q-Hook Length Formulas for Signed Labeled Forests

William Y.C. Chen[1], Oliver X.Q. Gao[2] and Peter L. Guo[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

emails: [1]chen@nankai.edu.cn, [2]oliver@cfc.nankai.edu.cn,
[3]lguo@cfc.nankai.edu.cn

Abstract. A signed labeled forest is defined as a (plane) forest labeled by $\{1, 2, \ldots, n\}$ along with minus signs associated to some vertices. Signed labeled forests can be viewed as an extension of signed permutations. We define the inversion number, the flag major index and the R-major index on signed labeled forests. They can be considered as type B analogues of the indices for labeled forests introduced by Björner and Wachs. The flag major index for signed labeled forests is based on the flag major index on signed permutations introduced by Adin and Roichman, whereas the R-major index for signed labeled forests is based on the R-major index that we introduce for signed permutations, which is closely related to the major defined by Reiner. We obtain q-hook length formulas by q-counting signed labelings of a given forest with respect to the above indices, from which we see that these three indices are equidistributed for signed labeled forests. Our formulas for the major indices and the inversion number are type B analogues of the formula due to Björner and Wachs. We also give a type D analogue with respect to the inversion number of even-signed labeled forests.

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1   Introduction

The inversion number and major index for the permutation group are two of the most important statistics which have received remarkable attention in the combinatorial

literature, see, e.g., [10, 11, 12, 13, 15, 17]. Extensions of these two statistics have been intensely studied. The present paper concerns two directions of such extensions: One is toward labeled forests duo to Björner and Wachs [8], and the other is toward the Coxeter groups (mainly types B and D), see, e.g., [1, 2, 3, 4, 5, 6, 9, 18].

For integers m, n $(m \leq n)$, we use $[m, n]$ to denote the interval $\{m, m+1, \ldots, n\}$. Denote by S_n the permutation group on $[1, n]$. Note that we use $[1, n]$, instead of $[n]$, to denote the set $\{1, 2, \ldots n\}$ before we shall use $[n]$ to denote the q-number $1 + q + q^2 + \cdots + q^{n-1}$. We shall represent a permutation $\pi \in S_n$ in one-line notation $\pi = \pi_1 \cdots \pi_n$. Then the descent set of π is defined by

$$\mathrm{Des}(\pi) = \{i \colon\ 1 \leq i \leq n - 1,\, \pi_i > \pi_{i+1}\}.$$

The inversion number and major index of π are defined by

$$\mathrm{inv}(\pi) = |\{(i, j) \colon\ 1 \leq i < j \leq n,\, \pi_i > \pi_j\}|,$$

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

The following classical formula is duo to MacMahon [17]

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]! = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)}, \tag{1.1}$$

where $[n] = 1 + q + \cdots + q^{n-1}$, and $[n]! = [1][2] \cdots [n]$.

The inversion number and the major index on permutations have been generalized by Björner and Wachs [8] to labeled forests. Let F be a (plane) forest with vertex set $V(F)$. The reason that we consider plane forests is that every vertex can be viewed as having a unique position in the sense that all the vertices are implicitly labeled. A labeling w of F is a bijection

$$w \colon\ V(F) \longrightarrow [1, n].$$

For each vertex $u \in F$, the hook length of u, denoted by h_u, is the size of the subtree rooted at u. When F is considered as a poset with roots at the top, the hook length of u equals the cardinality of the principle ideal $\{v \in F \colon\ v \leq_F u\}$ where $\leq_F$ is the order relation. Björner and Wachs [8] defined the descent set of a labeled forest as given below,

$$\mathrm{Des}(F, w) = \{u \in F \colon\ w(u) > w(v),\, v \text{ is the parent of } u\},$$

and

$$\mathrm{inv}(F, w) = |\{(u, v) \colon\ u >_F v,\, w(u) < w(v)\}|, \tag{1.2}$$

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$$\mathrm{maj}(F, w) = \sum_{u \in \mathrm{Des}(F,w)} h_u. \tag{1.3}$$

If F is a linear tree, then we get a permutation by reading the labels bottom up. The descent set, the inversion number and the major index for a linear tree coincide with the corresponding indices for permutations. Björner and Wachs [8] derived the following q-hook length formula by q-counting all labelings of a fixed forest with respect to the inversion number and the major index, which reduces to the formula (1.1) by restricting F to be a linear tree.

Theorem 1.1 (Björner and Wachs [8], Theorem 1.3) *Let F be a forest of size n. Then*

$$\sum_{(F,w)} q^{\mathrm{inv}(F,w)} = \frac{n!}{\prod_{u \in F} h_u} \prod_{u \in F} [h_u] = \sum_{(F,w)} q^{\mathrm{maj}(F,w)}, \tag{1.4}$$

where w ranges over all labelings of F.

In this paper, we shall be concerned with signed labelings of a (plane) forest. The signed permutation group B_n is the group of bijections σ on the set $[-n, n]\backslash\{0\}$ such that

$$\sigma(-i) = -\sigma(i)$$

for $i \in [-n, n]\backslash\{0\}$. Recall that B_n is also known as the hyperoctahedral group of rank n, or the Coxeter group of type B_n. For $\sigma \in B_n$, we write σ in the one-line notation $\sigma = \sigma_1 \cdots \sigma_n$, where $\sigma_i = \sigma(i)$ for $i \in [1, n]$. In the language of Coxeter groups, B_n is the Coxeter group of type B_n with respect to the generating set $\{s_0, s_1, \ldots, s_{n-1}\}$, where s_i, $i = 1, 2, \ldots, n-1$, are the simple transpositions

$$s_i = (1, \ldots, i-1, i+1, i, i+2, \ldots, n)$$

and s_0 is the sign change

$$s_0 = (-1, 2, \ldots, n).$$

The length function of an element π in a Coxeter group, denoted by $\ell(\pi)$, is the minimum number of generators that occur in its factorizations, see, Björner and Brenti [7], namely,

$$\ell(\pi) = \min\{r: \ \pi = s_{i_1} s_{i_2} \cdots s_{i_r}, \ s_{i_1}, s_{i_2}, \ldots, s_{i_r} \text{ are generarors}\}.$$

For B_n, there exists a simple combinatorial interpretation for the length function. Let $\sigma = \sigma_1 \cdots \sigma_n$ be a signed permutation, and define

$$\mathrm{n}_1(\sigma) = |\{\sigma_i: \ 1 \le i \le n, \ \sigma_i < 0\}|, \tag{1.5}$$

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$$\mathrm{n}_2(\sigma) = \big|\{\{i,j\}\colon\ 1 \le i, j \le n,\ \sigma_i + \sigma_j < 0\}\big|. \tag{1.6}$$

Then the length function of σ is given by

$$\ell_B(\sigma) = \mathrm{inv}(\sigma) + \mathrm{n}_1(\sigma) + \mathrm{n}_2(\sigma), \tag{1.7}$$

which can be seen as the inversion number for signed permutations, see, Biagioli [4], or Björner and Brenti [7]. The following length generating function is well-known.

