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Finite Differences of the Logarithm of the Partition Function

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Abstract. Let $p(n)$ denote the partition function. DeSalvo and Pak proved that $\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}$ for $n \geq 2$, as conjectured by Chen. Moreover, they conjectured that a sharper inequality $\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}$ holds for $n \geq 45$. In this paper, we prove the conjecture of Desalvo and Pak by giving an upper bound for $-\Delta^2 \log p(n-1)$, where Δ is the difference operator with respect to n. We also show that for given $r \geq 1$ and sufficiently large n, $(-1)^{r-1}\Delta^r \log p(n) > 0$. This is analogous to the positivity of finite differences of the partition function. It was conjectured by Good and proved by Gupta that for given $r \geq 1$, $\Delta^r p(n) > 0$ for sufficiently large n.

Keywords: partition function, log-concavity, finite difference, the Lambert W function, the Hardy-Ramanujan-Rademacher formula

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1 Introduction

A partition of positive integer n is a nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. Let $p(n)$ denote the number of partitions of n. In particular, we set $p(0) = 1$. The Hardy-Ramanujan-

Rademacher formula for $p(n)$ states that

$$p(n) = \frac{\sqrt{12}}{24n-1}\sum_{k=1}^{N} A_k(n)\sqrt{k}\left[\left(1-\frac{k}{\mu(n)}\right)e^{\mu(n)/k} + \left(1+\frac{k}{\mu(n)}\right)e^{-\mu(n)/k}\right] + R_2(n,N),$$

where $A_k(n)$ is an arithmetic function, $R_2(n,N)$ is the remainder term and

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \tag{1.1}$$

see, for example, Hardy and Ramanujan [11], Rademacher [18]. Note that $A_1(n) = 1$ and $A_2(n) = (-1)^n$ for $n \geq 1$. Lehmer [14, 15] gave the following error bound

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}}\left[\left(\frac{N}{\mu(n)}\right)^3 \sinh\frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2\right],$$

which is valid for all positive integers n and N.

Employing Rademacher's convergent series and Lehmer's error bound, DeSalvo and Pak [8] proved the following inequality conjectured by Chen [6].

Theorem 1.1. *For $n \geq 2$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}. \tag{1.2}$$

The above relation has been improved by DeSalvo and Pak [8].

Theorem 1.2. *For $n \geq 7$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{1.3}$$

They also proposed the following conjecture.

Conjecture 1.3. *For $n \geq 45$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{1.4}$$

It should be mentioned that by using Lehmer's error bound for the remainder term of $p(n)$, Bessenrodt and Ono [5] proved the following inequality.

Theorem 1.4. *For any integers a, b satisfying $a, b > 1$ and $a + b > 9$, we have*

$$p(a)p(b) > p(a+b).$$

In this paper, we shall prove Conjecture 1.3 by giving an upper bound for $-\Delta^2 \log p(n-1)$ for $n \geq 5000$. Moreover, for any given r, we give an upper bound for $(-1)^{r-1}\Delta^r \log p(n)$.

In 1977, Good [9] conjectured that $\Delta^r p(n)$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive. Using the Hardy-Rademacher series [19] for $p(n)$, Gupta [10] proved that for any given r, $\Delta^r p(n) > 0$ for sufficiently large n. In 1988, Odlyzko [16] proved the conjecture of Good and obtained the following asymptotic formula for $n(r)$:

$$n(r) \sim \frac{6}{\pi^2} r^2 \log^2 r \quad \text{as } r \to \infty.$$

Knessl and Keller [12, 13] obtained an approximation $n(r)'$ for $n(r)$ for which $|n(r)' - n(r)| \leq 2$ up to $r = 75$. Almkvist [2, 3] proved that $n(r)$ satisfies certain equations.

By using the bounds of the modified Bessel function of the first kind, we shall prove that for any given $r \geq 1$, there exists a positive integer $n(r)$ such that $(-1)^{r-1}\Delta^r \log p(n) > 0$ for $n \geq n(r)$.

2 Proof of Conjecture 1.3

In this section, we give a proof of Conjecture 1.3 by using an inequality of DeSalvo and Pak [8]. Let

$$p_2(n) = 2\log p(n) - \log p(n-1) - \log p(n+1),$$

DeSalvo and Pak have shown that for $n \geq 50$,

$$\begin{aligned} p_2(n) \ < \ & \frac{24\pi}{(24(n-1)-1)^{3/2}} + \frac{288\pi(-3+\pi\sqrt{24(n-1)-1})}{(24(n-1)-1)^{3/2}(-6+\pi\sqrt{24(n-1)-1})^2} \\ & - \frac{864}{(24(n+1)-1)^2} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}. \end{aligned} \tag{2.1}$$

We shall give an estimate of the right hand side of (2.1), leading to a proof of the conjecture.

Proof of Conjecture 1.3. The conjecture can be restated as follows

$$p_2(n) < \log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right), \tag{2.2}$$

where $n \geq 45$. We proceed to give an estimate of each term of the right hand side of (2.1).

We begin with the first term of the right hand side of (2.1). We claim that for $n \geq 50$,

$$\frac{24\pi}{(24(n-1)-1)^{3/2}} < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2 + \frac{3}{2n^{5/2}}. \tag{2.3}$$

For $0 < x \leq \frac{1}{48}$, it can be easily checked that

$$\frac{1}{(1-x)^{3/2}} < 1 + \frac{3}{2}x + \frac{3}{8}x^{3/2}. \tag{2.4}$$

For $n \geq 50$, we have $\frac{25}{24n} \leq \frac{1}{48}$, and hence we can apply (2.4) to deduce that

$$\begin{aligned} \frac{24\pi}{(24(n-1)-1)^{3/2}} &= \frac{24\pi}{(24n)^{3/2}\left(1-\frac{25}{24n}\right)^{3/2}} \\ &< \frac{24\pi}{(24n)^{3/2}}\left(1 + \frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{3/2}\right). \end{aligned} \tag{2.5}$$

For $n \geq 50$, we have

$$\begin{aligned} \frac{3}{8}\left(\frac{25}{24n}\right)^{3/2} &< \frac{3}{8}\left(\frac{25}{24}\right)^{3/2}\frac{1}{50^{1/2}n}, \\ \frac{24\pi}{(24n)^{3/2}} &< \frac{24\pi}{(24)^{3/2}50^{1/2}n}. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{24\pi}{(24n)^{3/2}}\left(\frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{3/2} + \frac{24\pi}{(24n)^{3/2}}\right) \\ &\leq \frac{24\pi}{(24n)^{3/2}n}\left(\frac{25}{16} + \frac{3}{8}\left(\frac{25}{24}\right)^{3/2}\frac{1}{50^{1/2}} + \frac{24\pi}{(24)^{3/2}50^{1/2}}\right) \\ &< \frac{3}{2n^{5/2}}. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain (2.3).

As for the second term of the right hand side of (2.1), it can be shown that for $n \geq 50$,

$$\frac{288\pi(-3+\pi\sqrt{24(n-1)-1})}{(24(n-1)-1)^{3/2}(-6+\pi\sqrt{24(n-1)-1})^2} < \frac{1}{2n^2} + \frac{1}{n^{5/2}}. \tag{2.7}$$

To this end, we need the following inequality for $\alpha \geq \frac{1}{2}$ and $0 < x \leq c < 1$,

$$\frac{1}{(1-x)^{\alpha}} \leq 1 + \left(\frac{1}{1-c}\right)^{\alpha+1} \alpha x. \tag{2.8}$$

Let

$$f(x) = \frac{1}{(1-x)^{\alpha}} - 1 - \left(\frac{1}{1-c}\right)^{\alpha+1} \alpha x.$$

For $\alpha \geq \frac{1}{2}$ and $0 \leq x \leq c < 1$, we see that

$$f'(x) = \frac{\alpha}{(1-x)^{\alpha+1}} - \left(\frac{1}{1-c}\right)^{\alpha+1} \alpha \leq 0.$$

Since $f(0) = 0$, we obtain that $f(x) \leq 0$ under the above assumption. This yields that $f(x) < 0$ for $0 < x \leq c < 1$ and $\alpha \geq \frac{1}{2}$, and hence (2.8) is proved.

