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Higher Order Turán Inequalities for the Partition Function

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Abstract.

The Turán inequalities and the higher order Turán inequalities arise in the study of Maclaurin coefficients of the real entire functions in the Laguerre-Pólya class. A sequence $\{a_n\}_{n\geq 0}$ of real numbers is said to satisfy the Turán inequalities or to be log-concave, if for $n\geq 1$, $a_n^2 - a_{n-1}a_{n+1} \geq 0$. It is said to satisfy the higher order Turán inequalities if for $n \geq 1$, $4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 \geq 0$. For the partition function $p(n)$, DeSalvo and Pak showed that for $n > 25$, the sequence $\{p(n)\}_{n>25}$ is log-concave, that is, $p(n)^2 - p(n-1)p(n+1) > 0$ for $n > 25$. It was conjectured by the first named author that $p(n)$ satisfies the higher order Turán inequalities for $n \geq 95$. In this paper, we prove this conjecture by using the Hardy-Ramanujan-Rademacher formula to derive an upper bound and a lower bound for $p(n+1)p(n-1)/p(n)^2$. Consequently, for $n \geq 95$, the Jensen polynomials $p(n-1)+3p(n)x+3p(n+1)x^2+p(n+2)x^3$ have only distinct real zeros. We conjecture that for any positive integer $m \geq 4$ there exists an integer $N(m)$ such that for $n \geq N(m)$, the Jensen polynomial associated with the sequence $(p(n), p(n+1), \cdots, p(n+m))$ has only real zeros. This conjecture was independently posed by Ono.

Keywords: the partition function, log-concavity, the higher order Turán inequalities, the Hardy-Ramanujan-Rademacher formula, the Jensen polynomials

AMS Classification: 05A20, 11P82

Suggested Running Title: Higher Order Turán Inequalities

1 Introduction

The objective of this paper is to prove the higher order Turán inequalities for the partition function $p(n)$ when $n \geq 95$, as conjectured in [6]. The Turán inequalities and the higher order Turán inequalities are related to the Laguerre-Pólya class of real entire functions [14, 44]. In this paper, a sequence $\{a_n\}_{n\geq 0}$ always means a sequence of real numbers, and it is said to satisfy the Turán inequalities or to be log-concave, if

$$a_n^2 - a_{n-1}a_{n+1} \geq 0, \tag{1.1}$$

for $n \geq 1$. The inequalities (1.1) are also called the Newton inequalities [8, 11, 35, 46]. We say that a sequence $\{a_n\}_{n\geq 0}$ satisfies the higher order Turán inequalities or cubic Newton inequalities if for $n \geq 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_na_{n+2}) - (a_na_{n+1} - a_{n-1}a_{n+2})^2 \geq 0, \tag{1.2}$$

see [14, 35, 41]. It is worth mentioning the double Turán inequalities and the higher order iterated Turán inequalities introduced by Csordas [10], see also [24]. Given a sequence $\{a_n\}_{n\geq 0}$, for $n \geq 1$, let

$$T_1(n) = a_n^2 - a_{n-1}a_{n+1},$$

and for $k \geq 2$ and $n \geq k$, let

$$T_k(n) = T_{k-1}(n)^2 - T_{k-1}(n-1)T_{k-1}(n+1).$$

A sequence $\{a_n\}_{n\geq 0}$ is said to satisfy the double Turán inequalities if $T_1(n) \geq 0$ for $n \geq 1$ and $T_2(n) \geq 0$ for $n \geq 2$. In general, $\{a_n\}_{n\geq 0}$ is said to satisfy the l-th order iterated Turán inequalities if for $1 \leq k \leq l$ and $n \geq k$, we have $T_k(n) \geq 0$. It should be noted that the above notion of higher order iterated Turán inequalities coincides with the notion of the higher order log-concavity introduced by Moll [34], see also [3, 4]. In the terminology of Moll, a sequence $\{a_n\}_{n\geq 0}$ satisfying the l-th order iterated Turán inequalities is called l-log-concave. A sequence $\{a_n\}_{n\geq 0}$ is said to be infinitely log-concave if it is l-log-concave for any $l \geq 1$.

It was conjectured by Boros and Moll [3] that for each $n \geq 0$, the sequence of binomial coefficients $\{\binom{n}{k}\}_{k=0}^{n}$ is infinitely log-concave. A stronger conjecture was proposed independently by Stanley (see [4]), Fisk [17] and McNamara and Sagan [32], which states that if $a_0 + a_1 x + \cdots + a_n x^n$ is a real-rooted polynomial with nonnegative coefficients, then so is the polynomial $b_0 + b_1 x + \cdots + b_n x^n$, where $b_0 = a_0$, $b_k = a_k^2 - a_{k+1}a_{k-1}$ for $1 \le k \le n-1$, and $b_n = a_n$. This conjecture has been proved by Brändén [4].

A real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.3}$$

is said to be in the Laguerre-Pólya class, denoted $\psi(x) \in \mathscr{L}\mathscr{P}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

where c, β, x_k are real numbers, $\alpha \ge 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. These functions are the only ones which are uniform limits of polynomials whose zeros are real. We refer to [29] and [42] for the background on the theory of the $\mathscr{L}\mathscr{P}$ class.

Jensen [22] proved that a real entire function $\psi(x)$ belongs to the $\mathscr{L}\mathscr{P}$ class if and only if for any positive integer m, the m-th associated Jensen polynomial

$$g_m(x) = \sum_{k=0}^{m} \binom{m}{k} \gamma_k x^k \tag{1.4}$$

has only real zeros. More properties of the Jensen polynomials can be found in [8, 11, 12].

Pólya and Schur [39] also obtained the above result based on multiplier sequences of the second kind. A real sequence $\{\gamma_k\}_{k\ge 0}$ is called a multiplier sequence of the second kind if for any nonnegative integer m and every real polynomial $\sum_{k=0}^{m} a_k x^k$ with only real zeros of the same sign, the polynomial $\sum_{k=0}^{m} a_k \gamma_k x^k$ has only real zeros. Pólya and Schur [39] proved that each multiplier sequence of the second kind satisfies the Turán inequalities. Moreover, they showed that a real entire function $\psi(x)$ belongs to the $\mathscr{L}\mathscr{P}$ class if and only if its Maclaurin coefficient sequence is a multiplier sequence of the second kind. This implies that the Maclaurin coefficients of $\psi(x)$ in the $\mathscr{L}\mathscr{P}$ class satisfy the Turán inequalities

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \ge 0 \tag{1.5}$$

for $k \ge 1$. In fact, (1.5) is a consequence of another property of the $\mathscr{L}\mathscr{P}$ class due to Pólya and Schur [39]: Let $\psi(x)$ be a real entire function in the $\mathscr{L}\mathscr{P}$ class. For any nonnegative integer n, the n-th derivative $\psi^{(n)}(x)$of $\psi(x)$ also belongs to the $\mathscr{L}\mathscr{P}$ class. It is readily seen that the m-th Jensen polynomial associated with $\psi^{(n)}(x)$ is

$$g_{m,n}(x) = \sum_{k=0}^{m} \binom{m}{k} \gamma_{k+n} x^k, \tag{1.6}$$

and hence it has only real zeros for any nonnegative integers m and n. In particular, taking $m=2$, for any nonnegative integer n, the real-rootedness of $g_{2,n}(x)$ implies the inequality (1.5), that is, the discriminant $4(\gamma_{n+1}^2 - \gamma_n\gamma_{n+2})$ is nonnegative.

Dimitrov [14] observed that for a real entire function $\psi(x)$ in the $\mathscr{L}\mathscr{P}$ class, the Maclaurin coefficients satisfy the higher order Turán inequalities

$$4(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \geq 0 \tag{1.7}$$

for $k \geq 1$. This fact follows from a theorem of Mařík [31] stating that if a real polynomial

$$\sum_{k=0}^{m} \binom{m}{k} a_k x^k \tag{1.8}$$

of degree $m \geq 3$ has only real zeros, then $a_0, a_1, \ldots, a_m$ satisfy the higher order Turán inequalities.