Theorem 1.2 (Humphreys [16, Section 3.15])

$$\sum_{\sigma \in B_n} q^{\ell_B(\sigma)} = [2][4]\cdots[2n]. \tag{1.8}$$

As a subgroup of B_n, the group of even-signed permutations, i.e., permutations with an even number of minus signs, is denoted by D_n. It is well known that D_n is the Coxeter group respect to the generating set $\{t_0, s_1, \ldots, s_{n-1}\}$, where s_i for $1 \le i \le n-1$ are defined as above and

$$t_0 = (-2, -1, 3, \ldots, n).$$

For any signed permutation σ, let

$$\mathrm{Neg}(\sigma) = \{1 \le i \le n\colon \sigma_i < 0\}.$$

The length function for D_n can be computed by the following combinatorial formula

$$\ell_D(\sigma) = \mathrm{inv}(\sigma) - \sum_{i \in \mathrm{Neg}(\sigma)} \sigma_i - \mathrm{n}_1(\sigma), \tag{1.9}$$

which can be considered as the inversion number of an even-signed permutation, see, Björner and Brenti [7, Section 8.2], or Biagioli [5]. It can be checked that

$$-\sum_{i \in \mathrm{Neg}(\sigma)} \sigma_i = \mathrm{n}_1(\sigma) + \mathrm{n}_2(\sigma).$$

So (1.9) can be reformulated as

$$\ell_D(\sigma) = \mathrm{inv}(\sigma) + \mathrm{n}_2(\sigma). \tag{1.10}$$

The length generating function for D_n is given by the following formula.

Theorem 1.3 (Humphreys [16, Section 3.15])

$$\sum_{\sigma \in D_n} q^{\ell_D(\sigma)} = [2][4]\cdots[2n-2][n]. \tag{1.11}$$

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Statistics on Coxeter groups that are equiditributed with the length function are called Mahonian. An important statistic on B_n is the flag major index introduced by Adin and Roichman [3], which is defined in terms of Coxeter elements and can be expressed combinatorially as

$$\mathrm{fmaj}(\sigma) = 2\,\mathrm{maj}(\sigma) + \mathrm{n}_1(\sigma). \tag{1.12}$$

The second Mahonian major statistic on B_n is the negative major index introduced by Adin, Brenti and Roichman [1], which has the following combinatorial description

$$\mathrm{nmaj}(\sigma) = \mathrm{maj}(\sigma) + \mathrm{n}_1(\sigma) + \mathrm{n}_2(\sigma). \tag{1.13}$$

There is another Mahonian statistic which is based on the major index defined by Reiner [18]. Under the following order

$$1 < \cdots < n < -n < \cdots < -1, \tag{1.14}$$

the descent set $\mathrm{Des}_R(\sigma)$ is defined as

$$\mathrm{Des}_R(\sigma) = \{i \in [1, n] \colon \sigma_i > \sigma_{i+1}\},$$

under the assumption that $\sigma_{n+1} = n$. Then the major index $\mathrm{maj}_R(\sigma)$ is given by

$$\mathrm{maj}_R(\sigma) = \sum_{i \in \mathrm{Des}_R(\sigma)} i. \tag{1.15}$$

Reiner [18] has shown that

$$\sum_{\sigma \in B_n} t^{\mathrm{n}_1(\sigma)} q^{\mathrm{maj}_R(\sigma)} = (1 + tq)^n [n]!. \tag{1.16}$$

While the index maj_R is not Mahonian, as observed by Biaginoli and Zeng [6], from (1.16) it follows that the index

$$2\,\mathrm{maj}_R(\sigma) - \mathrm{n}_1(\sigma) \tag{1.17}$$

is equidistributed with the flag major index (1.12). They also gave a proof of this fact by justifying the following relation

$$\mathrm{maj}_R(\sigma) = \mathrm{maj}(\sigma) + \mathrm{n}_1(\sigma).$$

We next define a new Mahonian index called R-major index for signed permutations which relies on the major index maj_B with respect to the natural order

$$-n < \cdots < -1 < 0 < 1 < \cdots < n. \tag{1.18}$$

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The index maj_B can be shown to be isomorphic to Reiner's major index (1.15) defined with respect to the order (1.14). The descent set $\mathrm{Des}_B(\sigma)$ is described in terms of the natural order; that is,

$$\mathrm{Des}_B(\sigma) = \{i \in [1, n] \colon \sigma_i > \sigma_{i+1}\}, \qquad (1.19)$$

where $\sigma_{n+1} = 0$. Let

$$\mathrm{maj}_B(\sigma) = \sum_{i \in \mathrm{Des}_B(\sigma)} i, \qquad (1.20)$$

and let $\mathrm{p}(\sigma)$ be the number of positive entries of σ. Then the R-major index is given by

$$\mathrm{rmaj}(\sigma) = 2\,\mathrm{maj}_B(\sigma) - \mathrm{p}(\sigma). \qquad (1.21)$$

It can be shown that the index $\mathrm{maj}_B(\sigma)$ in (1.20) is isomorphic to the index $\mathrm{maj}_R(\sigma)$ in (1.15) of Reiner, see Lemma 4.7. Under this correspondence, the index $\mathrm{rmaj}(\sigma)$ in (1.21) is isomorphic to the index $2\,\mathrm{maj}_R(\sigma) - \mathrm{n}_1(\sigma)$ in (1.17). It should be mentioned that we prefer the R-major index formulation based on the the natural order (1.18) because it is consistent with the order of the flag major index and the negative major index and it seems to be easier to describe.

In this paper, we extend the three indices on signed permutations, i.e., the inversion number, the flag major index, and the R-major index, to signed labeled forests. The inversion number for signed labeled forests is motivated by the inversion number (1.2) and the length function (1.7). The flag major index for signed labeled forests is an extension of the major indices (1.3) for forests and (1.12) for signed permutations, whereas the R-major index for signed labeled forests stems from the major indices (1.3) and (1.21).

We obtain q-hook length formulas by q-counting signed labelings of a given forest with respect to the above indices, from which we see that these three indices are equidistributed for signed labeled forests. Our formulas for the major indices and the inversion number are type B analogues of the formula due to Björner and Wachs. We also define the inversion number on even-signed labeled forests in connection with the inversion number (1.2) and the length function (1.10) for even-signed permutations.

This paper is organized as follows. In Section 2, we give the definitions of the inversion number and the two major indices on singed labeled forests. The main results are described in this section. We also include a sketch of the proof of the q-hook length formula for the flag major index for signed labeled forests. Section 3 is devoted to the proofs of the generating function formulas for the inversion numbers on signed labeled forests and even-signed labeled forests. In Section 4, we consider the generating function of the R-major index. To this end, we define (P, w)-partitions of type B which can be viewed as a type B extension of the usual (P, w)-partitions

as introduced by Stanley [21]. In Section 5, we give a bijection which establishes the connection between the flag major index and the R-major index on signed labeled forests. Section 6 gives some further questions.