The left hand side of (2.7) can be rewritten as

$$\frac{144\pi^2\sqrt{24n-25}}{(24n-25)^{3/2}(-6+\pi\sqrt{24n-25})^2} + \frac{288\pi(-3+\frac{\pi}{2}\sqrt{24n-25})}{(24n-25)^{3/2}(-6+\pi\sqrt{24n-25})^2},$$

which can be simplified to

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2} + \frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)}. \tag{2.9}$$

Setting $x = \frac{25}{24n}$, $\alpha = 2$ and $c = \frac{1}{48}$, for $n \geq 50$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we find that for $n \geq 50$,

$$\frac{1}{\left(1-\frac{25}{24n}\right)^2} \leq 1 + \left(\frac{48}{47}\right)^3 \frac{25}{12n}. \tag{2.10}$$

Setting $x = \frac{6}{\pi\sqrt{24n-25}}$, $\alpha = 2$ and $c = \frac{1}{15}$, for $n \geq 50$, we also have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. Again, using (2.8), we see that for $n \geq 50$,

$$\frac{1}{\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2} < 1 + \left(\frac{15}{14}\right)^3 \frac{6}{\pi\sqrt{24n-25}} < 1 + \frac{24}{\pi\sqrt{24n-25}}. \tag{2.11}$$

Combining (2.10) and (2.11), we deduce that for $n \geq 50$,

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2}$$

$$\leq \frac{1}{4n^2}\left(1+\left(\frac{48}{47}\right)^3\frac{25}{12n}\right)\left(1+\frac{24}{\pi\sqrt{24n-25}}\right). \tag{2.12}$$

It is easily seen that

$$\frac{24}{\pi\sqrt{24n-25}} = \frac{24}{\pi(24n)^{1/2}}\frac{1}{\left(1-\frac{25}{24n}\right)^{1/2}}. \tag{2.13}$$

Setting $x=\frac{25}{24n}$, $\alpha=\frac{1}{2}$ and $c=\frac{1}{48}$, for $n\geq 50$, we have $0<x<c<1$ and $\alpha\geq\frac{1}{2}$. By (2.8), for $n\geq 50$, we get

$$\frac{1}{\left(1-\frac{25}{24n}\right)^{1/2}} < 1+\left(\frac{48}{47}\right)^{3/2}\frac{25}{48n}. \tag{2.14}$$

Combining (2.12), (2.13) and (2.14), we find that for $n\geq 50$,

$$\begin{aligned}&\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2}\\ &\quad\leq \frac{1}{4n^2}\left(1+\left(\frac{48}{47}\right)^3\frac{25}{12n}\right)\left(1+\frac{24}{\pi(24n)^{1/2}}\left(1+\left(\frac{48}{47}\right)^{3/2}\frac{25}{48n}\right)\right).\end{aligned} \tag{2.15}$$

The right hand side of (2.15) can be expanded as follows

$$\begin{aligned}&\frac{1}{4n^2}+\frac{\sqrt{6}}{2\pi n^{5/2}}+\frac{25}{48n^3}\left(\frac{48}{47}\right)^3+\frac{25\sqrt{6}}{96\pi n^{7/2}}\left(\frac{48}{47}\right)^{3/2}\\ &\quad+\frac{25\sqrt{6}}{24\pi n^{7/2}}\left(\frac{48}{47}\right)^3+\frac{25^2\sqrt{24}}{48^2\pi n^{9/2}}\left(\frac{48}{47}\right)^{9/2}.\end{aligned} \tag{2.16}$$

Clearly, for $\alpha>\frac{5}{2}$ and $n\geq 50$,

$$\frac{1}{n^\alpha}\leq\frac{1}{50^{\alpha-5/2}n^{5/2}},$$

which implies that for $n\geq 50$,

$$\frac{1}{n^3}\leq\frac{1}{50^{1/2}n^{5/2}}, \tag{2.17}$$

$$\frac{1}{n^{7/2}}\leq\frac{1}{50n^{5/2}}, \tag{2.18}$$

$$\frac{1}{n^{9/2}}\leq\frac{1}{50^2n^{5/2}}. \tag{2.19}$$

Applying (2.17), (2.18) and (2.19) to the last four terms of (2.16), we obtain that for $n \geq 50$,

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2} < \frac{1}{4n^2} + \frac{1}{2n^{5/2}}. \tag{2.20}$$

For the second term of (2.9). Setting $x = \frac{6}{\pi\sqrt{24n-25}}$, $\alpha = 1$ and $c = \frac{1}{15}$, for $n \geq 50$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we see that for $n \geq 50$,

$$\frac{1}{1-\frac{6}{\pi\sqrt{24n-25}}} < 1 + \left(\frac{15}{14}\right)^2 \frac{6}{\pi\sqrt{24n-25}} < 1 + \frac{12}{\pi\sqrt{24n-25}}. \tag{2.21}$$

Using (2.21) and the same argument as in the derivation of (2.20), it can be shown that for $n \geq 50$,

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)} < \frac{1}{4n^2} + \frac{1}{2n^{5/2}}. \tag{2.22}$$

In view of (2.20) and (2.22), we arrive at (2.7).

To estimate the third term of the right hand side of (2.1), we aim to show that for $n \geq 50$,

$$-\frac{864}{(24(n+1)-1)^2} < \frac{1}{2n^{5/2}} - \frac{3}{2n^2}. \tag{2.23}$$

It is easily verified that for $\alpha \geq 1/2$ and $0 \leq x \leq 1$,

$$1 \geq \frac{1}{(1+x)^\alpha} \geq 1 - \alpha x. \tag{2.24}$$

So for $n \geq 50$, we have

$$\frac{1}{\left(1+\frac{23}{24n}\right)^2} \geq 1 - \frac{23}{12n}.$$

Consequently, for $n \geq 50$,

$$-\frac{864}{(24(n+1)-1)^2} = -\frac{3}{2n^2\left(1+\frac{23}{24n}\right)^2} \leq \frac{23}{8n^3} - \frac{3}{2n^2} \leq \frac{1}{2n^{5/2}} - \frac{3}{2n^2}.$$

Utilizing the above upper bounds (2.3), (2.7) and (2.23) for the three terms of the right hand side of (2.1), we conclude that for $n \geq 50$,

$$p_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2 - \frac{1}{n^2} + \frac{3}{n^{5/2}} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}.$$

Next we show that for $n \geq 5000$,

$$p_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2. \tag{2.25}$$

Clearly, for $n \geq 100$,

$$-\frac{1}{n^2} + \frac{3}{n^{5/2}} < -\frac{2}{3n^2}.$$

To prove that for $n \geq 5000$,

$$-\frac{2}{3n^2} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}} < 0, \tag{2.26}$$

let

$$g(x) = -\frac{2}{3x^2} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2x}{3}}}.$$

The equation $g(x) = 0$ has two solutions

$$\begin{aligned} x_1 &= \frac{2400}{\pi^2}\left(W_0\left(-\frac{\pi\sqrt{2}}{40 \cdot 3^{3/4}}\right)\right)^2, \\ x_2 &= \frac{2400}{\pi^2}\left(W_{-1}\left(-\frac{\pi\sqrt{2}}{40 \cdot 3^{3/4}}\right)\right)^2, \end{aligned}$$

where $W_0(z)$ and $W_{-1}(z)$ are two branches of Lambert W function $W(z)$, see Corless, Gonnet, Hare, Jeffrey and Knuth [7]. More explicitly, we have $x_1 \approx 0.64$ and $x_2 \approx 4996.47$. It can be checked that $g(5000) < 0$. Thus for $x \geq 5000$,

$$g(x) < 0.$$

This proves (2.26). Hence (2.25) holds.

