On Permutations with Bounded Drop Size

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Abstract. The maximum drop size of a permutation π of \([n] = \{1, 2, \ldots, n\}\) is defined to be the maximum value of \(i - \pi(i)\). Chung, Claesson, Dukes and Graham found polynomials \(P_k(x)\) that can be used to determine the number of permutations of \([n]\) with \(d\) descents and maximum drop size at most \(k\). Furthermore, Chung and Graham gave combinatorial interpretations of the coefficients of \(Q_k(x) = x^kP_k(x)\) and \(R_{n,k}(x) = Q_k(x)(1 + x + \cdots + x^k)^{n-k}\), and raised the question of finding a bijective proof of the symmetry property of \(R_{n,k}(x)\). In this paper, we construct a map \(\varphi_k\) on the set of permutations with maximum drop size at most \(k\). We show that \(\varphi_k\) is an involution and it induces a bijection in answer to the question of Chung and Graham. The second result of this paper is a proof of a unimodality conjecture of Hyatt concerning the type \(B\) analogue of the polynomials \(P_k(x)\).

Keywords: descent polynomial, unimodal polynomial, maximum drop size

AMS Subject Classifications: 05A05, 05A15

1 Introduction

This paper is concerned with the study of permutations of \([n] = \{1, 2, \ldots, n\}\) having \(d\) descents and maximum drop size at most \(k\). Let this number be denoted by \(E^k(n, d)\). Chung, Claessson, Dukes and Graham found polynomials \(P_k(x)\) that can be used to determine the number \(E^k(n, d)\). They proved that the polynomials \(P_k(x)\) are unimodal.
Furthermore, Chung and Graham obtained combinatorial interpretations for the polynomials $Q_k(x) = x^k P_k(x)$ and $R_{n,k}(x) = Q_k(x)(1 + x + \cdots + x^n)^{n-k}$, and asked for a combinatorial interpretation of the symmetry property of $R_{n,k}(x)$. The first result of this paper is to present a bijection in answer to the question of Chung and Graham. The second result of this paper is a proof of a conjecture of Hyatt [7] on the unimodality of the type $B$ analogue of the polynomials $P_k(x)$.

Let us give an overview of notation and terminology. Let $S_n$ denote the set of permutations of $[n]$. For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ in $S_n$, we say that a number $1 \leq i \leq n-1$ is a descent of $\pi$ if $\pi_i > \pi_{i+1}$. The descent set of $\pi \in S_n$, denoted by $\text{Des}(\pi)$, is defined by

$$\text{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}.$$ 

Let $\text{des}(\pi)$ denote the number of descents of $\pi \in S_n$. An excedance of $\pi$ is an index $i$ such that $\pi_i > i$ and a drop of $\pi$ is an index $i$ such that $i > \pi_i$. It is well-known that the number of excedances and the number of descents are equidistributed over $S_n$. It is clear that the number of excedances and the number of drops have the same distribution over $S_n$. If $i$ is a drop of a permutation $\pi \in S_n$, then we define the drop size to be $i - \pi_i$. The maximum drop size of $\pi$ is

$$\text{maxdrop}(\pi) = \max\{i - \pi_i : 1 \leq i \leq n\}.$$ 

For example, let $\pi = 43562187$. The set of excedances of $\pi$ is given by $\{1, 2, 3, 4, 7\}$, the set of drops of $\pi$ is given by $\{5, 6, 8\}$, $\text{des}(\pi) = 4$, and $\text{maxdrop}(\pi) = 5$.

Diaconis and Graham [5] studied the permutation statistic “Spearman’s disarray”, which is related to the drop size. This statistic, called “total displacement” by Knuth [8], is defined as

$$\sum_{i=1}^{n} |\pi_i - i| = 2 \sum_{\pi_i > i} (\pi_i - i) = 2 \sum_{i > \pi_i} (i - \pi_i).$$ 

Petersen and Tenner [9] introduced a permutation statistic called the depth in terms of factorizations of the elements into products of reflections. It turns out that the depth of a permutation is half of its total displacement.

Chung, Claesson, Dukes and Graham [3] obtained a polynomial $P_k(x)$ that can be used to determine the number $E_k(n, d)$ of permutations of $[n]$ with $d$ descents and maximum drop size at most $k$. Let $A_{n,k}$ denote the set of permutations of $[n]$ with maximum drop size at most $k$. The $k$-maxdrop-restricted descent polynomial is defined by

$$A_{n,k}(y) = \sum_{\pi \in A_{n,k}} y^{\text{des}(\pi)} = \sum_{d \geq 0} E_k(n, d) y^d.$$ 

Clearly, for $k \geq n$, we have $A_{n,k} = S_n$ and $A_{n,k}(y)$ becomes the Eulerian polynomial

$$A_n(y) = \sum_{\pi \in S_n} y^{\text{des}(\pi)}.$$
Notice that here we have adopted the definition of the Eulerian polynomial as used by Chung et al. [3], which differs from the definition given in Stanley [10] by a factor of $y$.

Chung, Claesson, Dukes and Graham [3] obtained the following recurrence relation for $A_{n,k}(y)$.

**Theorem 1.1** (Chung, Claesson, Dukes and Graham, [3]) For $n,k \geq 0$,

$$A_{n+k+1,k}(y) = \sum_{i=1}^{k+1} \binom{k+1}{i} (y-1)^{i-1} A_{n+k+1-i,k}(y),$$

where $A_{i,k}(y) = A_i(y)$ for $0 \leq i \leq k$.

Using the recurrence relation for $A_{n,k}(y)$ in Theorem 1.1, Chung, Claesson, Dukes and Graham introduced the polynomials

$$P_k(x) = \sum_{l=0}^{k} A_{k-l}(x^{k+1})(x^{k+1} - 1)^l \sum_{i=l}^{k} \binom{i}{l} x^{-i}, \quad (1.1)$$

and derived the following expression for $A_{n,k}(y)$ which can be used to determine the number $E^k(n,d)$.

**Theorem 1.2** (Chung, Claesson, Dukes and Graham,[3]) For $n,k \geq 0$,

$$A_{n,k}(y) = \sum_d \beta_k((k+1)d)y^d, \quad (1.2)$$

where

$$\sum_j \beta_k(j)x^j = P_k(x) \left( \frac{1-x^{k+1}}{1-x} \right)^{n-k}. \quad (1.3)$$

By the definition of $A_{n,k}(y)$, one sees from the above theorem that $E^k(n,d)$ equals the coefficient of $x^{(k+1)d}$ in

$$P_k(x)(1 + x + x^2 + \cdots + x^k)^{n-k}.$$  

We say a sequence $(s_1, s_2, \ldots, s_n)$ is unimodal if there exists an integer $1 \leq t \leq n$ such that $s_1 \leq s_2 \leq \cdots \leq s_t$ and $s_t \geq s_{t+1} \geq \cdots \geq s_n$. A polynomial is said to be unimodal if the sequence of its coefficients is unimodal. Chung, Claesson, Dukes and Graham [3] proved that the polynomial $P_k(x)$ is unimodal for all $k$.

Furthermore, Chung and Graham [3] found combinatorial interpretations of the coefficients of the polynomials $Q_k(x) = x^k P_k(x)$ and $R_{n,k}(x) = Q_k(x)(1+x+\cdots+x^k)^{n-k}$. They used the notation $\left\{\begin{array}{c} n \\ i \end{array}\right\}$ for the number of permutations $\pi \in S_n$ such that $\text{des}(\pi) = i$. 