2 Main results

In this section, we give the definitions of the inversion number, the flag major index and the R-major index on singed (even-signed) labeled forests as aforementioned in the introduction, and outline the main results of this paper.

Let F be a forest. Denote by $B_n(F)$ (resp., $D_n(F)$) the set of signed (resp., even-signed) labeled forests with the underlying forest F. We use (F, w) to denote a signed labeling of F. The motivation to consider types B and D analogues for signed labeled forests is the observation that the number $\mathrm{n}_2(\sigma)$ in (1.6) has a natural extension to signed labeled forests; that is, for $(F, w) \in B_n(F)$, we may define

$$\mathrm{n}_2(F, w) = |\{(u, v) \colon u <_F v, w(u) + w(v) < 0\}|. \tag{2.1}$$

Let $\mathrm{n}_1(F, w)$ be the number of negative labels of w. So we define the inversion number for $(F, w) \in B_n(F)$ as

$$\mathrm{inv}_B(F, w) = \mathrm{inv}(F, w) + \mathrm{n}_1(F, w) + \mathrm{n}_2(F, w), \tag{2.2}$$

and for $(F, w) \in D_n(F)$ the inversion number is defined by

$$\mathrm{inv}_D(F, w) = \mathrm{inv}(F, w) + \mathrm{n}_2(F, w). \tag{2.3}$$

The first two theorems assert that the inversion numbers (2.2) and (2.3) lead to types B and D analogues of the length generating functions (1.8) and (1.11) respectively.

Theorem 2.1 *Let F be a forest of size n. Then*

$$\sum_{(F, w) \in B_n(F)} q^{\mathrm{inv}_B(F, w)} = \frac{n!}{\prod_{u \in F} h_u} \prod_{u \in F} [2h_u]. \tag{2.4}$$

Theorem 2.2 *Let F be a forest of size n. Then*

$$\sum_{(F, w) \in D_n(F)} q^{\mathrm{inv}_D(F, w)} = \frac{n!}{2 \prod_{u \in F} h_u} \prod_{u \in F} (1 + q^{h_u - 1})[h_u]. \tag{2.5}$$

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Note that Theorem 1.2 and Theorem 1.3 can be deduced respectively from Theorem 2.1 and Theorem 2.2 by restricting F to a linear order.

We next turn to two major indices for signed labeled forests. The first is based on the flag major index for signed permutations, namely, for $(F, w) \in B_n(F)$, we define

$$\mathrm{fmaj}(F, w) = 2 \, \mathrm{maj}(F, w) + \mathrm{n}_1(F, w). \tag{2.6}$$

The following theorem shows that the flag major index (2.6) is equidistributed with the inversion number (2.2) for signed labeled forests.

Theorem 2.3 *Let F be a forest of size n. Then*

$$\sum_{(F, w) \in B_n(F)} q^{\mathrm{fmaj}(F, w)} = \frac{n!}{\prod_{u \in F} h_u} \prod_{u \in F} [2h_u]. \tag{2.7}$$

On the other hand, it will be seen that Theorem 2.3 can be deduced from Theorem 1.1 and a decomposition of the signed permutation group. We sketch the proof below. Let

$$T_n = \{\sigma_1 \cdots \sigma_n \in B_n \colon \sigma_1 < \cdots < \sigma_n\}.$$

Then the signed permutation group B_n has the following decomposition

$$B_n = \bigcup_{\pi \in S_n} \{\sigma \pi \colon \sigma \in T_n\}, \tag{2.8}$$

which is the multiplicative decomposition of a Coxeter group into a parabolic subgroup and its minimal coset representatives, see, e.g., Björner and Brenti [7] or Humphreys [16]. Such a decomposition has been used by Adin et al. [1, Proposition 3.1] to prove the Mahonian property of the negative major index (1.13).

Sketch of the proof of Theorem 2.3. Let w_0 be a decreasing labeling of F, i.e., a labeling of F such that $w_0(u) > w_0(v)$ whenever $u > v$. We now define a bijection $\psi \colon (F, w) \longmapsto \sigma$ from $B_n(F)$ to B_n such that $\sigma_i = w(i)$ for $1 \leq i \leq n$. Let

$$T_n(F) = \{\psi^{-1}(\sigma) \colon \sigma \in T_n\},$$

and let $U_n(F) = \psi^{-1}(S_n)$ be the set of ordinary labelings of F. From the decomposition (2.8), we get

$$B_n(F) = \bigcup_{w \in U_n(F)} \{\tau w \colon \tau \in T_n(F)\}, \tag{2.9}$$

where $\tau w(i) = \tau(w(i))$ for any vertex $i \in F$.

Let u be the parent of v, and let $w_0(u) = i_u$ and $w_0(v) = i_v$. It is not hard to verify that $\tau w(i_u) > \tau w(i_v)$ if and only if $w(i_u) > w(i_v)$. This implies that τw and w have the same descent set. Similarly, we see that $n_1(F, \tau w) = n_1(F, \tau)$. Thus

$$
\begin{aligned}
\mathrm{fmaj}(F, \tau w) &= 2\,\mathrm{maj}(F, \tau w) + n_1(F, \tau w) \\
&= 2\,\mathrm{maj}(F, w) + n_1(F, \tau).
\end{aligned}
\tag{2.10}
$$

Again, by the decomposition (2.10) and Theorem 1.1, we obtain

$$
\begin{aligned}
\sum_{(F,\,w)\in B_n(F)} q^{\mathrm{fmaj}(F,\,w)} &= \sum_{w\in U_n(F)} \sum_{\tau\in T_n(F)} q^{2\,\mathrm{maj}(F,\,w)+n_1(F,\,\tau)} \\
&= \sum_{\tau\in T_n(F)} q^{n_1(F,\,w)}\, \frac{n!}{\prod_{u\in F} h_u} \prod_{u\in F}[h_u]_{q^2} \\
&= (1+q)^n \frac{n!}{\prod_{u\in F} h_u} \prod_{u\in F}[h_u]_{q^2} \\
&= \frac{n!}{\prod_{u\in F} h_u} \prod_{u\in F}\left([h_u]_{q^2}(1+q)\right) \\
&= \frac{n!}{\prod_{u\in F} h_u} \prod_{u\in F}[2h_u],
\end{aligned}
$$

as desired. ∎

The second major index for signed labeled forests is the R-major index. For $(F, w) \in B_n(F)$, we define the descent set as

$$
\mathrm{Des}_B(F, w) = \mathrm{Des}(F, w) \bigcup \{u \in F \colon u \text{ is a root of } F \text{ with a positive label}\}.
$$

Let

$$
\mathrm{maj}_B(F, w) = \sum_{u\in\mathrm{Des}_B(F,w)} h_u.
\tag{2.11}
$$

Denote by $\mathrm{p}(F, w)$ the number of positive labels of w. Then the R-major index is defined by

$$
\mathrm{rmaj}(F, w) = 2\,\mathrm{maj}_B(F, w) - \mathrm{p}(F, w).
\tag{2.12}
$$

The following theorem shows that the R-major index is equidistributed with the inversion number for signed labeled forests.

