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The Rogers-Ramanujan-Gordon Theorem for Overpartitions

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Abstract. Let $B_{k,i}(n)$ be the number of partitions of n with certain difference condition and let $A_{k,i}(n)$ be the number of partitions of n with certain congruence condition. The Rogers-Ramanujan-Gordon theorem states that $B_{k,i}(n) = A_{k,i}(n)$. Lovejoy obtained an overpartition analogue of the Rogers-Ramanujan-Gordon theorem for the cases $i = 1$ and $i = k$. We find an overpartition analogue of the Rogers-Ramanujan-Gordon theorem in the general case. Let $D_{k,i}(n)$ be the number of overpartitions of n satisfying certain difference condition and $C_{k,i}(n)$ be the number of overpartitions of n whose non-overlined parts satisfy certain congruences condition. We show that $C_{k,i}(n) = D_{k,i}(n)$. By using a function introduced by Andrews, we obtain a recurrence relation which implies that the generating function of $D_{k,i}(n)$ equals the generating function of $C_{k,i}(n)$. We also find a generating function formula of $D_{k,i}(n)$ by using Gordon marking representations of overpartitions, which can be considered as an overpartition analogue of an identity of Andrews for ordinary partitions.

Keywords: overpartition, the Rogers-Ramanujan-Gordon theorem, the Gordon marking of an overpartition

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1 Introduction

In this paper, we obtain the Rogers-Ramanujan-Gordon theorem for overpartitions. Furthermore, by introducing the Gordon marking of an overpartition, we find a generating function formula which can be considered as an overpartition analogue of an identity of Andrews. Notice that the identity of Andrews implies the Rogers-Ramanujan-Gordon theorem for ordinary partitions, see Kurşungöz [12].

An overpartition is a partition for which the first occurrence of a part may be overlined. For example, $(\overline{7}, 7, 6, \overline{5}, 2, \overline{1})$ is an overpartition of 28. There are many q-series identities that have combinatorial interpretations in terms of overpartitions, see, for example, Corteel and Lovejoy [8]. Furthermore, overpartitions possess many analogous properties of ordinary partitions, see Lovejoy [13, 15]. For example, various overpartition analogues of the Rogers-Ramanujan-Gordon theorem have been obtained by Corteel and Lovejoy [9], Corteel, Lovejoy and Mallet [10] and Lovejoy [13, 14, 16, 17].

Let us recall that Gordon [11] found the following combinatorial generalization of the Rogers-Ramanujan identities [18], which has been called the Rogers-Ramanujan-Gordon theorem, see Andrews [1].

Theorem 1.1 *(Rogers-Ramanujan-Gordon) Let $B_{k,i}(n)$ denote the number of partitions of n for the form $b_1 + b_2 + \cdots + b_s$, where $b_j \geq b_{j+1}$, $b_j - b_{j+k-1} \geq 2$ and at most $i-1$ of the b_j are equal to 1 and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i (mod\ 2k+1)$. Then for all $n \geq 0$,*

$$A_{k,i}(n) = B_{k,i}(n).$$

Lovejoy [13] obtained overpartition analogues of the above Rogers-Ramanujan-Gordon theorem for $i = k$ and $i = 1$.

Theorem 1.2 *Let $\overline{B}_k(n)$ denote the number of overpartitions of n of the form $y_1 + y_2 + \cdots + y_s$, such that $y_j - y_{j+k-1} \geq 1$ if y_j is overlined and $y_j - y_{j+k-1} \geq 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of n into parts not divisible by k. Then $\overline{A}_k(n) = \overline{B}_k(n)$.*

Theorem 1.3 *Let $\overline{D}_k(n)$ denote the number of overpartitions of n of the form $z_1 + z_2 + \cdots + z_s$, such that 1 cannot occur as a non-overlined part, and where $z_j - z_{j+k-1} \geq 1$ if z_j is overlined and $z_j - z_{j+k-1} \geq 2$ otherwise. Let $\overline{C}_k(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k$. Then $\overline{C}_k(n) = \overline{D}_k(n)$.*

The first result of this paper is to give an overpartition analogue of the Rogers-Ramanujan-Gordon theorem in the general case.

Theorem 1.4 *For $k \geq i \geq 1$, let $D_{k,i}(n)$ denote the number of overpartitions of n of the form $d_1 + d_2 + \cdots + d_s$, such that 1 can occur as a non-overlined part at most $i-1$ times, and where $d_j - d_{j+k-1} \geq 1$ if d_j is overlined and $d_j - d_{j+k-1} \geq 2$ otherwise. For $k > i \geq 1$, let $C_{k,i}(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm i$ modulo $2k$ and let $C_{k,k}(n)$ denote the number of overpartitions of n with parts not divisible by k. Then $C_{k,i}(n) = D_{k,i}(n)$.*

It is clear that Theorem 1.4 contains Theorems 1.2 and 1.3 as special cases for $i = k$ and $i = 1$. To be more specific, $\overline{B}_k(n)$ and $\overline{A}_k(n)$ in Theorem 1.2 are $D_{k,k}(n)$ and $C_{k,k}(n)$ in Theorem 1.4, $\overline{D}_k(n)$ and $\overline{C}_k(n)$ in Theorem 1.3 are $D_{k,1}(n)$ and $C_{k,1}(n)$ in Theorem 1.4.

We will give an algebraic proof of Theorem 1.4 in the next section by showing that the generating function of $D_{k,i}(n)$ equals the generating function of $C_{k,i}(n)$. It is evident that the generating function of $C_{k,i}(n)$ equals

$$\sum_{n\geq 0} C_{k,i}(n) q^n = \frac{(-q)_\infty (q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q)_\infty}. \tag{1.1}$$

In fact, we shall prove a stronger result on a refinement of the generating function of $D_{k,i}(n)$.

The generating function versions of Theorem 1.1 for $k = 2$ are the Rogers-Ramanujan identities

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}, \tag{1.2}$$

and

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q^1, q^4; q^5)_\infty}. \tag{1.3}$$

Note that the left hand sides of (1.2) and (1.3) can be interpreted as the generating functions for $B_{2,1}(n)$ and $B_{2,2}(n)$ respectively. As a generalization of the Rogers-Ramanujan identities, Andrews [2] obtained the following theorem.

Theorem 1.5 *For $k \geq i \geq 1$,*

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}. \tag{1.4}$$

The sum on the left hand side of (1.4) can be viewed as the generating function for $B_{k,i}(n)$. Andrews proved that the both sides of (1.4) satisfy the same recurrence relation.

While it is easy to give combinatorial interpretations of the left hand sides of (1.2) and (1.3), it does not seem to be trivial to show that the left hand side of (1.4) is the generating function for $B_{k,i}(n)$. Kurşungöz [12] provided a combinatorial explanation of the left hand side of (1.4) by introducing the notion of the Gordon marking of a partition. More precisely, he obtained the following formula for the generating function of $B_{k,i}(m,n)$, where $B_{k,i}(m,n)$ denotes the number of partitions enumerated by $B_{k,i}(n)$ that have m parts.

Theorem 1.6 *For $k \geq i \geq 1$,*

$$\sum_{m,n\geq 0} B_{k,i}(m,n)x^m q^n = \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \tag{1.5}$$

The second result of this paper is the following formula for the generating function of the number $D_{k,i}(m,n)$ of overpartitions enumerated by $D_{k,i}(n)$ that have m parts. We shall give a combinatorial proof of this identity by using the Gordon marking representations of overpartitions.

Theorem 1.7 *For $k \geq i \geq 1$, we have*

$$\begin{aligned}\sum_{n=0}^{\infty} & D_{k,i}(m,n)x^m q^n \\ &= \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}},\end{aligned} \tag{1.6}$$

where assume that $N_k = 0$.

By setting $x = 1$ in (1.6), we obtain the generating function for $D_{k,i}(n)$ which is the left hand side of (1.7). By Theorem 1.4, we are led to the following theorem which can be seen as an overpartition analogue of Andrews' identity (1.4).

Theorem 1.8 *For $k \geq i \geq 1$,*

$$\begin{aligned}\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} & \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} \\ &= \frac{(-q)_\infty(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty}.\end{aligned} \tag{1.7}$$

It is clear that the generating function for $C_{k,i}(n)$ equals the right hand side of (1.7). Hence identity (1.7) can be viewed as the generating function version of Theorem 1.4. It should be noticed that the approach of Andrews to (1.4) for ordinary partitions does not seem to apply to the above identity (1.7) for overpartitions.

The special case of identity (1.7) for $i = 1$ was derived by Chen, Sang and Shi [7] by using Andrews' multiple series transformation [3]. In this case, the left hand side of (1.7) has a combinatorial interpretation in terms of the generating function of the number of anti-lecture hall compositions of n with the first entry not exceeding $2k-2$.

The special case of (1.7) for $i = k$ was obtained by Corteel and Lovejoy [8] also by using Andrews' multiple series transformation. In this case, the left hand side of (1.7) has a combinatorial interpretation in terms of the number of overpartitions whose Frobenius representation has a top row with at most $k-2$ Durfee squares in its associated partition.

However, for $2 \leq i \leq k-1$, identity (1.7) does not seem to be a consequence of Andrews' multiple series transformation. It should be mentioned that for $i = 1, k$, the combinatorial interpretation of the left hand side of (1.7) as a Rogers-Ramanujan-Gordon theorem for overpartitions as in Theorem 1.4 is different from the interpretation in terms of anti-lecture hall compositions given in Chen, Sang and Shi [7] or the Frobenius representations given in Corteel and Lovejoy [8].

This paper is organized as follows. In Section 2, we give an algebraic proof of Theorem 1.4 by showing that $C_{k,i}(n)$ and $D_{k,i}(n)$ satisfy the same recurrence relation. In Section 3, we introduce the notion of the Gordon marking of an overpartition. To prove Theorem 1.7, we divide the set of overpartitions enumerated by $D_{k,i}(m,n)$ into two subsets. In Section 4, we define the first reduction operation and the first dilation operation. Based on the these two operations we give the first bijection for the proof of Theorem 1.7. In Section 5, we introduce the second reduction operation and the second dilation operation on the Gordon marking representations of overpartitions. Then we give the second bijection for the proof of Theorem 1.7. In Section 6, we give the third bijection for the proof of Theorem 1.7. In Section 7, we complete the proof of the Theorem 1.7.

