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A Context-free Grammar for the Ramanujan-Shor Polynomials

William Y.C. Chen[1] and Harold R.L. Yang[2]

[1]Center for Applied Mathematics
Tianjin University
Tianjin 300072, P. R. China

[2]School of Science
Tianjin University of Technology and Education
Tianjin 300222, P. R. China

emails: [1]chenyc@tju.edu.cn, [2]yangruilong@mail.nankai.edu.cn

Dedicated to Professor Joseph P. S. Kung

Abstract

The polynomials $\psi_k(r,x)$ were introduced by Ramanujan. Berndt, Evans and Wilson obtained a recurrence relation for $\psi_k(r,x)$. Shor introduced polynomials related to improper edges of a rooted tree, leading to a refinement of Cayley's formula. Zeng realized that the polynomials of Ramanujan coincide with the polynomials of Shor, and that the recurrence relation of Shor coincides with the recurrence relation of Berndt, Evans and Wilson. These polynomials also arise in the work of Wang and Zhou on the orbifold Euler characteristics of the moduli spaces of stable curves. Dumont and Ramamonjisoa found a context-free grammar G to generate the number of rooted trees on n vertices with k improper edges. Based on the grammar G, we find a grammar H for the Ramanujan-Shor polynomials. This leads to a formal calculus for these polynomials. In particular, we obtain a grammatical derivation of the Berndt-Evans-Wilson-Shor recursion. We also provide a grammatical approach to the Abel identities and a grammatical explanation of the Lacasse identity.

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1 Introduction

For integers $1 \leq k \leq r+1$, Ramanujan [14] defined the polynomials $\psi_k(r,x)$ by the following relation:

$$\sum_{k=0}^{\infty} \frac{(x+k)^{r+k} e^{-u(x+k)} u^k}{k!} = \sum_{k=1}^{r+1} \frac{\psi_k(r,x)}{(1-u)^{r+k}}, \tag{1.1}$$

and derived the recurrence relation:

$$\psi_k(r+1,x) = (x-1)\psi_k(r,x-1) + \psi_{k-1}(r+1,x) - \psi_{k-1}(r+1,x-1), \tag{1.2}$$

where $\psi_1(0,x) = 1$, $\psi_0(r,x) = 0$ and $\psi_k(r,x) = 0$ for $k > r+1$. Berndt, Evans and Wilson [1] obtained another recurrence relation for $1 \leq k \leq r+1$,

$$\psi_k(r,n) = (n-r-k+1)\psi_k(r-1,n) + (r+k-2)\psi_{k-1}(r-1,n). \tag{1.3}$$

By setting $u = 0$ in (1.1), Ramanujan deduced the identity for $r \geq 1$,

$$\sum_{k=1}^{r+1} \psi_k(r,x) = x^r. \tag{1.4}$$

Zeng [21] observed that the polynomials $\psi_k(r,x)$ coincide with the polynomails introduced by Shor [17]. Let

$$Q_{n,k}(x) = \psi_{k+1}(n-1,x+n). \tag{1.5}$$

Then (1.4) can be rewritten as

$$\sum_{k=0}^{n-1} Q_{n,k}(x) = (x+n)^{n-1}, \tag{1.6}$$

(1.3) can be recast as

$$Q_{n,k}(x) = (x-k+1)Q_{n-1,k}(x+1) + (n+k-2)Q_{n-1,k-1}(x+1), \tag{1.7}$$

and (1.2) can be restated as

$$Q_{n,k}(x) = (x+n-1)Q_{n-1,k}(x) + Q_{n,k-1}(x) - Q_{n,k-1}(x-1). \tag{1.8}$$

Shor [17] defined the polynomials $Q_{n,k}(x)$ in a slightly different notation. To be more specific, for nonnegative integers n, k, x, the numbers $Q_{n,k}(x)$ are defined by the initial conditions $Q_{1,0}(x) = 1$, and $Q_{1,k}(x) = 0$ for $k \geq 1$, and the recurrence relation for $n \geq 1$,

$$Q_{n,k}(x) = (x+n-1)Q_{n-1,k}(x) + (n+k-2)Q_{n-1,k-1}(x), \tag{1.9}$$

where we set $Q_{n,-1}(x) = 0$ for $n \geq 1$. It is worth mentioning that the recursion (1.7) can be deduced from the recursion (1.8) and the recursion (1.9). More precisely, subtracting (1.9) from (1.8) and substituting k by $k+1$, we get

$$Q_{n,k}(x) = Q_{n,k}(x-1) + (n+k-1)Q_{n-1,k}(x). \tag{1.10}$$

Subtracting (1.10) by (1.9) and substituting x by $x+1$, we are led to the recurrence relation (1.7).

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Shor showed that, for a positive integer x, $xQ_{n-x,k}(x)$ equals the number of forests on $[n]$ rooted at $\{1, 2, \ldots, x\}$ with k improper edges, and Shor established the relation (1.6) as a refinement of Cayley's formula. For a rooted tree T and a vertex v of T, we use T_v to denote the subtree of T rooted at v, namely, the subtree of T consisting of all the descendants of v, where v is considered a descendant of itself. An edge (u, v) of T, with v being the child of u, is called an improper edge if there exists a vertex in T_v that is smaller than u.

Shor [17] noticed that $Q_{n,k}(x)$ is a polynomial in x and that the recurrence relation (1.7) implies the identity (1.6). So we call $Q_{n,k}(x)$ the Ramanujan-Shor polynomials.

Recently, Wang and Zhou [19] showed that the orbifold Euler characteristic of the moduli space of stable curves of genus zero with n marked points turns out to be the Ramanujan-Shor polynomial $Q_{n-1,k+1}(x)$ evaluated at $x = -1$. As proved by Dumont and Ramamonjisoa [7], $Q_{n,k}(-1)$ equals the number of rooted trees on $[n]$ with k improper edges with the vertex 1 being a leaf.

Zeng [21] found the following interpretations of the polynomials $Q_{n,k}(x)$ in terms of improper edges of trees on $[n+1]$ with root 1:

$$Q_{n,k}(x) = \sum_{T \in F_{n+1,k}} x^{\deg_T(1)-1}, \tag{1.11}$$

where $F_{n,k}$ denotes the set of trees on $[n]$ with k improper edges and with root 1, and $\deg_T(1)$ denotes the degree of the vertex 1 in T. The degree of a vertex v in a rooted tree T is defined to be the number of children of v. Zeng also showed that the polynomials $Q_{n,k}(x)$ can be interpreted by the number of improper edges of rooted trees (not necessarily rooted at 1) on $[n]$, namely,

$$Q_{n,k}(x) = \sum_{T \in R_{n,k}} (x+1)^{\deg_T(1)}, \tag{1.12}$$

where $R_{n,k}$ denotes the set of rooted trees on $[n]$ with k improper edges.

Shor asked the question of finding a combinatorial interpretation of (1.7), which we call the Berndt-Evans-Wilson-Shor recursion. By using the above interpretation of $Q_{n,k}(x)$ given by Zeng [21], Chen and Guo [4] found a combinatorial proof of (1.7). A simpler bijection was given by Guo [9].