Theorem 2.4 *Let F be a forest of size n. Then*

$$
\sum_{(F,\,w)\in B_n(F)} q^{\mathrm{rmaj}(F,\,w)} = \frac{n!}{\prod_{u\in F} h_u} \prod_{u\in F}[2h_u].
\tag{2.13}
$$

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3   The inversion numbers for types B and D

In this section, we are aimed to prove Theorems 2.1 and 2.2, both of which can be deduced from the following theorem involving the weighted counting of signed labeled forest with respect to the number of inversions and the number of negative labels.

Theorem 3.1 *Let F be a forest of size n. Then*

$$\sum_{(F,w)\in B_n(F)} t^{n_1(F,w)} q^{\text{inv}_B(F,w)} = \frac{n!}{\prod_{u\in F} h_u} \prod_{u\in F}(1+tq^{h_u})[h_u]. \tag{3.1}$$

It is clear that the constant terms on both sides of (3.1) lead to the q-hook length formula of Björner and Wachs as stated in Theorem 1.1.

Proof. We proceed by induction on n. Assume that F is a forest consisting of k trees $T_1, \ldots, T_k$. Let us consider the following two cases.

Case 1. Suppose that $k > 1$ and T_i has n_i vertices for $1 \le i \le k$. Each signed labeling of F corresponds to a k-tuple consisting of a distribution of the n labels (some of which may have minus signs) into k trees with each tree T_i receiving n_i labels. We see that

$$\sum_{(F,w)} t^{n_1(F,w)} q^{\text{inv}_B(F,w)} = \binom{n}{n_1,\ldots,n_k} \prod_{i=1}^k \sum_{(T_i,w_i)} t^{n_1(T_i,w_i)} q^{\text{inv}_B(T_i,w_i)}.$$

Applying the induction hypothesis yields

$$\sum_{(F,w)} t^{n_1(F,w)} q^{\text{inv}_B(F,w)} = \binom{n}{n_1,\ldots,n_k} \prod_{i=1}^k \frac{n_i!}{\prod_{u\in T_i} h_u} \prod_{u\in T_i}(1+tq^{h_u})[h_u]$$

$$= \frac{n!}{\prod_{u\in F} h_u} \prod_{u\in F}(1+tq^{h_u})[h_u].$$

Case 2. Suppose that $k = 1$. Let F' be the forest obtained from F by removing the root u. Every signed labeling w of F corresponds to a pair $(w(u), w')$, where w' is the signed labeling of F' induced by w. Since w is a signed labeling, we need to consider the two subcases $w(u) > 0$ and $w(u) < 0$.

Assume that $w(u) > 0$. Then let $w(u) = i$, where $1 \le i \le n$. We get

$$\text{inv}(F,w) = \text{inv}(F',w') + |\{v \in F' : w'(v) > i\}|,$$

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$$n_1(F, w) = n_1(F', w'),$$

$$n_2(F, w) = n_2(F', w') + |\{v \in F' \colon w'(v) + i < 0\}|.$$

It is easy to check

$$\{v \in F' \colon w'(v) > i\} \sqcup \{v \in F' \colon w'(v) + i < 0\} = \{v \colon i < |w'(v)| \le n\},$$

where $\sqcup$ denotes the disjoint union. Thus we find

$$\mathrm{inv}_B(F, w) = \mathrm{inv}_B(F', w') + n - i.$$

Assume that $w(u) < 0$. Then let $w(u) = -i$, where $1 \le i \le n$. We have

$$\mathrm{inv}(F, w) = \mathrm{inv}(F', w') + |\{v \in F' \colon w'(v) > -i\}|,$$

$$n_1(F, w) = n_1(F', w') + 1,$$

$$n_2(F, w) = n_2(F', w') + |\{v \in F' \colon w'(v) - i < 0\}|.$$

We need to determine the numbers c_1 and c_2 defined by

$$c_1 = |\{v \in F' \colon w'(v) > -i\}|, \quad c_2 = |\{v \in F' \colon w'(v) - i < 0\}|.$$

(i). Consider the index j with $i < j \le n$. If j is in the labeling set of w, then j contributes 1 to c_1 and 0 to c_2. If $-j$ is in the labeling set of w, then $-j$ contributes 0 to c_1 and 1 to c_0.

(ii). Consider the index j with $1 \le j < i$. If j is in the labeling set of w, then j contributes 1 to both c_1 and c_2. If $-j$ is in the labeling set of w, then $-j$ also contributes 1 to both c_1 and c_2.

It follows that

$$|\{v \in F' \colon w'(v) > -i\} + |\{v \in F' \colon w'(v) - i < 0\}| = (n - i) + 2(i - 1) = n - 2 + i,$$

which gives

$$\mathrm{inv}_B(F, w) = \mathrm{inv}_B(F', w') + c_1 + c_2 + 1$$

$$= \mathrm{inv}_B(F', w') + n - 1 + i.$$

Combining the above two cases $w(u) > 0$ and $w(u) < 0$ and using the induction hypothesis, we deduce that

$$\sum_{(F, w)} t^{n_1(F, w)} q^{\mathrm{inv}_B(F, w)}$$

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$$= \sum_{i=1}^{n} q^{n-i} \sum_{(F',w')} t^{n_1(F',w')} q^{\mathrm{inv}_B(F',w')} + \sum_{i=1}^{n} tq^{n-1+i} \sum_{(F',w')} t^{n_1(F',w')} q^{\mathrm{inv}_B(F',w')}$$

$$= (1 + q + \cdots + q^{n-1} + t(q^n + \cdots + q^{2n-1})) \sum_{(F',w')} t^{n_1(F',w')} q^{\mathrm{inv}_B(F',w')},$$

$$= (1 + tq^n)[n] \sum_{(F',w')} t^{n_1(F',w')} q^{\mathrm{inv}_B(F',w')}$$

$$= (1 + tq^n)[n] \frac{(n-1)!}{\prod_{u \in F'} h_u} \prod_{u \in F'} (1 + tq^{h_u})[h_u]$$

$$= \frac{n!}{\prod_{u \in F} h_u} \prod_{u \in F} (1 + tq^{h_u})[h_u],$$

as claimed. ∎

Once Theorem 3.1 is established, it is easy to derive Theorem 2.1 by taking $t = 1$ in (3.1) and by observing that

$$(1 + q^n)[n] = [2n].$$

We are now ready to give a proof of Theorem 2.2.