2 An algebraic proof of Theorem 1.4

In this section, we give an algebraic proof of Theorem 1.4, that is, $C_{k,i}(n) = D_{k,i}(n)$ for any $k \geq i \geq 1$. We shall use a series $H_{k,i}(a; x; q)$ introduced by Andrews [1, 2], which is defined by

$$H_{k,i}(a;x;q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1-x^i q^{2ni})(axq^{n+1})_\infty (1/a)_n}{(q)_n (xq^n)_\infty}. \tag{2.1}$$

In his algebraic proof of the Rogers-Ramanujan-Gordon theorem, Andrews used the function $J_{k,i}(a;x;q)$ constructed based on $H_{k,i}(a;x;q)$,

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) - axqH_{k,i-1}(a;xq;q). \tag{2.2}$$

Lovejoy [15] proved Theorem 1.2 and Theorem 1.3 also by using $J_{k,i}(a;x;q)$ for special values of a and x. More precisely, he showed the generating function of $\overline{A}_k(n)$ and $\overline{C}_k(n)$, namely, $C_{k,k}(n)$ and $C_{k,1}(n)$, are given by the functions $J_{k,k}(-1;1;q)$ and $J_{k,1}(-1/q;1;q)$. As pointed out by Lovejoy, the approach of using the function $J_{k,i}(a;x;q)$ does not seem to apply to the general case, since for $i \neq 1, k$, the functions $J_{k,i}(-1;1;q)$ and $J_{k,i}(-1/q;1;q)$ do not appear to be expressible as single infinite products.

We find that for overpartitions the function $H_{k,i}(a;x;q)$ itself is the right choice to prove that $C_{k,i}(n) = D_{k,i}(n)$ for all $k \geq i \geq 1$. In fact, we shall show that the generating function of $C_{k,i}(n)$ can be expressed in terms of $H_{k,i}(a;x;q)$ for special values of a and x. To explain the fact that the generating functions of $C_{k,k}(n)$ and $C_{k,1}(n)$ can also be expressed by $J_{k,k}(-1;1;q)$ and $J_{k,1}(-1/q;1;q)$, we have the observations

$$J_{k,k}(-1;1;q) = H_{k,k}(-1/q;q;q), \tag{2.3}$$

and

$$J_{k,1}(a;x;q) = H_{k,1}(a;xq;q). \tag{2.4}$$

Andrews [1, 4] showed that the generating function of $B_{k,i}(m,n)$ can be expressed by $J_{k,i}(a;x;q)$:

$$\sum_{m,n\geq 0} B_{k,i}(m,n)x^m q^n = J_{k,i}(0;x;q). \tag{2.5}$$

We shall give the following theorem which involves a refinement of the number $D_{k,i}(n)$. Recall that $D_{k,i}(m,n)$ is the number of overpartitions enumerated by $D_{k,i}(n)$ with m parts. As will be seen, once the generating function of $D_{k,i}(m,n)$ is obtained, it is easy to derive the generating function of $D_{k,i}(n)$ by using Jacobi's triple product identity.

Theorem 2.1 *For $k \geq i \geq 1$, we have*

$$\sum_{m,n\geq 0} D_{k,i}(m,n)x^m q^n = H_{k,i}(-1/q;xq;q). \tag{2.6}$$

Proof. We define

$$W_{k,i}(x;q) = H_{k,i}(-1/q;xq;q), \tag{2.7}$$

and

$$W_{k,i}(x;q) = \sum_{m,n=-\infty}^{\infty} W_{k,i}(m,n)x^m q^n. \tag{2.8}$$

By the recurrence relation of $H_{k,i}(a;x;q)$, one can derive a recurrence relation of $W_{k,i}(m,n)$. It is easy to give a combinatorial interpretation of $D_{k,i}(m,n)-D_{k,i-1}(m,n)$. This yields a recurrence relation of $D_{k,i}(m,n)$ which coincides with a recurrence relation of $W_{k,i}(m,n)$.

Recall that $H_{k,i}(a;x;q)$ satisfies the following recurrence relation, see Andrews [4, Lemma 7.1],

$$H_{k,i}(a;x;q) - H_{k,i-1}(a;x;q) = x^{i-1}H_{k,k-i+1}(a;xq;q) - ax^i qH_{k,k-i}(a;xq;q). \tag{2.9}$$

Substituting $a = -1/q$ and $x = xq$ into (2.9), we obtain

$$W_{k,i}(x;q) - W_{k,i-1}(x;q) = (xq)^i W_{k,k-i}(xq;q) + (xq)^{i-1}W_{k,k-i+1}(xq;q). \tag{2.10}$$

Our goal is to prove that $D_{k,i}(m,n)$ equals $W_{k,i}(m,n)$. In doing so, we shall show that $D_{k,i}(m,n)$ and $W_{k,i}(m,n)$ satisfy the same recurrence relation with the same initial values, where $W_{k,i}(m,n)$ is the coefficient of $x^m q^n$ in the expansion of $W_{k,i}(x;q)$, as given by (2.8).

Clearly, we have the initial values $W_{k,i}(0,0) = 1$ for $k \geq i \geq 1$ and $W_{k,0}(m,n) = 0$ for $k \geq 1, m,n \geq 0$. Moreover, we assume that $W_{k,i}(m,n) = 0$ if m or n is zero but not both, and $W_{k,i}(m,n) = 0$ if m or n is negative. From (2.10) it is easily seen that

$$W_{k,i}(m,n) - W_{k,i-1}(m,n) = W_{k,k-i}(m-i,n-m) + W_{k,k-i+1}(m-i+1,n-m), \tag{2.11}$$

Thus $W_{k,i}(m,n)$ can be defined by the recurrence relation (2.11) along with the initial values.

Next we wish to find a recurrence relation of $D_{k,i}(m,n)$. It can be verified that $D_{k,i}(m,n)$ has the initial values $D_{k,i}(0,0) = 1$ for $k \geq i \geq 1$ and $D_{k,0}(m,n) = 0$ for $k \geq 1, m,n \geq 0$. Clearly, if exactly one of m and n is zero, then $D_{k,i}(m,n) = 0$. If one of m and n is negative, then $D_{k,i}(m,n) = 0$. Hence $D_{k,i}(m,n)$ has the same initial values as $W_{k,i}(m,n)$. It remains to prove that

$$D_{k,i}(m,n) - D_{k,i-1}(m,n) = D_{k,k-i}(m-i,n-m) + D_{k,k-i+1}(m-i+1,n-m). \tag{2.12}$$

From the definition of $D_{k,i}(m,n)$, one sees that $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ equals the number of overpartitions enumerated by $D_{k,i}(m,n)$ such that the non-overlined part 1 appears exactly $i-1$ times. We shall divide the overpartitions enumerated by $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ into two classes so that we can give a combinatorial interpretation of the right hand side of (2.12).

Let S_1 be the set of overpartitions enumerated by $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ that contain a part $\overline{1}$, and let S_2 be the set of overpartitions enumerated by $D_{k,i}(m,n) - D_{k,i-1}(m,n)$ that do not contain the part $\overline{1}$. We shall show that the number of overpartitions in S_1 equals $D_{k,k-i}(m-i, n-m)$ and the number of overpartitions in S_2 equals $D_{k,k-i+1}(m-i+1, n-m)$.

Let λ be an overpartition in S_1. So λ has i parts equal to 1 or $\overline{1}$. Removing these i parts, we obtain an overpartition that contains neither 1 nor $\overline{1}$. Subtracting 1 from each part of the resulting overpartition, we get an overpartition λ'. More precisely, by subtracting 1 from $\overline{r}$ we mean to change $\overline{r}$ to $\overline{r-1}$. From the definition of $D_{k,i}(m,n)$, we find that the parts 1, $\overline{1}$ and 2 occur at most $k-1$ times. Notice that the number of occurrences of 1 and $\overline{1}$ in λ equals i. Thus, 2 appear at most $k-i-1$ times in λ. So after the subtraction, the part 1 appears at most $k-i-1$ times in λ'. By the definition of $D_{k,i}(m,n)$, we deduce that the resulting overpartition λ' is enumerated by $D_{k,k-i}(m-i, n-m)$. Moreover, it is readily seen that every overpartition enumerated by $D_{k,k-i}(m-i, n-m)$ can be constructed by the above procedure.

For an overpartition λ in S_2, there are exactly $i-1$ parts equal to 1 in λ, so the part 2 occurs at most $k-i$ times in λ. Removing the $i-1$ parts 1 and subtracting 1 from each of the remaining parts, we get an overpartition λ'. It can be seen that the part 1 appears $k-i$ times in λ'. By the definition of $D_{k,k-i+1}(m-i+1, n-m)$, we find that λ' is enumerated by $D_{k,k-i+1}(m-i+1, n-m)$. Conversely, every overpartition enumerated by $D_{k,k-i+1}(m-i+1, n-m)$ can be constructed from an overpartition λ in S_2.

So we have proved relation (2.12), which implies that $D_{k,i}(m,n) = W_{k,i}(m,n)$ for all $k \geq i \geq 1$, and $m, n \geq 0$, since $D_{k,i}(m,n)$ and $W_{k,i}(m,n)$ have the same initial values. Thus the generating function of $D_{k,i}(m,n)$ equals $W_{k,i}(x;q)$. This completes the proof. ∎

We are ready to prove Theorem 1.4. Let us compute the generating function of $D_{k,i}(n)$. Setting $x=1$ in Theorem 2.1, we obtain that

$$\begin{aligned} H_{k,i}(-1/q;q;q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{kn^2+kn-in}(1-q^{(2n+1)i})(-q^{n+1})_\infty(-q)_n}{(q)_n(q^{n+1})_\infty} \\ &= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{kn^2+kn-in}(1-q^{(2n+1)i}) \end{aligned}$$

$$= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+kn-in}.$$

In view of Jacobi's triple product identity, we find that

$$\sum_{n\geq 0} D_{k,i}(n) q^n = \frac{(q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty (-q)_\infty}{(q)_\infty}, \tag{2.13}$$

which implies that $C_{k,i}(n) = D_{k,i}(n)$. This completes the proof of Theorem 1.4.

