Proof of Moll’s Minimum Conjecture

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Abstract. Let $d_i(m)$ denote the coefficients of the Boros-Moll polynomials. Moll’s minimum conjecture states that the sequence \{i(i+1)((d_2^2(m) - d_{i-1}(m)d_{i+1}(m)))\}_{1 \leq i \leq m} attains its minimum at $i = m$ with $2^{-2m}m(m+1)\binom{2m}{m}$. This conjecture is stronger than the log-concavity conjecture proved by Kauers and Paule. We give a proof of Moll’s conjecture by utilizing the spiral property of the sequence \{i!d_i(m)\}_{0 \leq i \leq m}, and the log-concavity of the sequence \{i!d_i(m)\}_{0 \leq i \leq m}.

Keywords: ratio monotonicity, log-concavity, Boros-Moll polynomials.

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1 Introduction

The objective of this note is to give a proof of Moll’s conjecture on the minimum value of a sequence involving the coefficients of the Boros-Moll polynomials which arise in the evaluation of the following quartic integral, see, [1–6, 11]. It has been shown that for any $a > -1$ and any nonnegative integer $m$,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^m+1} dx = \frac{\pi}{2^{m+3/2}(a+1)m+1/2} P_m(a),$$

where

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k.$$  (1.1)
Write $P_m(a)$ as
\[ P_m(a) = \sum_{i=0}^{m} d_i(m)a^i. \] (1.2)

The polynomials $P_m(a)$ are called the Boros-Moll polynomials. By (1.2), $d_i(m)$ can be expressed as
\[ d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m - 2k}{m - k} \binom{m + k}{k} \binom{k}{i}. \] (1.3)

From the above formula (1.3) one sees that the coefficients $d_i(m)$ are positive. Boros and Moll [3,4] have proved that for $m \geq 2$ the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is unimodal and the maximum entry appears in the middle, that is,
\[ d_0(m) < d_1(m) < \cdots < d_{\left[\frac{m}{2}\right]}(m) < d_{\left[\frac{m}{2}\right]+1}(m) > \cdots > d_m(m). \]

Moll [11] conjectured that the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is log-concave for $m \geq 2$. Kauers and Paule [9] have proved this conjecture by using a computer algebra approach. Chen and Xia [8] have shown that the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ satisfies the strongly ratio monotone property which implies the log-concavity and the spiral property. Chen and Gu [7] have proved that the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ satisfies the reverse ultra log-concavity. They have also proved that the sequence $\{\text{i!}d_i(m)\}_{0 \leq i \leq m}$ is log-concave.

In fact, Moll [10,12] proposed a stronger conjecture than the log-concavity conjecture. He formulated his conjecture in terms of the numbers $b_i(m)$ as defined by
\[ b_i(m) = \sum_{k=i}^{m} 2^k \binom{2m - 2k}{m - k} \binom{m + k}{k} \binom{k}{i}. \] (1.4)

Clearly, $b_i(m) = 2^{2m}d_i(m)$ and the log-concavity of $d_i(m)$ is equivalent to that of $b_i(m)$.

**Conjecture 1.1.** Given $m \geq 2$, for $1 \leq i \leq m$,
\[ (m + i)(m + 1 - i)b_{i-1}^2(m) + i(i + 1)b_i^2(m) - i(2m + 1)b_{i-1}(m)b_i(m), \]
attains its minimum at $i = m$ with $2^{2m}m(m + 1)\binom{2m}{m}^2$.

We will give a proof of the above conjecture by using the spiral property of $\{d_i(m)\}_{0 \leq i \leq m}$ and the log-concavity of $\{\text{i!}d_i(m)\}_{0 \leq i \leq m}$. 

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2 Proof of Moll’s Minimum Conjecture

As pointed out by Moll [12], his conjecture implies that \( \{d_i(m)\}_{0 \leq i \leq m} \) is log-concave for \( m \geq 2 \). To see this, we may employ a recurrence relation to reformulate his conjecture by using the three terms \( d_{i-1}(m) \), \( d_i(m) \) and \( d_{i+1}(m) \). Recall that Kauers and Paule [9] and Moll [12] have independently derived the following recurrence relation for \( 1 \leq i \leq m \),

\[
i(i-1)d_i(m) = (i-1)(2m+1)d_{i-1}(m) - (m+2-i)(m+i-1)d_{i-2}(m)
\]

(2.1)

Note that we have adopted the convention that \( d_i(m) = 0 \) for \( i < 0 \) or \( i > m \). From (2.1) and the relation \( d_i(m) = 2^{-2m}b_i(m) \), it follows that

\[
(m+i)(m+1-i)b_{i-1}^2(m) + i(i+1)b_i^2(m) - i(2m+1)b_{i-1}(m)b_i(m)
\]

\[
= i(i+1)(b_i^2(m) - b_{i+1}(m)b_{i-1}(m)).
\]

Thus, Moll’s conjecture can be restated as follows.

**Theorem 2.1.** Given \( m \geq 2 \), for \( 1 \leq i \leq m \), \( i(i+1)(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)) \) attains its minimum at \( i = m \) with \( 2^{-2m}m(m+1)(2^m_m)^2 \).

Chen and Xia [8] have shown that the Boros-Moll polynomials satisfy the ratio monotone property which implies the log-concavity and the spiral property.