Using (2.25), we shall show that inequality (2.2) in the theorem holds for $n \geq 5000$. It is easily verified that for $x > 0$,

$$x(1-x) < \log(1+x). \tag{2.27}$$

Let

$$h(x) = \log(1+x) - x + x^2.$$

For $x \geq 0$, we see that

$$h'(x) = \frac{x + 2x^2}{1+x} \geq 0.$$

Since $h(0) = 0$, we have $h(x) > 0$ for $x > 0$. Combining (2.25) and (2.27), we deduce that for $n \geq 5000$,

$$p_2(n) < \log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right).$$

Since DeSalvo and Pak [8] have verified the above relation for $45 \leq n \leq 8000$, we reach the conclusion that inequality (2.2) holds for $n \geq 45$ and hence the proof is complete. ∎

3 An upper bound for $(-1)^{r-1}\Delta^r \log p(n)$

The conjecture of DeSalvo and Pak can be formulated as an upper bound for $2\log p(n) - \log p(n-1) - \log p(n+1)$, namely, for $n \geq 45$,

$$-\Delta^2 \log p(n-1) < \log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right), \tag{3.1}$$

where Δ is the difference operator as given by $\Delta f(n) = f(n+1) - f(n)$.

In this section, we give an upper bound for $(-1)^{r-1}\Delta^r \log p(n)$. When $r = 2$, this upper bound reduces to the above relation (3.1). In the following theorem, we adopt the notation $(a)_k$ for the rising factorial, namely, $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

Theorem 3.1. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1} \frac{1}{(n+1)^{r-\frac{1}{2}}}\right).$$

In the proof of the above theorem, we shall use Hardy-Ramanujan-Rademacher series for $n \geq 1$,

$$p(n) = 2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty} A_k(n) k^{-5/2} L_{3/2}\left(\frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\right), \tag{3.2}$$

and the following estimate for $A_k(n)$,

$$|A_k(n)| \leq 2k^{3/4}, \tag{3.3}$$

see Rademacher [19]. In particular, we have $A_1(n) = 1$ and $A_2(n) = (-1)^n$. The function $L_\nu(x)$ in (3.2) is defined by

$$L_\nu(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!\Gamma(m+\nu+1)}, \tag{3.4}$$

where $\Gamma(m+\nu+1)$ is the Gamma function.

With the notation of $\mu(n)$ as in (1.1), we have

$$\frac{\pi^2}{6}\left(n - \frac{1}{24}\right) = \frac{\mu^2(n)}{4},$$

and so (3.2) can be rewritten as

$$p(n) = 2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty} A_k(n) k^{-5/2} L_{3/2}\left(\frac{\mu^2(n)}{4k^2}\right), \tag{3.5}$$

Denote the kth summand in (3.5) by $f_k(n)$, namely,

$$f_k(n) = 2\pi \left(\frac{\pi}{12}\right)^{3/2} A_k(n) k^{-5/2} L_{3/2}\left(\frac{\mu^2(n)}{4k^2}\right). \tag{3.6}$$

Writing (3.5) as

$$p(n) = f_1(n)\left(1 + \frac{f_2(n)}{f_1(n)}\right)\left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n) + f_2(n)}\right). \tag{3.7}$$

It is known that

$$L_{3/2}(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx}\left(\frac{\sinh 2\sqrt{x}}{\sqrt{x}}\right),$$

see Abramowitz and Stegun [1] or Almkvist [2]. Since $A_1(n) = 1$, $f_1(n)$ can be expressed as

$$f_1(n) = \frac{\sqrt{12}}{24n - 1}\left[\left(1 - \frac{1}{\mu(n)}\right) e^{\mu(n)} + \left(1 + \frac{1}{\mu(n)}\right) e^{-\mu(n)}\right]. \tag{3.8}$$

Recall $A_2(n) = (-1)^n$, by (3.4) and (3.6) we obtain that for $n \geq 1$,

$$f_1(n) - |f_2(n)| = 2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{m=0}^{\infty} \left(\frac{1}{4^m} - \frac{1}{2^{5/2} 16^m}\right) \frac{\mu^{2m}(n)}{m!\Gamma(m + 5/2)}.$$

Clearly, $\frac{1}{4^m} - \frac{1}{2^{5/2}16^m} > 0$ for $m \geq 0$. Hence for $n \geq 1$,

$$f_1(n) - |f_2(n)| > 0, \tag{3.9}$$

which implies that for $n \geq 1$, $f_1(n)$ is positive and

$$f_1(n) + f_2(n) > 0.$$

It is also clear that, for $n \geq 1$, both of $\mu(n) - 1$ and $1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n)+f_2(n)}$ are positive. Applying (3.8) to (3.7), we obtain that for $n \geq 1$

$$\begin{aligned}\log p(n) = &\log \frac{\pi^2}{6\sqrt{3}} - 3\log \mu(n) + \log(\mu(n) - 1) + \mu(n) \\ &+ \log\left(1 + \frac{\mu(n) + 1}{\mu(n) - 1} e^{-2\mu(n)}\right) + \log\left(1 + \frac{f_2(n)}{f_1(n)}\right) \\ &+ \log\left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n) + f_2(n)}\right).\end{aligned}$$

Hence

$$(-1)^{r-1}\Delta^r \log p(n) = H_r + F_1 + F_2 + F_3, \tag{3.10}$$

where

$$\begin{aligned}
H_r &= (-1)^{r-1}\Delta^r\left(-3\log\mu(n)+\log(\mu(n)-1)+\mu(n)\right),\\
F_1 &= (-1)^{r-1}\Delta^r\log\left(1+\frac{\mu(n)+1}{\mu(n)-1}e^{-2\mu(n)}\right),\\
F_2 &= (-1)^{r-1}\Delta^r\log\left(1+\frac{f_2(n)}{f_1(n)}\right),\\
F_3 &= (-1)^{r-1}\Delta^r\log\left(1+\frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n)+f_2(n)}\right).
\end{aligned}$$

Let

$$G_r = F_1+F_2+F_3. \tag{3.11}$$

To estimate $(-1)^{r-1}\Delta^r\log p(n)$, we shall give upper bounds for H_r and G_r. We first consider G_r.

Theorem 3.2. *For $n\geq 50$, we have*

$$|G_r| < 5\cdot 2^{r+\frac{1}{2}}e^{-\frac{\mu(n)}{2}}. \tag{3.12}$$

To prove Theorem 3.2, we recall a monotone property of the ratio of two power series, see Ponnusamy and Vuorinen [17]. We also need a lower bound and an upper bound on the ratio of $L_\nu(x)$ and $L_\nu(y)$, which can be deduced from known bounds on the ratio of two modified Bessel functions of the first kind.

Proposition 3.3. *Suppose that the power series*

$$f(x)=\sum_{m=0}^{\infty}\alpha_m x^m \quad \textit{and} \quad g(x)=\sum_{m=0}^{\infty}\beta_m x^m$$

both converge for $|x|<\infty$ and $\beta_m>0$ for all $m>0$. Then the function $\frac{f(x)}{g(x)}$ is strictly decreasing for $x>0$ if the sequence $\{\alpha_m/\beta_m\}_{m=0}^{\infty}$ is strictly decreasing.