3 The Gordon marking of an overpartition

In this section, we introduce the notion of the Gordon marking of an overpartition and give an outline of the proof of the generating function formula for $D_{k,i}(m,n)$ as stated in Theorem 1.7. To compute the generating function of $D_{k,i}(m,n)$, we divide the set enumerated by $D_{k,i}(m,n)$ into two classes $U_{k,i}(m,n)$ and $I_{k,i}(m,n)$. Let $F_{k,i}(m,n)$ be the number of overpartitions in $U_{k,i}(m,n)$. By two simple bijections we can express the generating function of $D_{k,i}(m,n)$ by the generating function of $F_{k,i}(m,n)$. We shall give the generating function of $F_{k,i}(m,n)$ in Theorem 3.3. As will be seen, we need three bijections to prove Theorem 3.3, which will be presented in Sections 4–6.

Notice that identity (1.4) of Andrews [2] is a generalization of the Rogers-Ramanujan identity. It is natural to ask whether there is an overpartition analogue of (1.4). The answer is given in Theorem 1.8. To this end, we shall give a combinatorial treatment of the generating function of $D_{k,i}(m,n)$ by introducing the notion of Gordon marking representations of overpartitions. Observe that the generating function of $D_{k,i}(m,n)$ stated in Theorem 1.7 is in the form of the left hand side of (1.4). Thus Theorem 1.8 can be deduced from Theorem 1.7 and Theorem 1.4.

Kurşungöz [12] introduced the notion of the Gordon marking of an ordinary partition and gave a combinatorial interpretation of identity (1.5). A Gordon marking of an ordinary partition λ is an assignment of positive integers (marks) to parts of λ such that any two equal parts, as well as any two nearly equal parts j and $j+1$ are assigned different marks, and the marks are as small as possible assuming that the marks are assigned to the parts in increasing order. For example, the Gordon marking of

$$\lambda = (1, 1, 2, 3, 4, 4, 5, 5, 6, 6, 8, 9)$$

can be expressed as follows

$$\lambda = \left[\begin{array}{ccccccccc} & & & & 5 & & & & \\ & 2 & & 4 & & 6 & & & \\ 1 & & & 4 & & 6 & & & 9 \\ 1 & & 3 & & 5 & & & 8 & \end{array}\right]\begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array}, \tag{3.1}$$

where the marks are listed outside the brackets, that is, the parts at the bottom are marked with 1, and the parts immediately next to the bottom line are marked by 2, and so on. The Gordon marking of a partition can be considered as a way to represent a partition. For this reason, the diagram (3.1) is called the Gordon marking representation of a partition.

We shall introduce the Gordon marking of an overpartition. In fact, the three bijections in the proof of Theorem 1.7 are constructed based on Gordon markings of overpartitions. The Gordon marking of an overpartition can be defined as follows. It is clear that this notion is an extension of the Gordon marking of an ordinary partition.

Definition 3.1 *The Gordon marking of an overpartition λ is an assignment of positive integers (marks) to parts of λ. We assign the marks to parts in the following order*

$$\overline{1} < 1 < \overline{2} < 2 < \cdots \tag{3.2}$$

such that the marks are as small as possible subject to the following conditions. If $\overline{j+1}$ is not a part of λ, then all the parts j, $\overline{j}$, and $j+1$ are assigned different integers. If λ contains an overlined part $\overline{j+1}$, then the smallest mark assigned to a part j or $\overline{j}$ can be used as the mark of $j+1$ or $\overline{j+1}$.

For example, given an overpartition

$$\lambda = (16, 13, 12, 12, 11, \overline{10}, \overline{8}, 8, 8, 7, \overline{6}, 6, 5, 5, 4, 2, 2, \overline{1}).$$

The Gordon marking of λ is

$$(\overline{1}_1, 2_2, 2_3, 4_1, 5_2, 5_3, \overline{6}_1, 6_2, 7_3, \overline{8}_1, 8_2, 8_3, \overline{10}_1, 11_2, 12_1, 12_3, 13_2, 16_1),$$

where the subscripts are the marks. The Gordon marking of λ can also be illustrated as

$$\lambda = \left[\begin{array}{cccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & & \\ & 2 & & 5 & 6 & & 8 & & 11 & & 13 & \\ \overline{1} & & 4 & & \overline{6} & & \overline{8} & \overline{10} & & 12 & & 16 \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}, \tag{3.3}$$

where the parts in the third row are marked by 1, the parts in the second row are marked by 2, and the parts in the first row are marked by 3.

It is not hard to see that the Gordon marking of any overpartition is unique. To compute the generating function of $D_{k,i}(m,n)$, let $T_{k,i}(m,n)$ denote the set of overpartitions enumerated by $D_{k,i}(m,n)$. We further classify $T_{k,i}(m,n)$ by considering whether the smallest part of an overpartition is overlined element. Keep in mind that the parts of an overpartition are ordered by (3.2). Let $U_{k,i}(m,n)$ denote the set of overpartitions in $T_{k,i}(m,n)$ for which the smallest part is overlined, and let $I_{k,i}(m,n)$ denote the set of overpartitions in $T_{k,i}(m,n)$ with non-overlined smallest part. Thus we have

$$T_{k,i}(m,n) = U_{k,i}(m,n) \cup I_{k,i}(m,n). \tag{3.4}$$

Let $F_{k,i}(m,n) = |U_{k,i}(m,n)|$ and $G_{k,i}(m,n) = |I_{k,i}(m,n)|$. Then we have

$$D_{k,i}(m,n) = F_{k,i}(m,n) + G_{k,i}(m,n). \tag{3.5}$$

Below is a relation between $F_{k,i}(m,n)$ and $G_{k,i}(m,n)$.

Lemma 3.2 *For $2 \leq i \leq k$, we have*

$$F_{k,i-1}(m,n) = G_{k,i}(m,n). \tag{3.6}$$

For $i = 1$, we have

$$G_{k,1}(m,n) = F_{k,k}(m,n-m). \tag{3.7}$$

Proof. For $i \geq 2$, there is a simple bijection between $U_{k,i-1}(m,n)$ and $I_{k,i}(m,n)$. For an overpartition $\lambda \in U_{k,i-1}(m,n)$, we change the smallest part $\overline{j}$ of λ to a non-overlined part j. Then we get an overpartition in $I_{k,i}(m,n)$. Conversely, we can change one of the smallest part j of an overpartition $\beta \in I_{k,i}(m,n)$ to an overlined part $\overline{j}$ to get an overpartition in $U_{k,i-1}(m,n)$. Clearly, this map is a bijection. Hence (3.6) holds for $i \geq 2$.

For $i = 1$, we shall show a bijection between $I_{k,1}(m,n)$ and $U_{k,k}(m,n-m)$. Substracting one from each part of overpartition λ in $I_{k,1}(m,n)$ and changing one of the smallest parts to an overlined part, we obtain an overpartition in $U_{k,k}(m,n-m)$. Conversely, for an overpartition μ in $U_{k,k}(m,n-m)$, we can switch the smallest part to a non-overlined part, and increase each part of μ by one (regardless of the overlines), so that we can get an overpartition in $I_{k,1}(m,n)$. So we arrive at (3.7). This completes the proof. ∎

By the above lemma, the generating function of $G_{k,i}(m,n)$ can be obtained from the generating function of $F_{k,i}(m,n)$. Moreover, from (3.5) it follows that the generating function of $D_{k,i}(m,n)$ can be deduced from $F_{k,i}(m,n)$. The following theorem gives the generating function of $F_{k,i}(m,n)$.

Theorem 3.3 *For $k \geq i \geq 1$,*

$$\sum_{n=0}^{\infty} F_{k,i}(m,n)x^m q^n = \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \tag{3.8}$$

To derive the generating function of $F_{k,i}(m,n)$, we shall further classify the set $U_{k,i}(m,n)$. Let $\lambda^{(r)}$ denote the partition that consists of all r-marked parts of λ. Let N_r be the number of r-marked parts (i.e. the number of parts in $\lambda^{(r)}$), and let $n_r = N_r - N_{r-1}$ for any positive integer r. Notice that for any overpartition λ enumerated by $D_{k,i}(m,n)$, the parts j, $\overline{j}$ and $j+1$ occur at most $k-1$ times in λ. It follows that the marks of λ do not exceed $k-1$. So we are led to consider the parameters $N_1,\ldots,N_{k-1}$ and $n_1,\ldots,n_{k-1}$ as the summation indices when we compute the generating function of $F_{k,i}(m,n)$. It also can be seen that $N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0$ and $n_1,n_2,\ldots,n_{k-1}\geq 0$. The detailed proof of Theorem 3.3 will be given in the next four sections.

4 The first bijection for the proof of Theorem 1.7

In this section, we classify the set $U_{k,i}(m,n)$ according to the parameters $N_1,\ldots,N_{k-1}$, and we give the first bijection for the proof of Theorem 1.7. Let $\sum_{i=1}^{k-1} N_i = m$, and let $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$ denote the set of overpartitions in $U_{k,i}(m,n)$ that have N_r r-marked parts for $1\leq r\leq k-1$. Let $P_{N_1,N_2,\ldots,N_{k-1};i}(n)$ denote the set of overpartitions in $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$ for which all the 1-marked parts are overlined. Set

$$U_{N_1,N_2,\ldots,N_{k-1};i} = \bigcup_{n\geq 0} U_{N_1,N_2,\ldots,N_{k-1};i}(n), \tag{4.1}$$

$$P_{N_1,N_2,\ldots,N_{k-1};i} = \bigcup_{n\geq 0} P_{N_1,N_2,\ldots,N_{k-1};i}(n). \tag{4.2}$$

More precisely, we shall give a bijection for the following relation.