3
and \( \pi_n = j \) and the notation \( \genfrac{[}{]}{0pt}{}{n}{i}^j_k \) for the number of permutations \( \pi \in A_{n,k} \) such that \( \text{des}(\pi) = i \) and \( \pi_n = j \). In this paper, we write \( E(n, i; j) \) for \( \genfrac{[}{]}{0pt}{}{n}{i}^j \) and \( E^k(n, i; j) \) for \( \genfrac{[}{]}{0pt}{}{n}{i}^j_k \).

**Theorem 1.3** (Chung and Graham, [4]) For \( n \geq 0 \),

\[
Q_n(x) = \sum_{0 \leq i, j \leq n} E(n + 1, i; j + 1) x^{(n+1)i+j}.
\]

**Theorem 1.4** (Chung and Graham, [4]) For \( n \geq k \geq 0 \),

\[
R_{n,k}(x) = \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq k} E^k(n + 1, i; n + 1 - k + j) x^{(k+1)i+j}.
\]

Chung and Graham [4] showed that the polynomials \( Q_n(x) \) and \( R_{n,k}(x) \) are symmetric. They constructed a bijection for the symmetry of \( Q_n(x) \), and they raised the question of finding a bijective proof of the symmetry of \( R_{n,k}(x) \). More precisely, the symmetry property of \( R_{n,k}(x) \) can be described as follows. Assume that

\[
R_{n,k}(x) = \sum_{r=0}^{(n+2)k} c_{n,k,r} x^r.
\]

The symmetry of \( R_{n,k}(x) \) states that for \( 0 \leq r \leq (n + 2)k \) and \( 0 \leq r' \leq (n + 2)k \) such that \( r + r' = (n + 2)k \), we have \( c_{n,k,r} = c_{n,k,r'} \). For example, for \( n = 4 \) and \( k = 2 \), we have

\[
R_{4,2}(x) = x^2 + 3x^3 + 7x^4 + 10x^5 + 12x^6 + 10x^7 + 7x^8 + 3x^9 + x^{10}.
\]

For \( 0 \leq r \leq (n + 2)k \), one can uniquely express \( r \) as \( r = (k + 1)i + j \), where \( 0 \leq i \leq n \) and \( 0 \leq j \leq k \). Thus Theorem 1.4 can be written as

\[
c_{n,k,r} = E^k(n + 1, i; n + 1 - k + j).
\]

Consequently, the symmetry of \( R_{n,k}(x) \) takes the following form.

**Theorem 1.5** (Chung and Graham, [3]) For \( n \geq k \geq 0 \), the polynomials \( R_{n,k}(x) \) are symmetric. In other words, for \( r = (k + 1)i + j \) and \( r' = (k + 1)i' + j' \) such that \( r + r' = (n + 2)k \), where \( 0 \leq i, i' \leq n \), \( 0 \leq j, j' \leq k \), we have

\[
E^k(n + 1, i; n + 1 - k + j) = E^k(n + 1, i'; n + 1 - k + j').
\]
\[ \pi \in A_{5,2} \text{ with } \text{des}(\pi) = 1 \text{ and } \pi_5 = 4 \]

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\[ \pi \in A_{5,2} \text{ with } \text{des}(\pi) = 2 \text{ and } \pi_5 = 5 \]

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Table 1.1: Permutations enumerated by \( E^2(5, 1; 4) \) and \( E^2(5, 2; 5) \).

As an example, let \( n = 4, k = 2, r = 4 \) and \( r' = 8 \). Writing \( r = 3 \cdot 1 + 1 \) and \( r' = 3 \cdot 2 + 2 \), by Theorem 1.4, we find that \( c_{4,2,4} = E^2(5, 1; 4) = 7 \) and \( c_{4,2,8} = E^2(5, 2; 5) = 7 \). Permutations enumerated by \( E^2(5, 1; 4) \) and \( E^2(5, 2; 5) \) are given in Table 1.1.

In Section 2, we construct a map \( \varphi_k \) on \( \Gamma_k \) by a recursive procedure, where \( \Gamma_k \) is the set of permutations with maximum drop size at most \( k \). Then, we prove that \( \varphi_k \) induces a bijection for Theorem 1.5.

In Section 3, we consider the unimodality of the type \( B \) analogue of the polynomials \( P_k(x) \). As pointed out by Chung et al. [3], the maxdrop statistic is related to the bubble sorting algorithm. Let \( B_n \) denote the type \( B \) Coxeter group of rank \( n \), that is, the group of signed permutations on \([n]\). Hyatt [7] found a natural way to extend the bubble sorting algorithm to signed permutations. Moreover, he introduced the notion of the maximum drop size of a signed permutation.

Recall that a signed permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \) can be viewed as a permutation of \([n]\) for which each element may be associated with a minus sign. We shall use the bar notation \( \bar{i} \) to signify an element \( i \) with a minus sign. The descent set of a signed permutation \( \pi \) is defined to be

\[ \text{Des}_B(\pi) = \{ i \in [0, n - 1] : \pi_i > \pi_{i+1} \} \]

where we assume that \( \pi_0 = 0 \), see Brenti [1]. Let \( \pi \) be a signed permutation in \( B_n \). The number of descents of \( \pi \) is denoted by \( \text{des}_B(\pi) \). Hyatt [7] defined the maximum drop size of \( \pi \) as given by

\[ \text{maxdrop}_B(\pi) = \max \left\{ \max \{ i - \pi_i : \pi_i > 0 \}, \max \{ i : \pi_i < 0 \} \right\} \]

For example, let \( \pi = 43562\bar{7}87 \). Then we have \( \text{des}_B(\pi) = 5 \) and \( \text{maxdrop}_B(\pi) = 6 \).

Let \( B_{n,k} \) denote the set of signed permutations of \([n]\) with maximum drop size at most \( k \), and let \( E^k_B(n, d) \) denote the number of signed permutations in \( B_{n,k} \) with \( d \) descents.

The type \( B \) \( k \)-maxdrop-restricted descent polynomial is defined by

\[ B_{n,k}(y) = \sum_{\pi \in B_{n,k}} y^{\text{des}_B(\pi)} = \sum_{d \geq 0} E^k_B(n, d) y^d. \]
When $k \geq n$, $B_{n,k} = B_n$ and $B_{n,k}(y)$ becomes the type $B$ Eulerian polynomial $B_n(y)$, which is defined by

$$B_n(y) = \sum_{\pi \in B_n} y^{des(\pi)}.$$

Hyatt [7] showed that $B_{n,k}(y)$ satisfied the following recurrence relation.

**Theorem 1.6** (Hyatt, [7]) For $n, k \geq 0$,

$$B_{n+k+1,k}(y) = \sum_{i=1}^{k+1} \binom{k+1}{i} (y-1)^{i-1} B_{n+k+1-i,k}(y),$$

where $B_{i,k}(y) = B_i(y)$ for $0 \leq i \leq k$.

Using the above recurrence relation for $B_{n,k}(y)$, Hyatt obtained the following type $B$ analogue of the polynomials $P_k(x)$,

$$T_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+1})(x^{k+1} - 1)^l \sum_{i=l}^{k} \binom{i}{l} x^{-i}, \quad (1.4)$$

which determines the number $E_B^k(n, d)$.

**Theorem 1.7** (Hyatt, [7]) For $n, k \geq 0$,

$$B_{n,k}(y) = \sum_d \gamma_k((k+1)d)y^d, \quad (1.5)$$

where

$$\sum_j \gamma_k(j)x^j = T_k(x) \left( \frac{1 - x^{k+1}}{1 - x} \right)^{n-k}. \quad (1.6)$$

The above theorem implies that $E_B^k(n, d)$ equals the coefficient of $x^{(k+1)d}$ in $T_k(x)(1 + x + x^2 + \cdots + x^{k})^{n-k}$.

The following conjecture was posed by Hyatt [7].