As noted in [6], for $k=1$, the polynomial in (1.7) coincides with an invariant

$$I(a_0,a_1,a_2,a_3) = 3a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2 + 6a_0a_1a_2a_3$$

of the cubic binary form

$$a_3x^3 + 3a_2x^2y + 3a_1xy^2 + a_0y^3. \tag{1.9}$$

In other words, the above invariant $I(a_0,a_1,a_2,a_3)$ can be rewritten as

$$I(a_0,a_1,a_2,a_3) = 4(a_1^2 - a_0a_2)(a_2^2 - a_1a_3) - (a_1a_2 - a_0a_3)^2. \tag{1.10}$$

We refer to Hilbert [20], Kung and Rota [25] and Sturmfels [43] for the background on the invariant theory of binary forms. Notice that $27I(a_0,a_1,a_2,a_3)$ is the discriminant of the cubic polynomial $a_3x^3 + 3a_2x^2 + 3a_1x + a_0$ [33]. For a real cubic polynomial, the discriminant is positive if and only if the three zeros are real and distinct. In general, for a real polynomial of degree greater than or equal to four, the discriminant is positive if and only if the number of non-real roots is multiple of four. More properties about discriminant can be found in [21] and [33].

Recall that for a real entire function $\psi(x)$ in the $\mathscr{L}\mathscr{P}$ class, its n-th derivative $\psi^{(n)}(x)$ is also a real entire function in the $\mathscr{L}\mathscr{P}$ class. Thus the real-rootedness of the cubic Jensen polynomial $g_{3,n}(x)$ associated with $\psi^{(n)}(x)$ implies the higher order Turán inequalities (1.7) of Dimitrov, that is, the discriminant $27I(\gamma_n,\gamma_{n+1},\gamma_{n+2},\gamma_{n+3})$ is nonnegative.

Let $f(x)$ be a real polynomial with degree m. If $x_1,x_2,\ldots,x_m$ are the roots of $f(x)$, let $S_k=\sum_{i=1}^m x_i^k$ denote the k-th Newton sums of $f(x)$ for $k\geq 1$ and $S_0=m$. Explicit expressions for these power sums via the coefficients of $f(x)$ are given by the Waring formulas [30]. The Hermite matrix of $f(x)$ is a symmetric $m\times m$ matrix, defined as

$$\mathscr{H}_m(f(x))=(S_{i+j-2})_{i,j=1,2,\cdots,m}.$$

It is a Hankel matrix whose entries are polynomials in the coefficients of $f(x)$. We write $D_k(f(x))$ for the leading principal minor of $\mathscr{H}_m(f(x))$ of order k. A theorem of Hermite [37] says that all the zeros of $f(x)$ are real if and only if all the leading principal minors of $\mathscr{H}_m(f(x))$ are nonnegative. By this theorem, one can state that much more (higher Turán) inequalities would need to be true in order for higher degree Jensen polynomials to have only real zeros. Considering $m=4$, for example, the Jensen polynomials $g_{4,k}(x)$, that is, for $k\geq 0$,

$$g_{4,k}(x)=\gamma_k+4\gamma_{k+1}x+6\gamma_{k+2}x^2+4\gamma_{k+3}x^3+\gamma_{k+4}x^4.$$

It can be calculated that for $k\geq 0$,

$$D_1(g_{4,k}(x))=4,\qquad D_2(g_{4,k}(x))=\frac{48\left(\gamma_{k+3}^2-\gamma_{k+2}\gamma_{k+4}\right)}{\gamma_{k+4}^2},$$

$$D_3(g_{4,k}(x))=\frac{192A}{\gamma_{k+4}^4},\qquad \det(\mathscr{H}_4(g_{4,k}(x)))=\frac{256(B^3-27C^2)}{\gamma_{k+4}^6},$$

where

$$\begin{aligned}
A= &\ 6\gamma_{k+2}^2\gamma_{k+3}^2+\gamma_k\gamma_{k+2}\gamma_{k+4}^2+14\gamma_{k+1}\gamma_{k+2}\gamma_{k+3}\gamma_{k+4}-9\gamma_{k+4}\gamma_{k+2}^3\\
&-8\gamma_{k+1}\gamma_{k+3}^3-3\gamma_{k+1}^2\gamma_{k+4}^2-\gamma_k\gamma_{k+4}\gamma_{k+3}^2,\\
B= &\ 3\gamma_{k+2}^2-4\gamma_{k+1}\gamma_{k+3}+\gamma_k\gamma_{k+4},\\
C= &\ \gamma_{k+2}^3-2\gamma_{k+1}\gamma_{k+2}\gamma_{k+3}-\gamma_k\gamma_{k+2}\gamma_{k+4}+\gamma_k\gamma_{k+3}^2+\gamma_{k+1}^2\gamma_{k+4}.
\end{aligned}$$

Notice that B,C and B^3-27C^2 are actually the invariants of the quartic binary form with the same coefficients as $g_{4,k}(x)$ [16, 20] and $256(B^3-27C^2)$ is the discriminant of $g_{4,k}(x)$. By Hermite's theorem, one of the necessary and sufficient conditions for $g_{4,k}(x)$ to eventually have only real zeros is that all the leading principal minors of $\mathscr{H}_4(g_{4,k}(x))$ are nonnegative, that is, for $k\geq 0$,

$$\gamma_{k+3}^2-\gamma_{k+2}\gamma_{k+4}\geq 0,\quad A\geq 0,\quad B^3-27C^2\geq 0. \tag{1.11}$$

Real entire functions in the $\mathscr{L}\mathscr{P}$ class with nonnegative Maclaurin coefficients also received much attention. Aissen, Schoenberg and Whitney [1] proved that if $\psi(x)$ is a real entire function in the $\mathscr{L}\mathscr{P}$ class with nonnegative Maclaurin coefficients, then the sequence $\{\gamma_k/k!\}$ associated with $\psi(x)$ forms a Pólya frequency sequence. An infinite sequence $\{a_n\}_{n\geq 0}$ of nonnegative numbers is called a Pólya frequency sequence (or a PF-sequence) if the matrix $(a_{i-j})_{i,j\geq 0}$ is a totally positive matrix, where $a_n = 0$ if $n < 0$, that is, all minors of $(a_{i-j})_{i,j\geq 0}$ have nonnegative determinants. More properties of totally positive matrices and PF-sequences can be found in [9, 23].

The $\mathscr{L}\mathscr{P}$ class is closely related to the Riemann hypothesis. Let ζ denote the Riemann zeta-function and Γ be the gamma-function. The Riemann ξ-function is defined by

$$\xi(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-z/2-1/4}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right), \tag{1.12}$$

see, for example, Boas [2]. It is well known that the Riemann ξ-function is an entire function of order one and can be represented in the following form[38]:

$$\frac{1}{8}\xi\left(\frac{x}{2}\right) = \sum_{k=0}^{\infty}(-1)^k \hat{b}_k \frac{x^{2k}}{(2k)!}, \quad k = 0,1,2,\cdots \tag{1.13}$$

where

$$\hat{b}_k = \int_0^{\infty} t^{2t}\Phi(t)\mathrm{d}t \quad \text{and} \quad \Phi(t) = \sum_{n=0}^{\infty}(2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t})\exp(-n^2\pi e^{4t}).$$