Proof of Theorem 2.2. Let

$$D_n(t,q) = \sum_{(F,w) \in B_n(F)} t^{n_1(F,w)} q^{\mathrm{inv}(F,w) + n_2(F,w)}.$$

By Theorem 3.1 we obtain

$$D_n(t,q) = \sum_{(F,w) \in B_n(F)} (t/q)^{n_1(F,w)} q^{\mathrm{inv}(F,w) + n_1(F,w) + n_2(F,w)}$$

$$= \sum_{(F,w) \in B_n(F)} (t/q)^{n_1(F,w)} q^{\mathrm{inv}_B(F,w)}$$

$$= \frac{n!}{\prod_{u \in F} h_u} \prod_{u \in F} (1 + tq^{h_u - 1})[h_u].$$

Since F has at least one leaf, there must exist a factor $(1 + tq^{h_u - 1}) = (1 + t)$ in $D_n(t,q)$. Hence $D_n(-1, q) = 0$, or, equivalently,

$$\sum_{i \text{ is even}} [t^i] D_n(t,q) = \sum_{i \text{ is odd}} [t^i] D_n(t,q), \tag{3.2}$$

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where $[t^i]$ denotes the coefficient of t^i in $D_n(t,q)$. Thus

$$\sum_{(F,w)\in D_n(F)} q^{\text{inv}_D(F,w)} = \sum_{i \text{ is even}} [t^i]D_n(t,q)$$

$$= \frac{D_n(1,q)}{2},$$

which coincides with the formula (2.5). $\blacksquare$

It would be interesting to give a combinatorial interpretation for (3.2).

4   The R-major index for signed labeled forests

In this section, we shall present a proof of Theorem 2.4. This theorem will be deduced from the following more general formula.

Theorem 4.1 *Let F be a forest of size n. Then*

$$\sum_{(F,w)\in B_n(F)} t^{\text{p}(F,w)} q^{\text{maj}_B(F,w)} = \frac{n!}{\prod_{u\in F} h_u}(1+tq)^n \prod_{u\in F}[h_u]. \tag{4.1}$$

In fact, the proof of the above Theorem relies on the formula for q-counting the major index maj_B of linear extensions of a signed labeled forest, which can be considered as type B analogue of the q-hook length formula of Björner and Wachs [8] for the q-counting of the major index of linear extensions.

Let us recall the definition of linear extensions of poset. For a poset P with n elements $x_1, \ldots, x_n$, linear extensions of P can be seen as permutations $x_{i_1} \cdots x_{i_n}$ such that $x_{i_j} <_P x_{i_k}$ implies $j < k$. A labeling of P is a bijection from $\{x_1, \ldots, x_n\}$ to $[1,n]$. For a poset P with a labeling w, we usually use a permutation $w(x_{i_1}) \cdots w(x_{i_n})$ to represent linear extension $x_{i_1} \cdots x_{i_n}$. Denote by $\mathcal{L}(P,w)$ the set of all such permutations. Björner and Wachs [8] obtained the following generating function for the major index of linear extensions of any given labeled forest, which reduces to the result of Stanley [21] when the labeling is decreasing.

Theorem 4.2 (Björner and Wachs [8], Theorem 1.2) *Let F be a forest of size n, and w a labeling of F. Then*

$$\sum_{\pi\in\mathcal{L}(F,w)} q^{\text{maj}(\pi)} = q^{\text{maj}(F,w)} \frac{[n]!}{\prod_{u\in F}[h_u]}. \tag{4.2}$$

When w is a signed labeling, $\mathcal{L}(F, w)$, as defined above, is a set of signed permutations. We obtain the following theorem which can be viewed as a type B analogue of Theorem 4.2.

Theorem 4.3 *Let F be a forest of size n, and w a signed labeling of F. Then*

$$\sum_{\sigma \in \mathcal{L}(F,w)} q^{\mathrm{maj}_B(\sigma)} = q^{\mathrm{maj}_B(F,w)} \frac{[n]!}{\prod_{u \in F}[h_u]}. \tag{4.3}$$

Though Theorem 4.3 is a type B analogue of Theorem 4.2, its proof does not seem to be straightforward. To prove Theorem 4.2, Björner and Wachs [8] defined the recursive labelings for forests. They first derived the following q-hook length formula

$$\sum_{\pi \in \mathcal{L}(F,w)} q^{\mathrm{inv}(\pi)} = q^{\mathrm{inv}(F,w)} \frac{[n]!}{\prod_{u \in F}[h_u]},$$

where w is a recursive labeling, see Björner and Wachs [8, Theorem 1.1]. Then they proved that Foata's bijection (see, Foata [10]) is invariant when restricted to the set of linear extensions of a forest with a recursive labeling, see Björner and Wachs [8, Theorem 2.2]. Moreover they observed that the inversion number and the major index of a labeled forest are equal for a recursive labeling, see Björner and Wachs [8, Lemma 2.3]. For recursive labelings, they obtained Theorem 4.2. Finally, they used a equivalence relation on labelings to extend the special case to the general case, see Björner and Wachs [8, Lemma 4.2], and eventually finished the proof of Theorem 4.2.

The way in which Björner and Wachs [8] proved Theorem 4.2 does not seem to apply to Theorem 4.3. To prove Theorem 4.3, we introduce (F, w)-partitions of type B which reduce to the (P, w)-partitions due to Stanley [21] when P is a forest and w is a signed labeling of P with labeling set $\{-1, -2, \ldots, -n\}$. When w is an ordinary labeling, our approach leads to a direct combinatorial proof of Theorem 4.2 of Björner and Wachs.

Let $\mathbb{N}$ be the set of nonnegative integers. A (F, w)-partition of type B is a map $f \colon V(F) \longrightarrow \mathbb{N}$ satisfying the following conditions

(1) $f(x) \leq f(y)$ if $x \geq_F y$;

(2) $f(x) < f(y)$ if $x >_F y$ and $w(x) < w(y)$;

(3) $f(u) \geq 1$ if u is a root of F with $w(u) > 0$.

We denote by $A_B(F, w)$ the set of (F, w)-partitions of type B. For $f \in A_B(F, w)$, let

$$|f| = \sum_{x \in F} f(x).$$

We shall compute the generating function for the q-counting of (F, w)-partitions of type B in two different ways. These two different countings lead to the following identity

$$\frac{\sum_{\sigma \in \mathcal{L}(F,w)} q^{\mathrm{maj}_B(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{q^{\mathrm{maj}_B(F,w)}}{\prod_{u \in F}(1-q^{h_u})}, \tag{4.4}$$

which is equivalent to the formula (4.3).