Theorem 4.1 *For $k \geq i \geq 1$, we have*

$$\sum_{\lambda \in U_{N_1,N_2,\cdots,N_{k-1};i}} x^{l(\lambda)} q^{|\lambda|} = (-q)_{N_1-1} \sum_{\alpha \in P_{N_1,N_2,\cdots,N_{k-1};i}} x^{l(\alpha)} q^{|\alpha|}, \tag{4.3}$$

where $l(\lambda)$ denotes the number of parts of λ.

Before we present the bijection for the above relation, we introduce a reduction operation based on the Gordon markings, which transforms an overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$ containing at least one non-overlined part with mark 1 to an overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$. This reduction operation preserves the number of r-marked parts for $r = 1, 2, \ldots, k-1$. Since we shall give another reduction operation in the next section, we call the reduction operation described below the first reduction operation.

The First Reduction Operation. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an overpartition of n containing at least one non-overlined part with mark 1. Assume that λ_j is the rightmost non-overlined part with mark 1. To be more precisely, for a part λ_j, we write $\lambda_j = \overline{a_j}$ to indicate that λ_j is an overline part and write $\lambda_j = a_j$ to indicate that λ_j is a non-overline part. Moreover, we say that a_j is the underlying part of λ_j. We consider two cases.

Case 1. There is a non-overlined part $a_j + 1$ of λ but there is no overlined 1-marked part $\overline{a_j + 1}$. First, we change the part λ_j to a 1-mark part $\overline{a_j}$. Then we choose the part $a_j + 1$ with the smallest mark, say r, and replace this r-marked part $a_j + 1$ with a r-marked part a_j. Since in λ r is the smallest mark of the parts $a_j + 1$ and the 1-marked a_j is non-overlined, by the definition of the Gordon marking of an overpartition, we deduce that either r is still the smallest mark of the parts with underlying part $a_j - 1$ or there are no parts with underlying part $a_j - 1$. In either case, we may place the new r-marked part a_j in a position with mark r.

If there is a 1-marked overlined part to the right of the $\overline{a_j}$, we switch it to a non-overlined part and we can see that the rightmost 1-marked nonoverlined part of the resulting overpartition is right to λ_j. If there are no 1-marked parts larger than a_j, we shall do nothing and in this case we can notice that the number of 1-marked overlined parts in the resulting overpartition is one more than it in λ. In either case, we denote the resulting overpartition by μ. Clearly, μ is an overpartition of $n - 1$. Moreover, it can be seen that μ contains the same number of r-marked parts as λ, for $1 \leq r \leq k-1$.

Case 2. Either an overlined part $\overline{a_j + 1}$ is a 1-marked part of λ or there are no parts with underlying part $a_j + 1$. In either case, we may change the part λ_j to a 1-marked overlined part $\overline{a_j - 1}$.

If there are 1-marked parts larger than a_j, then they are all overlined parts because of the choice of λ_j. In this case we switch the overlined 1-marked part next to λ_j

to a non-overlined part. Let μ denote the resulting overpartition. It is easily seen that in this case the rightmost non-overlined part in μ is right to the part λ_j and μ has the same number of 1-marked overlined parts and the same number of 1-marked nonoverlined parts as λ.

It remains to consider the case when there are no 1-marked parts larger than a_j. In this case, no operation is needed and we set μ to be the overpartition obtained in the previous step. It is clear that μ has one more 1-marked overlined parts and one less 1-marked non-overlined parts than λ.

In either case, one can deduce that μ is an overpartition of $n-1$ with the same number of r-marked parts as λ, for $1 \le r \le k-1$.

For example, let λ be an overpartition in $U_{7,6,5;1}(135)$ as given below

$$\left[\begin{array}{cccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & & \\ & 2 & & 5 & 6 & & 8 & & 11 & & \mathbf{13} & \\ \overline{1} & & 4 & & \overline{6} & & \overline{8} & \overline{10} & & \mathbf{12} & & \overline{\mathbf{15}} \end{array}\right]\begin{array}{l} 3 \\ 2 \\ 1 \end{array}.$$

The part $\mathbf{12}$ with mark 1 is the λ_j as in the description of the reduction operation, since it is the rightmost non-overlined part with mark 1. Notice that 13 is not a 1-marked part of λ, but $\mathbf{13}$ is a 2-marked part. By the operation in Case 1, we change the 1-marked part $\mathbf{12}$ to a part $\overline{\mathbf{12}}$, then we change the 2-marked part $\mathbf{13}$ to $\mathbf{12}$ and place it in a position with mark 2. Then we switch $\overline{\mathbf{15}}$ to $\mathbf{15}$. After the reduction operation by choose λ_j to be 1-marked 12, we get an overpartition μ in $U_{7,6,5;1}(134)$

$$\left[\begin{array}{ccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & \\ & 2 & & 5 & 6 & & 8 & & 11 & \mathbf{12} & \\ \overline{1} & & 4 & & \overline{6} & & \overline{8} & \overline{10} & & \overline{\mathbf{12}} & \mathbf{15} \end{array}\right]\begin{array}{l} 3 \\ 2 \\ 1 \end{array}.$$

Let us apply the reduction operation to above overpartition μ. The part $\mathbf{15}$ is the rightmost non-overlined part with mark 1 in μ and there are no parts greater than 15. So we need to apply the operation in Case 2. By changing $\mathbf{15}$ to $\overline{\mathbf{14}}$, we obtain an overpartition in $U_{7,6,5;1}(133)$

$$\left[\begin{array}{ccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & \\ & 2 & & 5 & 6 & & 8 & & 11 & 12 & \\ \overline{1} & & 4 & & \overline{6} & & \overline{8} & \overline{10} & & \overline{12} & \overline{\mathbf{14}} \end{array}\right]\begin{array}{l} 3 \\ 2 \\ 1 \end{array}.$$

Indeed, the above reduction operation is reversible. This implies that there is a bijection for the relation in Theorem 4.1. We shall give the dilation operation as the inverse of the reduction operation, and we shall call it the first dilation operation. In fact, there are two types of dilation operations depending on the choice of the position where the operation will take place.

The First Dilation Operation. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$. For a part λ_j, we use a_j to denote the underlying part of λ_j.

We proceed to determine the part λ_j which tells where the dilation operation will take place. There are two types of the dilation operation. If there are no 1-marked parts next to the rightmost overlined part λ_j, then we may choose λ_j and we shall say that the operation is of type A. If there is at least one overlined part such that the next 1-marked part is non-overlined, then we choose the rightmost one to be λ_j. For this choice, we say that the dilation operation is of type B. It should be mentioned that it is possible that we can apply two types of operations to an overpartition. For each overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$, we can apply at least one of the two types of the dilation operation. As will be seen, in the proof of Theorem 4.1 we need to consider how to apply the two types of the dilation operation.

Case 1: There are two parts of the same mark with underlying parts a_j and $a_j - 1$, we denote this same mark by r. It should be noticed that there are no 1-marked parts with underlying part $a_j + 1$ because of the choice of λ_j. We change λ_j to a non-overlined part a_j and replace the r-marked part a_j by an r-marked part $a_j + 1$.

If there are 1-marked parts with underlying parts greater than a_j, we consider the leftmost one, which must be non-overlined, and we change this non-overlined part to an 1-marked overlined part. Denote the resulting overpartition by μ. Clearly, the rightmost 1-marked overlined part to the left of a non-overlined part in μ must be to the left of λ_j in λ. Moreover, μ has the same number of 1-marked overlined parts and the same number of 1-marked non-overlined parts as λ.

We now turn to the case when there are no 1-marked parts with underlying parts greater than a_j. In this case no operation is required and we denote the overpartition obtained so far by μ. Notice that μ has one less 1-marked overlined parts and one more 1-marked non-overlined parts than λ.

In either case, one can deduce that μ is an overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}(n+1)$ with the same number of r-marked parts as λ, for $1 \leq r \leq k-1$.

Case 2: There are no two parts with underlying parts a_j and $a_j - 1$ that have the same mark. We see that there is no 1-marked part with underlying part $a_j + 1$ because of the choice of λ_j. We change λ_j to a non-overlined part a_j with mark 1. We denote by r the largest mark of the parts equal to a_j, and replace the r-marked non-overlined part a_j with an r-marked non-overlined part $a_j + 1$. Since r is the largest mark of the parts equal to a_j and $a_j + 1$ is not a 1-marked part of λ, we see that $a_j + 1$ cannot be a part with a mark not exceeding r. So we may place the new part equal to $a_j + 1$ in a position of mark r.

If there is a 1-marked non-overlined part next to λ_j, we switch this non-overlined part to an overlined part. Let μ denote the resulting overpartition. It is easily seen that in this case μ has the same number of 1-marked overlined parts and the same number of 1-marked non-overlined parts as λ.

We still need to consider the case when there are no parts next to λ_j, In this case, we just denote the resulting overpartition by μ. Clearly, μ has one more 1-marked non-overlined parts and one less 1-marked overlined parts than λ.

In either case, we see that μ is an overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}(n+1)$ with the same number of r-marked parts as λ, for $1 \leq r \leq k-1$.

It is easily checked that the first reduction operation is the inverse of the first dilation operation. More precisely, we have the following property.

Theorem 4.2 *The dilation operation of Type A is the inverse of the reduction operation which increases the number of overlined parts in λ, whereas the dilation operation of Type B is the inverse of the reduction operation which preserves the number of overlined parts in λ.*

We are now ready to present the proof of Theorem 4.1.