It is easy to see that (1.11) is equivalent to the interpretation of $Q_{n,k}(x)$ given by Shor. As pointed out by Shor [17], for a positive integer r, $rQ_{n,k}(r)$ equals the number of forests on $[n+r]$ rooted at $\{1, 2, \ldots, r\}$ with a total number of k improper edges. Let F be such a forest counted by $rQ_{n,k}(r)$. Let T_i be the tree in F rooted at i, where $1 \leq i \leq r$. For each T_i, removing the root i and coloring the subtrees of T_i with color i, we get a forest on $\{r+1, r+2, \ldots, r+n\}$ with each tree colored by one of colors $1, 2, \ldots, r$. After relabeling, this leads to a forest on $[n]$ with each tree associated with one of the colors $1, 2, \ldots, r$. Let $U_{n,k}$ denote the set of forests of rooted trees on $[n]$ with k improper edges. For a forest F in $U_{n,k}$, let $\mathrm{tree}(F)$ denote the number of trees in F. By the above argument, we find that

$$rQ_{n,k}(r) = \sum_{F \in U_{n,k}} r^{\mathrm{tree}(F)},$$

which is equivalent to (1.11), since a forest F in $U_{n,k}$ gives rise to a rooted tree T in $F_{n+1,k}$ by adding a new root 0.

Utilizing Shor's recursive procedure to construct rooted trees, Dumont and Ramamonjisoa [7] found a context-free grammar to enumerate rooted trees with a given number of improper edges. They defined a grammar G by the following substitution rules:

$$G\colon A \to A^3S,\ S \to AS^2.$$

Let D denote the formal derivative with respect to G. Dumont and Ramamonjisoa showed that, for $n \geq 1$,

$$D^{n-1}(AS) = A^n S^n \sum_{k=0}^{n-1} b(n,k) A^k,$$

where $b(n,k)$ denotes the number of rooted trees on $[n]$ with k improper edges. Note that $b(n,k) = Q_{n,k}(0)$.

Based on the Dumont-Ramamonjisoa grammar, we obtain a grammar H to generate the Ramanujan-Shor polynomials $Q_{n,k}(x)$. Let

$$H\colon a \to axy, x \to xyw, y \to y^3w, w \to yw^2,$$

and let D denote the formal derivative with respect to H. For $n \geq 1$, we obtain the following relation

$$D^n(a) = axy^n w^{n-1} \sum_{k=0}^{n-1} Q_{n,k}(xw^{-1}) y^k.$$

With the aid of the grammar H, we are led to a simple derivation of the Berndt-Evans-Wilson-Shor recursion in the form of (1.10).

It turns out that the grammar H can also be used to derive the Abel identities. As will be seen, the Abel identities can be deduced from the Leibnitz formula with respect to the grammar H.

Riordan [15] defined the sum

$$A_n(x_1, x_2; p, q) = \sum_{k=0}^{n} \binom{n}{k} (x_1 + k)^{k+p} (x_2 + n - k)^{n-k+q},$$

where $n \geq 1$ and the parameters p, q are integers. He found closed formulas of $A_n(x_1, x_2; p, q)$ for some p and q. These identities were called the Abel identities or the Abel-type identities since the case $(p, q) = (-1, 0)$ corresponds to the classical Abel identity. We give a grammar H' based on the grammar H and show that the summations $A_n(x_1, x_2; p, q)$ can be evaluated by using the grammar H'. Using this approach, closed forms can be deduced for $A_n(x_1, x_2; -1, 0)$, $A_n(x_1, x_2; -1, -1)$ and $A_n(x_1, x_2; -2, 0)$ and $A_n(x_1, x_2; -2, -2)$. The case of $A_n(x_1, x_2; -2, -2)$ seems to be new.

We conclude this paper with a grammatical explanation of the identity

$$n^{n+1} = \sum_{k=1}^{n} \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} j^j k^k (n-j-k)^{n-j-k}.$$

We call this identity the Lacasse identity. It was conjectured by Lacasse [10] in the study of the PAC-Bayesian machine learning theory. Since then, several proofs have been found. For example, Sun [18] gave a derivation by using the umbral calculus, Younsi [20] found a proof with the aid of the Abel identity, Prodinger [13] provided a justification based on Cauchy's integral formula, Gessel [8] proved the identity by means of the Lagrange inversion formula, and Chen, Peng and Yang [5] obtained a combinatorial interpretation in terms of triply rooted trees.

This paper is organized as follows. In Section 2, we give an overview of the Dumont-Ramamonjisoa grammar and introduce a grammatical labeling of labeled trees. In Section 3, we find a grammar H to generate the Ramanujan-Shor polynomials. Section 4 is devoted to a proof of the Berndt-Evans-Wilson-Shor recursion by using the grammar H. In Section 5, we consider grammatical derivations of Abel identities. We also provide a grammatical explanation of the Lacasse identity.

2 The Dumont-Ramamonjisoa Grammar

In this section, we give an overview of the context-free grammar introduced by Dumont and Ramamonjisoa [7] to generate rooted trees. The approach of using context-free grammars to study combinatorial polynomials was introduced in [2]. Further studies can be found in [3,6,7,11,12]. A context-free grammar G over an alphabet A is defined to be a set of substitution rules. Given a context-free grammar, one may define a formal derivative D as a differential operator on polynomials or Laurent polynomials in A, that is, D is a linear operator satisfying the relation

$$D(uv) = D(u)v + uD(v),$$

and in general the Leibnitz formula

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u)D^{n-k}(v). \tag{2.1}$$

Dumont and Ramamonjisoa [7] defined the following grammar

$$G\colon A \to A^3S, S \to AS^2. \tag{2.2}$$

Let D denote the formal derivative with respect to the grammar G. Notice that D can also be viewed as the operator

$$D = A^3S\frac{\partial}{\partial A} + AS^2\frac{\partial}{\partial S}.$$

Dumont and Ramamonjisoa established a connection between the grammar G and the enumeration of rooted trees on $[n]$ with k improper edges. The notion of an improper edge of a rooted tree was introduced by Shor. Let T be a rooted tree on $[n]$. An edge of T is represented by a pair (u, v) of vertices with v being a child of u. We say that an edge (u, v) of T is improper if there exists a descendant of v that is smaller than u, bearing in mind that any vertex of T is considered as a descendant of itself; otherwise, (u, v) is called a proper edge. Recall that $b(n, k)$ denotes the number of rooted trees on $[n]$ with k improper edges. Dumont and Ramamonjisoa obtained the following relation.