Setting $z = -x^2$ in (1.13), we are led to an entire function of order $1/2$, denoted $\xi_1(z)$, that is,

$$\xi_1(z) = \sum_{k=0}^{\infty}\hat{\gamma}_k \frac{z^k}{k!}, \tag{1.14}$$

where

$$\hat{\gamma}_k = \frac{k!}{(2k)!}\hat{b}_k, \;\; k = 0,1,2,\cdots.$$

Thus, the Riemann hypothesis holds if and only if $\xi_1(z)$ belongs to the $\mathscr{L}\mathscr{P}$ class. We note that $\xi_1(z)$ has no positive zeros since $\hat{\gamma}_k$ is positive for $k \geq 0$. For a real entire function $\psi(x)$ as defined in (1.3), it is well known that $\psi(x)$ belongs to $\mathscr{L}\mathscr{P}$ if and only if the Jensen polynomials $g_{m,n}(x)$ (1.6) have only real zeros [12, 37]. Let $\hat{g}_{m,n}(x)$ denote the Jensen polynomials associated with $\xi_1(z)$, that is,

$$\hat{g}_{m,n}(x) = \sum_{k=0}^{m}\binom{m}{k}\hat{\gamma}_{k+n}x^k, \quad m,n = 0,1,2,\cdots,$$

then Riemann hypothesis is equivalent to the statement that $\hat{g}_{m,n}(x)$ has only real zeros for $m \geq 1$ and $n \geq 0$. For $m = 2$ and $n \geq 0$, $\hat{g}_{2,n}(x)$ has only real zeros if and only if the discriminant of $\hat{g}_{2,n}(x)$ is nonnegative, that is, for $k \geq 1$,

$$\hat{\gamma}_k^2 - \hat{\gamma}_{k-1}\hat{\gamma}_{k+1} \geq 0. \tag{1.15}$$

The above inequalities (1.15) were proved by Csordas, Norfolk and Varga [11]. This shows that for each $n \geq 0$, $\hat{g}_{2,n}(x)$ has only real zeros. For $m = 3$ and $n \geq 0$, the real-rootness of $\hat{g}_{3,n}(x)$ can be deduced from the higher order Turán inequalities (1.7), that is, for $k \geq 1$,

$$4(\hat{\gamma}_k^2 - \hat{\gamma}_{k-1}\hat{\gamma}_{k+1})(\hat{\gamma}_{k+1}^2 - \hat{\gamma}_k\hat{\gamma}_{k+2}) - (\hat{\gamma}_k\hat{\gamma}_{k+1} - \hat{\gamma}_{k-1}\hat{\gamma}_{k+2})^2 \geq 0,$$

as proved by Dimitrov and Lucas [15]. Recently, Griffin, Ono, Rolen, and Zagier [18] proved that for each $m \geq 1$, $\hat{g}_{m,n}(x)$ has only real zeros with at most finitely many exceptions n.

Let us now turn to the partition function. A partition of a positive integer n is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. A sequence $\{a_k\}_{k\geq 0}$ satisfying the Turán inequalities, that is, $a_k^2 - a_{k-1}a_{k+1} \geq 0$ for $k \geq 1$, is also called log-concave. DeSalvo and Pak [13] proved the log-concavity of the partition function for $n > 25$ as well as the following inequality as conjectured in [5]: For $n \geq 2$,

$$\frac{p(n-1)}{p(n)}\left(1 + \frac{1}{n}\right) > \frac{p(n)}{p(n+1)}. \tag{1.16}$$

DaSalvo and Pak also conjectured that for $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{1.17}$$

Chen, Wang and Xie [7] gave an affirmative answer to this conjecture.

It was conjectured in [6] that for large n, the partition function $p(n)$ satisfies many inequalities pertaining to invariants of a binary form. Here we mention two of them.

Conjecture 1.1. *For $n \geq 95$, the higher order Turán inequalities*

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_na_{n+2}) - (a_na_{n+1} - a_{n-1}a_{n+2})^2 \geq 0 \tag{1.18}$$

hold for $a_n = p(n)$.

The following conjecture is a higher order analogue of (1.17).

Conjecture 1.2. *Let*

$$u_n = \frac{p(n+1)p(n-1)}{p(n)^2}. \tag{1.19}$$

For $n \geq 2$,

$$4(1-u_n)(1-u_{n+1}) < \left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right)(1-u_nu_{n+1})^2.$$

The objective of this paper is to prove Conjecture 1.1. In fact, we shall prove the following equivalent form.

Theorem 1.3. *Let u_n be defined as in* (1.19). *For $n \geq 95$,*

$$4(1-u_n)(1-u_{n+1})-(1-u_nu_{n+1})^2>0. \tag{1.20}$$

The above theorem can be restated as follows.

Theorem 1.4. *For any $n \geq 95$, the cubic polynomial*

$$p(n-1)+3p(n)x+3p(n+1)x^2+p(n+2)x^3$$

has three distinct real zeros.

In general, we propose the following conjecture.

Conjecture 1.5. *For any positive integer $m \geq 4$, there exists a positive integer $N(m)$ such that for any $n \geq N(m)$, the Jensen polynomial*

$$\sum_{k=0}^{m}\binom{m}{k}p(k+n)x^k$$

has only real zeros.

The above conjecture was independently proposed by Ono [36]. For fixed degree m and large n, recently, Griffin, Ono, Rolen, and Zagier [18] proved that this conjecture is true. In fact, they showed that for suitable entire functions and certain sequences, the associated Jensen polynomials have only real and distinct zeros with at most finite exceptions. To be more precisely, they defined the normalized Jensen polynomials by changing the variable of Jensen polynomials and proved that for large n, the normalized Jensen polynomials were small perturbations of Hermite polynomials $H_m(x)$. Since all the roots of $H_m(x)$ are real and distinct [45], the real parts of the roots of the normalized Jensen polynomials are distinct for large n, which implies that all the roots are real and distinct. According to the definition of normalized Jensen polynomials, it is easy to see that for large n, all the roots of such Jensen polynomials are also real and distinct.

Assume that $N(m)$ is the minimum value in Conjecture 1.5. Larson and Wagner [26] showed that $N(3)=94$, $N(4)=206$, and $N(5)=381$, and that $N(m) \leq (3m)^{24m}(50m)^{3m^2}$. They also gave a proof of Conjecture 1.2.

2 Bounding u_n

In this section, we give an upper bound and a lower bound for

$$u_n=\frac{p(n+1)p(n-1)}{p(n)^2},$$

as defined in (1.19). DeSalvo and Pak [13] proved that for $n > 25$,

$$1-\frac{1}{n+1} < u_n < 1.$$

On the other hand, Chen, Wang and Xie [7] showed that for $n \geq 45$,

$$1-\frac{\pi}{\sqrt{24}n^{3/2}+\pi} < u_n.$$

Nevertheless, the above bounds for u_n are not sharp enough for the purpose of proving Theorem 1.3. To state our bounds for u_n, we adopt the following notation as used in [28]:

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}. \tag{2.1}$$

For convenience, let

$$x = \mu(n-1), \quad y = \mu(n), \quad z = \mu(n+1), \quad w = \mu(n+2). \tag{2.2}$$

Define

$$f(n) = e^{x-2y+z}\frac{\left(x^{10}-x^9-1\right)y^{24}\left(z^{10}-z^9-1\right)}{x^{12}\left(y^{10}-y^9+1\right)^2 z^{12}}, \tag{2.3}$$

$$g(n) = e^{x-2y+z}\frac{\left(x^{10}-x^9+1\right)y^{24}\left(z^{10}-z^9+1\right)}{x^{12}\left(y^{10}-y^9-1\right)^2 z^{12}}. \tag{2.4}$$

Then we have the following bounds for u_n.

Theorem 2.1. *For $n \geq 1207$,*

$$f(n) < u_n < g(n). \tag{2.5}$$

In order to give a proof of Theorem 2.1, we need the following upper bound and lower bound for $p(n)$.

Lemma 2.2. *Let*

$$B_1(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{10}}\right),$$

$$B_2(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{10}}\right),$$

then for $n \geq 1206$,

$$B_1(n) < p(n) < B_2(n). \tag{2.6}$$

The proof of Lemma 2.2 relies on the Hardy-Ramanujan-Rademacher formula [19, 40] for $p(n)$ as well as Lehmer's error bound for the remainder of this formula. The Hardy-Ramanujan-Rademacher formula reads

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] + R_2(n,N), \tag{2.7}$$

where $A_k(n)$ is an arithmetic function and $R_2(n,N)$ is the remainder term, see, for example, Rademacher [40]. Lehmer [27, 28] gave the following error bound:

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)}\right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2 \right], \tag{2.8}$$

which is valid for all positive integers n and N.