**Conjecture 1.8** (Hyatt, [7]) The polynomial $T_k(x)$ is unimodal for $k \geq 0$.

The second result of this paper is a proof of the above conjecture, which will be given in Section 3.
2 Combinatorial proof of the symmetry of \( R_{n,k}(x) \)

In this section, we give a combinatorial proof of Theorem 1.5. For \( k \geq 0 \), let \( \Gamma^k \) be the set of permutations with maximum drop size at most \( k \). We construct a map \( \varphi_k \) on \( \Gamma^k \) by a recursive procedure. We shall prove that \( \varphi_k \) is an involution on \( \Gamma^k \) and it induces a bijection for Theorem 1.5.

To describe the map \( \varphi_k \), we begin with some notation. Given \( \pi \in S_n \) and \( 1 \leq i \leq n+1 \), let \( \pi \leftarrow i \) denote the permutation \( \mu = \mu_1 \mu_2 \cdots \mu_{n+1} \) in \( S_{n+1} \) that is obtained from \( \pi \) by adding \( i \) at the end of \( \pi \) and increasing the elements \( i, i+1, \ldots, n \) by 1. For example, \( 3421 \leftarrow 3 = 45213 \).

For \( n \geq 1 \), let \( \pi = \pi_1 \pi_2 \cdots \pi_n \) be a permutation in \( \Gamma^k \). The permutation \( \varphi_k(\pi) \) is recursively constructed as follows. If \( n = 1 \), define \( \varphi_k(1) = 1 \). We now assume that \( n \geq 2 \). Let \( i = \text{des}(\pi) \) and \( j = \pi_n - n + k \). Assume that \( \pi' \) is the permutation of \([n-1]\) that is order isomorphic to \( \pi_1 \pi_2 \cdots \pi_{n-1} \). In other words, write \( \pi = \pi' \leftarrow \pi_n \). In order to recursively construct \( \varphi_k(\pi) \), it is necessary to verify that \( \text{maxdrop}(\pi') \leq k \), that is, \( t - \pi_t' \leq k \) for \( 1 \leq t \leq n-1 \). We consider two cases. If \( \pi_t' = \pi_t \), then \( t - \pi_t' = t - \pi_t \leq k \). If \( \pi_t' = \pi_t - 1 \), by the definition of \( \pi' \), we get \( \pi_t > \pi_n \). Thus \( t - \pi_t' = t + 1 - \pi_t \leq n - \pi_n \leq k \). So \( \pi' \) is a permutation of length \( n - 1 \) in \( \Gamma^k \). This enables us to define

\[
\varphi_k(\pi) = \varphi_k(\pi') \leftarrow (n - k + j'),
\]

where \( j' \) is uniquely determined by \( n, k, i \) and \( j \), as given below

\[
i' = \left\lfloor \frac{(n+1)k - (k+1)i - j}{k+1} \right\rfloor, \tag{2.1}
\]

\[
j' = (n+1)k - (k+1)i - j - (k+1)i'. \tag{2.2}
\]

For example, let \( \pi = 12354 \). It can be checked that \( \pi \in \Gamma^1 \). So we also have \( \pi \in \Gamma^2 \). To demonstrate that the map \( \varphi_k \) is indeed dependent on \( k \), let us compute \( \varphi_2(\pi) \) and \( \varphi_1(\pi) \). To compute \( \varphi_2(\pi) \), we have \( i = \text{des}(\pi) = 1 \) and \( j = \pi_5 - 5 + 2 = 1 \). By relations (2.1) and (2.2), we get \( i' = 2 \) and \( j' = 2 \). Write \( \pi = \pi' \leftarrow \pi_5 = 1234 \leftarrow 4 \).

By the definition of the map \( \varphi_2 \), we get \( \varphi_2(\pi) = \varphi_2(\pi') \leftarrow 5 \). We now turn to \( \varphi_2(\pi') \). Repeating the above process, we obtain that \( \pi'' = 123 \), \( \pi''' = 12 \) and \( \pi''' = 1 \). It follows that \( \varphi_2(\pi''') = 1, \varphi_2(\pi'''') = 1, \varphi_2(\pi''''') = 321, \varphi_2(\pi''''') = 321 \). So we find that \( \varphi_2(\pi) = 32145 \). Similarly, we obtain that \( \varphi_1(\pi) = 21534 \). It can be seen that \( \varphi_2(\pi) \neq \varphi_1(\pi) \).

The following theorem states that for \( k \geq 0 \), \( \varphi_k \) is an involution, that is, for any \( \pi \in \Gamma^k \), we have \( \varphi_k^2(\pi) = \pi \).

**Theorem 2.1** For \( k \geq 0 \), the map \( \varphi_k \) is an involution on \( \Gamma^k \).

To prove the above theorem, we need the following property of the map \( \varphi_k \). Let \( \Gamma^k(n, i; j) \) denote the set of permutations on \([n]\) enumerated by \( E^k(n, i; n-k+j) \), that
is, the set of permutations on \([n]\) with maximum drop size at most \(k\) such that the descent number equals \(i\) and the last element equals \(n - k + j\).

**Theorem 2.2** For \(n \geq 1\), \(n \geq k \geq 0\), \(0 \leq i \leq n - 1\), \(0 \leq j \leq k\) and a permutation \(\pi\) in \(\Gamma^k(n, i; j)\), we have \(\varphi_k(\pi) \in \Gamma^k(n, i'; j')\), where \(i'\) and \(j'\) are given by relations (2.1) and (2.2).

**Proof.** We proceed by induction on \(n\). For \(n = 1\), we have \(1 \in \Gamma^k(1, 0; k)\). By (2.1) and (2.2), we deduce that \(i' = 0\) and \(j' = k\). Clearly, \(\varphi_k(1) \in \Gamma^k(1, 0; k)\) for any \(k \geq 0\). This proves the case for \(n = 1\). Assume that the theorem holds for \(n - 1\), where \(n \geq 2\). We aim to show that it is valid for \(n\).

Write \(\pi = \pi_1 \pi_2 \cdots \pi_n\) and assume that \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}\) is the permutation of \([n-1]\) that is order isomorphic to \(\pi_1 \pi_2 \cdots \pi_{n-1}\), that is, \(\pi = \sigma \leftarrow \pi_n\). Denote \(\varphi_k(\pi)\) by \(\beta = \beta_1 \beta_2 \cdots \beta_n\). By the recursive construction of \(\varphi_k\), we have

\[
\beta = \varphi_k(\sigma) \leftarrow (n - k + j') ,
\]

where \(j'\) is given by (2.1) and (2.2).

To show that \(\beta \in \Gamma^k(n, i'; j')\), denote \(\varphi_k(\sigma)\) by \(\alpha = \alpha_1 \alpha_2 \cdots \alpha_{n-1}\). Let

\[
\begin{align*}
    s &= \text{des}(\sigma), \\
    t &= \sigma_{n-1} - n + 1 + k, \\
    s' &= \left\lfloor \frac{nk - s(k + 1) - t}{k + 1} \right\rfloor, \\
    t' &= nk - s(k + 1) - t - s'(k + 1).
\end{align*}
\]

In the above notation, we have \(\sigma \in \Gamma^k(n - 1, s'; t')\). By the induction hypothesis, \(\alpha \in \Gamma^k(n - 1, s'; t')\). This implies that \(\text{maxdrop}(\alpha) \leq k\). It can be seen from (2.3) that \(\beta_n = n - k + j'\) and \(\beta_i \geq \alpha_i\) for \(1 \leq i \leq n - 1\), so that \(\text{maxdrop}(\beta) \leq \text{max\{maxdrop}(\alpha), k - j'\}\). It follows that \(\text{maxdrop}(\beta) \leq k\).