Theorem 2.1 *For $n \geq 1$,*

$$D^{n-1}(AS) = A^n S^n \sum_{k=0}^{n-1} b(n,k) A^k. \tag{2.3}$$

For example, for $n = 1, 2, 3$, we have

$$\begin{aligned}
D^0(AS) &= AS, \\
D^1(AS) &= D(A)S + AD(S) = A^2S^2(1 + A), \\
D^2(AS) &= D(D(AS)) = A^3S^3(2 + 4A + 3A^2).
\end{aligned}$$

Dumont and Ramamonjisoa gave a proof of the above theorem by showing that the coefficients of $D^n(AS)$ satisfy the recurrence relation (1.9) of Shor. More precisely, let $s(n,k)$ denote the coefficient of $A^{n+k}S^n$ in $D^{n-1}(AS)$, Dumont and Ramamonjisoa proved that

$$s(n,k) = (n-1)s(n-1,k) + (n+k-2)s(n-1,k-1),$$

which is equivalent to the relation (1.9) for the case $x = 0$.

Here we present a proof in the language of a grammatical labeling of rooted trees, which was introduced in [3]. Let R_n denote the set of rooted trees on $[n]$ and let F_n denote the set of rooted trees on $[n]$ with root 1. Recall that $R_{n,k}$ is the set of rooted trees in R_n with k improper edges. Shor [17] provided a construction of a rooted tree in R_n by adding the vertex n into a tree in R_{n-1}. For a better understanding of the construction, let us consider the following procedure to delete the vertex n from a rooted tree T in R_n to obtain a rooted tree T' in R_{n-1}. For a rooted tree $T \in R_n$ and a vertex u in T, we adopt the notation $\beta_T(u)$, or simply $\beta(u)$, for the minimum vertex among the vertices in the subtree of T rooted at u.

1. Case 1: n is a leaf in T. Delete the vertex n.

2. Case 2: n is not a leaf. Assume that n has t children $b_1, b_2, \ldots, b_t$. We may further assume that

   $$\beta(b_1) < \beta(b_2) < \cdots < \beta(b_t).$$

   Contract the edge (n, b_t) and relabel the resulting vertex by b_t.

Conversely, one can construct a rooted tree T on $[n]$ with k or $k+1$ improper edges from a rooted tree T' on $[n-1]$ with k improper edges. There are four operations to add the vertex n to T'.

1. Adding n to the tree T' as a child of an arbitrary vertex v, we obtain a tree $T \in R_{n,k}$ with n being a leaf.

2. Splitting a proper edge (i, j) into (i, n) and (n, j), we obtain a tree $T \in R_{n,k+1}$. In this case, the degree of n equals one.

3. Splitting an improper edge (i, j) into (i, n) and (n, j), we also obtain a tree $T \in R_{n,k+1}$. In this case, the degree of n also equals one.

4. Choose an improper edge (v, b_j) in T', where v has t children $b_1, b_2, \ldots, b_t$ listed in the increasing order of their β-values. We relabel v by n and make v a child of n. Moreover, assign $b_1, \ldots, b_j$ to be the children of n and assign $b_{j+1}, \ldots, b_t$ to be the children of v. Then we are led to a tree $T \in R_{n,k+1}$. In this case, the degree of n in T is at least two.

As will be seen, the above construction is closely related to the grammar G. To demonstrate this connection, we introduce a grammatical labeling of rooted trees. We may view a rooted tree T on $[n]$ as a rooted tree $\hat{T}$ on $\{0, 1, 2, \ldots, n\}$ with 0 being the root with only one child. Clearly, the edge below the root 0 of $\hat{T}$ is always a proper edge. Moreover, we represent an improper edge by double edges, called the left edge and the right edge. The idea of using double edges to represent an improper edge is due to Dumont and Ramamonjisoa [7]. We label a vertex of $\hat{T}$ except for 0 by S and label an edge of $\hat{T}$ by A. In other words, a proper edge of T is labeled by A and an improper edge of T is labeled by A^2. The weight of T is defined by the product of the labels attached to $\hat{T}$, denoted by $w(T)$. Apparently, for any tree T in $R_{n,k}$, we have $w(T) = A^{n+k}S^n$.

Figure 2.1 illustrates all rooted trees on $\{1, 2, 3\}$, where the improper edges are represented by double edges, and the vertex 0 is added at the top of each tree in R_3.

The following relation is a restatement of Theorem 2.1.

Theorem 2.2 *For $n \geq 1$,*

$$D^{n-1}(AS) = \sum_{T \in R_n} w(T). \tag{2.4}$$

In view of the above grammatical laleling of rooted trees, it can be seen that the four cases in Shor's construction of a tree T' on $[n]$ from a tree on $[n-1]$ correspond to the substitution rules in G. Instead of giving a detailed proof, let us use an example to demonstrate the correspondence.

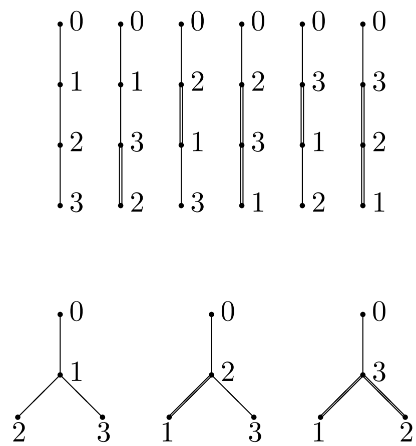


Figure 2.1: Rooted trees in R_3.

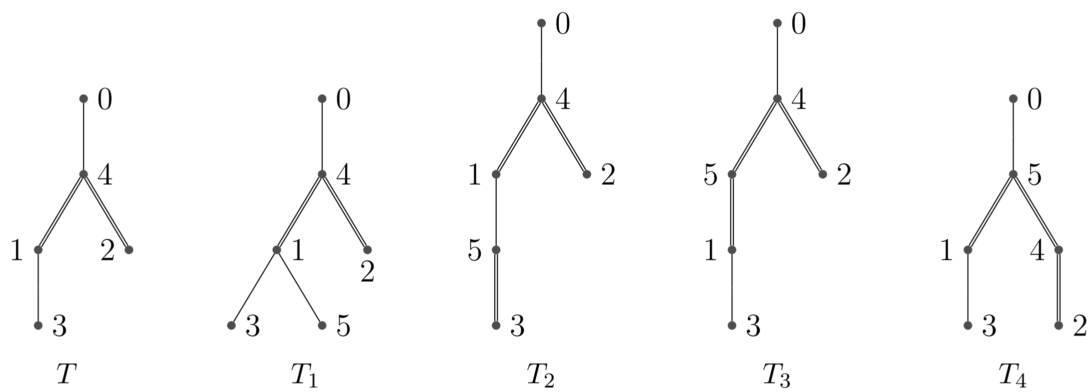


Figure 2.2: An example for the operator D

In Figure 2.2, T is a rooted tree on $\{1,2,3,4\}$. The weight of T is $w(T) = A^6S^4$. The trees T_1, T_2, T_3 and T_4 are obtained from T in the four cases of Shor's construction.

Case 1: T_1 is obtained from T by adding the vertex 5 as a leaf. Comparing the weights of T_1 and T, it can be seen that this operation corresponds to the substitution rule $S \to AS^2$. Notice that the label S indicates where one can apply this operation.