Let $I_\nu(x)$ be the modified Bessel function of the first kind as given by

$$I_\nu(x)=\left(\frac{x}{2}\right)^{\nu}\sum_{m=0}^{\infty}\frac{\left(\frac{x^2}{4}\right)^m}{m!\Gamma(m+\nu+1)},$$

see Watson [20]. It is known that for $\nu\geq 1/2$ and $0<x<y$, $I_\nu(x)$ increases with x and

$$e^{x-y}\left(\frac{x}{y}\right)^{\nu} < \frac{I_\nu(x)}{I_\nu(y)} < e^{x-y}\left(\frac{y}{x}\right)^{\nu},$$

see Baricz [4, inequalities 2.2 and 2.4]. For $x > 0$, from (3.4) we see that $L_\nu(x)$ can be expressed by $I_\nu(x)$,

$$L_\nu(x) = x^{-\nu/2} I_\nu(2\sqrt{x}).$$

Thus the above properties of $I_\nu(x)$ can be restated in terms of $L_\nu(x)$.

Proposition 3.4. *For $\nu \geq 1/2$ and $0 < x < y$, we have*

$$e^{2\sqrt{x}-2\sqrt{y}} < \frac{L_\nu(x)}{L_\nu(y)} < e^{2\sqrt{x}-2\sqrt{y}} \left(\frac{y}{x}\right)^\nu.$$

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Since $|G_r| \leq |F_1| + |F_2| + |F_3|$, in order to estimate G_r, we shall estimate $|F_1|$, $|F_2|$ and $|F_3|$. By the definition of $f_k(n)$, we have

$$|f_k(n)| = 2\pi \left(\frac{\pi}{12}\right)^{3/2} |A_k(n)| k^{-5/2} L_{3/2}\left(\frac{\mu(n)^2}{4k^2}\right).$$

It follows from (3.3) that for $n \geq 1$,

$$|f_k(n)| \leq 4\pi \left(\frac{\pi}{12}\right)^{3/2} k^{-7/4} L_{3/2}\left(\frac{\mu(n)^2}{4k^2}\right),$$

which yields that

$$\sum_{k=3}^{\infty} |f_k(n)| \leq 4\pi \left(\frac{\pi}{12}\right)^{3/2} \zeta(7/4) L_{3/2}\left(\frac{\mu(n)^2}{36}\right), \tag{3.13}$$

where $\zeta(x)$ is the Riemann zeta function. For convenience, we denote by $g(n)$ the right hand side of the above inequality, so that (3.13) becomes

$$\sum_{k=3}^{\infty} |f_k(n)| \leq g(n). \tag{3.14}$$

To estimate F_1, F_2 and F_3, we shall make use of the monotonicity of $\frac{\mu(n)+1}{\mu(n)-1}e^{-2\mu(n)}$, $\frac{|f_2(n)|}{f_1(n)}$ and $\frac{g(n)}{f_1(n)-|f_2(n)|}$. It is easily seen that $\frac{\mu(n)+1}{\mu(n)-1}e^{-2\mu(n)}$ decreases with n for $n \geq 1$, since $\frac{y+1}{y-1}e^{-2y}$ decreases with y for $y > 0$ and $\mu(n)$ increases with n. By (3.6), we have

$$\frac{|f_2(n)|}{f_1(n)} = \frac{L_{3/2}(\mu^2(n)/16)}{2^{5/2} L_{3/2}(\mu^2(n)/4)}.$$

The ratio of coefficients of x^m in $L_{3/2}(\mu^2(n)/16)$ and $L_{3/2}(\mu^2(n)/4)$ is $\frac{4^m}{16^m}$. By Proposition 3.3, we see that $\frac{L_{3/2}(y/16)}{L_{3/2}(y/4)}$ decreases with y for $y > 0$. Notice that $\mu^2(x)$ increases with x for $x \geq 1$. So $\frac{L_{3/2}(\mu^2(x)/16)}{L_{3/2}(\mu^2(x)/4)}$ decreases with x for $x \geq 1$. This implies that $\frac{|f_2(n)|}{f_1(n)}$ decreases with n.

Next we prove the monotonicity of $\frac{g(n)}{f_1(n)-|f_2(n)|}$. Recall that

$$\frac{g(n)}{f_1(n) - |f_2(n)|} = \frac{2\zeta(7/4)L_{3/2}(\mu^2(n)/36)}{L_{3/2}(\mu^2(n)/4) - 2^{-5/2}L_{3/2}(\mu^2(n)/16)}.$$

The ratio of coefficients of x^m in $L_{3/2}(y/36)$ and $L_{3/2}(y/4) - 2^{-5/2}L_{3/2}(y/16)$ equals

$$\frac{\frac{1}{36^m}}{\frac{1}{4^m} - \frac{1}{2^{5/2}16^m}},$$

which decreases with m for $m \geq 0$. By Proposition 3.3, we deduce that for $y > 0$,

$$\frac{L_{3/2}(y/36)}{L_{3/2}(y/4) - 2^{-5/2}L_{3/2}(y/16)}$$

decreases with y. Hence $\frac{g(n)}{f_1(n)-|f_2(n)|}$ decreases with n for $n \geq 1$.

Using the above monotone properties, we proceed to derive upper bounds for $|F_1|$, $|F_2|$ and $|F_3|$. It is known that for $0 < x < 1$,

$$\log(1-x) \geq \frac{-x}{1-x}, \tag{3.15}$$

$$|\log(1 \pm x)| \leq -\log(1-x), \tag{3.16}$$

see also DeSalvo and Pak [8].

We first estimate F_1. Since

$$\Delta^r f(n) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(n+k),$$

we have

$$F_1 = \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \log\left(1 + \frac{\mu(n+k)+1}{\mu(n+k)-1} e^{-2\mu(n+k)}\right).$$

It follows that

$$|F_1| \leq \sum_{k=0}^{r} \binom{r}{k} \log\left(1 + \frac{\mu(n+k)+1}{\mu(n+k)-1} e^{-2\mu(n+k)}\right). \tag{3.17}$$

By the monotonicity of $\frac{\mu(n)+1}{\mu(n)-1}e^{-2\mu(n)}$, we see that for $n \geq 1$ and $0 \leq k \leq r$,

$$\log\left(1 + \frac{\mu(n+k)+1}{\mu(n+k)-1} e^{-2\mu(n+k)}\right) \leq \log\left(1 + \frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}\right). \tag{3.18}$$

Applying (3.18) to (3.17), we find that for $n \geq 1$,

$$|F_1| \leq 2^r \log\left(1 + \frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}\right).$$

Since $\log(1+x) \leq x$ for $x \geq 0$, we see that for $n \geq 1$,

$$|F_1| \leq 2^r \frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}. \tag{3.19}$$

To estimate F_2, we begin with the following expression

$$F_2 = \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \log\left(1 + \frac{f_2(n+k)}{f_1(n+k)}\right). \tag{3.20}$$

It follows from (3.9) that

$$0 < 1 - \frac{|f_2(n)|}{f_1(n)} < 1.$$

Using (3.16), we find that for $n \geq 1$,

$$\left|\log\left(1 + \frac{f_2(n+k)}{f_1(n+k)}\right)\right| \leq -\log\left(1 - \frac{|f_2(n+k)|}{f_1(n+k)}\right). \tag{3.21}$$

Combining (3.20) and (3.21), we obtain that for $n \geq 1$,

$$|F_2| \leq -\sum_{k=0}^{r} \binom{r}{k} \log\left(1 - \frac{|f_2(n+k)|}{f_1(n+k)}\right).$$

