 \left[\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; -qx \right].$$

Let

$$H_{n,k}(x) = q^{\binom{n}{2}} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} (-qx)^k$$

be the k-th summand, and let D_q denote the q-difference operator, that is,

$$D_q f(x) = \frac{f(xq) - f(x)}{(q-1)x}.$$

By applying the extended q-Zeilberger algorithm to the similar terms

$$xD_q H_{n,k}(x),\ D_q H_{n+1,k}(x),\ D_q H_{n,k}(x),\ D_q H_{n-1,k}(x),$$

we obtain

$$xD_qH_n(x) = \frac{1-q^n}{1-q^{n+1}}D_qH_{n+1}(x) + q^{n-2}(1-q^n)D_qH_{n-1}(x).$$

Using the similar terms

$$D_qH_{n,k}(x),\ H_{n+1,k}(x),\ H_{n,k}(x),\ H_{n-1,k}(x),$$

the extended q-Zeilberger algorithm gives the relation

$$D_qH_n(x) = \frac{1-q^n}{1-q}H_{n-1}(x).$$

Example 4.2 The q-Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x) = \frac{1}{(q;q)_n}\,{}_2\phi_1\left[\begin{array}{c} q^{-n}, -x \\ 0 \end{array}\middle| q; q^{n+\alpha+1}\right].$$

It is known that

$$D_qL_n^{(\alpha)}(x) = -\frac{q^{(\alpha+1)}}{1-q}L_{n-1}^{(\alpha+1)}(xq),$$

see [9, Section 3.21]. This relation can be easily verified by using the extended q-Zeilberger algorithm. Notice that the right hand side of the above identity involves shifts of three parameters n, α and x. By choosing other shifts of parameters, we obtain the following identity

$$D_qL_n^{(\alpha)}(x) = \frac{1}{(1+x)(q-1)}(L_n^{(\alpha+1)}(x) - L_n^{(\alpha)}(x)). \tag{4.2}$$

The right hand side involves only the shift of the parameter α, where we do not require the coefficients to be x-free.

As the last example of this paper, we consider a q-series identity due to Jackson. In view of the terminating condition of the q-Zeilberger algorithm, there exists a recurrence relation for the summation on the left hand side. However, the q-Zeilberger algorithm, which was implemented by Koepf [3] and Riese [12], did not seem to be practically efficient to deliver a recurrence relation. Our algorithm gives a recurrence relation with multiple parameters.

Example 4.3 Jackson's terminating q-analogue of Dixon's sum reads

$${}_3\phi_2\left[\begin{array}{c} q^{-2n}, b, c \\ q^{1-2n}/b, q^{1-2n}/c \end{array}\middle| q; \frac{q^{2-n}}{bc}\right] = \frac{(b,c;q)_n(q,bc;q)_{2n}}{(q,bc;q)_n(b,c;q)_{2n}}, \tag{4.3}$$

see [6, p. 237]. Let $f(n,b,c)$ be the sum on the left hand side and

$$F_{n,k}(b,c) = \frac{(q^{-2n};q)_k(b;q)_k(c;q)_k}{(q;q)_k(q^{1-2n}/b;q)_k(q^{1-2n}/c;q)_k}\left(\frac{q^{2-n}}{bc}\right)^k$$

be the k-th summand. By applying the extended q-Zeilberger algorithm to the terms

$$F_{n,k}(b,c),\ F_{n-1,k}(bq,c),\ F_{n-1,k}(b,cq),\ F_{n-1,k}(bq,cq),$$

we deduce that

$$f(n,b,c) = \alpha_1 f(n-1,bq,c) + \alpha_2 f(n-1,b,cq) + \alpha_3 f(n-1,bq,cq), \tag{4.4}$$

where

$$\alpha_1 = -\frac{(q+cq^n)(c^2q^{2n}-q)(-q+cq^n)(bq^{2n}c-q)}{q^n(cq^{2n}-q)(cq^{2n}-q^2)(-c+b)c},$$

$$\alpha_2 = \frac{(q^{2n}b^2-q)(q+q^nb)(-q+q^nb)(bq^{2n}c-q)}{(-c+b)q^n(bq^{2n}-q)(bq^{2n}-q^2)b},$$

and

$$\alpha_3 = -\frac{(cq^nb-1)(-q+cq^nb)}{bcq^n}.$$

Then (4.3) can be justified by verifying that the right hand side satisfies the same recurrence relation since the identity holds for $n=0$.

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