It remains to verify that \(\text{des}(\beta) = i'\). In view of (2.3), it suffices to check that \(i' = s' + 1\) when \(\alpha_{n-1} \geq \beta_n\) and \(i' = s'\) when \(\alpha_{n-1} < \beta_n\). Since \(\beta_n = n - k + j'\) and \(\alpha_{n-1} = n - 1 - k + t'\), we need to show that \(i' = s' + 1\) when \(j' - t' \leq -1\) and \(i' = s'\) when \(j' - t' > -1\). To this end, we need the following four relations (2.8)-(2.11).

By the definition \(t\), we have \(0 \leq t \leq k\). Since \(0 \leq j \leq k\), we find that 

\[
- k \leq j - t \leq k.
\]

Similarly,

\[
- k \leq j' - t' \leq k.
\]
By (2.2) and (2.7), we see that
\[
\begin{align*}
i(k + 1) + j + i'(k + 1) + j' &= (n + 1)k, \quad (2.10) \\
s(k + 1) + t + s'(k + 1) + t' &= nk. \quad (2.11)
\end{align*}
\]

Since \(i = \text{des}(\pi), s = \text{des}(\sigma)\) and \(\pi = \sigma \leftarrow \pi_n\), we have \(i = s\) or \(i = s + 1\). So there are two cases.

Case 1: \(i = s\), so \(\pi_{n-1} < \pi_n\), and so \(j - t > -1\). By (2.10) and (2.11),
\[
(i' - s')(k + 1) = k - (j - t) - (j' - t').
\]

If \(j' - t' \leq -1\), by (2.9), we see that \(k \geq 1\). By (2.8) and the assumption \(j - t > -1\), we deduce that \(-1 < j - t \leq k\). By (2.9) and the assumption \(j' - t' \leq -1\), we find that \(-k \leq j' - t' \leq -1\). It follows that \((i' - s')(k + 1) \in [1, 2k]\), where \(k \geq 1\). Hence we arrive at the assertion that \(i' = s' + 1\).

If \(j' - t' > -1\), by (2.9), we find that \(-1 < j' - t' \leq k\). By (2.8) and the assumption \(j - t > -1\), we get \(-1 < j - t \leq k\). Thus, \((i' - s')(k + 1) \in [-k, k]\). So we deduce that \(i' = s'\).

Case 2: \(i = s + 1\), so \(\pi_{n-1} > \pi_n\), and so \(j - t \leq -1\). By (2.8) and the assumption \(j - t \leq -1\), we deduce that \(k \geq 1\). It follows from (2.10) and (2.11) that
\[
(i' - s')(k + 1) = -1 - (j - t) - (j' - t').
\]

If \(j' - t' \leq -1\), we claim that \(k \geq 2\). Assume to the contrary that \(k = 1\). By (2.8) and (2.9), we obtain that \(j' - t' = -1\) and \(j - t = -1\). By (2.12), we deduce that \(2(i' - s') = 1\), a contradiction. This proves that \(k \geq 2\). Using (2.8) and the assumption \(j - t \leq -1\), we find that \(-k \leq j - t \leq -1\). Similarly, we have \(-k \leq j' - t' \leq -1\). It follows that \((i' - s')(k + 1) \in [1, 2k - 1]\), where \(k \geq 2\). So we reach the conclusion that \(i' = s'\).

If \(j' - t' > -1\), by (2.9), we deduce that \(-1 < j' - t' \leq k\). By (2.8) and the assumption \(j - t \leq -1\), we find that \(-k \leq j - t \leq -1\). It follows that \((i' - s')(k + 1) \in [-k, k - 1]\), where \(k \geq 1\). This implies that \(i' = s'\).

Up to now, we have shown that \(i' = s' + 1\) when \(j' - t' \leq -1\) and \(i' = s'\) when \(j' - t' > -1\). This yields that \(\text{des}(\beta) = i'\), and hence the proof is complete.

We are now ready to finish the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let \(\pi = \pi_1 \pi_2 \cdots \pi_n\) be a permutation in \(\Gamma^k\), we aim to show that \(\varphi_k^2(\pi) = \pi\). We proceed by induction on \(n\). When \(n = 1\), it is obvious that \(\varphi_k^2(1) = 1\). So the theorem is valid for \(n = 1\). Assume that the theorem holds for \(n - 1\), where \(n \geq 2\), that is, for any permutation \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}\), we have \(\varphi_k^2(\sigma) = \sigma\). Denote \(\varphi_k^2(\pi)\) by \(\gamma = \gamma_1 \gamma_2 \cdots \gamma_n\).
To prove that $\gamma = \pi$, write $\pi = \sigma \prec \pi_n$, where $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$. Let $i = \text{des}(\pi)$ and $j = \pi_n - n + k$; that is, $\pi$ is a permutation in $\Gamma^k(n, i, j)$. By Theorem 2.2, we know that $\varphi_k(\pi) = \varphi_k(\sigma \prec (n - k + j)) \in \Gamma^k(n, i'; j')$, where $i'$ and $j'$ are given by (2.1) and (2.2). By the construction of $\varphi_k$, we have

$$\varphi_k(\pi) = \varphi_k(\sigma \prec (n - k + j)) = \varphi_k(\sigma) \prec (n - k + j').$$  \hspace{1cm} (2.13)

Let $i''$ and $j''$ be the integers obtained from $i'$ and $j'$ by using (2.1) and (2.2). A direct computation indicates that $i'' = i$ and $j'' = j$. Applying (2.13) twice yields that

$$\gamma = \varphi_k^2(\pi) = \varphi_k^2(\sigma) \prec (n - k + j).$$

But the induction hypothesis says that $\varphi_k^2(\sigma) = \sigma$, so we get

$$\gamma = \sigma \prec (n - k + j) = \pi.$$

This completes the proof. \hfill \blacksquare

To conclude this section, we notice that when restricted to $\Gamma^k(n, i; j)$ the map $\varphi_k$ serves as a combinatorial interpretation of Theorem 1.5 with $n + 1$ replaced by $n$. For $n \geq 1$, $n \geq k \geq 0$, $r = (k + 1)i + j$ and $r' = (k + 1)i' + j'$ such that $r + r' = (n + 1)k$, $0 \leq i, i' \leq n - 1$ and $0 \leq j, j' \leq k$, it is easy to see that the integers $i'$ and $j'$ are uniquely determined by $n, k, i, j$, as given by relations (2.1) and (2.2). Combining Theorems 2.1 and 2.2, we are led to the following bijection.

**Theorem 2.3** For $n \geq 1$, $n \geq k \geq 0$, $r = (k + 1)i + j$ and $r' = (k + 1)i' + j'$ such that $r + r' = (n + 1)k$, $0 \leq i, i' \leq n - 1$ and $0 \leq j, j' \leq k$, $\varphi_k$ induces a bijection from $\Gamma^k(n, i; j)$ to $\Gamma^k(n, i'; j')$.

### 3 The unimodality of $T_k(x)$

In this section, we prove a conjecture of Hyatt [7] on the unimodality of a type $B$ analogue of the polynomials $P_k(x)$. Let $B_n$ be the set of signed permutations on $[n]$. For $\pi \in B_n$, Hyatt defined the maximum drop size of $\pi$ as follows. We say $\pi$ has a drop at position $i$ if $i > \pi(i)$. If $\pi$ has a drop at position $i$, the drop size at this position is defined to be $\min\{i - \pi(i), i\}$. The type $B$ maximum drop size of $\pi$, denoted $\text{maxdrop}_B(\pi)$, is the maximum value of all drop sizes of $\pi$; that is,

$$\text{maxdrop}_B(\pi) = \max \left\{ \max\{i - \pi_i : \pi_i > 0\}, \max\{i : \pi_i < 0\} \right\}.$$

Based on the type $B$ descent number and the maximum drop size of a signed permutation, for $k \geq 0$, Hyatt introduced a type $B$ analogue of the polynomial $P_k(x)$, denoted $T_k(x)$. 