Proof of Lemma 2.2. Consider the Hardy-Ramanujan-Rademacher formula (2.7) for $N = 2$, and note that $A_1(n) = 1$ and $A_2(n) = (-1)^n$ for any positive integer n. Hence (2.7) can be rewritten as

$$p(n) = \frac{\sqrt{12} e^{\mu(n)}}{24n-1} \left(1 - \frac{1}{\mu(n)} + T(n)\right), \tag{2.9}$$

where

$$T(n) = \frac{(-1)^n}{\sqrt{2}} \left(\left(1 - \frac{2}{\mu(n)}\right) e^{-\mu(n)/2} + \left(1 + \frac{2}{\mu(n)}\right) e^{-3\mu(n)/2} \right) + \left(1 + \frac{1}{\mu(n)}\right) e^{-2\mu(n)} + (24n-1) R_2(n,2) / \sqrt{12} e^{\mu(n)}. \tag{2.10}$$

In order to prove (2.6), we proceed to use Lehmer's error bound in (2.8) to show that for $n > 1520$,

$$|T(n)| < \frac{1}{\mu(n)^{10}}. \tag{2.11}$$

Let us begin with the first three terms in (2.10). Evidently, for $n \geq 1$,

$$0 < \frac{1}{\mu(n)} < \frac{1}{2},$$

so that

$$\left(1 - \frac{2}{\mu(n)}\right) e^{-\mu(n)/2} < e^{-\mu(n)/2}, \tag{2.12}$$

$$\left(1+\frac{2}{\mu(n)}\right)e^{-3\mu(n)/2} < 2e^{-3\mu(n)/2}, \tag{2.13}$$

$$\left(1+\frac{1}{\mu(n)}\right)e^{-2\mu(n)} < 2e^{-2\mu(n)}. \tag{2.14}$$

As for the last term in (2.10), we claim that for $n > 350$,

$$\left|\frac{(24n-1)R_2(n,2)}{\sqrt{12}e^{\mu(n)}}\right| < e^{-\mu(n)/2}. \tag{2.15}$$

Applying (2.8) with $N = 2$, we obtain that

$$\begin{aligned}
&\left|\frac{(24n-1)R_2(n,2)}{\sqrt{12}e^{\mu(n)}}\right| = \left|\frac{36\mu(n)^2R_2(n,2)}{\sqrt{12}\pi^2e^{\mu(n)}}\right| \\
&< \frac{\mu(n)^2e^{-\mu(n)}}{2^{2/3}} + \frac{12\sqrt[3]{2}e^{-\mu(n)/2}}{\mu(n)} - \frac{12\sqrt[3]{2}e^{-3\mu(n)/2}}{\mu(n)} - 12\sqrt[3]{2}e^{-\mu(n)} \\
&< \frac{\mu(n)^2e^{-\mu(n)}}{2^{2/3}} + \frac{12\sqrt[3]{2}e^{-\mu(n)/2}}{\mu(n)} \\
&< \frac{24e^{-\mu(n)/2}}{\mu(n)} + \mu(n)^2e^{-\mu(n)}.
\end{aligned} \tag{2.16}$$

To bound the first term in (2.16), we find that for $n > 350$,

$$\frac{24e^{-\mu(n)/2}}{\mu(n)} < \frac{e^{-\mu(n)/2}}{2}, \tag{2.17}$$

which simplifies to

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1} > 48, \tag{2.18}$$

which is true for $n > 350$. Concerning the second term in (2.16), it will be shown that for $n > 22$,

$$\mu(n)^2e^{-\mu(n)} < \frac{e^{-\mu(n)/2}}{2}, \tag{2.19}$$

which can be rewritten as

$$\frac{e^{\mu(n)/4}}{\mu(n)/4} > 4\sqrt{2}. \tag{2.20}$$

Let

$$F(t) = \frac{e^t}{t}. \tag{2.21}$$

Since $F'(t) = e^t(t-1)/t^2 > 0$ for $t > 1$, $F(t)$ is increasing for $t > 1$. Thus,

$$F\left(\frac{\mu(n)}{4}\right) = \frac{e^{\mu(n)/4}}{\mu(n)/4} > F(3) = \frac{e^3}{3} > 4\sqrt{2}.$$

Here we have used the fact that for $n > 22$, $\mu(n)/4 > 3$. This proves (2.20). Applying the estimates (2.17) and (2.19) to (2.16), we reach (2.15).

Taking all the above estimates into account, we deduce that for $n > 350$,

$$|T(n)| < 6e^{-\mu(n)/2}. \tag{2.22}$$

To obtain (2.11), we have only to show that for $n > 1520$,

$$6e^{-\mu(n)/2} < \frac{1}{\mu(n)^{10}}, \tag{2.23}$$

which can be recast as

$$\frac{e^{\mu(n)/20}}{\mu(n)/20} > 20\sqrt[10]{6}. \tag{2.24}$$

Since $\mu(n)/20 > 5$ for $n > 1520$, by the monotone property of $F(t)$, we have that for $n > 1520$,

$$F\left(\frac{\mu(n)}{20}\right) = \frac{e^{\mu(n)/20}}{\mu(n)/20} > F(5) = \frac{e^5}{5} > 20\sqrt[10]{6},$$

as asserted by (2.24). Thus (2.11) follows from (2.22) and (2.23). In other words, for $n > 1520$,

$$-\frac{1}{\mu(n)^{10}} < T(n) < \frac{1}{\mu(n)^{10}}. \tag{2.25}$$

Substituting (2.9) into (2.25), we see that (2.6) holds for $n > 1520$. It is routine to check that (2.6) is true for $1206 \le n \le 1520$, and hence the proof is complete. ∎

We are now ready to prove Theorem 2.1 by Lemma 2.2.

Proof of Theorem 2.1. Since $B_1(n)$ and $B_2(n)$ are all positive for $n \ge 1$, using the bounds for $p(n)$ in (2.6), we find that for $n \ge 1207$,

$$\frac{B_1(n-1)B_1(n+1)}{B_2(n)^2} < \frac{p(n-1)p(n+1)}{p(n)^2} < \frac{B_2(n-1)B_2(n+1)}{B_1(n)^2}.$$

This proves (2.5). ∎

3 An inequality on $f(n)$ and $g(n)$

In this section, we establish an inequality between $f(n)$ and $g(n+1)$ which will be used in the proof of Theorem 1.3.

Theorem 3.1. *For $n \geq 2$,*

$$g(n+1) < f(n) + \frac{110}{\mu(n-1)^5}. \tag{3.1}$$

Proof. Let $\mu(n)$ be defined as in (2.1), that is,

$$\mu(n) = \frac{\pi\sqrt{24n-1}}{6},$$

and let

$$x = \mu(n-1), \quad y = \mu(n), \quad z = \mu(n+1), \quad w = \mu(n+2),$$

as defined in (2.2).

Since $x \geq 0$ for $n \geq 2$, we proceed to show that for $n \geq 2$,

$$f(n)x^5 - g(n+1)x^5 + 110 > 0.$$

Let

$$\alpha(t) = t^{10} - t^9 + 1, \qquad \beta(t) = t^{10} - t^9 - 1. \tag{3.2}$$

By the definitions of $f(n)$ and $g(n)$ as given in (2.3) and (2.4), we obtain that

$$f(n)x^5 - g(n+1)x^5 + 110 = \frac{-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 110t_3}{t_3}, \tag{3.3}$$

where

$$t_1 = x^{12}z^{36}\alpha(y)^3\alpha(w), \tag{3.4}$$

$$t_2 = y^{36}w^{12}\beta(x)\beta(z)^3, \tag{3.5}$$

$$t_3 = x^7y^{12}z^{12}w^{12}\alpha(y)^2\beta(z)^2. \tag{3.6}$$

Since $t_3 > 0$ for $n \geq 2$, (3.1) is equivalent to

$$-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 110t_3 > 0, \tag{3.7}$$

for $n \geq 2$. To verify (3.7), we shall estimate $t_1, t_2, t_3, e^{w+y-2z}$ and e^{x-2y+z} in terms of x. Noting that for $n \geq 2$,

$$y = \sqrt{x^2 + \frac{2\pi^2}{3}}, \qquad z = \sqrt{x^2 + \frac{4\pi^2}{3}}, \qquad w = \sqrt{x^2 + 2\pi^2}, \tag{3.8}$$

we have

$$y = x + \frac{\pi^2}{3x} - \frac{\pi^4}{18x^3} + \frac{\pi^6}{54x^5} - \frac{5\pi^8}{648x^7} + \frac{7\pi^{10}}{1944x^9} - \frac{7\pi^{12}}{3888x^{11}} + o\left(\frac{1}{x^{12}}\right),$$

$$z = x + \frac{2\pi^2}{3x} - \frac{2\pi^4}{9x^3} + \frac{4\pi^6}{27x^5} - \frac{10\pi^8}{81x^7} + \frac{28\pi^{10}}{243x^9} - \frac{28\pi^{12}}{243x^{11}} + o\left(\frac{1}{x^{12}}\right),$$

$$w = x + \frac{\pi^2}{x} - \frac{\pi^4}{2x^3} + \frac{\pi^6}{2x^5} - \frac{5\pi^8}{8x^7} + \frac{7\pi^{10}}{8x^9} - \frac{21\pi^{12}}{16x^{11}} + o\left(\frac{1}{x^{12}}\right).$$