Lemma 4.4 *Let F be a forest and w a signed labeling of F. Then*

$$\sum_{f \in A_B(F,w)} q^{|f|} = \frac{q^{\mathrm{maj}_B(F,w)}}{\prod_{u \in F}(1-q^{h_u})}. \tag{4.5}$$

Proof. For each $f \in A_B(F, w)$ and $u \in \mathrm{Des}_B(F, w)$, define $f_u \colon V(F) \longrightarrow \mathbb{N}$ as

$$f_u(x) = \begin{cases} f(x) - 1, & \text{if } x \leq_F u, \\ f(x), & \text{otherwise.} \end{cases}$$

Assume $\mathrm{Des}_B(F, w) = \{u_1, u_2, \ldots, u_k\}$. Define recursively $f_{u_1 u_2 \cdots u_k} = (f_{u_1 u_2 \cdots u_{k-1}})_{u_k}$. Since $f_{uv} = f_{vu}$ then $f_{u_1 u_2 \cdots u_k}$ is independent of the order of $u_1, \ldots, u_k$. Thus $f_{u_1 u_2 \cdots u_k}$ is well defined. It is easy to see the resulting $f_{u_1 u_2 \cdots u_k}$ are maps from $V(F)$ to $\mathbb{N}$ satisfying only condition (1). Such maps are formally called P-partitions where P is a forest, see Stanley [20, Chapter 4]. In such a way, for fixed (F, w) we establish a bijection $\varphi \colon f \longmapsto f_{u_1 u_2 \cdots u_k}$ between the following two sets

$$\varphi \colon A_B(F, w) \longrightarrow \{F\text{-partitions}\}, \tag{4.6}$$

which, for each $f \in A_B(F, w)$, satisfies

$$|f| = \sum_{u \in \mathrm{Des}_B(F, w)} h_u + |\varphi(f)|$$

$$= \mathrm{maj}_B(F, w) + |\varphi(f)|. \tag{4.7}$$

Stanley [21, Proposition 22.1] has proved (see Sagan [19] for a combinatorial proof)

$$\sum_g q^{|g|} = \frac{1}{\prod_{u \in F}(1-q^{h_u})}, \tag{4.8}$$

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where the sum ranges over all F-partitions. (4.7) together with (4.8) implies (4.5). This completes the proof. ∎

Lemma 4.4 gives one formulation for the generating function of $A_B(F, w)$. To give the other, we go on by extending a fundamental result concerning the usual (P, w)-partitions. For $\sigma = \sigma_1 \cdots \sigma_n \in B_n$, a map $f \colon \{\sigma_1, \ldots, \sigma_n\} \longrightarrow \mathbb{N}$ is said to be σ-compatible if it satisfies the following conditions

(i) $f(\sigma_1) \geq f(\sigma_2) \geq \cdots \geq f(\sigma_n)$;

(ii) For $i \in [1, n-1]$, $f(\sigma_i) > f(\sigma_{i+1})$ if $\sigma_i > \sigma_{i+1}$;

(iii) $f(\sigma_n) \geq 1$ if $\sigma_n > 0$.

Let A_σ^B denote the set of all σ-compatible maps. Then we have the following decomposition.

Lemma 4.5 *Let F be a forest with a signed labeling w. Then*

$$A_B(F, w) = \biguplus_{\sigma \in \mathcal{L}(F, w)} A_\sigma^B. \tag{4.9}$$

Note that if w is a signed labeling with the labeling set $\{-1, -2, \ldots, -n\}$ then Lemma 4.5 coincides with the decomposition of Stanley [21, Theorem 6.2] where the poset is a forest.

Proof. By the definition of (F, w)-partitions of type B and the decomposition of Stanley [21, Theorem 6.2], it is not hard to see for each $f \in A_B(F, w)$ there exists a unique linear extension $\sigma \in \mathcal{L}(F, w)$ such that $f \in A_\sigma^B$. It suffices to verify that for each $\sigma \in \mathcal{L}(F, w)$, if $f \in A_\sigma^B$ then $f \in A_B(F, w)$. In other words, it is necessary to show that $f(\sigma_i) \geq 1$ whenever $\sigma_i > 0$. Note that condition (iii) ensures this holds in the case of $\sigma_n > 0$. So we are left with the case $\sigma_n < 0$. Assuming $\sigma_n < 0$, let σ_j be the rightmost entry of σ such that $\sigma_j > 0$. Then we have $\sigma_j > \sigma_{j+1}$ since $\sigma_{j+1} < 0$. Given condition (ii), we see that $f(\sigma_j) \geq 1$. Therefore we have $f(\sigma_i) \geq 1$ whenever $\sigma_i > 0$. This completes the proof. ∎

Lemma 4.6 *Let F be a forest and w a signed labeling of F. Then*

$$\sum_{f \in A_B(F, w)} q^{|f|} = \frac{\sum_{\sigma \in \mathcal{L}(F, w)} q^{\mathrm{maj}_B(\sigma)}}{(1-q)(1-q^2) \cdots (1-q^n)}. \tag{4.10}$$

Proof. For $f \in A_\sigma^B$ and $1 \le i \le n - 1$, let

$$p_i = f(\sigma_i) - f(\sigma_{i+1}),$$

and let $p_n = f(\sigma_n)$. Clearly,

$$|f| = p_1 + 2p_2 + \cdots + np_n.$$

Under the conditions (i), (ii), and (iii), we see that $p_i > 0$ if $i \in \mathrm{Des}_B(\sigma)$ and $p_i \ge 0$ otherwise. Hence

$$\sum_{f \in A_\sigma^B} q^{|f|} = \sum_{(p_1, \ldots, p_n)} q^{p_1 + 2p_2 + \cdots + np_n} = q^{\mathrm{maj}_B(\sigma)} \sum_{\substack{(q_1, \ldots, q_n) \\ q_i \ge 0}} q^{q_1 + 2q_2 + \cdots + nq_n}$$

$$= \frac{q^{\mathrm{maj}_B(\sigma)}}{(1 - q)(1 - q^2) \cdots (1 - q^n)}.$$

Now, using the decomposition (4.9), we get (4.10). ∎

Combining Lemma 4.4 and Lemma 4.6, we deduce Theorem 4.3. We now need one more lemma for the proof of Theorem 4.1, which shows that the R-major index for singed permutations is equidistributed with Reiner's major index (1.15).

Lemma 4.7 *For $n \ge 0$, we have*

$$\sum_{\sigma \in B_n} t^{\mathrm{p}(\sigma)} q^{\mathrm{maj}_B(\sigma)} = (1 + tq)^n [n]!. \tag{4.11}$$

Proof. We will give a bijection $\psi \colon \sigma \longmapsto \tau$ on B_n such that $\mathrm{maj}_B(\sigma) = \mathrm{maj}_R(\tau)$ and $\mathrm{p}(\sigma) = \mathrm{n}_1(\tau)$. Then the formula (4.11) follows from Reiner's formula (1.16). For $\sigma = \sigma_1 \cdots \sigma_n \in B_n$, let $\{\sigma_{i_1}, \ldots, \sigma_{i_k}\}_<$ (resp., $\{\sigma_{j_1}, \ldots, \sigma_{j_l}\}_<$) be the set of positive (resp., negative) entries of σ, where the subscript $_<$ means $\sigma_{i_1} < \cdots < \sigma_{i_k}$. Define $\tau_{i_s} = -\sigma_{i_{k+1-s}}$ for $1 \le s \le k$, and $\tau_{j_t} = -\sigma_{j_{l+1-t}}$ for $1 \le t \le l$. Let $\tau = \tau_1 \cdots \tau_n$. It is clear that $\mathrm{p}(\sigma) = \mathrm{n}_1(\tau)$. Then it can be checked that $\mathrm{Des}_B(\sigma) = \mathrm{Des}_R(\tau)$. The details are omitted. Thus ψ is the required bijection. This completes the proof. ∎

We are now ready to prove Theorem 4.1. We use a similar technique as given by Björner and Wachs [8] for deriving the generating function for the major index of a given forest with the ordinary labelings.