Case 2: T_2 is obtained from T by splitting the proper edge $(1,3)$ into $(1,5)$ and $(5,3)$. This operation also corresponds to the substitution rule $A \to A^3S$.

Case 3: T_3 is obtained from T by splitting the left edge $(4,1)$ into two edges $(4,5)$ and $(5,1)$. This operation corresponds to the substitution rule $A \to A^3S$.

Case 4: T_4 is obtained by adding 5 to T via the following procedure: 4 is relabeled by 5, a new vertex 4 is added as a child of 5, the subtree rooted by 1 and the subtree rooted by 2 are assigned as a child of 5 and a child of 4, respectively. It can be seen that this operation also corresponds to the substitution rule $A \to A^3S$.

The above argument is sufficient to be formalized as a proof of relation (2.4).

As suggested by the referee, one can use a slightly different grammar to give a more intuitive understanding of the recursive construction of Shor. More precisely, proper edges and improper edges can be distinguished by two different labels. We may use A to label a proper edge and use B to label an improper edge. Then the Shor construction leads to the following grammar H:

$$\{A \to ABS, B \to 2B^2S, S \to AS^2\}$$

and we have for $n \geq 1$,

$$D^{n-1}(AS) = \sum_{T \in R_n} A^{\mathrm{prop}(T)+1} B^{\mathrm{imp}(T)} S^n. \tag{2.5}$$

Indeed, one can set $B = A^2$ to reduce H to G, since

$$D(A^2) = 2AD(A) = 2A^4S$$

is consistent with the substitution rule $B \to 2B^2S$. Moreover, if we set $B = A^2$, then (2.5) becomes (2.4) since

$$A^{\mathrm{prop}(T)+1} B^{\mathrm{imp}(T)} S^n = A^{\mathrm{prop}(T)+2\mathrm{imp}(T)+1} S^n = A^{n+\mathrm{imp}(T)} S^n,$$

which equals $w(T)$. Here we adopt the grammar G and the double edge representation of improper edges in accordance with the notation of Dumont and Ramamonjisoa.

3 A Grammar for the Ramanujan-Shor Polynomials

In this section, we give a grammar H to generate the Ramanujan-Shor polynomials $Q_{n,k}(x)$. Define

$$H : a \to axy, x \to xyw, y \to y^3w, w \to yw^2. \tag{3.1}$$

Recall that we use F_n to denote the set of rooted trees on $[n]$ with root 1. Furthermore, let $F_{n,k}$ denote the set of rooted trees on $[n]$ with root 1 and k improper edges. For $T \in F_{n,k}$, we label a proper edge by y, and represent each improper edge of T by double edges, which are both labeled by y. Meanwhile, we label the vertex 1 by a, label each child of the vertex 1 by x and label other vertices by w, so that for $T \in F_{n+1,k}$, the weight of T equals

$$w(T) = ax^{\deg_T(1)} y^{n+k} w^{n-\deg_T(1)}. \tag{3.2}$$

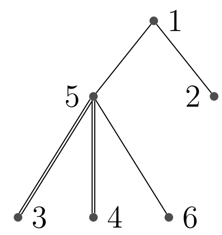


Figure 3.3: A rooted tree $T \in F_{6,2}$

Figure 3.3 illustrates a rooted tree in $F_{6,2}$ with weight $w(T) = ax^2y^7w^3$.

Let D denote the formal derivative with respect to the grammar H. Recall that F_n is the set of rooted tree on $[n]$ with root 1. The next theorem shows that D can be used to generate the sum of weights of rooted trees in F_n.

Theorem 3.1 *For $n \geq 1$,*

$$D^n(a) = \sum_{T \in F_n} w(T). \tag{3.3}$$

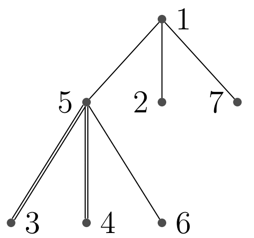
To prove the above relation, it is sufficient to observe that the substitution rules in H correspond to the changes of labels in Shor's construction according to the above labeling scheme.

Figure 3.4 gives three rooted trees T_1, T_2 and T_3 obtained from the tree T in Figure 3.3 by adding the vertex 7 as a leaf as in Case 1 of Shor's construction. Since 7 is child of the root 1, $w(T_1)$ is obtained from $w(T)$ by applying the substitution rule $a \to axy$. Similarly, 7 is a child of 2 in T_2, and $w(T_2)$ is obtained from $w(T)$ by utilizing the rule $x \to xyw$. Since 7 is a child of 6 in T_3, $w(T_3)$ is obtained from $w(T)$ by the rule $w \to yw^2$.

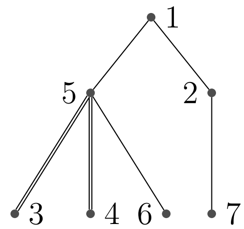
For Case 2, Case 3 and Case 4 in Shor's construction, the changes of weights of the resulting trees can be characterized by the rule $y \to y^3w$, just like the rule $A \to A^3S$ in the Dumont-Ramamonjisoa grammar.

Let $V_n(x, y)$ denote the generating function of $Q_{n,k}(x)$, that is,

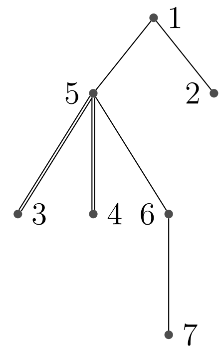
$$V_n(x, y) = \sum_{k=0}^{n-1} Q_{n,k}(x)y^k = \sum_{T \in F_{n+1}} x^{\deg_T(1)-1} y^{\mathrm{imp(T)}}. \tag{3.4}$$



(a) $w(T_1) = ax^3y^8w^3$



(b) $w(T_2) = ax^2y^8w^4$



(c) $w(T_3) = ax^2y^8w^4$

Figure 3.4: The action of D

For $n = 1, 2, 3$, we have

$$V_1(x, y) = 1,$$
$$V_2(x, y) = y + x + 1,$$
$$V_3(x, y) = 3y^2 + (3x + 4)y + x^2 + 3x + 2.$$

We now come to a relationship between the grammar H and the polynomials $V_n(x, y)$.

Theorem 3.2 *For $n \geq 1$,*

$$D^n(a) = axy^nw^{n-1}V_n(xw^{-1}, y). \tag{3.5}$$

For $n = 1, 2, 3$, we have

$$\begin{aligned} D(a) &= axy = axyV_1(xw^{-1}, y), \\ D^2(a) &= ax^2y^2 + axy^2w + axy^3w = axy^2w(xw^{-1} + 1 + y) = axy^2wV_2(xw^{-1}, y), \\ D^3(a) &= ax^3y^3 + 3ax^2y^3w + 3ax^2y^4w + 2axy^3w^2 + 4axy^4w^2 + 3axy^5w^2 \\ &= axy^3w^2\left(x^2w^{-2} + 3xw^{-1} + 2 + (3xw^{-1} + 4)y + 3y^2\right) \\ &= axy^3w^2V_3(xw^{-1}, y). \end{aligned}$$

We end this section with a grammatical derivation of the relation (1.9) of Shor.