It is readily checked that for $x \geq 4$,

$$y_1 < y < y_2, \tag{3.9}$$

$$z_1 < z < z_2, \tag{3.10}$$

$$w_1 < w < w_2, \tag{3.11}$$

where

$$y_1 = x + \frac{\pi^2}{3x} - \frac{\pi^4}{18x^3} + \frac{\pi^6}{54x^5} - \frac{5\pi^8}{648x^7} + \frac{7\pi^{10}}{1944x^9} - \frac{7\pi^{12}}{3888x^{11}}, \tag{3.12}$$

$$y_2 = x + \frac{\pi^2}{3x} - \frac{\pi^4}{18x^3} + \frac{\pi^6}{54x^5} - \frac{5\pi^8}{648x^7} + \frac{7\pi^{10}}{1944x^9}, \tag{3.13}$$

$$z_1 = x + \frac{2\pi^2}{3x} - \frac{2\pi^4}{9x^3} + \frac{4\pi^6}{27x^5} - \frac{10\pi^8}{81x^7} + \frac{28\pi^{10}}{243x^9} - \frac{28\pi^{12}}{243x^{11}}, \tag{3.14}$$

$$z_2 = x + \frac{2\pi^2}{3x} - \frac{2\pi^4}{9x^3} + \frac{4\pi^6}{27x^5} - \frac{10\pi^8}{81x^7} + \frac{28\pi^{10}}{243x^9}, \tag{3.15}$$

$$w_1 = x + \frac{\pi^2}{x} - \frac{\pi^4}{2x^3} + \frac{\pi^6}{2x^5} - \frac{5\pi^8}{8x^7} + \frac{7\pi^{10}}{8x^9} - \frac{21\pi^{12}}{16x^{11}}, \tag{3.16}$$

$$w_2 = x + \frac{\pi^2}{x} - \frac{\pi^4}{2x^3} + \frac{\pi^6}{2x^5} - \frac{5\pi^8}{8x^7} + \frac{7\pi^{10}}{8x^9}. \tag{3.17}$$

With these bounds of y, z and w in (3.9), (3.10) and (3.11), we are now in a position to estimate $t_1, t_2, t_3, e^{w+y-2z}$ and e^{x-2y+z} in terms of x.

First, we consider t_1, t_2, and t_3. By the definition of $\alpha(t)$,

$$\alpha(w) = w^{10} - w^9 + 1.$$

Noting that $w^9 = (x^2 + 2\pi^2)^4 \sqrt{x^2 + 2\pi^2}$, which involves a radical, to give a feasible estimate for w^9 without a radical, we may make use of (3.11) to deduce that for $x \geq 4$,

$$w_1 w^8 < w^9 < w_2 w^8.$$

Let

$$\eta_1 = w^{10} - w_1 w^8 + 1,$$

so that for $x \geq 4$,

$$\alpha(w) < \eta_1. \tag{3.18}$$

Similarly, set

$$\begin{aligned}
\eta_2 =& y^{30} - 3y_1 y^{28} + 3y^{28} - y_1 y^{26} + 3y^{20} - 6y_1 y^{18} + 3y^{18} + 3y^{10} - 3y_1 y^8 + 1,\\
\eta_3 =& z^{30} - 3z_2 z^{28} + 3z^{28} - z_2 z^{26} - 3z^{20} + 6z_1 z^{18} - 3z^{18} + 3z^{10} - 3z_2 z^8 - 1,\\
\eta_4 =& y^{20} - 2y_2 y^{18} + y^{18} + 2y^{10} - 2y_2 y^8 + 1,\\
\eta_5 =& z^{20} - 2z_2 z^{18} + z^{18} - 2z^{10} + 2z_1 z^8 + 1.
\end{aligned}$$

Then we have for $x \geq 4$,

$$\alpha(y)^3 < \eta_2, \qquad \beta(z)^3 > \eta_3, \qquad \alpha(y)^2 > \eta_4, \qquad \beta(z)^2 > \eta_5. \tag{3.19}$$

Employing the relations in (3.18) and (3.19), we deduce that for $x \geq 4$,

$$t_1 = x^{12} z^{36} \alpha(y)^3 \alpha(w) < x^{12} z^{36} \eta_1 \eta_2, \tag{3.20}$$

$$t_2 = (x^{10} - x^9 - 1) y^{36} w^{12} \beta(z)^3 > (x^{10} - x^9 - 1) y^{36} w^{12} \eta_3, \tag{3.21}$$

$$t_3 = x^7 y^{12} z^{12} w^{12} \alpha(y)^2 \beta(z)^2 > x^7 y^{12} z^{12} w^{12} \eta_4 \eta_5. \tag{3.22}$$

We continue to estimate e^{w+y-2z} and e^{z+x-2y}. Applying (3.9), (3.10) and (3.11) to $w+y-2z$, we see that for $x \geq 4$,

$$w + y - 2z < w_2 + y_2 - 2z_1, \tag{3.23}$$

which implies that

$$e^{w+y-2z} < e^{w_2+y_2-2z_1}. \tag{3.24}$$

In order to give a feasible upper bound for e^{w+y-2z}, we define

$$\Omega(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720}, \tag{3.25}$$

and it can be proved that for $t < 0$,

$$e^t < \Omega(t). \tag{3.26}$$

Note that

$$w_2 + y_2 - 2z_1 = -\frac{\pi^4(108x^8 - 216\pi^2 x^6 + 375\pi^4 x^4 - 630\pi^6 x^2 - 224\pi^8)}{972x^{11}} < 0$$

holds for $x \geq 5$, since

$$108x^8 - 216\pi^2 x^6 > 0$$

for $x > \sqrt{2}\pi \approx 4.443$, and

$$375\pi^4 x^4 - 630\pi^6 x^2 - 224\pi^8 > 0$$

for $x > \frac{\pi}{5}\sqrt{\sqrt{2443/3}+21} \approx 4.422$. Thus, by (3.26), we obtain that for $x \geq 5$,

$$e^{w_2+y_2-2z_1} < \Omega(w_2+y_2-2z_1). \tag{3.27}$$

Combining (3.24) and (3.27) yields that for $x \geq 5$,

$$e^{w+y-2z} < \Omega(w_2+y_2-2z_1). \tag{3.28}$$

Similarly, applying (3.9), (3.10) and (3.11) to $z+x-2y$, we find that for $x \geq 4$,

$$z_1+x-2y_2 < z+x-2y, \tag{3.29}$$

so that

$$e^{z+x-2y} > e^{z_1+x-2y_2}. \tag{3.30}$$

Define

$$\omega(t) = 1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}. \tag{3.31}$$