10
Recall that the type $B$ Eulerian polynomials are associated with the type $B$ descent number of a signed permutation, which are given by

$$B_n(y) = \sum_{\pi \in B_n} y^{\text{des}_B(\pi)}.$$  

The polynomials $T_k(x)$ are defined by

$$T_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+1})(x^{k+1} - 1)^l \sum_{i=l}^{k} \binom{i}{l} x^{-i}.$$  

Let $E^B_k(n, d)$ be the number of signed permutations on $[n]$ with $d$ type $B$ descents and type $B$ maximum drop size at most $k$. For $k \geq 0$, Hyatt showed that $E^B_k(n, d)$ equals the coefficient of $x^{(k+1)d}$ in $T_k(x)(1 + x + x^2 + \cdots + x^k)^{n-k}$, and he conjectured that $T_k(x)$ is unimodal.

To prove this conjecture, we define the polynomials $H_k(x)$ as given by

$$H_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{2k+2})(x^{2k+2} - 1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2k+1-s}$$

$$+ \sum_{l=0}^{k} B_{k-l}(x^{-2k-2})(x^{-2k-2} - 1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2(k+1)^2+s}.$$  

As will be shown that the sequence of coefficients of $T_k(x)$ is a subsequence of those of $H_k(x)$. Thus the unimodality of $T_k(x)$ follows from the unimodality of $H_k(x)$.

Let $\tilde{T}_k(x) = x^kT_k(x)$, that is,

$$\tilde{T}_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+1})(x^{k+1} - 1)^l \sum_{i=l}^{k} \binom{i}{l} x^{k-i}.$$  

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\tilde{T}_k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$x + 2x^2 + x^3$</td>
</tr>
<tr>
<td>2</td>
<td>$x^2 + 4x^3 + 6x^4 + 6x^5 + 4x^6 + 2x^7 + x^8$</td>
</tr>
</tbody>
</table>
| 3   | $x^3 + 8x^4 + 12x^5 + 18x^6 + 23x^7 + 32x^8 + 32x^9 + 28x^{10} + 23x^{11}$  
\hphantom{|} $+ 8x^{12} + 4x^{13} + 2x^{14} + x^{15}$ |

Table 3.2: The polynomials $\tilde{T}_k(x)$ for $0 \leq k \leq 3$.

For $0 \leq k \leq 3$, the polynomials $\tilde{T}_k(x)$ are given in Table 3.2. Analogous to the array representation of $Q_k(x)$ given by Chung and Graham [4], we define an array representation of $\tilde{T}_k(x)$. For $0 \leq i \leq k+1$ and $0 \leq j \leq k$, the $(i, j)$-entry $t_k(i, j)$ is set to be the
coefficient of $x^{(k+1)i+j}$ of $\tilde{T}_k(x)$, that is,
\begin{equation}
\tilde{T}_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{k} t_k(i, j)x^{(k+1)i+j}.
\end{equation}

Similarly, we can arrange the coefficients of $H_k(x)$ in a $(k + 2) \times 2(k + 1)$ array $h_k$ so that
\begin{align*}
H_k(x) &= \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h_k(i, j)x^{2(k+1)i+j}.
\end{align*}

In fact, for any $k \geq 0$, $h_k$ can be obtained from $t_k$ as described in the following lemma.

**Lemma 3.1** For $k \geq 0$, $h_k$ can be obtained by rotating $t_k$ 180 degrees (in either direction), and adjoining the rotated array to the left side of $t_k$.

For example, the array $h_2$ can be obtained from the array $t_2$ by the following operations. First, rotate the array $t_2$ 180 degrees. Then adjoin this rotated array to the left side of $t_2$. Table 3.3 gives the array $t_2$ and Table 3.4 illustrates the corresponding array $h_2$.

| 0 0 1 | 0 0 0 0 0 1 |
| 4 6 6 | 1 2 4 4 6 6 |
| 4 2 1 | 6 6 4 4 2 1 |
| 0 0 0 | 1 0 0 0 0 0 |

Table 3.3: The array $t_2$  
Table 3.4: The array $h_2$

To prove Lemma 3.1, we need the following property.

**Lemma 3.2** For $k \geq 0$, define
\begin{equation}
F_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+2})(x^{k+2} - 1)^l \sum_{i=l}^{k} \binom{i}{l} x^{k+1-i}.
\end{equation}

Arrange the coefficients of $F_k(x)$ in a $(k + 2) \times (k + 2)$ array $f_k$ so that
\begin{align*}
F_k(x) &= \sum_{i=0}^{k+1} \sum_{j=0}^{k+1} f_k(i, j)x^{(k+2)i+j}.
\end{align*}

Then the array $f_k$ can be obtained from $t_k$ by adjoining a column of zeros to the left of $t_k$. 

12
Proof. To prove that \( f_k \) can be obtained from \( t_k \) by inserting a column of zeros in front of \( t_k \), we proceed to verify that \( f_k(i, 0) = 0 \) for \( 0 \leq i \leq k + 1 \) and \( f_k(i, j + 1) = t_k(i, j) \) for \( 0 \leq i \leq k + 1 \) and \( 0 \leq j \leq k \).

For convenience, for \( 0 \leq l \leq k \), let
\[
U_l(t) = B_{k-l}(t)(t-1)^l,
\]
\[
V_l(t) = \sum_{i=l}^{k} \binom{i}{l} t^{k-i}.
\]

Notice that \( U_l(t) \) is a polynomial in \( t \) of degree \( k \) and \( V_l(t) \) is a polynomial in \( t \) of degree at most \( k \).

From the expression (3.4) of \( F_k(x) \), we see that
\[
F_k(x) = \sum_{l=0}^{k} xU_l(x^{k+2})V_l(x).
\]

Since \( U_l(x^{k+2}) \) can be seen as a polynomial in \( x^{k+2} \) and the degree of \( V_l(x) \) is at most \( k \), we deduce that the coefficient of \( x^{(k+2)i} \) in \( F_k(x) \) equals zero for \( 0 \leq i \leq k+1 \). Hence \( f_k(i, 0) = 0 \) for \( 0 \leq i \leq k + 1 \).

Next we prove that \( t_k(i, j) = f_k(i, j + 1) \) for \( 0 \leq i \leq k + 1 \) and \( 0 \leq j \leq k \). We shall adopt the common notation \( [x^l] p(x) \) for the coefficient of \( x^l \) in a polynomial \( p(x) \). It suffices to show that
\[
[x^{(k+1)i+j}] \widetilde{T}_k(x) = [x^{(k+2)i+j+1}] F_k(x). \tag{3.5}
\]

From the expression (3.2) of \( \widetilde{T}_k(x) \), it follows that
\[
\widetilde{T}_k(x) = \sum_{l=0}^{k} U_l(x^{k+1})V_l(x).
\]

Recalling that \( V_l(x) \) is a polynomial in \( x \) of degree at most \( k \), for \( 0 \leq i \leq k + 1 \) and \( 0 \leq j \leq k \), it is easily checked that
\[
[x^{(k+1)i+j}] \widetilde{T}_k(x) = \sum_{l=0}^{k} \left( [x^{(k+1)i}] U_l(x^{k+1}) \right) \left( [x^j] V_l(x) \right)
\]
\[= \sum_{l=0}^{k} \left( [t^l] U_l(t) \right) \left( [x^j] V_l(x) \right). \tag{3.6}
\]

Similarly, we have
\[
[x^{(k+2)i+j+1}] F_k(x) = \sum_{l=0}^{k} \left( [x^{(k+2)i}] U_l(x^{k+2}) \right) \left( [x^{j+1}] V_l(x) \right).
\]
\[\sum_{l=0}^{k} \left( [x^{(k+2)i}] U_l(x^{k+2}) \right) \left( [x^j] V_l(x) \right)\]

\[= \sum_{l=0}^{k} \left( [t^l] U_l(t) \right) \left( [x^j] V_l(x) \right). \quad (3.7)\]

Hence (3.5) follows from (3.6) and (3.7). So we arrive at the conclusion that \(f_k(i, j+1) = t_k(i, j)\) for \(0 \leq i \leq k + 1\) and \(0 \leq j \leq k\). This completes the proof.