Theorem 3.3 *For $n \geq 2$ and $1 \leq k \leq n-2$, we have*

$$Q_{n,k}(x) = (x+n-1)Q_{n-1,k}(x) + (n+k-2)Q_{n-1,k-1}(x). \tag{3.6}$$

Proof. For $n \geq 1$, by the definition of $V_n(x,y)$, (3.5) can be written as

$$D^n(a) = (xw^{-1})ay^n w^n \sum_{k=0}^{n-1} y^k Q_{n,k}(xw^{-1}). \tag{3.7}$$

For $n \geq 2$, substituting n by $n-1$, (3.7) takes the form

$$D^{n-1}(a) = (xw^{-1})ay^{n-1}w^{n-1} \sum_{k=0}^{n-2} y^k Q_{n-1,k}(xw^{-1}). \tag{3.8}$$

Since

$$D(xw^{-1}) = xyw \cdot w^{-1} - x \cdot w^{-2}yw^2 = 0, \tag{3.9}$$

that is, xw^{-1} is a constant with respect to D, we find that

$$D\left(y^k Q_{n-1,k}(xw^{-1})\right) = Q_{n-1,k}(xw^{-1})D(y^k) = ky^{k+2}wQ_{n-1,k}(xw^{-1}).$$

Meanwhile,

$$D\left(ay^{n-1}w^{n-1}\right) = axy^n w^{n-1} + (n-1)ay^{n+1}w^n + (n-1)ay^n w^n.$$

Therefore, applying the operator D to both sides of (3.8) yields

$$\begin{aligned} D^n(a) = (xw^{-1})ay^n w^n \Bigg\{ & \sum_{k=0}^{n-2} ky^{k+1} Q_{n-1,k}(xw^{-1}) \\ & + (xw^{-1} + (n-1)y + (n-1)) \sum_{k=0}^{n-2} y^k Q_{n-1,k}(xw^{-1}) \Bigg\}. \end{aligned} \tag{3.10}$$

For $n \geq 2$ and $0 \leq k \leq n-2$, comparing the coefficients of $axy^{n+k}w^n$ on the right-hand sides of (3.7) and (3.10), we deduce that

$$Q_{n,k}(xw^{-1}) = (xw^{-1}+n-1)Q_{n-1,k}(xw^{-1}) + (n+k-2)Q_{n-1,k-1}(xw^{-1}). \tag{3.11}$$

Setting $w=1$ completes the proof. ∎

4 The Berndt-Evans-Wilson-Shor Recursion

The section is devoted to a grammatical derivation of the Berndt-Evans-Wilson-Shor recursion (1.7). To this end, we establish a grammatical expression for $V_n(r+x,y)$, where r is a nonnegative integer. Recall that $V_n(x,y)$ is the generating function of $Q_{n,k}(x)$ as defined by (3.4).

Theorem 4.1 *For $n \geq 1$ and $r \geq 0$,*

$$D^n(ax^r) = ax^r y^n w^n (r + xw^{-1}) V_n(r + xw^{-1}, y). \tag{4.1}$$

To prove the above relation, we give a combinatorial interpretation of $V_n(r+x,y)$ based on Zeng's interpretation of $Q_{n,k}(x)$ in terms of the set $F_{n+1,k}$ of rooted trees on $[n+1]$ with root 1 and with k improper edges. We define $F_n^{(r)}$ to be the set of rooted trees on $[n]$ with root 1 for which each child of the root is colored by one of the colors $b, w_1, w_2, \ldots, w_r$, where b stands for the black color, and $w_1, w_2, \ldots, w_r$ are considered white colors.

We now define a grammatical labeling of a rooted tree $\bar{T} \in F_{n+1}^{(r)}$. First, represent an improper edge of $\bar{T}$ by double edges, and denote the resulting tree by $\hat{T}$. Then the root of $\hat{T}$ is labeled by ax^r, a black vertex is labeled by x and each of the remaining vertices is labeled by w. Moroever, each edge of $\hat{T}$ is labeled by y. In other words, as far as $\bar{T}$ is concerned, a proper edge is labeled by y and an improper edge is labeled by y^2. For $\bar{T} \in F_n^{(r)}$, we have

$$w(\bar{T}) = ax^{\mathrm{black}(\bar{T})+r} w^{n-\mathrm{black}(\bar{T})} y^{n+\mathrm{imp}(\bar{T})}, \tag{4.2}$$

where $\mathrm{black}(\bar{T})$ denotes the number of black vertices in $\bar{T}$.

Using the above labeling scheme, the right-hand side of (4.1) can be expressed as follows.

Theorem 4.2 *For $n \geq 0$ and $r \geq 0$,*

$$ax^r y^n w^n (r + xw^{-1}) V_n(r + xw^{-1}, y) = \sum_{\bar{T} \in F_{n+1}^{(r)}} w(\bar{T}). \tag{4.3}$$

Proof. By (4.2), we see that

$$\sum_{\bar{T}\in F_{n+1}^{(r)}} w(\bar{T}) = \sum_{\bar{T}\in F_{n+1}^{(r)}} a x^{\mathrm{black}(\bar{T})+r} w^{n-\mathrm{black}(\bar{T})} y^{n+\mathrm{imp}(\bar{T})}$$

$$= a x^r y^n w^n \sum_{k=0}^{n-1} y^k \sum_{\bar{T}\in F_{n+1,k}^{(r)}} x^{\mathrm{black}(\bar{T})} w^{-\mathrm{black}(\bar{T})}.$$

Given a rooted tree $T \in F_{n+1,k}$, one can construct a rooted tree $\bar{T}$ in $F_{n+1,k}^{(r)}$ by assigning the color b to some children of the root 1 and one of the r white colors to each remaining children of the root 1. Thus

$$\sum_{\bar{T}\in F_{n+1,k}^{(r)}} x^{\mathrm{black}(\bar{T})} w^{-\mathrm{black}(\bar{T})} = \sum_{T\in F_{n+1,k}} \sum_{i=0}^{\deg_T(1)} \binom{\deg_T(1)}{i} (xw^{-1})^i r^{\deg_T(1)-i}$$

$$= \sum_{T\in F_{n+1,k}} (r + xw^{-1})^{\deg_T(1)},$$

which can be expressed as $Q_{n,k}(r + xw^{-1})$ according to the interpretation (1.11) of $Q_{n,k}(x)$. It follows that

$$\sum_{\bar{T}\in F_{n+1}^{(r)}} w(T) = a x^r y^n w^n (1 + xw^{-1}) \sum_{k=0}^{n-1} y^k Q_{n,k}(r + xw^{-1}),$$

as claimed. ∎

The following theorem establishes a connection between the grammar H and the sum of weights of rooted trees in $F_n^{(r)}$.