It is true that for $t < 0$,

$$\omega(t) < e^t. \tag{3.32}$$

We now give a lower bound for $e^{z_1+x-2y_2}$. Since

$$\begin{aligned} z+x-2y &= \sqrt{x^2+\frac{4\pi^2}{3}}+x-2\sqrt{x^2+\frac{2\pi^2}{3}} \\ &= \frac{-\left(\sqrt{x^2+4\pi^2/3}-x\right)^2}{\sqrt{x^2+4\pi^2/3}+x+2\sqrt{x^2+2\pi^2/3}}, \end{aligned}$$

which is negative for $n \geq 2$, by (3.29), we deduce that for $x \geq 4$,

$$z_1+x-2y_2 < 0. \tag{3.33}$$

Thus, applying (3.32) to (3.33) gives us that for $x \geq 4$,

$$e^{z_1+x-2y_2} > \omega(z_1+x-2y_2). \tag{3.34}$$

Combining (3.30) and (3.34), we find that for $x \geq 4$,

$$e^{z+x-2y} > \omega(z_1+x-2y_2). \tag{3.35}$$

Using the above bounds for $t_1, t_2, t_3, e^{w+z-2y}$ and e^{z+x-2y}, we obtain that for $x \geq 5$,

$$\begin{aligned}
&-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 110t_3 \\
&\quad > -\Omega(w_2+y_2-2z_1)x^{12}z^{36}\eta_1\eta_2 + \omega(z_1+x-2y_2)(x^{10}-x^9-1)y^{36}w^{12}\eta_3 \\
&\qquad + 110x^7y^{12}z^{12}w^{12}\eta_4\eta_5.
\end{aligned} \tag{3.36}$$

To verify (3.7), we show that for $x \geq 358$,

$$\begin{aligned}
&-\Omega(w_2+y_2-2z_1)x^{12}z^{36}\eta_1\eta_2 + \omega(z_1+x-2y_2)(x^{10}-x^9-1)y^{36}w^{12}\eta_3 \\
&\qquad + 110x^7y^{12}z^{12}w^{12}\eta_4\eta_5 > 0.
\end{aligned} \tag{3.37}$$

Substituting y, z and w with $\sqrt{x^2+2\pi^2/3}$, $\sqrt{x^2+4\pi^2/3}$ and $\sqrt{x^2+2\pi^2}$ respectively, the left hand side of the inequality (3.37) can be expressed as $H(x)/G(x)$, where

$$H(x) = \sum_{k=0}^{171} a_k x^k$$

and

$$G(x) = 3968620165647335477675708742853516263948288 0x^{88}.$$

Here we just list the values of a_{169}, a_{170} and a_{171}:

$$\begin{aligned}
a_{169} &= 73492966030506212549550161904694745628672 0 \\
&\qquad \times \left(35640 + 261360\pi^2 - 194\pi^6 - 249\pi^8\right), \\
a_{170} &= 58794372824404970039640129523755796502937 60\left(7\pi^6 - 2970\right), \\
a_{171} &= 44095779618303727529730097142816847377203 20\left(990 - \pi^6\right),
\end{aligned}$$

which are all positive.

Given that $G(x)$ is always positive, we aim to prove that $H(x) > 0$. Apparently, $x \geq 2$ for $n \geq 2$ and hence

$$H(x) \geq \sum_{k=0}^{170} -|a_k|x^k + a_{171}x^{171}. \tag{3.38}$$

Moreover, numerical evidence indicates that for any $0 \leq k \leq 168$,

$$-|a_k|x^k > -a_{169}x^{169} \tag{3.39}$$

holds for $x \geq 181$. It follows that for $x \geq 181$,

$$\sum_{k=0}^{170} -|a_k|x^k + a_{171}x^{171} > (-170a_{169} - a_{170}x + a_{171}x^2)x^{169}. \tag{3.40}$$

Combining (3.38) and (3.40), we obtain that for $x \geq 181$,

$$H(x) > (-170a_{169} - a_{170}x + a_{171}x^2)x^{169}. \tag{3.41}$$

Thus, $H(x)$ is positive provided

$$-170a_{169} - a_{170}x + a_{171}x^2 > 0, \tag{3.42}$$

which is true if

$$x > \frac{\sqrt{a_{170}^2 + 680a_{169}a_{171}} + a_{170}}{2a_{171}} \approx 357.867.$$

Hence we conclude that $H(x)$ is positive when $x \geq 358$. This proves (3.37).

Combining (3.36) and (3.37), we find that for $x \geq 358$, or equivalently, for $n \geq 19480$, (3.7) holds, that is,

$$-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 110t_3 > 0. \tag{3.43}$$

For $2 \leq n \leq 19480$, (3.43) can be directly verified. This completes the proof. ∎

4 An inequality on u_n and $f(n)$

In this section, we present an inequality on u_n and $f(n)$ that is also needed in the proof of Theorem 1.3.

Theorem 4.1. *Let u_n be defined as* (1.19)*, that is,*

$$u_n = \frac{p(n+1)p(n-1)}{p(n)^2}.$$

For $0 < t < 1$*, let*

$$Q(t) = \frac{3t + 2\sqrt{(1-t)^3} - 2}{t^2}. \tag{4.1}$$

Then for $n \geq 85$,

$$f(n) + \frac{110}{\mu(n-1)^5} < Q(u_n). \tag{4.2}$$

The proof of this theorem is based on the following Lemma, which gives an upper bound of $f(n)$. Recall that

$$f(n)=e^{x-2y+z}\frac{\left(x^{10}-x^{9}-1\right)y^{24}\left(z^{10}-z^{9}-1\right)}{x^{12}\left(y^{10}-y^{9}+1\right)^{2}z^{12}},$$

where x,y,z,w are defined in (2.2).

Lemma 4.2. *Let $\Omega(t)$ be defined as in* (3.25)*, that is,*

$$\Omega(t)=1+t+\frac{t^{2}}{2}+\frac{t^{3}}{6}+\frac{t^{4}}{24}+\frac{t^{5}}{120}+\frac{t^{6}}{720},$$

and let y_1,y_2,z_1 and z_2 be defined as in (3.12)*,* (3.13)*,* (3.14)*, and* (3.15)*. For $n\geq 4$, we have*

$$f(n)<\frac{\Omega(x-2y_1+z_2)y^{24}(x^{10}-x^{9}-1)\left(z^{10}-z^{8}z_1-1\right)}{x^{12}z^{12}(y^{20}-2y^{18}y_2+y^{18}+2y^{10}-2y^{8}y_2+1)}<1. \tag{4.3}$$

Proof. To prove (4.3), we proceed to give estimates of the factors $\left(y^{10}-y^{9}+1\right)^{2}$, $z^{10}-z^{9}-1$ and e^{x-2y+z} that appear in $f(n)$. The third inequality in (3.19) gives an estimate of $\left(y^{10}-y^{9}+1\right)^{2}$, that is, for $x\geq 4$,

$$\left(y^{10}-y^{9}+1\right)^{2}>y^{20}-2y_2y^{18}+y^{18}+2y^{10}-2y_2y^{8}+1. \tag{4.4}$$

Using the bounds for y and z as given in (3.9) and (3.10), we are led to the following estimates for $z^{10}-z^{9}-1$ and e^{x-2y+z} when $x\geq 4$,

$$z^{10}-z^{9}-1<z^{10}-z^{8}z_1-1, \tag{4.5}$$

$$e^{x-2y+z}<e^{x-2y_1+z_2}. \tag{4.6}$$

To give an upper bound for $e^{x-2y_1+z_2}$, write

$$x-2y_1+z_2=-\frac{\pi^{4}\left(216x^{8}-216\pi^{2}x^{6}+210\pi^{4}x^{4}-210\pi^{6}x^{2}-7\pi^{8}\right)}{1944x^{11}}. \tag{4.7}$$

For $x>\pi$, we have

$$216x^{8}-216\pi^{2}x^{6}>0,$$

and for $x>\pi\sqrt{(\sqrt{17/15}+1)/2}\approx 3.192$, we have

$$210\pi^{4}x^{4}-210\pi^{6}x^{2}-7\pi^{8}>0.$$

Therefore, it follows from (4.7) that for $x\geq 4$,

$$x-2y_1+z_2<0,$$

which, together with (3.26), yields that for $x \geq 4$,

$$e^{x-2y_1+z_2} < \Omega(x-2y_1+z_2). \tag{4.8}$$

Combining (4.6) and (4.8), we find that for $x \geq 4$,

$$e^{x-2y+z} < \Omega(x-2y_1+z_2). \tag{4.9}$$

By means of the estimates in (4.4), (4.5) and (4.9), we arrive at the first inequality in (4.3).