In view of the monotonicity of $\frac{|f_2(n)|}{f_1(n)}$, we see that for $n \geq 1$,

$$|F_2| \leq -2^r \log\left(1 - \frac{|f_2(n)|}{f_1(n)}\right).$$

Hence, by (3.15), we obtain that for $n \geq 1$,

$$|F_2| \leq 2^r \frac{|f_2(n)|}{f_1(n) - |f_2(n)|}. \tag{3.22}$$

To estimate F_3, we use the following expression

$$F_3 = \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \log\left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n+k)}{f_1(n+k) + f_2(n+k)}\right). \tag{3.23}$$

By Proposition 3.4, we find that for $n \geq 1$

$$2^{-\frac{5}{2}}e^{-\frac{\mu(n)}{2}} < \frac{|f_2(n)|}{f_1(n)} < \sqrt{2}e^{-\frac{\mu(n)}{2}}, \tag{3.24}$$

and

$$2\zeta(7/4)e^{-\frac{2\mu(n)}{3}} < \frac{g(n)}{f_1(n)} < 54\zeta(7/4)e^{-\frac{2\mu(n)}{3}}. \tag{3.25}$$

Consequently, for $n \geq 1$,

$$\frac{|f_2(n)|}{f_1(n)} + \frac{g(n)}{f_1(n)} < \sqrt{2}e^{-\frac{\mu(n)}{2}} + 54\zeta(7/4)e^{-\frac{2\mu(n)}{3}}. \tag{3.26}$$

For $n \geq 50$, it can be checked that

$$\sqrt{2}e^{-\frac{\mu(n)}{2}} + 54\zeta(7/4)e^{-\frac{2\mu(n)}{3}} < 1. \tag{3.27}$$

Combining (3.26) and (3.27), we obtain that for $n \geq 50$,

$$\frac{|f_2(n)|}{f_1(n)} + \frac{g(n)}{f_1(n)} < 1,$$

or equivalently,

$$f_1(n) - |f_2(n)| - g(n) > 0. \tag{3.28}$$

Combining (3.14) and (3.28), we see that for $n \geq 50$,

$$f_1(n) - |f_2(n)| - |\sum_{k\geq 3}^{\infty} f_k(n)| > 0,$$

which can be rewritten as

$$1 \geq 1 - \frac{\left|\sum_{k\geq 3}^{\infty} f_k(n)\right|}{f_1(n) - |f_2(n)|} > 0.$$

Thus, we can use (3.16) to deduce that for $n \geq 50$,

$$\left|\log\left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n) + f_2(n)}\right)\right| \leq -\log\left(1 - \frac{|\sum_{k\geq 3}^{\infty} f_k(n)|}{f_1(n) - |f_2(n)|}\right). \tag{3.29}$$

Since $-\log(1-x)$ is increasing for $x > -1$, according to (3.14) and (3.29), we deduce that for $n \geq 50$,

$$-\log\left(1 - \frac{|\sum_{k\geq 3}^{\infty} f_k(n)|}{f_1(n) - |f_2(n)|}\right) < -\log\left(1 - \frac{g(n)}{f_1(n) - |f_2(n)|}\right). \tag{3.30}$$

Combining (3.29) and (3.30), we see that for $n \geq 50$,

$$\left|\log\left(1+\frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n)+f_2(n)}\right)\right| \leq -\log\left(1-\frac{g(n)}{f_1(n)-|f_2(n)|}\right). \tag{3.31}$$

It follows from (3.23) and (3.31) that for $n \geq 50$,

$$|F_3| \leq -\sum_{k=0}^{r}\binom{r}{k}\log\left(1-\frac{g(n+k)}{f_1(n+k)-|f_2(n+k)|}\right).$$

Based on the monotonicity of $\frac{g(n)}{f_1(n)-|f_2(n)|}$, we find that for $n \geq 50$,

$$|F_3| \leq -2^r \log\left(1-\frac{g(n)}{f_1(n)-|f_2(n)|}\right).$$

Hence, by (3.15), we obtain that for $n \geq 50$,

$$|F_3| \leq 2^r \frac{g(n)}{f_1(n)-|f_2(n)|-g(n)}. \tag{3.32}$$

By Proposition 3.4, we see that for $n \geq 1$

$$2^{\frac{7}{2}}\zeta(7/4)e^{-\frac{\mu(n)}{6}} < \frac{g(n)}{|f_2(n)|} < 27\sqrt{2}\zeta(7/4)e^{-\frac{\mu(n)}{6}}. \tag{3.33}$$

In view of (3.19) and (3.24), we obtain that for $n \geq 50$,

$$\frac{|F_1|}{F_4} < 2^{\frac{5}{2}}\frac{\mu(n)+1}{\mu(n)-1}e^{-\frac{3}{2}\mu(n)}, \tag{3.34}$$

where F_4 is defined by

$$F_4 = 2^r \frac{|f_2(n)|}{f_1(n)}.$$

As a consequence of (3.22) and (3.24), it can be checked that for $n \geq 50$,

$$\frac{|F_2|}{F_4} < \frac{1}{1-\sqrt{2}e^{-\frac{\mu(n)}{2}}}. \tag{3.35}$$

Applying (3.24), (3.25) and (3.33) to (3.32), we obtain that for $n \geq 50$,

$$\frac{|F_3|}{F_4} < \frac{27\sqrt{2}\zeta(7/4)}{e^{\frac{\mu(n)}{6}}-\sqrt{2}e^{-\frac{\mu(n)}{3}}-54\zeta(7/4)e^{-\frac{\mu(n)}{2}}}. \tag{3.36}$$

Combining (3.34), (3.35) and (3.36), we conclude that for $n \geq 50$,

$$|F_1|+|F_2|+|F_3| < 5F_4. \tag{3.37}$$

It follows from (3.24) that for $n \geq 1$,

$$F_4 < 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{3.38}$$

Thus (3.37) and (3.38) lead to an upper bound for $|F_1| + |F_2| + |F_3|$. This completes the proof. ∎

To prove Theorem 3.1, we still need to estimate H_r and we shall use two relations due to Odlyzko [16] on the relations between the higher order differences and derivatives.

Proposition 3.5. *Let r be a positive integer. Suppose that $f(x)$ is a function with infinite continuous derivatives for $x \geq 1$, and $(-1)^{k-1} f^{(k)}(x) > 0$ for $k \geq 1$. Then for $r > 1$,*

$$(-1)^{r-1} f^{(r)}(x+r) \leq (-1)^{r-1} \Delta^r f(x) \leq (-1)^{r-1} f^{(r)}(x).$$

Proof of Theorem 3.1. First, we treat the case $r = 1$, which states that for $n \geq 12$,

$$\Delta \log p(n) < \log \left(1 + \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}}\right). \tag{3.39}$$

Since we have estimated $|G_r|$, we only need to estimate H_r for $r = 1$. By Proposition 3.5, we have

$$H_1 \leq \frac{2\pi}{\sqrt{24n-1}} - \frac{36}{24(n+1)-1} + \frac{12}{(24n-1)(1-\frac{6}{\pi\sqrt{24n-1}})}. \tag{3.40}$$

We claim that for $n \geq 50$,

$$H_1 < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{1}{n+1} + \frac{5}{4\,(n+1)^{3/2}}. \tag{3.41}$$

We proceed to estimate each term of the right hand side of (3.40). For the first term, we need to show that for $n \geq 50$,

$$\frac{2\pi}{\sqrt{24n-1}} < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{3}{2(n+1)}. \tag{3.42}$$

Setting $x = \frac{25}{24(n+1)}$, $\alpha = 1/2$ and $c = \frac{1}{48}$, for $n \geq 50$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. It follows from (2.8) that for $n \geq 50$,

$$\frac{2\pi}{\sqrt{24n-1}} = \frac{2\pi}{\sqrt{24}\,(n+1)^{1/2}\left(1 - \frac{25}{24(n+1)}\right)^{1/2}}$$

$$\leq \quad \frac{2\pi}{\sqrt{24}\,(n+1)^{1/2}}\left(1+\left(\frac{48}{47}\right)^{3/2}\frac{25}{48(n+1)}\right).$$

This proves (3.42).