Theorem 4.3 *For $n \geq 1$ and $r \geq 0$,*

$$D^n(ax^r) = \sum_{\bar{T}\in F_{n+1}^{(r)}} w(\bar{T}).$$

The proof is similar to that of Theorem 3.1. The operation of adding n as a black child of the root 1 can be described by the substitution rule $a \to axy$ and the operation of adding n as a white child of the root 1 corresponds to the rule $x \to xyw$.

We now give a grammatical derivation of the Berndt-Evans-Wilson-Shor recursion for $Q_{n,k}(x)$, that is, for $n \geq 1$ and $0 \leq k \leq n-1$,

$$Q_{n,k}(1 + x) = Q_{n,k}(x) + (n + k - 1)Q_{n-1,k}(1 + x). \tag{4.4}$$

Note that $Q_{1,0}(x) = 1$ and $Q_{n,k}(x) = 0$ if $k \geq n$ or $k < 0$.

Our proof relies on the generating function with respect to the grammar H. For a Laurent polynomial w of the variables in the alphabet V, the exponential generating function of w with respect to D is defined by

$$\mathrm{Gen}(w,t) = \sum_{n\geq 0} D^n(w)\frac{t^n}{n!}.$$

We have the following properties:

$$\mathrm{Gen}'(w,t) = \mathrm{Gen}(D(w),t), \tag{4.5}$$

$$\mathrm{Gen}(w+v,t) = \mathrm{Gen}(w,t) + \mathrm{Gen}(v,t), \tag{4.6}$$

$$\mathrm{Gen}(wv,t) = \mathrm{Gen}(w,t)\mathrm{Gen}(v,t), \tag{4.7}$$

where $\mathrm{Gen}'(w,t)$ stands for the differentiation of $\mathrm{Gen}(w,t)$ with respect to t, and v is also a Laurant polynomial of the variables in the alphabet V, see [2].

We are now in a position to present a grammatical proof of (4.4). It is easily seen that (4.4) follows from the following relation for $n \geq 1$,

$$\begin{aligned}
&axy^{n+1}w^n(1+xw^{-1})V_n(1+xw^{-1},y)\\
&\quad = axy^{n+1}w^n(1+xw^{-1})V_n(xw^{-1},y)\\
&\qquad + axw^n(1+xw^{-1})\sum_{k=0}^{n-2}(n+k-1)Q_{n-1,k}(1+xw^{-1})y^{n+k+1}.
\end{aligned} \tag{4.8}$$

Invoking (4.1) for $n \geq 1$ and $r = 0$, we obtain that for $n \geq 1$,

$$D^n(a) = axy^nw^{n-1}V_n(xw^{-1},y). \tag{4.9}$$

Again, utilizing (4.1) for $n \geq 1$ and $r = 1$, we find that

$$D^n(ax) = axy^nw^n(1+xw^{-1})V_n(1+xw^{-1},y), \tag{4.10}$$

and so

$$D^{n-1}(ax) = axy^{n-1}w^{n-1}(1+xw^{-1})V_{n-1}(1+xw^{-1},y). \tag{4.11}$$

Thus (4.8) can be rewritten as

$$yD^n(ax) = yw(1+xw^{-1})D^n(a) + y^3w\frac{\partial(D^{n-1}(ax))}{\partial y}. \tag{4.12}$$

Expanding (4.11) as

$$D^{n-1}(ax) = axy^{n-1}w^{n-1}(1+xw^{-1})\sum_{k=0}^{n-2} Q_{n-1,k}(1+xw^{-1})y^k,$$

we see that

$$\begin{aligned} axy\frac{\partial(D^{n-1}(ax))}{\partial a} &= xyD^{n-1}(ax), \\ xyw\frac{\partial(D^{n-1}(ax))}{\partial x} &= ywD^{n-1}(ax), \\ yw^2\frac{\partial(D^{n-1}(ax))}{\partial w} &= (n-1)ywD^{n-1}(ax). \end{aligned}$$

Notice that

$$D = axy\frac{\partial}{\partial a} + xyw\frac{\partial}{\partial x} + y^3w\frac{\partial}{\partial y} + yw^2\frac{\partial}{\partial w},$$

so that

$$D^n(ax) = xyD^{n-1}(ax) + nywD^{n-1}(ax) + y^3w\frac{\partial(D^{n-1}(ax))}{\partial y}, \tag{4.13}$$

and therefore, (4.12) is equivalent to

$$(y-1)D^{n-1}(D(ax)) + (nyw+xy)D^{n-1}(ax) = yw(1+xw^{-1})D^{n-1}(D(a)), \tag{4.14}$$

for $n \geq 1$. In terms of the generating functions, (4.14) can be reformulated as

$$\begin{aligned} &(xy+yw)\text{Gen}(ax,t) + (y-1+tyw)\text{Gen}(axyw+ax^2y,t) \\ &\qquad = (xy+yw)\text{Gen}(axy,t). \end{aligned} \tag{4.15}$$

Let

$$\begin{aligned} A(t) = (y-1+tyw)&\text{Gen}(axyw+ax^2y,t) \\ &+ (xy+yw)\text{Gen}(ax,t) - (xy+yw)\text{Gen}(axy,t). \end{aligned}$$

Since $D(xw^{-1}) = 0$ as given in (3.9), we have

$$A(t) = (1+xw^{-1})\text{Gen}(axyw,t)\left(y-1+tyw+yw\text{Gen}(y^{-1}w^{-1}-w^{-1},t)\right).$$

It remains to show that

$$y-1+tyw+yw\text{Gen}(y^{-1}w^{-1}-w^{-1},t) = 0. \tag{4.16}$$

Observe that

$$D(y^{-1}w^{-1} - w^{-1}) = -y^{-2}w^{-1}y^3w - y^{-1}w^{-2}yw^2 + w^{-2}yw^2 = -1.$$

Hence

$$\mathrm{Gen}(y^{-1}w^{-1} - w^{-1}, t) = y^{-1}w^{-1} - w^{-1} - t, \tag{4.17}$$

which proves (4.16), so that $A(t)$ vanishes. This completes the proof. ∎

5 The Abel Identities

In this section, we present a grammatical approach to the Abel identites. We establish an expression of $D^n(ax^ry)$ in terms of rooted trees on $[n]$. Recall that the set of rooted trees on $[n]$ is denoted by R_n.

For a rooted tree $T \in R_n$, we may construct a rooted tree $\bar{T}$ by coloring each child of the vertex 1 by one of the colors $b, w_1, w_2, \ldots, w_r$. It should be noted that 1 is not necessarily the root of T. Let $R_n^{(r)}$ denote the set of rooted trees on $[n]$ for which the children of 1 are colored as described above.

We need the following grammatical labeling for a rooted tree $\bar{T} \in R_{n,k}^{(r)}$: First, represent $\bar{T}$ as a rooted tree $\hat{T}$ on $\{0, 1, \ldots, n\}$ with root 0, and represent an improper edge by double edges. Label the vertex 1 by ax^r, label a black vertex by x and label each of the remaining vertices by w. Moreover, each edge in $\hat{T}$ is labeled by y. Thus the weight of $\bar{T}$ is given by

$$w(\bar{T}) = ax^{\mathrm{black}(\bar{T})+r}w^{n-1-\mathrm{black}(\bar{T})}y^{n+\mathrm{imp}(\bar{T})}. \tag{5.1}$$

Using the same argument as in the proof of Theorem 4.3, we are led to the following relation.