Proof of Theorem 4.1. We aim to establish the relation (4.1) by evaluating the double sum

$$\sum_{(F,w) \in B_n(F)} \sum_{\sigma \in \mathcal{L}(F,w)} t^{\mathrm{p}(F,w)} q^{\mathrm{maj}_B(\sigma)}$$

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in two different ways. By Lemma 4.3, we have

$$\sum_{(F,w)\in B_n(F)}\sum_{\sigma\in\mathcal{L}(F,w)}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(\sigma)}=\sum_{(F,w)\in B_n(F)}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(F,w)}\frac{[n]!}{\prod_{u\in P}[h_u]}$$

$$=\frac{[n]!}{\prod_{u\in P}[h_u]}\sum_{(F,w)\in B_n(F)}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(F,w)}.$$

On the other hand, we may compute the above double sum by exchanging the order of summation. Let χ denote the indicator function which equals 1 when the statement is true and 0 otherwise. Then we have

$$\sum_{(F,w)\in B_n(F)}\sum_{\sigma\in\mathcal{L}(F,w)}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(\sigma)}=\sum_{(F,w)\in B_n(F)}\sum_{\sigma\in B_n}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(\sigma)}\chi(\sigma\in\mathcal{L}(F,w))$$

$$=\sum_{\sigma\in B_n}q^{\mathrm{maj}_B(\sigma)}\sum_{(F,w)\in B_n(F)}t^{\mathrm{p}(F,w)}\chi(\sigma\in\mathcal{L}(F,w))$$

$$=\sum_{\sigma\in B_n}q^{\mathrm{maj}_B(\sigma)}t^{\mathrm{p}(\sigma)}\sum_{(F,w)\in B_n(F)}\chi(\sigma\in\mathcal{L}(F,w)).$$

Recall that for any permutation $\pi\in S_n$, Björner and Wachs [8] have shown that there are

$$\frac{n!}{\prod_{u\in F}h_u}$$

ordinary labelings w such that $\pi\in\mathcal{L}(F,w)$. Clearly, this counting argument also applies to a signed permutation $\sigma\in B_n$. Consequently,

$$\sum_{(F,w)\in B_n(F)}\sum_{\sigma\in\mathcal{L}(F,w)}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(\sigma)}=\sum_{\sigma\in B_n}q^{\mathrm{maj}_B(\sigma)}t^{\mathrm{p}(\sigma)}\frac{n!}{\prod_{u\in F}h_u}$$

$$=\frac{n!}{\prod_{u\in F}h_u}\sum_{\sigma\in B_n}q^{\mathrm{maj}_B(\sigma)}t^{\mathrm{p}(\sigma)}.$$

By Lemma 4.7, we get

$$\sum_{(F,w)\in B_n(F)}\sum_{\sigma\in\mathcal{L}(F,w)}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(\sigma)}=\frac{n!}{\prod_{u\in F}h_u}(1+tq)^n[n]!.$$

The above double counting gives the following relation

$$\frac{[n]!}{\prod_{u\in F}[h_u]}\sum_{(F,w)\in B_n(F)}t^{\mathrm{p}(F,w)}q^{\mathrm{maj}_B(F,w)}=\frac{n!}{\prod_{u\in F}h_u}(1+tq)^n[n]!,$$

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which is equivalent to (4.1). This completes the proof. ∎

Based on Theorem 4.1, it is easy to derive Theorem 2.4.

Proof of Theorem 2.4. Setting $q \to q^2$ and $t \to q^{-1}$ in (4.1), the left-hand side becomes

$$\sum_{(F,w) \in B_n(F)} (q^{-1})^{\mathrm{p}(F,w)} q^{2\mathrm{maj}_B(F,w)} = \sum_{(F,w) \in B_n(F)} q^{2\mathrm{maj}_B(F,w) - \mathrm{p}(F,w)}$$

$$= \sum_{(F,w) \in B_n(F)} q^{\mathrm{rmaj}(F,w)}$$

and the right-hand side can be written as

$$\frac{n!}{\prod_{u \in F} h_u}(1 + q^{-1}q^2)^n \prod_{u \in F} [h_u]_{q^2} = \frac{n!}{\prod_{u \in F} h_u} \prod_{u \in F} [2h_u].$$

This completes the proof. ∎

5   A correspondence

From Theorem 2.4 and Theorem 2.3, one sees that the R-major index and the flag major index are equidistributed for signed labeled forests. One is naturally led to the question of finding a correspondence that explains the equidistribution property. This is the objective of this section to provide such a correspondence. Of course, this bijection can be considered as an alternative proof of Theorem 2.4.

Define a bijection $\phi\colon (F,w) \longmapsto (F,w')$ on $B_n(F)$ as follows. For each vertex $u \in F$,

(1) $w'(u)$ has the same sign with $-w(u)$,

(2) $|w'(u)| = n + 1 - |w(u)|$.

Theorem 5.1 *The above map ϕ is a bijection with the following property*

$$\mathrm{rmaj}(F,w) = \mathrm{fmaj}(F,w').$$

Proof. The theorem holds if we can show that

$$\mathrm{maj}_B(F,w) = \mathrm{maj}(F,w') + \mathrm{p}(F,w) \quad \text{and} \quad \mathrm{p}(F,w) = \mathrm{n}(F,w'). \tag{5.1}$$

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We proceed to prove (5.1) by induction on the number of vertices of F. Without loss of generality, we may assume that F is a tree.

If F has only one vertex, it is easy to check (5.1). So we may assume that F has at least two vertices. Let u_0 be the root of F, and $C(u_0)$ the set of children of u_0. By the definition of ϕ, we see that $\mathrm{p}(F, w) = \mathrm{n}(F, w')$. Now we claim that

$$\mathrm{maj}_B(F, w) = \mathrm{maj}(F, w') + \mathrm{p}(F, w). \tag{5.2}$$

Here are two cases.