To prove the second inequality in (4.3), recall the expressions of y and z in (3.8), namely,

$$y = \sqrt{x^2 + \frac{2\pi^2}{3}}, \qquad z = \sqrt{x^2 + \frac{4\pi^2}{3}}.$$

It can be checked that

$$\Omega(x-2y_1+z_2)y^{24}(x^{10}-x^9-1)\left(z^{10}-z^8z_1-1\right) = \frac{I(x)}{x^{77}},$$

and

$$x^{12}z^{12}(y^{20}-2y^{18}y_2+y^{18}+2y^{10}-2y^8y_2+1) = N(x),$$

where $I(x)$ is a polynomial in x of degree 121 and $N(x)$ is a polynomial in x of degree 44. Thus we may assume that

$$\frac{\Omega(x-2y_1+z_2)y^{24}(x^{10}-x^9-1)\left(z^{10}-z^8z_1-1\right)}{x^{12}z^{12}(y^{20}-2y^{18}y_2+y^{18}+2y^{10}-2y^8y_2+1)} = \frac{K(x)}{J(x)},$$

where $K(x)$ and $J(x)$ are both polynomials of degree 121. Write

$$K(x) = \sum_{k=0}^{121} b_k x^k, \qquad J(x) = \sum_{k=0}^{121} c_k x^k. \tag{4.10}$$

Here are the values of b_k and c_k for $116 \leq k \leq 121$:

$$b_{116} = -1398983398232765780459520\pi^4\left(5181+41\pi^2\right),$$

$$b_{117} = 2518170116818978404827136 0\pi^2\left(21+151\pi^2\right),$$

$$b_{118} = -4196950194698297341378560\pi^2\left(258+\pi^2\right),$$

$$c_{116} = -719776958390757994046423040 0\pi^4,$$

$$c_{117} = 75545103504569352144814080\pi^2\left(7+50\pi^2\right),$$

$$c_{118} = -10828131502321607140756684 80\pi^2,$$

$$b_{119} = c_{119} = 1259085058409489202413568 0\left(3 + 44\pi^2\right),$$

$$b_{120} = c_{120} = -75545103504569352144814080,$$

$$b_{121} = c_{121} = 37772551752284676072407040.$$

We claim that for $x \geq 135$,

$$J(x) > 0, \tag{4.11}$$

and

$$J(x) - K(x) > 0. \tag{4.12}$$

It can be shown that for $0 \leq k \leq 118$,

$$-|c_k|x^k > -c_{119}x^{119}, \tag{4.13}$$

when

$$x > \pi\sqrt{\frac{6\left(7 + 50\pi^2\right)}{3 + 44\pi^2}} \approx 8.232.$$

It follows that for $x \geq 9$,

$$J(x) > (-120c_{119} + c_{120}x + c_{121}x^2)x^{119}. \tag{4.14}$$

Since

$$-120c_{119} + c_{120}x + c_{121}x^2 > 0 \tag{4.15}$$

when

$$x > 1 + \sqrt{11\left(11 + 160\pi^2\right)} \approx 133.255,$$

we find that $J(x) > 0$ for $x \geq 134$.

Similarly, to prove (4.12), we observe that for $0 \leq k \leq 115$,

$$-|c_k - b_k|x^k > -(c_{116} - b_{116})x^{116} \tag{4.16}$$

when

$$x > \frac{1}{2}\pi\sqrt{\frac{5616 + 3127\pi^2}{108 + 123\pi^2}} \approx 8.232.$$

Let

$$\theta(x) = -117(c_{116} - b_{116}) + (c_{117} - b_{117})x + (c_{118} - b_{118})x^2.$$

Then (4.16) implies that for $x \geq 9$,

$$J(x) - K(x) = \sum_{k=0}^{118}(c_k - b_k)x^{118} > \theta(x)x^{116}. \tag{4.17}$$

Given that $\theta(x)$ is positive when $x \geq 135$, we arrive at (4.12).

Combining (4.11) and (4.12), we deduce that the second inequality (4.3) is valid for $x \geq 135$, or equivalently, for $n \geq 2771$. The case for $4 \leq n \leq 2771$ can be directly verified, and hence the proof is complete. ∎

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Recall that the theorem states that for $n \geq 85$,

$$f(n) + \frac{110}{\mu(n-1)^5} < Q(u_n). \tag{4.18}$$

It can be checked that (4.18) is true for $85 \leq n \leq 35456$. We now show that (4.18) is true for $n \geq 35457$. Recall that

$$Q(t) = \frac{3t + 2\sqrt{(1-t)^3} - 2}{t^2},$$

and so

$$Q'(t) = \frac{t\left(\sqrt{1-t} - 3\right) - 4\sqrt{1-t} + 4}{t^3}. \tag{4.19}$$

Setting $t = 1 - \tau$, we get

$$Q'(t) = \frac{1}{\left(\sqrt{\tau} + 1\right)^3},$$

thus (4.19) can be rewritten as

$$Q'(t) = \frac{1}{\left(\sqrt{1-t} + 1\right)^3}. \tag{4.20}$$

As $Q'(t)$ is positive for $0 < t < 1$, $Q(t)$ is increasing for $0 < t < 1$. By Theorem 2.1, we know that $f(n) < u_n$ for $n \geq 1207$, so that for $n \geq 1207$,

$$Q(f(n)) < Q(u_n). \tag{4.21}$$

Thus (4.18) is justified if we can prove that for $n \geq 35457$,

$$f(n) + \frac{110}{\mu(n-1)^5} < Q(f(n)). \tag{4.22}$$

Let

$$\vartheta(t) = Q(t) - t = \frac{3t + 2\sqrt{(1-t)^3} - t^3 - 2}{t^2}. \tag{4.23}$$

In this notation, (4.22) says that for $n \geq 35457$,

$$\vartheta(f(n)) > \frac{110}{\mu(n-1)^5}. \tag{4.24}$$

To prove the above inequality, we shall use the polynomials $J(x)$ and $K(x)$ as given by (4.10). More specifically,

$$K(x) = \sum_{k=0}^{121} b_k x^k, \qquad J(x) = \sum_{k=0}^{121} c_k x^k.$$

Note that

$$\vartheta'(t) = -\frac{t^3 + 4(1-t)^{3/2} + 3t + 3t\sqrt{1-t} - 4}{t^3}. \tag{4.25}$$

Setting $t = 1 - \tau$, (4.25) becomes

$$\vartheta'(t) = -\frac{\sqrt{\tau}\left(\tau + 3\sqrt{\tau} + 3\right)}{\left(\sqrt{\tau}+1\right)^3},$$

leading to the expression

$$\vartheta'(t) = -\frac{\sqrt{1-t}\left(3\sqrt{1-t} + 4 - t\right)}{\left(\sqrt{1-t}+1\right)^3},$$

which is negative for $0 < t < 1$. Thus, $\vartheta(t)$ is decreasing for $0 < t < 1$.