Proof of Theorem 4.1. Based on the reduction operation, we shall establish a bijection φ between $U_{N_1,N_2,\ldots,N_{k-1};i}$ and $P_{N_1,N_2,\ldots,N_{k-1};i} \times D_{N_1}$, where D_{N_1} denotes the set of ordinary partitions with distinct parts such that each part is less than N_1. Let λ be an overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}$. We shall give a procedure to construct $\varphi(\lambda)$, which is a pair (α,β), where α is an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}$ and β is a partition in D_{N_1}.

Step 1. Set $\alpha=\lambda$, $\beta=\phi$ and $t=1$. If there are no non-overlined 1-marked parts in α, go to Step 3; Otherwise, go to Step 2.

Step 2. If the largest 1-marked part of α is overlined, then apply the first reduction operation on α. If there are still non-overlined 1-marked parts in α, then set t to $t+1$ and repeat this step; Otherwise, go to Step 3.

If the largest 1-marked part of α is non-overlined, then add t to β as a new part and apply the first reduction operation on α. Reset t to 1 and repeat this step if there are still non-overlined 1-marked parts in α; Otherwise, go to Step 3.

Step 3. Set $\varphi(\lambda)=(\alpha,\beta)$.

Evidently, α is an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}$ and $|\lambda|=|\alpha|+|\beta|$. It remains to prove that the parts of β are less than N_1. Let

$$\lambda_1^{(1)} < \lambda_2^{(1)} < \cdots < \lambda_{N_1}^{(1)}$$

denote the 1-marked parts of λ. Moreover, suppose that there are s non-overlined 1-marked parts of λ, which are denoted by

$$\lambda_{i_1}^{(1)} < \lambda_{i_2}^{(1)} < \cdots < \lambda_{i_s}^{(1)}.$$

Examining Step 2 of the above procedure, we see that after applying the operation in Step 2 to the rightmost non-overlined part such that it is the largest 1-marked part of α, the number of non-overlined part decreases by one. So we find that for each non-overlined 1-marked part $\lambda_{i_t}^{(1)}$, we can iterate Step 2 $N_1 - i_t + 1$ times in order to decrease the number of non-overlined parts by one and add $N_1 - i_t + 1$ to β as a new part. Hence we deduce that $\beta = (N_1 - i_1 + 1, N_1 - i_2 + 1, \ldots, N_1 - i_s + 1)$. Recall that the smallest 1-marked part of an overpartition in $U_{N_1,N_2,\ldots,N_{k-1};i}$ is always overlined. It follows that $N_1 - i_t + 1 < N_1$, for $1 \leq t \leq s$. So β is a partition in D_{N_1}.

Next we give a brief description of the inverse of φ. The detailed proof is omitted because it is a straightforward verification.

Let α be an overpartition in $P_{N_1,N_2,\ldots N_k;i}$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_s)$ be a partition with distinct parts and $\beta_1 \leq N_1 - 1$. We shall give a procedure to construct $\varphi^{-1}(\alpha, \beta)$, which is an overpatition λ in $U_{N_1,N_2,\ldots N_k;i}$.

Step 1. Set $\lambda = \alpha$. Let s be the number of parts in β.

Step 2. For t from 1 to s, apply the dilation operation of type A to λ. Then the dilation operation of type B will be applied $\beta_t - 1$ times to λ. Now we get an overpartion λ in $U_{N_1,N_2,\ldots N_k;i}$. It can be checked that $\lambda_{N_1-\beta_1}^{(1)}, \ldots, \lambda_{N_1-\beta_s}^{(1)}$ are the non-overlined 1-marked parts of λ.

To prove that $\varphi^{-1}(\varphi(\lambda)) = \lambda$, we need the fact that the first reduction operation and the first dilation operation are inverses of each other. This completes the proof. ∎

To demonstrate the above bijection we give an example. Let λ be the overpartition as given in (3.3), that is,

$$\lambda = \left[\begin{array}{cccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & & \\ & 2 & & 5 & 6 & & 8 & & 11 & & 13 & \\ \overline{1} & & 4 & & \overline{6} & & \overline{8} & \overline{10} & & 12 & & 16 \end{array}\right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

First we set $\alpha = \lambda$, $\beta = \phi$ and $t = 1$. Notice that the greatest 1-marked part of α is non-overlined. So we let $\beta = (1)$ and set $t = 1$. Applying the first reduction operation, we have

$$\alpha = \left[\begin{array}{ccccccccccccccc} & 2 & & & 5 & & 7 & 8 & & & & 12 & & & \\ & 2 & & & 5 & 6 & & 8 & & & 11 & & 13 & & \\ \overline{1} & & & 4 & & \overline{6} & & \overline{8} & & \overline{10} & & 12 & & & \overline{15} \end{array}\right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}.$$

Since the greatest non-overlined 1-marked part is 12 which is not the greatest 1-marked part, we apply the first reduction operation on α and let $t = 2$. Then we get

$$\alpha = \left[\begin{array}{ccccccccccccccc} & 2 & & & 5 & & 7 & 8 & & & & 12 & & & \\ & 2 & & & 5 & 6 & & 8 & & & 11 & 12 & & & \\ \overline{1} & & & 4 & & \overline{6} & & \overline{8} & & \overline{10} & & \overline{12} & & & 15 \end{array}\right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}.$$

Now the rightmost non-overlined 1-marked part is 15 and it is the greatest 1-marked part. So we apply the reduction operation and let $\beta = (2, 1)$. Now we should reset $t = 1$. Then we get

$$\alpha = \left[\begin{array}{cccccccccccccc} & 2 & & & 5 & & 7 & 8 & & & & 12 & & \\ & 2 & & & 5 & 6 & & 8 & & & 11 & 12 & & \\ \overline{1} & & & 4 & & \overline{6} & & \overline{8} & & \overline{10} & & \overline{12} & & \overline{14} \end{array}\right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}.$$

In order to get an overpartition with no non-overlined 1-marked parts, we still need to apply the reduction operation 6 times. The details are omitted. Finally, we obtain

$$\alpha = \left[\begin{array}{ccccccccccccc} & 2 & & & 5 & 6 & & 8 & & & & 12 & \\ & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & \\ \overline{1} & & & \overline{4} & & \overline{6} & \overline{7} & & & \overline{10} & \overline{11} & & \overline{13} \end{array}\right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}, \tag{4.4}$$

and $\beta = (6, 3, 1)$. Thus we have constructed a pair (α, β), where α is an overpartition such that all 1-marked parts overlined, β is partition in D_7. Moreover, we have $|\lambda| = |\alpha| + |\beta|$.

5 The second bijection for the proof of Theorem 1.7

In this section, we introduce a class of overpartitions in $P_{N_1,N_2,\ldots,N_{k-1};i}$, which will be denoted by $Q_{N_1,N_2,\ldots,N_{k-1};i}$. We aim to relate the generating function of $P_{N_1,N_2,\ldots,N_{k-1};i}$ to that of $Q_{N_1,N_2,\ldots,N_{k-1};i}$. To define the set $Q_{N_1,N_2,\ldots,N_{k-1};i}$, we observe that for any $\lambda \in P_{N_1,N_2,\ldots,N_{k-1};i}(n)$ and for any $1 \le t \le n$, we have

$$f_t(\lambda) + f_{\bar{t}}(\lambda) + f_{t+1}(\lambda) \le k - 1, \tag{5.1}$$

where $f_t(\lambda)$ denotes the number of occurrences of t in λ. We define the set $Q_{N_1,N_2,\ldots,N_{k-1};i}$ as the set of overparitions λ in $P_{N_1,N_2,\ldots,N_{k-1};i}$ for which the equality holds in (5.1), namely,

$$f_t(\lambda) + f_{\bar{t}}(\lambda) + f_{t+1}(\lambda) = k - 1 \tag{5.2}$$

for any positive integer t which is smaller than the greatest $(k-1)$-marked part. It should be mentioned that Bressoud [5, 6] obtained a generalization of the Rogers-Ramanujan identities by considering ordinary partitions λ that satisfy the equality in (5.2), namely,

$$f_t(\lambda) + f_{t+1}(\lambda) = k - 1. \tag{5.3}$$

Set

$$Q_{N_1,N_2,\ldots,N_{k-1};i} = \bigcup_{n\geq 0} Q_{N_1,N_2,\ldots,N_{k-1};i}(n).$$

The following theorem establishes a relation between the generating function of $P_{N_1,N_2,\ldots,N_{k-1};i}$ and the generating function of $Q_{N_1,N_2,\ldots,N_{k-1};i}$.

Theorem 5.1 *For $N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0$, we have*

$$\sum_{\alpha\in P_{N_1,N_2,\ldots,N_{k-1};i}} x^{l(\alpha)}q^{|\alpha|} = \frac{1}{(q)_{N_{k-1}}} \sum_{\gamma\in Q_{N_1,N_2,\ldots,N_{k-1};i}} x^{l(\gamma)}q^{|\gamma|}. \tag{5.4}$$

To prove the above theorem, we shall give a bijection based on a reduction operation and a dilation operation which are called the second reduction and the second dilation. The second reduction transforms an overpartition α in $P_{N_1,N_2,\ldots,N_{k-1};i}(n) \setminus Q_{N_1,N_2,\ldots,N_{k-1};i}(n)$ to an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$. More precisely, this operation requires the choice of a $(k-1)$-marked part α_j whose underlying part is t satisfying one of the following two conditions

1. There are no parts with underlying part $t-1$;
2. There is a part with underlying part $t-1$ and

$$f_{t-2}(\alpha) + f_{\overline{t-2}}(\alpha) + f_{t-1}(\alpha) < k - 1. \tag{5.5}$$

By the definitions of $P_{N_1,N_2,\ldots,N_{k-1};i}(n)$ and $Q_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$, it is not difficult to see that for any α in $P_{N_1,N_2,\ldots,N_{k-1};i}(n)\setminus Q_{N_1,N_2,\ldots,N_{k-1};i}(n)$, there exists a $(k-1)$-marked part α_j satisfying one of the above conditions.