For the second term of the right hand side of (3.40), for $n \geq 50$, we have

$$-\frac{36}{24(n+1)-1} < -\frac{3}{2(n+1)}. \tag{3.43}$$

For the last term of the right hand side of (3.40), using the same argument as in the derivation of (2.20), we obtain that for $n \geq 50$,

$$\frac{12}{(24n-1)(1-\frac{6}{\pi\sqrt{24n-1}})} < \frac{1}{2(n+1)} + \frac{1}{2\,(n+1)^{3/2}}. \tag{3.44}$$

Combining (3.42), (3.43) and (3.44), we obtain (3.41).

By the estimate of H_1 in (3.41) and the estimate of G_1 in (3.12), we obtain that for $n \geq 50$,

$$\Delta \log p(n) < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{1}{n+1} + \frac{5}{4\,(n+1)^{3/2}} + 10\sqrt{2}e^{-\frac{\pi}{12}\sqrt{(24n-1)}}.$$

Notice that for $n \geq 200$,

$$\frac{5}{4\,(n+1)^{3/2}} < \frac{12-\pi^2}{24(n+1)},$$

and for $n \geq 50$,

$$10\sqrt{2}e^{-\frac{\pi}{12}\sqrt{(24n-1)}} < \frac{12-\pi^2}{24(n+1)}.$$

Hence, for $n \geq 200$,

$$\Delta \log p(n) < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{\pi^2}{12(n+1)}. \tag{3.45}$$

Moreover, it can be easily checked that for $x > 0$,

$$x\left(1-\frac{x}{2}\right) < \log(1+x).$$

Thus, for $n \geq 1$,

$$\frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{\pi^2}{12(n+1)} < \log\left(1+\frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}}\right).$$

Combining the above relation and (3.45), we reach (3.39) for $n \geq 200$.

It can be checked that (3.39) is valid for $12 \leq n \leq 200$, and so Theorem 3.1 holds for $r = 1$.

We now turn to the case $r \geq 2$. We proceed to show that there exists an integer $n(r)$ such that for $n \geq n(r)$,

$$(-1)^{r-1}\Delta^r \log p(n) < U_r, \tag{3.46}$$

where

$$U_r = \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\left(1-\frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right).$$

Since $x(1-x) < \log(1+x)$ for $x > 0$, we have that for $n \geq 1$,

$$U_r < \log\left(1+\frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right).$$

Thus (3.46) implies Theorem 3.1 for $r \geq 2$.

By (3.10), we see that for $n \geq 1$,

$$(-1)^{r-1}\Delta^r \log p(n) \leq H_r + |G_r|.$$

To prove (3.46), it suffices to show that for $n \geq n(r)$

$$H_r + |G_r| < U_r. \tag{3.47}$$

Since Theorem 3.2 gives the upper bound for $|G_r|$, we need an upper bound for H_r. Recall that for $n \geq 1$,

$$H_r = (-1)^{r-1}\Delta^r\left(-3\log\mu(n) + \log(\mu(n)-1) + \mu(n)\right). \tag{3.48}$$

For $x \geq 1$, write

$$\log(\mu(x)-1) = \log\mu(x) - \sum_{k=1}^{\infty}\frac{1}{k\mu(x)^k}.$$

By exchanging the order of two summations with one being finite, it can be seen that for $x \geq 1$,

$$\Delta^r \log(\mu(x)-1) = \Delta\log\mu(n) - \sum_{k=1}^{\infty}\Delta^r\left(\frac{1}{k\mu(n)^k}\right).$$

Hence (3.48) implies that for $n \geq 1$,

$$H_r = (-1)^{r-1}\Delta^r\left(\mu(n) - 2\log\mu(n)\right) - \sum_{k=1}^{\infty}(-1)^{r-1}\Delta^r\left(\frac{1}{k\mu(n)^k}\right).$$

The rth derivatives of $\mu(x) = \frac{\pi}{6}\sqrt{24x-1}, \log\mu(x)$ and $\mu(x)^{-k}$ are given as follows,

$$\begin{aligned} \mu^{(r)}(x) &= \frac{(-1)^{r-1}(\frac{1}{2})_{r-1}24^r\pi}{12(24x-1)^{r-\frac{1}{2}}}, \\ \log^{(r)}(\mu(x)) &= \frac{(-1)^{r-1}(r-1)!24^r}{(24x-1)^r}, \\ \left(\frac{1}{\mu^k}\right)^{(r)} &= \left(\frac{k}{2}\right)_r \frac{(-144)^r}{\pi^k(24x-1)^{\frac{k}{2}+r}}. \end{aligned}$$

Therefore, the functions $\mu(x) = \frac{\pi}{6}\sqrt{24x-1}, \log\mu(x)$ and $-\mu(x)^{-k}$ satisfy the conditions of Proposition 3.5 for $r \geq 1$ and $k \geq 1$. Hence,

$$\begin{aligned} H_r \leq\; & \frac{(\frac{1}{2})_{r-1}24^r\pi}{12(24n-1)^{r-\frac{1}{2}}} - \frac{(r-1)!24^r}{(24(n+r)-1)^r} \\ & + \sum_{k=1}^{\infty}\left(\frac{k}{2}\right)_r \frac{144^r}{k\pi^k(24n-1)^{\frac{k}{2}+r}}. \end{aligned} \tag{3.49}$$

To bound the first term of (3.49), we note that

$$\frac{(\frac{1}{2})_{r-1}24^r\pi}{12(24n-1)^{r-\frac{1}{2}}} = \frac{(\sqrt{6}\pi\frac{1}{2})_{r-1}}{(n+1)^{r-\frac{1}{2}}\left(1-\frac{25}{24(n+1)}\right)^{r-\frac{1}{2}}}.$$

We claim that for $n \geq 48r-3$,

$$\frac{\sqrt{6}\pi(\frac{1}{2})_{r-1}}{6(n+1)^{r-\frac{1}{2}}\left(1-\frac{25}{24(n+1)}\right)^{r-\frac{1}{2}}} \leq U_r + \frac{a_1}{(n+1)^{r+\frac{1}{2}}}, \tag{3.50}$$

where

$$a_1 = \left(\frac{1}{2}\right)_{r-1}\left(\frac{48}{47}\right)^{r+\frac{1}{2}}(2r-1)\frac{25\pi}{24^{\frac{3}{2}}} + \frac{\pi^2}{6}\left(\left(\frac{1}{2}\right)_{r-1}\right)^2\frac{1}{(48r-2)^{r-\frac{3}{2}}}.$$

Setting $x = \frac{25}{24(n+1)}$, $\alpha = r-1/2$ and $c = \frac{1}{48}$, for $n \geq 48r-3$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. Invoking (2.8), we find that for $n \geq 48r-3$,

$$\frac{1}{\left(1-\frac{25}{24(n+1)}\right)^{r-1/2}} \leq 1 + \left(\frac{48}{47}\right)^{r+1/2}\frac{25(2r-1)}{48(n+1)}.$$

It follows that for $n \geq 48r - 3$,

$$\frac{\sqrt{6}\pi(\frac{1}{2})_{r-1}}{6(n+1)^{r-\frac{1}{2}}\left(1-\frac{25}{24(n+1)}\right)^{r-\frac{1}{2}}}$$

$$\leq U_r + \frac{\pi^2\left(\left(\frac{1}{2}\right)_{r-1}\right)^2}{6(n+1)^{2r-1}} + \frac{25\pi(2r-1)\left(\frac{1}{2}\right)_{r-1}\left(\frac{48}{47}\right)^{r+\frac{1}{2}}}{24^{3/2}\,(n+1)^{r+1/2}}.$$

It is easily seen that for $n \geq 48r - 3$,

$$\frac{1}{(n+1)^{2r-1}} \leq \frac{1}{(n+1)^{r+1/2}\,(48r-2)^{r-3/2}}.$$

So we arrive at (3.50).