**Theorem 2.2.** Let \( m \geq 2 \) be an integer. The sequence \( \{d_i(m)\}_{0 \leq i \leq m} \) is strictly ratio monotone, that is,

\[
d_m(m) < d_{m-1}(m) < \cdots < d_{m-i}(m) < d_{m-i-1}(m) < \cdots < d_{m-[\frac{m}{2}]}(m) < 1,
\]

\[
d_0(m) < d_1(m) < \cdots < d_{i-1}(m) < d_i(m) < d_{m-i}(m) < d_{m-i-1}(m) < \cdots < d_{m-[\frac{m}{2}]}(m) < 1.
\]

As a consequence of Theorem 2.2, the spiral property of \( \{d_i(m)\}_{0 \leq i \leq m} \) can be stated as follows.

**Corollary 2.3.** (Chen and Xia [8]) For \( m \geq 2 \), the sequence \( \{d_i(m)\}_{0 \leq i \leq m} \) is spiral, that is,

\[
d_m(m) < d_0(m) < d_{m-1}(m) < d_1(m) < d_{m-2}(m) < \cdots < d_{[\frac{m}{2}]}(m).
\]

(2.2)
Chen and Gu [7] have shown that \( \{ild_i(m)\}_{0 \leq i \leq m} \) is log-concave. This property can be recast in the following form.

**Theorem 2.4.** For \( m \geq 2 \) and \( 1 \leq i \leq m - 1 \),
\[
 id_i^2(m) > (i + 1)d_{i+1}(m)d_{i-1}(m). 
\] (2.3)

We are now ready to present a proof of Theorem 2.1.

**Proof.** First, it follows from (1.3) that
\[
m(m + 1)d_m^2(m) = 2^{-2m}m(m + 1)\left(\frac{2m}{m}\right)^2. 
\] (2.4)

We now proceed to show that for \( 1 \leq i \leq m - 1 \),
\[
i(i + 1) (d_i^2(m) - d_{i+1}(m)d_{i-1}(m)) > m(m + 1)d_m^2(m). 
\] (2.5)

We first consider the case \( 1 \leq i \leq m - 2 \). By (2.3), we find that
\[
i(i + 1) (d_i^2(m) - d_{i+1}(m)d_{i-1}(m)) > i(i + 1)d_i^2(m) - i^2d_i^2(m) = id_i^2(m). 
\] (2.6)

Using the spiral property (2.2), we see that for \( 1 \leq i \leq m - 2 \),
\[
id_i^2(m) \geq d_i^2(m) > d_{m-1}(m). 
\] (2.7)

Combining (2.6) and (2.7), we get
\[
i(i + 1) (d_i^2(m) - d_{i+1}(m)d_{i-1}(m)) > d_{m-1}^2(m). 
\] (2.8)

On the other hand, by direct computation we may deduce from (1.3) that
\[
d_{m-1}(m) = \frac{2m + 1}{2}d_m(m). 
\] (2.9)

By (2.8) and (2.9), we have for \( 1 \leq i \leq m - 2 \),
\[
i(i + 1) (d_i^2(m) - d_{i+1}(m)d_{i-1}(m)) 
\geq \left(\frac{2m + 1}{2}\right)^2 d_m^2(m) > m(m + 1)d_m^2(m), 
\] (2.10)
and hence (2.5) is true for \( 1 \leq i \leq m - 2 \). It remains to consider the case \( i = m - 1 \).

Again, by (1.3) we find that
\[
d_{m-1}(m) = 2^{-m-1}(2m + 1)\left(\frac{2m}{m}\right); 
\] (2.11)
\[
d_{m-2}(m) = 2^{-m-2}\frac{(m - 1)(4m^2 + 2m + 1)}{2m - 1}\left(\frac{2m}{m}\right). 
\] (2.12)
From (2.4), (2.11) and (2.12), we deduce that
\[ m(m-1)\left( d_{m-1}^2(m) - d_m(m)d_{m-2}(m) \right) \]
\[ = m(m-1)2^{-2m} \left( \frac{2m}{m} \right)^2 \left( \frac{(2m+1)^2}{4} - \frac{(m-1)(4m^2 + 2m + 1)}{4(2m-1)} \right) \]
\[ = \frac{m(4m^2 + 6m - 1)}{4(2m-1)} m(m-1)2^{-2m} \left( \frac{2m}{m} \right)^2 \]
\[ > m(m+1)2^{-2m} \left( \frac{2m}{m} \right)^2 = m(m+1)d_m^2(m). \] (2.13)

Thus (2.5) holds for \( i = m-1 \), and so it holds for \( 1 \leq i \leq m-1 \). This completes the proof.

We conclude with the following ratio monotonicity conjecture. If it is true, it would imply that the sequence \( \{i(i+1)(d_i^2(m) - d_{i+1}(m)d_{i-1}(m))\}_{1 \leq i \leq m} \) is both spiral and log-concave for \( m \geq 2 \).

**Conjecture 2.5.** The sequence \( \{i(i+1)(d_i^2(m) - d_{i+1}(m)d_{i-1}(m))\}_{1 \leq i \leq m} \) is strongly ratio monotone.

For example, for \( m = 8 \), we have
\[ P_8(a) = \frac{4023459}{32768} + \frac{3283533}{4096} a + \frac{9804465}{4096} a^2 + \frac{8625375}{2048} a^3 + \frac{9695565}{2048} a^4 \]
\[ + \frac{1772199}{512} a^5 + \frac{819819}{512} a^6 + \frac{109395}{256} a^7 + \frac{6435}{128} a^8. \]
Let \( c_i = i(i+1)(d_i^2(8) - d_{i+1}(8)d_{i-1}(8)) \) for \( 1 \leq i \leq 8 \). One can verify that
\[ \frac{c_8}{c_1} < \frac{c_7}{c_2} < \frac{c_6}{c_3} < \frac{c_5}{c_4} < 1 \quad \text{and} \quad \frac{c_1}{c_7} < \frac{c_2}{c_6} < \frac{c_3}{c_5} < 1. \]

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**References**