We are now ready to give a proof of Lemma 3.1.

**Proof of Lemma 3.1.** Write \(H_k(x)\) as

\[H_k(x) = H'_k(x) + H''_k(x),\]

where

\[H'_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{2k+2})(x^{2k+2} - 1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2k+1-s}, \quad (3.8)\]

\[H''_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{-2k-2})(x^{-2k-2} - 1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2(k+1)^2+s}. \quad (3.9)\]

Assume \(H'_k(x)\) has an array representation \(h'_k\) such that

\[H'_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h'_k(i, j) x^{2(k+1)i+j},\]

and \(H''_k(x)\) has an array representation \(h''_k\) such that

\[H''_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h''_k(i, j) x^{2(k+1)i+j}.\]

Clearly, we have \(h_k = h'_k + h''_k\). Using Lemma 3.2 repeatedly, we deduce that \(h'_k\) can be obtained form \(t_k\) by adjoining \(k + 1\) columns of zeros to the left side of \(t_k\). Table 3.5 gives an example of \(h'_k\) for \(k = 2\).

From the expression (3.8) of \(H'_k(x)\) and the expression (3.9) of \(H''_k(x)\), we see that

\[H'_k(x) = H'(x^{-1}) x^{2(k+1)(k+2)-1}.\]

Hence, in the array representation, we deduce that \(h''_k\) can be obtained from \(h'_k\) by rotating \(h'_k\) 180 degrees. For example, the array \(h''_2\) in Table 3.6 is constructed from the array \(h'_2\) in Table 3.5.
By the fact that $h_k = h'_k + h''_k$ and the constructions of $h'_k$ and $h''_k$, we see that the first $k + 1$ columns of $h_k$ can be obtained from $t_k$ by a rotation of 180 degrees and $t_k$ remains to be the last $k + 1$ columns of $h_k$. This completes the proof.

As a consequence of Lemma 3.1, we have the following property.

**Corollary 3.3** For $k \geq 0$, the polynomial $H_k(x)$ is symmetric.

In the array representation, the symmetry of $H_k(x)$ means that for $0 \leq i \leq k + 1$ and $0 \leq j \leq 2k + 1$,

$$h_k(i, j) = h_k(k + 1 - i, 2k + 1 - j). \quad (3.10)$$

It is clear from Lemma 3.1 that the coefficients of $T_k(x)$ form a subsequence of those of $H_k(x)$. We shall prove that for $k \geq 0$, $H_k(x)$ is unimodal.

**Theorem 3.4** The polynomial $H_k(x)$ is unimodal for all $k \geq 0$.

To prove Theorem 3.4 we introduce the polynomials $G_k(x)$ which will be used to derive a recurrence relation of the coefficients of $H_k(x)$.

Based on the definition (3.1) of $H_k(x)$, we define

$$G_k(x) = \frac{1}{x} \sum_{l=0}^{k} B_{k-l}(x^{2k+4})(x^{2k+4} - 1)^l \sum_{s=l}^{k} \left( \frac{s}{l} \right) x^{2k+3-s}$$

$$+ \sum_{l=0}^{k} B_{k-l}(x^{-2k-4})(x^{-2k-4} - 1)^l \sum_{s=l}^{k} \left( \frac{s}{l} \right) x^{2(k+1)(k+2)+s} \quad (3.11)$$

Let $g_k$ be an array representation of $G_k(x)$ such that

$$G_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+3} g_k(i, j)x^{2(k+2)i+j}.$$

Let $g_k$ be an array representation of $G_k(x)$ such that

$$G_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+3} g_k(i, j)x^{2(k+2)i+j}.$$

We claim that the array $g_k$ can be obtained from $h_k$ by adding a column of zeros after the $(k+1)$-st column and adding a column of zeros after the $2(k+1)$-st column of $h_k$. The verification of this fact is similar to that of Lemma 3.1 hence the details are omitted. Table 3.7 gives the array $g_2$. 

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 6 & 6 \\
0 & 0 & 0 & 4 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

Table 3.5: The array $h'_2$ 

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 4 & 0 & 0 & 0 \\
6 & 6 & 4 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

Table 3.6: The array $h''_2$ 

15
Lemma 3.5 For $k \geq 0$, we have

$$H_{k+1}(x) = G_k(x) \cdot (x + x^2 + \cdots + x^{2k+4})$$ \hspace{1cm} (3.12)

Proof. We aim to show that

$$(1-x) \cdot H_{k+1}(x) = x G_k(x) \cdot (1-x^{2k+4}),$$ \hspace{1cm} (3.13)

which is equivalent to (3.12). By the definition of $H_k(x)$ in (3.1), we see that $(1-x) \cdot H_{k+1}(x)$ equals

$$(1-x) \sum_{l=0}^{k+1} B_{k+1-l}(x^{2k+4})(x^{2k+4} - 1)^l \sum_{s=l}^{k+1} \binom{s}{l} x^{2k+3-s}$$

$$+(1-x) \sum_{l=0}^{k+1} B_{k+1-l}(x^{-2k-4})(x^{-2k-4} - 1)^l \sum_{s=l}^{k+1} \binom{s}{l} x^{2(k+2)^2+s}$$

$$=(1-x) \sum_{l=1}^{k+1} B_{k+1-l}(x^{2k+4})(x^{2k+4} - 1)^l \sum_{s=l}^{k+1} \binom{s}{l} x^{2k+3-s}$$

$$+(1-x) \sum_{l=1}^{k+1} B_{k+1-l}(x^{-2k-4})(x^{-2k-4} - 1)^l \sum_{s=l}^{k+1} \binom{s}{l} x^{2(k+2)^2+s}$$

$$+(1-x) B_{k+1}(x^{2k+4}) \sum_{s=0}^{k+1} x^{2k+3-s} + (1-x) B_{k+1}(x^{-2k-4}) \sum_{s=0}^{k+1} x^{2(k+2)^2+s}$$

$$=- \sum_{l=0}^{k} B_{k-l}(x^{2k+4})(x^{2k+4} - 1)^{l+1} \sum_{s=l}^{k} \binom{s}{l} x^{2k+3-s}$$

$$+ \sum_{l=0}^{k} B_{k-l}(x^{-2k-4})(x^{-2k-4} - 1)^{l+1} \sum_{s=l}^{k} \binom{s}{l} x^{2(k+2)^2+s+1}$$

$$+ \sum_{l=0}^{k} B_{k-l}(x^{2k+4})(x^{2k+4} - 1)^{l+1} \binom{k+1}{l+1} x^{k+2}$$

$$- \sum_{l=0}^{k} B_{k-l}(x^{-2k-4})(x^{-2k-4} - 1)^{l+1} \binom{k+1}{l+1} x^{(k+2)(2k+5)}$$
\[ B_{k+1}(x^{2k+4})x^{k+2}(1 - x^{k+2}) + B_{k+1}(x^{-2k-4})x^{2(k+2)^2}(1 - x^{k+2}). \]  