As for the second term of (3.49), notice that

$$\frac{(r-1)!24^r}{(24(n+r)-1)^r} = \frac{(r-1)!}{(n+1)^r\left(1-\frac{24r-25}{24(n+1)}\right)^r},$$

and for $n \geq 48r - 3$,

$$0 < \frac{24r-25}{24(n+1)} < 1.$$

Consequently, for $n \geq 48r - 3$,

$$\frac{(r-1)!24^r}{(24(n+r)-1)^r} \geq \frac{(r-1)!}{(n+1)^r}. \tag{3.51}$$

Next we proceed to estimate the last term of (3.49). It can be checked that

$$\sum_{k=1}^{\infty}\binom{k}{2}_r \frac{144^r}{k\pi^k(24n-1)^{\frac{k}{2}+r}} = \sum_{k=1}^{\infty}\binom{k}{2}_r \frac{6^r}{k\pi^k 24^{\frac{k}{2}}\,(n+1)^{\frac{k}{2}+r}\left(1-\frac{25}{24(n+1)}\right)^{\frac{k}{2}+r}}.$$

We aim to show that for $n \geq 48r - 3$,

$$\sum_{k=1}^{\infty}\binom{k}{2}_r \frac{6^r}{k\pi^k 24^{\frac{k}{2}}\,(n+1)^{\frac{k}{2}+r}\left(1-\frac{25}{24(n+1)}\right)^{\frac{k}{2}+r}} \leq \frac{a_2+a_3}{(n+1)^{r+\frac{1}{2}}}, \tag{3.52}$$

where

$$a_2 = \sum_{k=1}^{\infty}\binom{k}{2}_r \left(\frac{1}{48r-2}\right)^{\frac{k-1}{2}} \frac{6^k}{k\pi^k 24^{\frac{k}{2}}},$$

$$a_3 = \sum_{k=1}^{\infty} \binom{\frac{k}{2}}{r+1} \left(\frac{1}{48r-2}\right)^{\frac{k+1}{2}} \left(\frac{48}{47}\right)^{\frac{k}{2}+r+1} \frac{25 \cdot 6^k (r+\frac{k}{2})}{k\pi^k 24^{\frac{k}{2}+1}}.$$

Note that for given r, a_2 and a_3 are convergent. Setting $x = \frac{25}{24(n+1)}$, $\alpha = k/2 + r$ and $c = \frac{1}{48}$, for $n \geq 48r - 3$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we find that for $n \geq 48r - 3$,

$$\frac{1}{\left(1 - \frac{25}{24(n+1)}\right)^{r-1/2}} \leq 1 + \left(\frac{48}{47}\right)^{k/2+r+1} \frac{25(2r+k)}{48(n+1)}. \tag{3.53}$$

Clearly, for $n \geq 48r - 3$ and $k \geq 1$,

$$\frac{1}{(n+1)^{k/2+r}} \leq \frac{1}{(n+1)^{r+1/2}(48r-2)^{\frac{k-1}{2}}}, \tag{3.54}$$

$$\frac{1}{(n+1)^{k/2+r+1}} \leq \frac{1}{(n+1)^{r+1/2}(48r-2)^{\frac{k+1}{2}}}. \tag{3.55}$$

Thus, (3.52) follows from (3.53), (3.54) and (3.55).

Combining (3.50), (3.51) and (3.52), we obtain that for $n \geq 48r - 3$,

$$H_r(n) < U_r - \frac{(r-1)!}{(n+1)^r} + \frac{a_1 + a_2 + a_3}{(n+1)^{r+\frac{1}{2}}}.$$

Let

$$u_1 = \frac{4(a_1 + a_2 + a_3)^2}{((r-1)!)^2}.$$

Notice that for given r ,a_1 is a finite number and $a_2 + a_3$ is convergent, so $a_1 + a_2 + a_3$ is a number for given r. It can be verified that for $n \geq u_1 + 1$,

$$\frac{a_1 + a_2 + a_3}{(n+1)^{r+\frac{1}{2}}} < \frac{(r-1)!}{2(n+1)^r}.$$

Thus, for $n \geq \max\{48r - 3, u_1 + 1\}$,

$$H_r(n) < U_r - \frac{(r-1)!}{2(n+1)^r}.$$

Employing the above inequality and (3.12), we deduce that for $n \geq \max\{50, 48r - 3, u_1 + 1\}$,

$$H_r + |G_r| < U_r - \frac{(r-1)!}{2(n+1)^r} + 5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}.$$

Observe that for $n \geq 1$,

$$\frac{1}{(n+1)^r} \geq \frac{\left(\frac{23}{48}\right)^r}{\left(n - \frac{1}{24}\right)^r}.$$

It follows that for $n \geq \max\{50, 48r-3, u_1+1\}$,

$$H_r + |G_r| < U_r - \frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(n-\frac{1}{24}\right)^r} + 5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{3.56}$$

To deduce (3.47) from (3.56), we consider the following equation

$$\frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(x-\frac{1}{24}\right)^r} = 5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}}. \tag{3.57}$$

Keep in mind that $\mu(x)$ is defined for $x \geq 1/24$. We claim that equation (3.57) has two real roots. Recall that the Lambert W function $W(z)$ is defined to be a function satisfying

$$W(z)e^{W(z)} = z, \tag{3.58}$$

for any complex number z, see Corless, Gonnet, Hare, Jeffrey and Knuth [7]. So a solution of (3.57) has the following form

$$x = \frac{1}{24} + \frac{6r^2}{\pi^2}\left(W\left(-\frac{\sqrt{46}\pi}{48r}\left(\frac{(r-1)!}{10\sqrt{2}}\right)^{\frac{1}{2r}}\right)\right)^2.$$

It is known that $W(z)$ is a multi-valued function. In particular, $W(z)$ has two real values $W_0(z)$ and $W_{-1}(z)$ for $-\frac{1}{e} < z < 0$. Using the following inequality, see Abramowitz and Stegun [1],

$$m! < \sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m+\frac{1}{12m}}, \tag{3.59}$$

we see that for $r \geq 2$,

$$\frac{\sqrt{46}\pi}{48r}\left(\frac{(r-1)!}{10\sqrt{2}}\right)^{\frac{1}{2r}} < \frac{1}{e}.$$

Hence (3.57) has two real roots. Let u_2 be the larger real root. Clearly, for sufficient large x,

$$5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}} < \frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(x-\frac{1}{24}\right)^r}.$$

It follows that for $n \geq u_2 + 1$,

$$5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}} < \frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(n-\frac{1}{24}\right)^r}. \tag{3.60}$$

Combining (3.56) and (3.60), we conclude that (3.47) holds for $n \geq n(r)$, where

$$n(r) = \max\{50, 48r-3, u_1+1, u_2+1\}.$$

This completes the proof for the case $r \geq 2$. ∎

4 The positivity of $(-1)^{r-1}\Delta^r \log p(n)$

In this section, we prove the positivity of $(-1)^{r-1}\Delta^r \log p(n)$ for $r \geq 1$ and sufficiently large n. This is analogous to the positivity of the differences of the partition function conjectured by Good [9] and proved by Gupta [10]. The proof relies on the estimates of H_r and G_r in the previous section.