Theorem 5.1 *For $n \geq 1$ and $r \geq 0$,*

$$D^{n-1}(ax^ry) = \sum_{\bar{T}\in R_n^{(r)}} w(\bar{T}). \tag{5.2}$$

Analogous to Theorem 4.1, there is a connection between $D^n(ax^ry)$ and $V_n(x, y)$.

Theorem 5.2 *For $n \geq 1$ and $r \geq 0$,*

$$D^{n-1}(ax^ry) = ax^ry^nw^{n-1}V_n(r + xw^{-1} - 1, y). \tag{5.3}$$

In the notation of $V_n(x,y)$, the relation (2.3) of Dumont and Ramamonjisoa takes the form

$$D^{n-1}(yw) = y^n w^n V_n(0,y). \tag{5.4}$$

Dumont and Ramamonjisoa [7] also obtained grammatical expressions of $V_n(1,y)$ and $V_n(-1,y)$: For $n \geq 1$,

$$D^n(w) = y^n w^{n+1} V_n(1,y), \tag{5.5}$$

$$D^n(y) = y^{n+1} w^n V_{n+1}(-1,y). \tag{5.6}$$

It can be checked that by setting $a = x = w$, the grammar H reduces to the grammar of Dumont and Ramamonjisoa. Meanwhile, (5.4) can be deduced from (5.3) by setting $a = x = w$ and $r = 0$ and (5.5) can be deduced from (4.1) by setting $a = x = w$ and $r = 0$.

We remark that (5.6) can also be justified by a grammatical labeling of rooted trees in the set $R_{n,k}[\deg_T(1) = 0]$ of rooted trees in $R_{n,k}$ in which the vertex 1 is a leaf. For a rooted tree T in $R_{n,k}[\deg_T(1) = 0]$, let $\hat{T}$ denote the tree obtained from T by adding a new root 0 and representing each improper edge by double edges. Label each vertex except for 1 by x and label each edge in $\hat{T}$ by y. Therefore, for a rooted tree in $R_{n,k}[\deg_T(1) = 0]$, we have

$$w(T) = y^{n+k} w^{n-1}. \tag{5.7}$$

Note that the vertex 1 is not endowed with any label. On the other hand, in Shor's construction, it is not allowed to add new vertices as children of the vertex 1. The argument for the proof of Theorem 2.2 implies that for $n \geq 1$,

$$D^n(y) = \sum_{T \in R_{n+1,k}[\deg_T(1)=0]} w(T).$$

Based on the interpretation (1.12) of $Q_{n,k}(x)$, we see that for $x = -1$,

$$Q_{n,k}(-1) = |R_{n+1,k}[\deg_T(1) = 0]|.$$

Thus it follows from (5.7) that

$$D^n(y) = y^{n+1} w^n \sum_{k=0}^{n-1} y^k Q_{n,k}(-1),$$

which is the right-hand side of (5.6).

The following relations are needed in the grammatical derivations of the Abel identities.

Theorem 5.3 *For $n \geq 1$,*

$$D^n(y)|_{y=w=1} = n^n, \tag{5.8}$$

$$D^n(yw)|_{y=w=1} = (n+1)^n, \tag{5.9}$$

$$D^n(ax^r)|_{a=y=w=1} = x^r(x+r)(x+r+n)^{n-1}, \tag{5.10}$$

$$D^n(ax^r y)|_{a=y=w=1} = x^r(x+r+n)^n. \tag{5.11}$$

Proof. In the notation of $V_n(x, y)$, the relation (1.6) can be rewritten as

$$V_n(x, 1) = (x+n)^{n-1}. \tag{5.12}$$

Setting $y = w = 1$ in (5.6), we obtain that

$$D^n(y)|_{y=w=1} = V_{n+1}(-1, 1),$$

which equals n^n according to (5.12). This proves (5.8). The rest of the relations in the theorem can be obtained from (5.4), (4.1) and (5.3), respectively. This completes the proof. ∎

The classical Abel identity states that for $n \geq 1$,

$$(x_1 + x_2 + n)^n = \sum_{k=0}^{n} \binom{n}{k} x_1(x_1 + k)^{k-1}(x_2 + n - k)^{n-k}. \tag{5.13}$$

The Abel identity can be justified by the umbral calculus, see Rota [16]. Below we give a proof by resorting to the grammar H.

Proof of (5.13). Let

$$\begin{aligned} H'\colon a_1 \to a_1 x_1 y,\ a_2 \to a_2 x_2 y,\ x_1 \to x_1 y w,\ x_2 \to x_2 y w, \\ y \to y^3 w,\ w \to y w^2, \end{aligned} \tag{5.14}$$

and let D denote the formal derivative associated with the grammar H'. Viewing a_1 as a and x_1 as x and applying (5.10) with respect to H, we get

$$D^n(a_1)|_{a_1=y=w=1} = x_1(x_1 + n)^{n-1}. \tag{5.15}$$

Similarly, invoking (5.11), we obtain that

$$D^n(a_2 y)|_{a_2=y=w=1} = (x_2 + n)^n. \tag{5.16}$$

Moreover, since

$$D(a_1 a_2) = a_1 a_2 (x_1 + x_2) y$$

and

$$D(x_1 + x_2) = (x_1 + x_2)yw,$$

treating a_1a_2 as a and $x_1 + x_2$ as x, we may apply (5.11) to deduce that

$$D^n(a_1a_2y)|_{a_1=a_2=y=w=1} = (x_1 + x_2 + n)^n. \tag{5.17}$$

Finally, (5.13) is follows from the Leibnitz formula

$$D^n(a_1a_2y) = \sum_{k=0}^{n} \binom{n}{k} D^k(a_1)D^{n-k}(a_2y),$$

together with the relations (5.15), (5.16) and (5.17). ∎

Let

$$A_n(x_1, x_2; p, q) = \sum_{k=0}^{n} \binom{n}{k} (x_1 + k)^{k+p}(x_2 + n - k)^{n-k+q},$$

where $n \geq 1$ and p, q are integers. The following relations are given by Riordan [15].