On the other hand, by the definition of \( G_k(x) \) in (3.11), we find that

\[
xG_k(x) \cdot (1 - x^{2k+4}) = - \sum_{l=0}^{k} B_{k-l}(x^{2k+4})(x^{2k+4} - 1)^{l+1} \sum_{s=l}^{k} \binom{s}{l} x^{2k+3-s}
\]

\[
+ \sum_{l=0}^{k} B_{k-l}(x^{-2k-4})(x^{-2k-4} - 1)^{l+1} \sum_{s=l}^{k} \binom{s}{l} x^{2(k+2)^2+s+1}.
\]

Comparing the above expression for \( xG_k(x) \cdot (1 - x^{2k+4}) \) and the first two summations in (3.14), to prove (3.13), it suffices to show that

\[
B_{k+1}(x^{2k+4})x^{2k+4} - B_{k+1}(x^{-2k-4})x^{2(k+2)^2} = 0.
\]

It is known that the type B Eulerian polynomial \( B_n(t) \) is a symmetric polynomial of degree \( n \), that is,

\[
B_n(t) = B_n(t^{-1})t^n,
\]

see Brenti [1]. Hence we have

\[
B_{k+1}(x^{2k+4})x^{2k+4} - B_{k+1}(x^{-2k-4})x^{2(k+2)^2} = 0.
\]

Thus (3.15) is equivalent to the following relation

\[
\sum_{l=0}^{k+1} B_{k+1-l}(x^{2k+4})(x^{2k+4} - 1)^{l} \binom{k+1}{l} x^{k+2}
\]

\[
- \sum_{l=0}^{k+1} B_{k+1-l}(x^{-2k-4})(x^{-2k-4} - 1)^{l} \binom{k+1}{l} x^{2(k+2)^2}.
\]  

(3.16)

Setting \( t = x^{2k+4} \) and \( n = k+1 \), (3.16) can be rewritten as

\[
\sum_{l=0}^{n} B_{n-l}(t)(t-1)^{l} \binom{n}{l} = \sum_{l=0}^{n} B_{n-l}(t^{-1})(t^{-1}-1)^{l} \binom{n}{l} t^{n+1}.
\]  

(3.17)

To prove (3.17), we need the following formula

\[
\sum_{n \geq 0} B_n(t) \frac{x^n}{n!} = \frac{(1 - t)e^{x(1-t)}}{1 - te^{2x(1-t)}},
\]  

(3.18)
which was obtained by Chow and Gessel [2]. Using (3.18), we get
\[
\sum_{n>1} \sum_{j=0}^{n} B_{n-j}(t)(t-1)^j \binom{n}{j} \frac{x^n}{n!}
= \left( \sum_{n \geq 0} B_n(t) \frac{x^n}{n!} \right) \left( \sum_{n \geq 0} (t-1)^n \frac{x^n}{n!} \right) - 1
= \frac{te^{2x(1-t)} - t}{1 - te^{2x(1-t)}}.
\] (3.19)

Similarly, using (3.18) we find that
\[
\sum_{n>1} \sum_{j=0}^{n} B_{n-j}(t^{-1})(t^{-1}-1)^j \binom{n}{j} \frac{t^{n+1}x^n}{n!}
= t \left( \sum_{n \geq 0} B_n(t^{-1}) \frac{x^n}{n!} \right) \left( \sum_{n \geq 0} (t-1)^n \frac{(tx)^n}{n!} \right) - t
= \frac{te^{2x(1-t)} - t}{1 - te^{2x(1-t)}}.
\] (3.20)

Combining (3.19) and (3.20), we arrive at (3.17). This completes the proof.

Based on Lemma 3.5 and the relationship between the array representation of \(H_k(x)\) and the array representation of \(G_k(x)\), we establish the following recurrence relations for the array representation of \(H_k(x)\).

**Corollary 3.6** For \(0 \leq i \leq k + 1\) and \(0 \leq j \leq k\), we have
\[
h_k(i,j) = h_{k-1}(i,0) + h_{k-1}(i,1) + \cdots + h_{k-1}(i,j-1)
+ h_{k-1}(i-1,j) + h_{k-1}(i-1,j+1) + \cdots + h_{k-1}(i-1,2k-1),
\] (3.21)
and for \(0 \leq i \leq k + 1\) and \(k + 1 \leq j \leq 2k + 1\), we have
\[
h_k(i,j) = h_{k-1}(i,0) + h_{k-1}(i,1) + \cdots + h_{k-1}(i,j-2)
+ h_{k-1}(i-1,j-1) + h_{k-1}(i-1,j) + \cdots + h_{k-1}(i-1,2k-1),
\] (3.22)
where we assume that \(h_k(i,j) = 0\) when \(i < 0\).

We are now in a position to complete the proof of Theorem 3.4.

**Proof of Theorem 3.4** We proceed by induction on \(k\). For \(k = 0\), by the expression (3.1) of \(H_k(x)\), we get \(H_0(x) = x + x^2\), which is unimodal. Assume that \(H_{k-1}(x)\) is unimodal, where \(k \geq 1\). We aim to prove that \(H_k(x)\) is unimodal.
Assume that \( k \geq 1 \). Let \((a_0, a_1, \ldots, a_{2k^2+2k-1})\) denote the sequence of coefficients of \( H_{k-1}(x) \). By the symmetry of \( H_{k-1}(x) \) as given in Corollary 3.3 we have \( a_i = a_{2k^2+2k-1-i} \). Hence, by the induction hypothesis, we have

\[
a_0 \leq a_1 \leq \cdots \leq a_{k^2+k-1}.
\] (3.23)

Assume that \((b_0, b_1, \ldots, b_{2k^2+6k+3})\) is the sequence of coefficients of \( H_k(x) \). By the symmetry of \( H_k(x) \), to prove that \( H_k(x) \) is unimodal, it suffices to prove that

\[
b_0 \leq b_1 \leq \cdots \leq b_{k^2+3k+1}.
\] (3.24)

Indeed, we can restate the above inequalities in terms of the array representation \( h_k \) of \( H_k(x) \). Recall that

\[
H_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h_k(i, j)x^{2(k+1)i+j}.
\]

Clearly, \( h_k(i, j) = b_{2(k+1)i+j} \) for \( 0 \leq i \leq k+1 \) and \( 0 \leq j \leq 2k+1 \). When \( k \) is odd, \( (3.24) \) can be restated as follows,

(i) \( h_k(i, j+1) - h_k(i, j) \geq 0 \) for \( 0 \leq i \leq \left\lfloor \frac{k+2}{2} \right\rfloor - 1 \) and \( 0 \leq j \leq 2k \);

(ii) \( h_k(i, j+1) - h_k(i, j) \geq 0 \) for \( i = \left\lfloor \frac{k+2}{2} \right\rfloor \) and \( 0 \leq j \leq k-1 \);

(iii) \( h_k(i, 0) - h_k(i-1, 2k+1) \geq 0 \) for \( 1 \leq i \leq \left\lfloor \frac{k+2}{2} \right\rfloor \).

Similarly, when \( k \) is even, \( (3.24) \) can be recast into the following assertions:

(iv) \( h_k(i, j+1) - h_k(i, j) \geq 0 \) for \( 0 \leq i \leq \frac{k}{2} \) and \( 0 \leq j \leq 2k \);

(v) \( h_k(i, 0) - h_k(i-1, 2k+1) \geq 0 \) for \( 1 \leq i \leq \frac{k}{2} \).