Case 1: $w(u_0) > 0$. We partition $C(u_0)$ into the following three subsets.

$$P_> = \{u \in C(u_0)\colon\ w(u) > w(u_0)\},$$
$$P_< = \{u \in C(u_0)\colon\ 0 < w(u) < w(u_0)\},$$
$$N = \{u \in C(u_0)\colon\ w(u) < 0\}.$$

For any vertex $u \in F$, let F_u be the subtree of F rooted at u. Then

$\mathrm{Des}_B(F, w)$

$$= \left(\bigcup_{u \in P_>} \mathrm{Des}_B(F_u, w_u)\right) \bigcup \left(\bigcup_{u \in P_<} \mathrm{Des}_B(F_u, w_u) \backslash P_<\right) \bigcup \left(\bigcup_{u \in N} \mathrm{Des}_B(F_u, w_u)\right) \bigcup \{u_0\},$$

where w_u is the signed labeling of F_u induced by w. So we have

$$\begin{aligned}
\mathrm{maj}_B(F, w) = &\sum_{u \in P_>} \mathrm{maj}_B(F_u, w_u) + \sum_{u \in P_<} \mathrm{maj}_B(F_u, w_u) \\
&+ \sum_{u \in N} \mathrm{maj}_B(F_u, w_u) + n - \sum_{u \in P_<} h_u.
\end{aligned} \tag{5.3}$$

Let us compute the major index of w'. Let

$$N_> = \{u \in C(u_0)\colon\ 0 > w'(u) > w'(u_0)\},$$
$$N_< = \{u \in C(u_0)\colon\ w'(u) < w'(u_0)\},$$
$$P = \{u \in C(u_0)\colon\ w'(u) > 0\}.$$

Then

$\mathrm{Des}(F, w')$

$$= \left(\bigcup_{u \in N_>} \mathrm{Des}_B(F_u, w'_u) \bigcup N_>\right) \bigcup \left(\bigcup_{u \in N_<} \mathrm{Des}(F_u, w'_u)\right) \bigcup \left(\bigcup_{u \in P} \mathrm{Des}(F_u, w'_u) \bigcup P\right),$$

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from which we deduce that

$$\mathrm{maj}(F, w') = \sum_{u \in N_>} \mathrm{maj}(F_u, w') + \sum_{u \in N_<} \mathrm{maj}(F_u, w')$$
$$+ \sum_{u \in P} \mathrm{maj}(F_u, w') + \sum_{u \in N_>} h_u + \sum_{u \in P} h_u. \tag{5.4}$$

By the definition of ϕ, it is not hard to verify

$$N_> = P_>, \quad N_< = P_< \quad \text{and} \quad P = N. \tag{5.5}$$

Therefore, by (5.3), (5.4), (5.5), and the induction hypothesis, we find that

$$\mathrm{maj}_B(F, w) = \sum_{u \in P_>} (\mathrm{maj}(F_u, w'_u) + \mathrm{p}(F_u, w_u)) + \sum_{u \in P_<} (\mathrm{maj}(F_u, w'_u) + \mathrm{p}(F_u, w_u))$$

$$+ \sum_{u \in N} (\mathrm{maj}(F_u, w'_u) + \mathrm{p}(F_u, w_u)) + n - \sum_{u \in P_<} h_u$$

$$= \sum_{u \in N_>} \mathrm{maj}(F_u, w'_u) + \sum_{u \in N_<} \mathrm{maj}(F_u, w'_u) + \sum_{u \in P} \mathrm{maj}(F_u, w'_u)$$

$$+ \mathrm{p}(F, w) - 1 + n - \sum_{u \in N_<} h_u.$$

$$= \mathrm{maj}(F, w') + \mathrm{p}(F, w) - 1 + n - \sum_{u \in N_<} h_u - \sum_{u \in N_>} h_u - \sum_{u \in P} h_u$$

$$= \mathrm{maj}(F, w') + \mathrm{p}(F, w) - 1 + n - \sum_{u \in C(u_0)} h_u,$$

which reduces to (5.2) since

$$\sum_{u \in C(u_0)} h_u = n - 1.$$

Case 2: $w(u_0) < 0$. We can use a similar argument to that for Case 1. So we reach the conclusion that (5.1) holds for any tree. This completes the proof. ∎

6   Concluding remarks

We conclude this paper with two questions. While we have derived the generating functions for the flag major index and the R-major index of signed labeled forests, it

would be interesting to give a suitable definition of the negative major index (1.13) for signed labeled forests. Intuitively, a natural choice would be

$$\text{nmaj}(F, w) = \text{maj}(F, w) + \text{n}_1(F, w) + \text{n}_2(F, w). \tag{6.1}$$

However, the above statistic is not equidistributed with the inversion number (2.2).

For type D_n permutations, Biagioli [5] defined the major index as follows

$$\text{dmaj}(\sigma) = \text{maj}(\sigma) + \text{n}_2(\sigma), \tag{6.2}$$

and shown it is equidistributed with the length function of D_n. However, the following statistic

$$\text{dmaj}(F, w) = \text{maj}(F, w) + \text{n}_2(F, w) \tag{6.3}$$

for labeled forests of type D_n is not equidistributed with the inversion number (2.3) of $D_n(F)$. We would pose the question of finding an appropriate major index for $D_n(F)$ which is equidistributed with the inversion number.

References

[1] R.M. Adin, F. Brenti and Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. 27 (2001), 210-224.

[2] R.M. Adin, F. Brenti and Y. Roichman, Equi-distribution over descent classes of the hyperoctahedral group, J. Combin. Theroy Ser. A 113 (2006), 917-933.

[3] R.M. Adin and Y. Roichman, The flag major index and group actions on polynomial rings, European J. Combin. 22 (2001), 431-446.

[4] R. Biagioli, Equidistribution of negative statistics and quotients of Coxeter groups of type B and D, Adv. Appl. Math. 41 (2008), 378-394.

[5] R. Biagioli, Major and descent statistics for the even-signed permutation group, Adv. Appl. Math. 31 (2003), 163-179.

[6] R. Biagioli and J. Zeng, On some analogues of Carlitz's identity for the hyperoctadedral group, arXiv:math.CO/0909.3961v1.

[7] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, Gran. Texts in Math., vol. 231, Springer-Verlag, Berlin, 2005.

[8] A. Björner and M. Wachs, q-Hook length formulas for forests, J. Combin. Theroy Ser. A 52 (1989), 165-187.

[9] C.O. Chow and I.M. Gessel, On the descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. 38 (2007), 275-301.

[10] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968), 236-240.

[11] D. Foata and M.P. Schützenberger, Major index and inversion number of permutations, Math. Nachr. 83 (1978), 143-159.

[12] A. Garsia, On the "maj" and "inv" q-analogues of Eulerian polynomials, Linear and Multilinear Algebra 8 (1979/80), 21-34.

[13] A. Garsia and I. Gessel, Permutation statistics and partitions, Adv. Math. 31 (1979), 288-305.

[14] I.M. Gessel and S. Seo, A Refinement of Cayley's formula for trees, Electron. J. Combin. 11 (2) (2006), R27.

[15] I.P. Goulden, A bijective proof of Stanley's shuffling theorem, Trans. Amer. Math. Soc. 57 (1985), 147-160.

[16] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[17] P.A. MacMahon, Combinatorial Analysis, Chelsea, New York, 1960.

[18] V. Reiner, Signed permutation statistics, European J. Combin. 14 (1993), 553-567.

[19] B. Sagan, Enumeration of partitions with hook lengths, European J. Combin. 3 (1982), 85-94.

[20] R.P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge University Press, Cambridge, UK, 1999.

[21] R.P. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc. 119 (1972).