Theorem 5.4 (Riordan [15]) *For $n \geq 1$, we have*

$$A_n(x_1, x_2; -1, -1) = x_1^{-1}x_2^{-1}(x_1 + x_2)(x_1 + x_2 + n)^{n-1} \tag{5.18}$$

and

$$\begin{aligned} &A_n(x_1, x_2; -2, 0) \\ &\quad = x_1^{-1}\left[(x_1 + 1)(x_1 + x_2 + n)^n - nx_1(x_1 + x_2 + n)^{n-1}\right]. \end{aligned} \tag{5.19}$$

Proof. Let H' denote the grammar given by (5.14), and let D denote the formal derivative associated with H'. Using the same reasoning as in the proof of (5.15), we see that

$$D^n(a_2)|_{a_2=y=w=1} = x_2(x_2 + n)^{n-1}. \tag{5.20}$$

Analogous to (5.17), we get

$$D^n(a_1a_2)|_{a_1=a_2=y=w=1} = (x_1 + x_2)(x_1 + x_2 + n)^{n-1}. \tag{5.21}$$

In view of (5.15), (5.20) and (5.21), we are led to (5.18) by applying the Leibnitz formula

$$D^n(a_1a_2) = \sum_{k=0}^{n} \binom{n}{k} D^k(a_1)D^{n-k}(a_2).$$

Set

$$s_1 = a_1y^{-1} + a_1x_1^{-1}w,$$

so that $D(s_1) = a_1x_1$. Analogous to (5.15), we find that for $n \geq 1$,

$$D^n(s_1)|_{a_1=y=w=1} = x_1(x_1+1)(x_1+n)^{n-2}. \tag{5.22}$$

Since

$$s_1a_2y = a_1a_2 + a_1a_2x_1^{-1}yw$$

and $x^{-1}w$ is a constant as shown in (3.9), we deduce that

$$D^n(s_1a_2y) = D^n(a_1a_2) + x_1^{-1}wD^n(a_1a_2y).$$

By the Leibnitz formula

$$D^n(s_1a_2y) = \sum_{k=0}^{n} \binom{n}{k} D^k(s_1)D^{n-k}(a_2y),$$

we find that

$$\sum_{k=0}^{n} \binom{n}{k} D^k(s_1)D^{n-k}(a_2y) = D^n(a_1a_2) + x^{-1}wD^n(a_1a_2y),$$

which yields (5.19) by applying (5.16), (5.17), (5.21) and (5.22). This completes the proof. ∎

We obtain a new Abel-type identity for the case $(p,q) = (-2,-2)$ by using the grammar H'.

Theorem 5.5 *For $n \geq 1$,*

$$\begin{aligned} A_n(x_1,x_2;-2,-2) = {} & \frac{(x_1+x_2)^3 - 3n(x_1+x_2) - 2n}{x_1x_2(x_1+1)(x_2+1)}(x_1+x_2+n)^{n-3} \\ & + \frac{(x_1+x_2)^2(x_1+x_2+1)}{x_1^2x_2^2(x_1+1)(x_2+1)}(x_1+x_2+n)^{n-2}. \end{aligned} \tag{5.23}$$

We need the following lemma.

Lemma 5.6 *Let D denote the formal derivative associated with the grammar H. For $n \geq 2$,*

$$D^n(ay^{-1})|_{a=y=w=1} = x(x+1)(x+n)^{n-2} - (x+n)^{n-1}. \tag{5.24}$$

For $n \geq 3$,

$$D^n(ay^{-2})|_{a=y=w=1} = (x^3 - 3nx - 2n)(x+n)^{n-3}. \tag{5.25}$$

Proof. Since $D(ay^{-1}) = ax - ayw$ and $D(x^{-1}w) = 0$ as given in (3.9), we have

$$D^n(ay^{-1}) = D^{n-1}(ax) - x^{-1}wD^{n-1}(axy).$$

Applying (5.10) and (5.11) with $r = 1$, we get

$$D^n(ay^{-1})|_{a=y=w=1} = x(x+1)(x+n)^{n-2} - (x+n)^{n-1}.$$

Since

$$D(ay^{-2}) = axy^{-1} - 2aw$$

and

$$D(axy^{-1}) = ax^2(1 + x^{-1}w) - axyw,$$

we see that for $n \geq 3$,

$$D^n(ay^{-2}) = (1 + x^{-1}w)D^{n-2}(ax^2) - x^{-1}wD^{n-2}(ax^2y) - 2x^{-1}wD^{n-1}(ax).$$

In light of (5.10) and (5.11), we find that

$$D^n(ay^{-2})|_{a=y=w=1} = x(x+1)(x+2)(x+n)^{n-3} - x(x+n)^{n-2} - 2(x+1)(x+n)^{n-2},$$

which implies (5.25). This complete the proof. ∎

Proof of Theorem 5.5. Assume that H' is the grammar given in (5.14) and D is the formal derivative with respect to H'. Let

$$s_1 = a_1y^{-1} + a_1x_1^{-1}w$$

and

$$s_2 = a_2y^{-1} + a_2x_2^{-1}w.$$

Clearly, $D(s_1) = a_1x_1$ and $D(s_2) = a_2x_2$. It follows from (5.10) that

$$D^n(s_2)|_{a_2=y=w=1} = x_2(1 + x_2)(x_2 + n)^{n-2}. \tag{5.26}$$

By the same argument as in the proof of (5.17), we deduce from (5.24) and (5.25) that

$$\begin{aligned} D^n(a_1a_2y^{-1})|_{a_1=a_2=y=w=1} &= (x_1 + x_2)(x_1 + x_2 + 1)(x_1 + x_2 + n)^{n-2} \\ &\quad - (x_1 + x_2 + n)^{n-1} \end{aligned}$$

and

$$D^n(a_1a_2y^{-2})|_{a_1=a_2=y=w=1} = ((x_1 + x_2)^3 - 3n(x_1 + x_2) - 2n)(x_1 + x_2 + n)^{n-3}.$$

Since $D(x_1w^{-1}) = D(x_2w^{-1}) = 0$, we get

$$\begin{aligned} D^n(s_1s_2) = D^n(a_1a_2y^{-2}) &+ (x_1^{-1} + x_2^{-1})wD^n(a_1a_2y^{-1}) \\ &+ (x_1x_2)^{-1}w^2D^n(a_1a_2). \end{aligned} \tag{5.27}$$

By the Leibnitz formula

$$D^n(s_1s_2) = \sum_{k=0}^{n} \binom{n}{k} D^k(s_1)D^{n-k}(s_2),$$

we obtain (5.23) by using (5.22), (5.26) and (5.27). This completes the proof. ∎

We conclude this paper with a grammatical explanation of the Lacasse identity.

Theorem 5.7 *For $n \geq 1$,*

$$n^{n+1} = \sum_{j=1}^{n} \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} j^j k^k (n-j-k)^{n-j-k}. \tag{5.28}$$

Proof. Because of the relation

$$k\binom{n}{k} = n\binom{n-1}{k-1},$$

(5.28) can be rewritten as

$$n^n = \sum_{j=1}^{n} \sum_{k=0}^{n-j} \binom{n-1}{j-1} \binom{n-j}{k} j^{j-1} k^k (n-k)^{n-k}. \tag{5.29}$$

Since $D(y) = y^3w$, we have

$$D^n(y) = D^{n-1}(y^3w) = \sum_{i+j+k=n-1} \binom{n-1}{i,j,k} D^i(y)D^j(yw)D^k(y). \tag{5.30}$$

Invoking (5.8) and (5.9) and setting $y = w = 1$, we see that (5.30) can be rewritten in the form of (5.29). This completes the proof. ∎

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