We now proceed to prove the above assertions. It follows from \( (3.21) \) that for \( 0 \leq i \leq k+1 \) and \( 0 \leq j \leq k-1 \),

\[
h_k(i, j+1) - h_k(i, j) = h_{k-1}(i, j) - h_{k-1}(i-1, j).
\] (3.25)

Using \( (3.22) \), we find that for \( 0 \leq i \leq k+1 \) and \( k+1 \leq j \leq 2k \),

\[
h_k(i, j+1) - h_k(i, j) = h_{k-1}(i, j-1) - h_{k-1}(i-1, j-1).
\] (3.26)

Moreover, by \( (3.21) \) and \( (3.22) \), it is easy to check that for \( 0 \leq i \leq k+1 \),

\[
h_k(i, k) = h_k(i, k+1),
\] (3.27)

\[
h_k(i, 0) = h_k(i-1, 2k+1).
\] (3.28)
We first consider the case when \( k \) is odd. To prove (i), we assume that \( 0 \leq i \leq \lfloor \frac{k+2}{2} \rfloor - 1 \) and \( 0 \leq j \leq 2k \). Here are three subcases. When \( 0 \leq j \leq k-1 \), we claim that 
\[
h_k(i, j + 1) - h_k(i, j) \geq 0.
\]
From (3.25) we see that
\[
h_k(i, j + 1) - h_k(i, j) = a_{2ki+j} - a_{2ki-2k+j}.
\]
Since \( 0 \leq i \leq \lfloor \frac{k+2}{2} \rfloor - 1 \) and \( 0 \leq j \leq k-1 \), noting \( 2 \lfloor \frac{k+2}{2} \rfloor = k+1 \), we find that
\[
2ki + j \leq 2k \left( \left\lfloor \frac{k+2}{2} \right\rfloor - 1 \right) + k - 1 = k^2 - 1.
\]
Clearly, we have \( 2ki + j \geq 2ki - 2k + j \). Thus we may use the induction hypothesis to deduce that \( a_{2ki+j} - a_{2ki-2k+j} \geq 0 \), which is equivalent to the claim.

When \( k+1 \leq j \leq 2k \), we claim that 
\[
h_k(i, j + 1) - h_k(i, j) \geq 0.
\]
By (3.26), we get
\[
h_k(i, j + 1) - h_k(i, j) = a_{2ki+j-1} - a_{2ki-2k+j-1}.
\]
Using the same argument as in the case when \( 0 \leq j \leq k-1 \), we deduce that
\[
2ki + j - 1 \leq 2k \left( \left\lfloor \frac{k+2}{2} \right\rfloor - 1 \right) + 2k - 1 = k^2 + k - 1.
\]
Similarly, we have \( 2ki + j - 1 \geq 2ki - 2k + j - 1 \). Hence we may use the induction hypothesis to deduce that \( a_{2ki+j-1} - a_{2ki-2k+j-1} \geq 0 \), as claimed.

Recall that \( h_k(i, k+1) = h_k(i, k) \) for \( 0 \leq i \leq k+1 \) as given in (3.27). On the other hand, when \( j = k \), assertion (i) becomes the relation 
\[
h_k(i, k+1) - h_k(i, k) \geq 0
\]
for \( 0 \leq i \leq \lfloor \frac{k+2}{2} \rfloor - 1 \), which is valid since the equality holds. Combining the above three cases, assertion (i) is proved.

To prove (ii), we assume that \( i = \lfloor \frac{k+2}{2} \rfloor \) and \( 0 \leq j \leq k-1 \). We claim that 
\[
h_k(i, j + 1) - h_k(i, j) \geq 0.
\]
By (3.25) and the symmetry relation (3.10), we find that
\[
h_k(i, j + 1) - h_k(i, j) = h_{k-1}(i, j) - h_{k-1}(i-1, j)
\]
\[
= h_{k-1}(k-i, 2k-1-j) - h_{k-1}(i-1, j)
\]
\[
= a_{2(k-i)+2k-1-j} - a_{2(k-1)+j}.
\]
Since \( i = \lfloor \frac{k+2}{2} \rfloor \) and \( 0 \leq j \leq k-1 \), we see that
\[
2k(k-i) + 2k - 1 - j \leq 2k \left( k - \left\lfloor \frac{k+2}{2} \right\rfloor \right) + 2k - 1 = k^2 + k - 1,
\]
and
\[
2k(k-i) + 2k - 1 - j \geq 2k(i-1) + j.
\]
Hence we may use the induction hypothesis to deduce that \( a_{2(k-i)+2k-1-j} - a_{2(k-1)+j} \geq 0 \). This proves the claim, and hence assertion (ii) holds.
Note that by (3.28), we have $h_k(i,0) = h_k(i - 1, 2k + 1)$ for $1 \leq i \leq \lfloor \frac{k+2}{2} \rfloor$. This proves assertion (iii).

Next we turn to the case when $k$ is even.

To prove (iv), we assume that $0 \leq i \leq \frac{k}{2}$ and $0 \leq j \leq 2k$. When $0 \leq i \leq \frac{k}{2}$ and $0 \leq j \leq k - 1$, we claim that $h_k(i, j + 1) - h_k(i, j) \geq 0$. By (3.25), we see that

$$h_k(i, j + 1) - h_k(i, j) = a_{2ki+j} - a_{2ki-2k+j}.$$ 

By the assumptions $0 \leq i \leq \frac{k}{2}$ and $0 \leq j \leq k - 1$, we see that

$$2ki + j \leq k^2 + k - 1.$$ 

So we may use the induction hypothesis to deduce that $a_{2ki+j} - a_{2ki-2k+j} \geq 0$. This proves the claim.

When $0 \leq i \leq \frac{k}{2} - 1$ and $k + 1 \leq j \leq 2k$, we claim that $h_k(i, j + 1) - h_k(i, j) \geq 0$. By (3.26), we find that

$$h_k(i, j + 1) - h_k(i, j) = a_{2ki+j-1} - a_{2ki-2k+j-1}.$$ 

By the assumptions $0 \leq i \leq \frac{k}{2} - 1$ and $k + 1 \leq j \leq 2k$, we see that

$$2ki + j - 1 \leq k^2 - 1.$$ 

Hence the induction hypothesis can be used to get $a_{2ki+j-1} - a_{2ki-2k+j-1} \geq 0$, which is equivalent to the claim.

When $i = \frac{k}{2}$ and $k + 1 \leq j \leq 2k$, we claim that $h_k(i, j + 1) - h_k(i, j) \geq 0$. By (3.26) and the symmetry relation (3.10), we find that

$$h_k(i, j + 1) - h_k(i, j) = h_{k-1}(i, j - 1) - h_{k-1}(i - 1, j - 1) = h_{k-1}(k - i, 2k - j) - h_{k-1}(i - 1, j - 1) = a_{2k(k-i)+2k-j} - a_{2k(i-1)+j-1}.$$ 

Using the assumptions $i = \frac{k}{2}$ and $k + 1 \leq j \leq 2k$, we get

$$2k(k - i) + 2k - j \leq k^2 + k - 1,$$

and

$$2k(k - i) + 2k - j \geq 2k(i - 1) + j - 1.$$ 

By the induction hypothesis, we obtain that $a_{2k(k-i)+2k-j} - a_{2k(i-1)+j-1} \geq 0$. This proves the claim.

Using the fact $h_k(i, k) = h_k(i, k + 1)$ for $0 \leq i \leq k + 1$ as given in (3.27), it can be easily checked that assertion (iv) is true for $j = k$. So we proved assertion (iv) for all
the cases of $j$. Clearly, by (3.28), we have $h_k(i, 0) = h_k(i - 1, 2k + 1)$ for $1 \leq i \leq \frac{k}{2}$. This confirms assertion (v), and so the proof is complete.

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**References**