The Second Reduction Operation. Let $\alpha=(\alpha_1,\ldots,\alpha_m)$ be an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}(n)\setminus Q_{N_1,N_2,\ldots,N_{k-1};i}(n)$. Let α_j be a $(k-1)$-marked part with underlying part t satisfying one of the above conditions.

If α_j satisfies Condition 1, that is, there are no parts with underlying part $t-1$, then there is an overlined part $\overline{t}$ since t is the underlying part of α_j. We replace $\overline{t}$ with a 1-marked overlined part $\overline{t-1}$.

If α_j satisfies Condition 2, write (5.5) as

$$\sum_{l=1}^{k-1}(f_{t-2}(\alpha^{(l)})+f_{\overline{t-2}}(\alpha^{(l)})+f_{t-1}(\alpha^{(l)}))<k-1, \tag{5.6}$$

where $\alpha^{(l)}$ is the overpartition consisting of the l-marked parts of α. So we can find the smallest mark $r\geq 2$ such that t is a part of mark r and

$$\sum_{l=1}^{r}(f_{t-2}(\alpha^{(l)})+f_{\overline{t-2}}(\alpha^{(l)})+f_{t-1}(\alpha^{(l)}))<r. \tag{5.7}$$

Replace the r-marked part t with an r-marked part $t-1$.

It can be seen that in either case we obtain the Gordon marking representation of an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$.

For example, let α be an overpartition in $P_{7,6,5;1}(126)$ as given below

$$\alpha=\begin{array}{l}\left[\begin{array}{cccccccccc} & 2 & 4 & 6 & & 8 & & & 12 & \\ & 2 & 4 & 6 & & 8 & 10 & & 12 & \\ \overline{1} & & \overline{\mathbf{4}} & \overline{6} & \overline{7} & & \overline{10} & \overline{11} & & \overline{13}\end{array}\right]\end{array}\begin{array}{l}3\\2\,.\\1\end{array}$$

Choosing α_j to be the 3-marked part 4, we see that the 1-marked part $\overline{4}$ satisfies Condition 1. Then we can replace $\overline{4}$ with a 1-marked $\overline{3}$ to transform α to an overpartition in $P_{7,6,5;1}(125)$:

$$\left[\begin{array}{ccccccccccc} & 2 & & 4 & 6 & & 8 & & & 12 & \\ & 2 & & \mathbf{4} & 6 & & 8 & 10 & & 12 & \\ \overline{1} & & \overline{3} & & \overline{6} & \overline{7} & & \overline{10} & \overline{11} & & \overline{13}\end{array}\right]\begin{array}{l}3\\2\,.\\1\end{array}$$

For the above overpartition, choosing the same α_j as before, we see that the 3-marked part 4 satisfies Condition 2. We further apply the reduction in this case. Clearly, 2 is the smallest mark satisfying Condition (5.7). So we can replace the 2-marked part 4 with a 2-marked part 3 to form an overpartition in $P_{7,6,5;1}(124)$:

$$\left[\begin{array}{ccccccccccc} & 2 & & 4 & 6 & & 8 & & & 12 & \\ & 2 & 3 & & 6 & & 8 & 10 & & 12 & \\ \overline{1} & & \overline{3} & & \overline{6} & \overline{7} & & \overline{10} & \overline{11} & & \overline{13}\end{array}\right]\begin{array}{l}3\\2\,.\\1\end{array}$$

The second dilation transforms an overpartition α in $P_{N_1,N_2,\ldots,N_{k-1};i}(n)$ to an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}(n-1) \setminus Q_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$. To be more specific, the operation starts with a choice of a $(k-1)$-marked part α_j subject to one of the following conditions:

1. The underlying part t of α_j satisfies

$$f_t(\alpha) + f_{\bar{t}}(\alpha) + f_{t+1}(\alpha) < k-1; \tag{5.8}$$

2. The underlying part t of α_j satisfies that

$$f_t(\alpha) + f_{\bar{t}}(\alpha) + f_{t+1}(\alpha) = k-1. \tag{5.9}$$

   Moreover, we have

$$f_{t+1}(\alpha) + f_{\overline{t+1}}(\alpha) + f_{t+2}(\alpha) < k-1. \tag{5.10}$$

It is easily seen that relation (5.10) holds for the largest $(k-1)$-marked part α_j of α with underlying part t. This implies there exists at least one $(k-1)$-marked part α_j satisfying one of the above two conditions. Our goal is to find a part of α with underlying part $t-1$ or t and we shall increase this underlying part by one.

The Second Dilation Operation. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}(n)$. Let α_j be a $(k-1)$-marked part with underlying part t for which one of the above two conditions holds.

We first consider the case when Condition 1 holds. Since t is the underlying part of α_j and $f_t(\alpha) < k-1$, we deduce that there exists a part with underlying part $t-1$. So we may assume that r is the largest mark of a part with underlying part $t-1$. If $r = 1$, we replace the 1-marked overlined part $\overline{t-1}$ with an 1-marked overlined part $\bar{t}$. If $r > 1$, we replace this r-marked non-overline part $t-1$ with an r-marked part t.

We now consider the case when Condition 2 holds. In this case, we observe that there is no $k-1$-marked part with underlying part $t+1$. Moreover, if (5.10) holds for $k = 2$, then we replace α_j with a 1-marked part $\overline{t+1}$. If (5.10) holds for $k > 2$, then we replace α_j with a $(k-1)$-marked part $t+1$.

In either case, we obtain the Gordon marking representation of an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}(n) \setminus Q_{N_1,N_2,\ldots,N_{k-1};i}(n)$.

It can be checked that the second reduction operation is the inverse of the second dilation operation. We are now ready to give a bijective proof of Theorem 5.1.

Proof of Theorem 5.1. Using the reduction operation, we shall establish a bijection ψ between $P_{N_1,N_2,\ldots,N_{k-1};i}$ and $Q_{N_1,N_2,\ldots,N_{k-1};i} \times R_{N_{k-1}}$, where $R_{N_{k-1}}$ denotes the set of ordinary partitions with at most N_{k-1} parts. Let α be an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}$. Assume that

$$\alpha_1^{(k-1)} < \alpha_2^{(k-1)} < \ldots < \alpha_{N_{k-1}}^{(k-1)}$$

are the $(k-1)$-marked parts of α.

Let us describe the procedure to construct $\psi(\alpha)$ by successively applying the second reduction operation. Keep in mind that $\psi(\alpha)$ is a pair (γ, δ), where γ is an overpartition in $Q_{N_1,N_2,\ldots,N_{k-1};i}$ and δ is a partition in $R_{N_{k-1}}$ such that $|\alpha| = |\gamma| + |\delta|$.

As discussed before, there always exists a $(k-1)$-marked part α_j which satisfies either Condition 1 or Condition 2 in the second reduction operation. We choose the smallest $(k-1)$-marked part which satisfies either Condition 1 or Condition 2. Assume that it is the l-th $(k-1)$-marked part of α, denoted $\alpha_l^{(k-1)}$. Notice that after applying reduction operation by choosing α_j to be $\alpha_l^{(k-1)}$, the $(l+1)$-th $(k-1)$-marked part $\alpha_{l+1}^{(k-1)}$ remains unchanged and it satisfies the Condition 1 or Condition 2. So can continue to apply the reduction operation by choosing α_j to be $\alpha_{l+1}^{(k-1)}$. Moreover, we can iterate this process with respect to the following $(k-1)$-marked parts $\alpha_l^{(k-1)}, \alpha_{l+1}^{(k-1)}, \ldots, \alpha_{N_{k-1}}^{(k-1)}$ to get an overpartition in $Q_{N_1,N_2,\ldots,N_{k-1};i}$. Meanwhile, during the above process we obtain an ordinary partition with at most N_{k-1} parts.

We now give a detailed description of the bijection ψ which consists of the following steps.

Step 1. Set $\delta = \phi$ and $t = 0$. We choose the smallest $(k-1)$-marked part $\alpha_l^{(k-1)}$ which satisfies either Condition 1 or Condition 2. If $l = 1$ and the number of parts with underlying part 1 is less than i, go to Step 2; Otherwise, set $v = l$ and go to Step 3.

Step 2. Recall that by the definition of $P_{N_1,N_2,\ldots,N_{k-1};i}$, i is the maximum number of occurrences of 1 and $\overline{1}$ in α. There are two cases. If $1 \leq i \leq k-1$, we repeatedly apply the reduction operation to α by choosing α_j to be $\alpha_1^{(k-1)}$ until α becomes an overpartition containing an overlined part $\overline{1}$ and $i-1$ non-overlined parts 1. If $i = k$, we repeatedly apply the reduction operation to α by choosing α_j to be $\alpha_1^{(k-1)}$ until α becomes an overpartition containing an overlined part $\overline{1}$ and $k-2$ non-overlined parts 1. In either case, let t be the number of the reduction operations that have been applied, and add t to δ as a new part. Set $v = 2$ and go to Step 3.

Step 3. For each s from v to N_{k-1}, we repeatedly apply the second reduction operations by choosing the $(k-1)$-marked part α_j to be $\alpha_s^{(k-1)}$ until $\alpha_s^{(k-1)}$ satisfies neither Condition 1 nor Condition 2. After each reduction we reset the resulting overpartition back to α. Let t be the number of reductions that have been applied. Add t to δ as a new part.

Step 4. Let $\gamma = \alpha$ and set $\psi(\alpha) = (\gamma, \delta)$.

It can be seen that γ is an overpartition in $Q_{N_1,N_2,\ldots,N_{k-1};i}$. Meanwhile, there are $N_{k-1}-l+1$ parts in δ. This implies that δ is a partition in $R_{N_{k-1}}$. Moreover we have $|\alpha| = |\gamma| + |\delta|$. An example is given after the proof.

Here is an outline of the inverse of ψ. Let γ be an overpartition in $Q_{N_1,N_2,\ldots N_{k-1};i}$ and δ be a partition with m parts, where $m \leq N_{k-1}$. Express the parts of δ as

$$\delta_1 \geq \cdots \geq \delta_m.$$

The following is a procedure to construct $\psi^{-1}(\gamma,\delta)$, which is an overpartition α in $P_{N_1,N_2,\ldots N_{k-1};i}$.