It can be seen from Lemma 4.2 that $0 < f(n) < K(x)/J(x) < 1$ for $n \geq 4$, so that for $n \geq 35457$,

$$\vartheta(f(n)) > \vartheta\left(\frac{K(x)}{J(x)}\right). \tag{4.26}$$

Because of (4.26), to verify (4.24), it is sufficient to show that for $n \geq 35457$,

$$\vartheta\left(\frac{K(x)}{J(x)}\right) > \frac{110}{x^5}. \tag{4.27}$$

This goal can be achieved by finding an estimate for $\vartheta\left(K(x)/J(x)\right)$. We first derive the following range of $K(x)/J(x)$ for $x \geq 134$,

$$\frac{\sqrt{5}-1}{2} < \frac{K(x)}{J(x)} < 1. \tag{4.28}$$

By Lemma (4.2), we know that $K(x)/J(x) < 1$ for $x \geq 4$ and $J(x) > 0$ for $x \geq 134$. To justify (4.28), we only need to show that for $x \geq 134$,

$$2K(x) - (\sqrt{5}-1)J(x) > 0. \tag{4.29}$$

Note that

$$b_{119} = c_{119}, \qquad b_{120} = c_{120}, \qquad b_{121} = c_{121},$$

and it can be shown that for $0 \leq k \leq 118$,

$$-\left|2b_k - (\sqrt{5}-1)c_k\right| x^k > -(3-\sqrt{5})c_{119}x^{119} \tag{4.30}$$

when

$$x > \pi\sqrt{\frac{\pi^2\left(\sqrt{5}+303\right)+42}{3+44\pi^2}} \approx 8.303.$$

It follows that for $x \geq 9$,

$$2K(x)-(\sqrt{5}-1)J(x) > (3-\sqrt{5})(-120c_{119}+c_{120}x+c_{121}x^2)x^{119}. \tag{4.31}$$

Since

$$-120c_{119}+c_{120}x+c_{121}x^2 > 0$$

for $x > \sqrt{11\left(11+160\pi^2\right)}+1 \approx 133.255$, we arrive at (4.29), and so (4.28) is proved.

The above range of $K(x)/J(x)$ enables us to bound $\vartheta\left(K(x)/J(x)\right)$. Recalling that

$$\vartheta(t) = \frac{3t+2\sqrt{(1-t)^3}-t^3-2}{t^2},$$

we obtain that

$$\vartheta(t)-(1-t)^{\frac{3}{2}} = -\frac{t^3+t^2(1-t)^{\frac{3}{2}}-2(1-t)^{\frac{3}{2}}-3t+2}{t^2}. \tag{4.32}$$

Set $t = 1-\tau$ to get

$$\vartheta(t)-(1-t)^{\frac{3}{2}} = -\frac{\tau^{\frac{3}{2}}\left(\tau+\sqrt{\tau}-1\right)}{\left(\sqrt{\tau}+1\right)^2},$$

Thus

$$\begin{aligned}
\vartheta(t)-(1-t)^{\frac{3}{2}} &= \frac{(1-t)^{\frac{3}{2}}\left(t-\sqrt{1-t}\right)}{\left(\sqrt{1-t}+1\right)^2} \\
&= \frac{(1-t)^{\frac{3}{2}}\left(t+\frac{\sqrt{5}+1}{2}\right)\left(t-\frac{\sqrt{5}-1}{2}\right)}{\left(\sqrt{1-t}+1\right)^2\left(\sqrt{1-t}+t\right)},
\end{aligned}$$

which is positive for $\frac{\sqrt{5}-1}{2} < t < 1$, and hence, for $\frac{\sqrt{5}-1}{2} < t < 1$ we have

$$\vartheta(t) > (1-t)^{\frac{3}{2}}. \tag{4.33}$$

In view of (4.28) and (4.33), we infer that for $x \geq 134$,

$$\vartheta\left(\frac{K(x)}{J(x)}\right) > \left(1-\frac{K(x)}{J(x)}\right)^{\frac{3}{2}}. \tag{4.34}$$

We continue to show that for $x \geq 483$,

$$\left(1-\frac{K(x)}{J(x)}\right)^{\frac{3}{2}} > \frac{110}{x^5}. \tag{4.35}$$

Since $J(x) > 0$ for $x \geq 134$, the above inequality can be reformulated as follows. For $x \geq 483$,

$$x^{10}(J(x)-K(x))^3 - 110^2 J(x)^3 > 0. \tag{4.36}$$

The left hand side of (4.36) is a polynomial of degree 364, so that we may write

$$x^{10}(J(x)-K(x))^3 - 110^2 J(x)^3 = \sum_{k=0}^{364} \gamma_k x^k. \tag{4.37}$$

The values of $\gamma_{364}, \gamma_{363}$ and γ_{362} are given below:

$$\begin{aligned}
\gamma_{364} &= 2^{72} 3^{105} 5^3 \pi^{12}, \\
\gamma_{363} &= -2^{73} 3^{107} 5^3 \left(490050 + \pi^{12}\right), \\
\gamma_{362} &= 2^{72} 3^{105} 5^3 \left(52925400 + 144\pi^{12} + 41\pi^{14}\right).
\end{aligned}$$

For $0 \leq k \leq 361$, we find that

$$-|\gamma_k| x^k > -\gamma_{362} x^{362}, \tag{4.38}$$

provided that

$$x > \frac{793881000 + 2328717600\pi^2 + 3996\pi^{12} + 4392\pi^{14} + \pi^{16}}{317552400 + 864\pi^{12} + 246\pi^{14}} \approx 20.126.$$

Thus, for $x \geq 21$,

$$x^{10}(J(x)-K(x))^3 - 110^2 J(x)^3 > (-363\gamma_{362} + \gamma_{363} x + \gamma_{364} x^2) x^{362},$$

which is positive, since

$$-363\gamma_{362} + \gamma_{363} x + \gamma_{364} x^2 > 0$$

as long as

$$x > \frac{\sqrt{1452\gamma_{362}\gamma_{364} + \gamma_{363}^2} - \gamma_{363}}{2\gamma_{364}} \approx 482.959.$$

Hence (4.35) is confirmed. Combining (4.34) and (4.35), we are led to (4.27). The proof is therefore complete. ∎

5 Proof of Theorem 1.3

In this section, we present a proof of Theorem 1.3 based on the intermediate inequalities in the previous sections. The theorem states that for $n \geq 95$.

$$4(1-u_n)(1-u_{n+1})-(1-u_nu_{n+1})^2 > 0, \tag{5.1}$$

where

$$u_n = \frac{p(n+1)p(n-1)}{p(n)^2}.$$

Proof of Theorem 1.3. We shall make use of the fact that $u_n < 1$ for $n \geq 26$, as proved by DeSalvo and Pak [13]. In order to prove (5.1), we define $\varphi(t)$ to be a function in t:

$$\varphi(t) = 4(1-u_n)(1-t)-(1-u_nt)^2. \tag{5.2}$$

Then (5.1) says that for $n \geq 95$,

$$\varphi(u_{n+1}) > 0. \tag{5.3}$$

For $95 \leq n \leq 1206$, (5.3) can be directly checked. We proceed to prove that (5.3) holds for $n \geq 1207$. Let $Q(t)$ be as defined in (4.1), that is,

$$Q(t) = \frac{3t+2\sqrt{(1-t)^3}-2}{t^2}.$$

We claim that $\varphi(t) > 0$ for $u_n < t < Q(u_n)$. Rewrite $\varphi(t)$ as

$$\varphi(t) = -u_n^2t^2+(6u_n-4)t-4u_n+3.$$

The equation $\varphi(t) = 0$ has two solutions:

$$P(u_n) = \frac{3u_n-2\sqrt{(1-u_n)^3}-2}{u_n^2}, \qquad Q(u_n) = \frac{3u_n+2\sqrt{(1-u_n)^3}-2}{u_n^2},$$

so that $\varphi(t) > 0$ for $P(u_n) < t < Q(u_n)$. Furthermore, we see that

$$\varphi(u_n) = (1-u_n)^3(u_n+3) > 0,$$

which implies $P(u_n) < u_n < Q(u_n)$. Therefore, $\varphi(t) > 0$ for $u_n < t < Q(u_n)$, as claimed.

To verify (5.3), it remains to show that for $n \geq 1207$,

$$u_n < u_{n+1} < Q(u_n). \tag{5.4}$$

Recall that $u_n < u_{n+1}$ holds for $n \geq 116$, as proved by Chen, Wang and Xie [7]. By Theorem 2.1, we know that $u_{n+1} < g(n+1)$ for $n \geq 1207$. But Theorem 3.1 asserts that for $n \geq 2$,

$$g(n+1) < f(n) + \frac{110}{\mu(n-1)^5}.$$

Furthermore, Theorem 4.1 states that for $n \geq 2$,

$$f(n) + \frac{110}{\mu(n-1)^5} < Q(u_n).$$

Thus we conclude that $u_{n+1} < Q(u_n)$ for $n \geq 1207$, as claimed. ∎

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