Theorem 4.1. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) > 0. \tag{4.1}$$

Proof. The case $r = 1$ is obvious since $p(n+1) > p(n)$ for $n \geq 1$. For $r = 2$, DeSalvo and Pak [8] have shown that sequence $p(n)$ is log-concave for $n > 25$, or equivalently, for $n \geq 25$,

$$-\Delta^2 \log p(n) > 0.$$

We now consider the case $r \geq 3$. Recall that

$$(-1)^{r-1}\Delta^r \log p(n) = H_r + G_r,$$

where H_r and G_r are given in (3.10)and (3.11). Hence, we see that for $r \geq 1$,

$$(-1)^{r-1}\Delta^r \log p(n) \geq H_r - |G_r|. \tag{4.2}$$

An upper bound for $|G_r|$ has been given in Theorem 3.2, so we only need a lower bound for H_r. By the definition of H_r, we find that

$$H_r = (-1)^{r-1}\Delta^r \left(\mu(n) - 2\log \mu(n) - \sum_{k=1}^{\infty} \frac{1}{k\mu(n)^k} \right). \tag{4.3}$$

Applying Proposition 3.5 to the right hand side of the above equation, we get

$$\begin{aligned} H_r \quad \geq \quad & \frac{(\frac{1}{2})_{r-1} 24^r \pi}{12(24(n+r)-1)^{r-\frac{1}{2}}} - \frac{(r-1)!24^r}{(24n-1)^r} \\ & + \sum_{k=1}^{\infty} \left(\frac{k}{2}\right)_r \frac{144^r}{k\pi^k (24(n+r)-1)^{\frac{k}{2}+r}}. \end{aligned} \tag{4.4}$$

The first term of the right hand side of (4.4) has the following lower bound for $n \geq 48r - 2$,

$$\frac{(\frac{1}{2})_{r-1} 24^r \pi}{12(24(n+r)-1)^{r-\frac{1}{2}}} \geq \frac{b_1}{n^{r-\frac{1}{2}}} - \frac{b_2}{n^r}, \tag{4.5}$$

where

$$b_1 = \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1},$$

$$b_2 = \frac{\pi\sqrt{48r-2}}{24^{\frac{3}{2}}}\left(\frac{1}{2}\right)_{r}.$$

Setting $x = \frac{24r-1}{24n}$ and $\alpha = r - 1/2$, for $n \geq 48r-2$, we have $0 < x < 1$ and $\alpha \geq \frac{1}{2}$. It follows from (2.24) that for $n \geq 48r-2$,

$$\frac{1}{\left(1+\frac{24r-1}{24n}\right)^{r-\frac{1}{2}}} \geq 1 - \frac{24r-1}{24n}\left(r-\frac{1}{2}\right),$$

or equivalently,

$$\frac{(\frac{1}{2})_{r-1}24^r\pi}{12(24(n+r)-1)^{r-\frac{1}{2}}} \geq \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r}\frac{24r-1}{24n^{r+\frac{1}{2}}}.$$

Observing that for $n \geq 48r-2$,

$$\frac{1}{n^{r+\frac{1}{2}}} \leq \frac{1}{\sqrt{48r-2}n^r},$$

we obtain (4.5) for $n \geq 48r-2$.

For the second term of the right hand side of (4.4), we claim that for $n \geq 48r-2$,

$$\frac{(r-1)!24^r}{(24n-1)^r} \leq \frac{b_3}{n^r}, \tag{4.6}$$

where

$$b_3 = (r-1)!\left(1+\frac{r}{24}\left(\frac{1}{48r-2}\right)\left(\frac{48}{47}\right)^{r+1}\right).$$

Setting $x = \frac{1}{24n}$, $\alpha = r$ and $c = \frac{1}{48}$, for $n \geq 48r-2$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we see that for $n \geq 48r-2$,

$$\frac{1}{\left(1-\frac{1}{24n}\right)^r} \leq 1 + \left(\frac{48}{47}\right)^{r+1}\frac{r}{24n}.$$

So we obtain (4.6) for $n \geq 48r-2$.

Since the last term of the right hand side of (4.4) is positive, combining (4.5) and (4.6), we deduce that for $n \geq 48r-2$,

$$H_r \geq \frac{b_1}{n^{r-\frac{1}{2}}} - \frac{b_2+b_3}{n^r}. \tag{4.7}$$

To derive a simpler expression for a lower bound of H_r, let

$$m_1 = \frac{4(b_2+b_3)^2}{b_1^2}.$$

Then we have that for $n \geq m_1 + 1$,

$$\frac{b_2+b_3}{n^r} < \frac{b_1}{2n^{r-\frac{1}{2}}}.$$

It follows that for $n \geq \max\{48r-2, m_1+1\}$,

$$H_r(n) > \frac{b_1}{2n^{r-\frac{1}{2}}}. \tag{4.8}$$

Combining (4.2) and (4.8), we find that for $n \geq \max\{50, 48r-2, m_1+1\}$,

$$(-1)^{r-1}\Delta^r \log p(n) > \frac{b_1}{2n^{r-\frac{1}{2}}} - 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{4.9}$$

Notice that for $r \geq 1$ and $n \geq 1$,

$$\frac{1}{n^{r-\frac{1}{2}}} \geq \frac{\left(\frac{23}{24}\right)^{r-\frac{1}{2}}}{\left(n-\frac{1}{24}\right)^{r-\frac{1}{2}}}.$$

Thus, for $n \geq \max\{50, 48r-2, m_1+1\}$,

$$(-1)^{r-1}\Delta^r \log p(n) > \left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2n^{r-\frac{1}{2}}} - 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{4.10}$$

To prove that the right hand side of (4.10) is positive for sufficiently large n, consider the following equation

$$\left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2x^{r-\frac{1}{2}}} = 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}}. \tag{4.11}$$

The solution of (4.11) can be expressed in terms of the Lambert W function, namely,

$$x = \frac{1}{24} + \frac{6\,(2r-1)^2}{\pi^2} W\left(-\frac{\sqrt{46}\pi}{24(2r-1)}\left(\frac{\pi\left(\frac{1}{2}\right)_{r-1}}{20\sqrt{6}}\right)^{\frac{1}{2r-1}}\right)^2. \tag{4.12}$$

For $r \geq 1$, we have $\left(\frac{1}{2}\right)_r < r!$. Using the estimate of $r!$ as given by (3.59), we obtain that for $r \geq 3$,

$$-\frac{1}{e} < -\frac{\sqrt{46}\pi}{24(2r-1)}\left(\frac{\pi\left(\frac{1}{2}\right)_{r-1}}{20\sqrt{6}}\right)^{\frac{1}{2r-1}} < 0.$$

Thus (4.11) has two real roots. Let m_2 be the larger real root of equation (4.11). Clearly, for sufficiently large x,

$$\left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2x^{r-\frac{1}{2}}} - 5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}} > 0. \tag{4.13}$$

It follows that for $n \geq m_2 + 1$,

$$\left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2n^{r-\frac{1}{2}}} - 5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}} > 0. \tag{4.14}$$

Let

$$n(r) = \max\{50, 48r - 2, m_1 + 1, m_2 + 1\}.$$

Combining (4.9) and (4.14), we conclude that for $n \geq n(r)$,

$$(-1)^{r-1} \Delta^r \log p(n) > 0. \tag{4.15}$$

This completes the proof. ∎

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