Step 1. Let $\alpha = \gamma$.

Step 2. For t from 1 to m, apply the dilation operation δ_t times by choosing α_j to be $\gamma^{(k-1)}_{N_{k-1}-t+1}$.

Step 3. Set $\psi^{-1}(\gamma,\delta) = \alpha$.

It can be verified that the map $\psi^{-1}(\gamma,\delta)$ is indeed the inverse of ψ. The details are omitted. So we have completed the proof of Theorem 5.1. ∎

We conclude this section with an example to demonstrate the above bijection. For $k = 4$ and $i = 1$, let α be an overpartition in $P_{7,6,5;1}(128)$ as given by (4.4), namely,

$$\alpha = \begin{array}{c} \left[\begin{array}{cccccccccc} 2 & & 5 & 6 & & 8 & & & 12 & \\ 2 & 4 & & 6 & & 8 & 10 & & 12 & \\ \overline{2} & \overline{4} & & \overline{6} & \overline{7} & & \overline{10} & \overline{11} & & \overline{13} \end{array}\right] \end{array} \begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

We apply the second reduction operation by choosing α_j to be the 3-marked part $\alpha_1^{(3)} = 2$. Then α is mapped to an overpartition containing a part $\overline{1}$ and no parts 1. Note that $i = 1$. Thus we cannot further apply the reduction by choosing α_j to be $\alpha_1^{(3)}$. Then we get $\delta = (1)$ and α is an overpartition in $P_{7,6,5;1}(127)$:

$$\alpha = \left[\begin{array}{ccccccccccc} & 2 & & 5 & 6 & & 8 & & & 12 & \\ & 2 & 4 & & 6 & & 8 & 10 & & 12 & \\ \overline{1} & & \overline{4} & & \overline{6} & \overline{7} & & \overline{10} & \overline{11} & & \overline{13} \end{array}\right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

Next we choose α_j to be $\alpha_2^{(3)}$. Then we can apply reduction three times to change the 3-marked part 5 to the 3-marked part 4, change the 1-marked part $\overline{4}$ to the 1-marked part $\overline{3}$, and change the 2-marked part 4 to the 2-marked part 3. After that

$\alpha_2^{(3)}$ no longer satisfies Condition 1 or Condition 2. Then we add 3 to δ as a new part to get $\delta = (3, 1)$ and α becomes an overpartition in $P_{7,6,5;1}(124)$:

$$\left[\begin{array}{ccccccccccccc} & 2 & & 4 & & 6 & & 8 & & & & 12 & \\ & 2 & 3 & & & 6 & & 8 & & 10 & & 12 & \\ \overline{1} & & \overline{3} & & & \overline{6} & \overline{7} & & & \overline{10} & \overline{11} & & \overline{13} \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

We continue to consider $\alpha_3^{(3)} = 6$ as a choice of α_j. We can apply reduction three times so that α becomes an overpartition in $P_{7,6,5;1}(121)$ as given below:

$$\left[\begin{array}{ccccccccccccc} & 2 & & 4 & 5 & & & 8 & & & & 12 & \\ & 2 & 3 & & 5 & & & 8 & & 10 & & 12 & \\ \overline{1} & & \overline{3} & & \overline{5} & & \overline{7} & & & \overline{10} & \overline{11} & & \overline{13} \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

Then add 3 as a new part to δ and get $\delta = (3, 3, 1)$.

For the remaining 3-marked parts 8 we can apply the reduction three times by choosing $\alpha_j = 8$. Finally, for the 3-marked part 12, we can apply the reduction seven times by choosing $\alpha_j = 12$. Thus we get $\delta = (7, 3, 3, 3, 1)$. In the mean time, α is mapped to an overpartition in $Q_{7,6,5;1}(111)$ as given by

$$\gamma = \left[\begin{array}{ccccccccccccc} & 2 & & 4 & 5 & & 7 & & 9 & & & & \\ & 2 & 3 & & 5 & & 7 & 8 & & & & 12 & \\ \overline{1} & & \overline{3} & & \overline{5} & \overline{6} & & \overline{8} & & & \overline{11} & & \overline{13} \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

6 The third bijection for the proof of Theorem 1.7

In this section, we give the third bijection for the proof of Theorem 1.7, which is between $Q_{N_1,\ldots,N_{k-1};i}$ and $Q_{N_1-1,\ldots,N_{k-1}-1;i}$. By this correspondence, we can derive a recurrence relation on $Q_{N_1,\ldots,N_{k-1};i}$, which yields the generating function of $Q_{N_1,\ldots,N_{k-1};i}$ as stated in the following theorem.

Theorem 6.1 *For $k \geq 2$ and $1 \leq i \leq k$, we have*

$$\sum_{\gamma \in Q_{N_1,\ldots,N_{k-1};i}} x^{l(\gamma)} q^{|\gamma|} = \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}} x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}}. \tag{6.1}$$

In order to prove the above theorem by induction, we need the following bijection.

Theorem 6.2 *For $N_{k-1} > 0$, there is a bijection between $Q_{N_1,\ldots,N_{k-1};i}(n)$ and $Q_{N_1-1,\ldots,N_{k-1}-1;i}(n-N_1-2N_2-\cdots-2N_{k-1}+i-1)$. In terms of generating functions, we have*

$$\sum_{\gamma\in Q_{N_1,\ldots,N_{k-1};i}} q^{|\gamma|} = q^{N_1+2N_2+\ldots+2N_{k-1}-i+1} \sum_{\gamma\in Q_{N_1-1,\ldots,N_{k-1}-1;i}} q^{|\gamma|}. \tag{6.2}$$

Proof. Assume that $N_{k-1} > 0$. We will give a bijection χ between $Q_{N_1,\ldots,N_{k-1};i}(n)$ with and $Q_{N_1-1,\ldots,N_{k-1}-1;i}(n-N_1-2N_2-\cdots-2N_{k-1}+i-1)$. Let γ be an overpartition in $Q_{N_1,\ldots,N_{k-1};i}(n)$. We proceed to construct $\chi(\gamma)$, which is an overpartition μ in $Q_{N_1-1,\ldots,N_{k-1}-1;i}(n-N_1-2N_2-\cdots-2N_{k-1}+i-1)$.

The idea of this bijection goes as follows. For each 1-marked part γ_j with underlying part a_j, we shall allocate a part with underlying part a_j subject to certain conditions. Then we increase this part by 1. Furthermore, for each 1-marked part, we remove the smallest part of each row in the Gordon marking representation of the resulting overpartition, and subtract 2 from the other parts. Here are the detailed description.

Step 1. Let $\mu=\gamma$.

Step 2. For i from N_1 to 1, let t be the underlying part of $\mu_i^{(1)}$.

If there are two parts of the same mark but with distinct underlying parts $t-1$ and t, we denote this mark by r. Then we change the r-marked part with underlying part t to an r-marked part with underlying part $t+1$;

Otherwise, we find the greatest mark r, such that there is an r-marked part with underlying part t. If $r=1$, replace the 1-marked overlined part $\overline{t}$ of μ with an 1-marked part $\overline{t+1}$. If $r>1$, replace the r-marked part t with an r-marked part $t+1$. Clearly, the sum of the parts of μ becomes $n+N_1$.

Step 3. Delete $\mu_1^{(1)},\ldots,\mu_1^{(k-1)}$ and subtract 2 from each part of μ.

From the definition of $Q_{N-1,\ldots,N_{k-1};i}$, the smallest part of each row is 1 or 2. Clearly, after Step 2 there are $i-1$ parts equal to 1 and $k-i$ parts equal to 2 in μ. So after Step 3 the sum of parts of μ equals

$$n+N_1-(i-1)-2(k-i)-2(N_1+\cdots+N_{k-1}-(k-1)) = n-N_1-2N_2-\cdots-2N_{k-1}+i-1.$$

Step 4. Let $\chi(\lambda)=\mu$.

It can be seen that after the above process we obtain the Gordon marking of an overpartition in $Q_{N_1-1,\ldots,N_{k-1}-1;i}(n-N_1-2N_2-\cdots-2N_{k-1}+i-1)$.

We now consider the inverse of χ. Let $\mu\in Q_{N_1-1,\ldots,N_{k-1}-1;i}(n)$. The following is a procedure to construct $\chi^{-1}(\mu)$, which is a partition γ in $Q_{N_1-1,\ldots,N_{k-1}-1;i}(n+N_1+2N_2+\ldots+2N_{k-1}-i+1)$.

Step 1. Let $\gamma = \mu$.

Step 2. Increase each part of γ by 2.

Step 3. If $i = 1$, we add 1-marked part $\overline{2}$, a 2-marked part 2, $\ldots$, and a $(k-1)$-marked part 2 to γ as new parts. If $i \geq 2$, we add a 1-marked part $\overline{1}$, $\ldots$, an $(i-1)$-marked part 1, an i-marked part 2, $\ldots$, and a $(k-1)$-marked part 2 to γ as new parts. Now γ contains $N_1 + 1$ parts with 1-marked.

Step 4. For j from 1 to $N_1 + 1$, let t be the underlying part of $\gamma_j^{(1)}$.

If $\overline{t+1}$ is a part of γ or there are no parts with underlying part $t+1$, then we replace the overlined 1-marked part $\overline{t}$ with a 1-marked part $\overline{t-1}$;

If $\overline{t+1}$ is not a part of γ but $t+1$ is a part of γ, then we choose the smallest mark r of parts with underlying part $t+1$, and replace this r-marked part $t+1$ with an r-marked part t.

Step 5. Set $\chi^{-1}(\mu) = \gamma$.

It can be verified that after the above steps we get the Gordon marking of an overpartition in $Q_{N_1-1,\ldots,N_{k-1}-1;i}(n + N_1 + 2N_2 + \ldots + 2N_{k-1} - i + 1)$.

It is routine to check that the map χ^{-1} is the inverse of χ. ∎

Here we give an example of the above bijection. Let $\gamma =$ $(\overline{1}, 1, \overline{2}, 2, 2, \overline{3}, 3, 3, 4, 4, \overline{5}, 5, 5, 5, \overline{6}, 6, 7, 7, 7, \overline{8}, 8, 8, \overline{9}, 9, 9, 10, 10, \overline{11}, 11, 12, 12, \overline{14}, 14, 15, \overline{17}, 17, 17)$ in $Q_{10,9,8,6,6;1}(311)$. Set $\mu = \gamma$. Below is the Gordon marking representation of μ

$$\left[\begin{array}{cccccccccccccccc}
 & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & & & \\
 & 2 & & 4 & \mathbf{5} & & 7 & & 9 & & 11 & & & & & \\
 & 2 & \mathbf{3} & & 5 & & 7 & & 9 & & 11 & & & & 15 & \mathbf{17} \\
\mathbf{1} & & 3 & & 5 & & 7 & \mathbf{8} & & 10 & \mathbf{11} & & & 14 & & 17 \\
\overline{1} & & \overline{3} & & \overline{5} & \overline{\mathbf{6}} & & \overline{8} & \overline{\mathbf{9}} & & \overline{11} & & \overline{\mathbf{13}} & \overline{\mathbf{14}} & & \overline{17}
\end{array}\right]\begin{array}{c}5\\4\\3\\2\\1\end{array}, \qquad (6.3)$$

where the parts in boldface are those we should move to the right in Step 2. After Step 2, μ is changed to

$$\left[\begin{array}{ccccccccccccccccc}
 & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & & & & \\
 & 2 & & 4 & & \mathbf{6} & 7 & & 9 & & 11 & & & & & & \\
 & 2 & & \mathbf{4} & 5 & & 7 & & 9 & & 11 & & & & 15 & & \mathbf{18} \\
 & \mathbf{2} & 3 & & 5 & & 7 & & \mathbf{9} & 10 & & \mathbf{12} & & 14 & & 17 & \\
\overline{1} & & \overline{3} & & \overline{5} & & \overline{\mathbf{7}} & \overline{8} & & \overline{\mathbf{10}} & \overline{11} & & & \overline{\mathbf{14}} & \overline{\mathbf{15}} & \overline{17} &
\end{array}\right]\begin{array}{c}5\\4\\3\\2\\1\end{array}.$$

Deleting the parts $\mu_1^{(1)}, \ldots, \mu_1^{(5)}$ and subtracting 2 from the other parts of μ, we get

$$\left[\begin{array}{cccccccccccccccc}
 & 2 & & 4 & & 6 & & 8 & & 10 & & & & & & \\
 & 2 & & 4 & 5 & & 7 & & 9 & & & & & & & \\
 & 2 & 3 & & 5 & & 7 & & 9 & & & & 13 & & & 16 \\
1 & & 3 & & 5 & & 7 & 8 & & 10 & & 12 & & & 15 & \\
\overline{1} & & \overline{3} & & \overline{5} & 6 & & \overline{8} & \overline{9} & & & \overline{12} & 13 & & \overline{15} &
\end{array}\right]\begin{array}{c} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}, \tag{6.4}$$

which is the Gordon marking representation of an overpartition in $Q_{9,8,7,5,5;1}(254)$. It can be checked that the above process is reversible.

The proof of Theorem 6.1. We use induction on k. For $k = 2$ and $i = 1$, the generating function of $Q_{N_1;1}$ is

$$\sum_{\lambda \in Q_{N_1;1}} q^{|\lambda|} = q^{\frac{(N_1+1)N_1}{2}}.$$

For $k = 2$ and $i = 2$, the generating function of $Q_{N_1;2}$ is

$$\sum_{\lambda \in Q_{N_1;2}} q^{|\lambda|} = q^{\frac{(N_1+1)N_1}{2}}.$$

So Theorem 6.1 holds for $k = 2$. Assume that it holds for $k - 1$, that is,

$$\sum_{\lambda \in Q_{N_1,\ldots,N_{k-2};i}} q^{|\lambda|} = \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-2}^2+N_{i+1}+\cdots+N_{k-2}}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\cdots(q)_{N_{k-3}-N_{k-2}}}.$$

We proceed to show that it holds for $Q_{N_1,\ldots,N_{k-1};i}$.

If $N_{k-1} = 0$, by the definitions of $Q_{N_1,\ldots,N_{k-2},0;i}$ and $P_{N_1,\ldots,N_{k-2};i}$, we find that

$$Q_{N_1,\ldots,N_{k-2},0;i} = P_{N_1,\ldots,N_{k-2};i}.$$

In view of Theorem 5.1, the generating function of $Q_{N_1,\ldots,N_{k-2},0;i}$ equals

$$\sum_{\lambda \in Q_{N_1,\ldots,N_{k-2},0;i}} q^{|\lambda|} = \frac{1}{(q)_{N_{k-2}}} \times \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-2}^2+N_{i+1}+\cdots+N_{k-2}}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\cdots(q)_{N_{k-3}-N_{k-2}}}. \tag{6.5}$$

If $N_{k-1} > 0$, applying Theorem 6.2 N_{k-1} times, we obtain that

$$\sum_{\lambda \in Q_{N_1,\ldots,N_{k-1};i}} q^{|\lambda|}$$

$$= q^{\frac{(2N_1-N_{k-1}+1)N_{k-1}}{2}+(2N_2-N_{k-1}+1)N_{k-1}+\cdots+(N_{k-1}+1)N_{k-1}-N_{k-1}i+N_{k-1}}$$
$$\times \sum_{\lambda\in Q_{N_1-N_{k-1},\ldots,N_{k-2}-N_{k-1},0;i}} q^{|\lambda|}. \tag{6.6}$$

Combining (6.5) and (6.6), we have for $1 \le i \le k-1$

$$\sum_{\lambda\in Q_{N_1,\ldots,N_{k-1};i}} q^{|\lambda|}$$
$$= q^{\frac{(2N_1-N_{k-1}+1)N_{k-1}}{2}+(2N_2-N_{k-1}+1)N_{k-1}+\cdots+(N_{k-1}+1)N_{k-1}-N_{k-1}i+N_{k-1}}$$
$$\times \sum_{\lambda\in Q_{N_1-N_{k-1},\ldots,N_{k-2}-N_{k-1},0;i}} q^{|\lambda|}$$
$$= \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\cdots(q)_{N_{k-2}-N_{k-1}}}.$$

Since for any overpartition in $Q_{N_1,\ldots,N_{k-1};i}$ the smallest 1-marked part is overlined, the non-overlined 1 can occur at most $k-2$ times. This implies that $Q_{N_1,\ldots,N_{k-1};k} = Q_{N_1,\ldots,N_{k-1};k-1}$. we have proved that identity (6.1) holds for $1 \le i \le k$, that is, Theorem 6.1 holds for k. This completes the proof. ∎

7 Proof of Theorem 1.7

In this section, we finish the proof of Theorem 1.7. Using the three bijections given in the previous sections, we can derive the generating function of $F_{k,i}(m,n)$ as stated in Theorem 3.3. Then we compute the generating function of $G_{k,i}(m,n)$ which leads to the generating function of $D_{k,i}(m,n)$. We first give the proof of Theorem 3.3.

Proof of Theorem 3.3. By Theorems 4.1, 5.1, and 6.1, we find that the generating function of $F_{k,i}(m,n)$ equals

$$\sum_{n=0}^{\infty} F_{k,i}(m,n)x^m q^n$$
$$= \sum_{N_1\ge N_2\ge\cdots\ge N_{k-1}\ge 0} \frac{(-q)_{N_1-1}}{(q)_{N_{k-1}}} \sum_{\lambda\in Q_{N_1,\ldots,N_{k-1};i}} x^{N_1+\cdots+N_{k-1}} q^{|\lambda|}$$
$$= \sum_{N_1\ge N_2\ge\cdots\ge N_{k-1}\ge 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}},$$

as claimed. ∎

Given the relation between $F_{k,i}(m,n)$ and $G_{k,i}(m,n)$ as stated in Lemma 3.2, we can derive the generating function of $G_{k,i}(m,n)$.

Theorem 7.1 *For $k \geq i \geq 1$,*

$$\sum_{n=0}^{\infty} G_{k,i}(m,n)x^m q^n = \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}(-q)_{N_1-1}x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \quad (7.1)$$

Proof. From relation (3.6), we deduce that for $2 \leq i \leq k$,

$$\begin{aligned}\sum_{m,n\geq 0} G_{k,i}(m,n)x^m q^n &= \sum_{m,n\geq 0} F_{k,i-1}(m,n)x^m q^n \\ &= \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}(-q)_{N_1-1}x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \quad (7.2)\end{aligned}$$

For $i=1$, from (3.7) it follows that

$$\sum_{m,n\geq 0} G_{k,1}(m,n)x^m q^n = \sum_{m,n\geq 0} F_{k,k}(m,n)(xq)^m q^n.$$

Using the generating function of $F_{k,k}(m,n)$, we obtain

$$\sum_{m,n\geq 0} G_{k,1}(m,n)x^m q^n = \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}(-q)_{N_1-1}x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \quad (7.3)$$

Observe that the above formulas (7.2) for $i>1$ and (7.3) for $i=1$ take the same form (7.1) as in the theorem. This completes the proof. ∎

We are now ready to finish the proof of Theorem 1.7.

Proof of Theorem 1.7. By the generating functions of $G_{k,i}(m,n)$ and $F_{k,i}(m,n)$ and relation (3.5), we find that

$$\begin{aligned}
&\sum_{m,n\geq 0} D_{k,i}(m,n)x^m q^n \\
&\quad = \sum_{m,n\geq 0} F_{k,i}(m,n)x^m q^n + \sum_{m,n\geq 0} G_{k,i}(m,n)x^m q^n \\
&\quad = \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}.
\end{aligned}$$

This completes the proof of Theorem 1.7. ∎

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