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Context-Free Grammars and Stable Multivariate Polynomials over Stirling Permutations

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Dedicated to Professor Peter Paule on the occasion of his 60th birthday

Abstract

Haglund and Visontai established the stability of the multivariate Eulerian polynomials as the generating functions of Stirling permutations, which serves as a unification of the results of Bóna, Brenti, Janson, Kuba, and Panholzer. Let $B_n(x)$ be the generating polynomials of the descent statistic over Legendre-Stirling permutations, and let $T_n(x) = 2^n C_n(x/2)$, where $C_n(x)$ are the second-order Eulerian polynomials. Haglund and Visontai proposed the problems of finding stable multivariate refinements of the polynomials $B_n(x)$ and $T_n(x)$. We provide solutions to these two problems by using context-free grammars. Moreover, the grammars enable us to obtain combinatorial interpretations of the multivariate polynomials in terms of Legendre-Stirling permutations and marked Stirling permutations.

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1 Introduction

This paper presents an approach to the construction of stable combinatorial polynomials from the perspective of context-free grammars. The framework of using context-free grammars to generate combinatorial polynomials was proposed in [9]. We find context-free grammars leading to stable multivariate polynomials over Legendre-Stirling permutations and marked Stirling permutations. These stable multivariate polynomials provide solutions to two problems raised by Haglund and Visontai [16] in their study of stable multivariate refinements of the second-order Eulerian polynomials.

Let us first give an overview of the second-order Eulerian polynomials. These polynomials were defined by Gessel and Stanley [13] as the generating functions of the descent statistic over Stirling permutations. Let $[n]_2$ denote the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. A permutation $\pi = \pi_1\pi_2 \cdots \pi_{2n-1}\pi_{2n}$ of $[n]_2$ is called a Stirling permutation if π satisfies the following condition: if $\pi_i = \pi_j$ then $\pi_k > \pi_i$ whenever $i < k < j$. For $1 \leq i \leq 2n$, we say that i is a descent of π if $i = 2n$ or $1 \leq i < 2n$ and $\pi_i > \pi_{i+1}$. Analogously, i is called an ascent of π if $i = 1$ or $1 < i \leq 2n$ and $\pi_{i-1} < \pi_i$. For the sake of consistency, we set $\pi_0 = \pi_{2n+1} = 0$. Let Q_n denote the set of Stirling permutations on $[n]_2$. Let $C(n, k)$ be the number of Stirling permutations of $[n]_2$ with k descents, and let

$$C_n(x) = \sum_{k=1}^{n} C(n, k)x^k.$$

Gessel and Stanley [13] showed that

$$\sum_{n=0}^{\infty} S(n + k, k)x^n = \frac{C_n(x)}{(1 - x)^{2k+1}},$$

where $S(n, k)$, as usual, denotes the Stirling number of the second kind. The numbers $C(n, k)$ are called the second-order Eulerian numbers by Graham et al. [14], and the polynomials $C_n(x)$ are called the second-order Eulerian polynomials by Haglund and Visontai [16]. Besides the connection with the enumeration of Stirling permutations, the second-order Eulerian number $C(n, k)$ has other combinatorial interpretations, such as the number of Riordan trapezoidal words of length n with k distinct letters [23], the number of rooted plane trees on $n + 1$ nodes with k leaves [18] and the number of matchings on $2n$ vertices with $n - k$ left-nestings [20].

The Stirling permutations were further studied by Bóna [1], Brenti [8], Janson [18] and Janson et al. [19]. Bóna [1] introduced the notion of a plateau of a Stirling permutation and studied the plateau statistic. Given a Stirling permutation $\pi = \pi_1\pi_2\ldots\pi_{2n} \in Q_n$, an index $1 < i \leq 2n$ is called a plateau of π if $\pi_{i-1} = \pi_i$. Bóna showed that the number of ascents, the number of descents and the number of plateaux have the same distribution over Q_n. Analogous to real-rootedness of the classical Eulerian polynomials, Bóna [1] proved the real-rootedness of the second-order Eulerian polynomials $C_n(x)$.

Theorem 1.1 (Bóna [1]) *For any positive integer n, the roots of the polynomial $C_n(x)$ are all real, distinct, and non-positive.*

It should be noted that the real-rootedness of $C_n(x)$ is essentially equivalent to the real-rootedness of the generating function of generalized Stirling permutations obtained by Brenti [8]. A permutation π of the multiset $\{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}$ is called a generalized Stirling permutation of rank n if π satisfies the same betweenness condition for a Stirling permutation. Let Q_n^* denote the set of generalized Stirling permutations of rank n. In particular, if $r_1 = r_2 = \cdots = r_n = r$ for some r, then π is called an r-Stirling permutation of order n. Let $Q_n(r)$ denote the set of r-Stirling permutations of order n. It is clear that 1-Stirling permutations are ordinary permutations and 2-Stirling permutations are Stirling permutations. Brenti [8] showed that the descent generating polynomials over Q_n^* have only real roots.

Janson [18] defined the following trivariate generating function

$$C_n(x, y, z) = \sum_{\pi \in Q_n} x^{\mathrm{des}(\pi)} y^{\mathrm{asc}(\pi)} z^{\mathrm{plat}(\pi)},$$

where $\mathrm{des}(\pi)$, $\mathrm{asc}(\pi)$, and $\mathrm{plat}(\pi)$ denote the number of descents, the number of ascents, and the number of plateaux of π, respectively, and proved that $C_n(x, y, z)$ is symmetric in x, y, z. This implies the equidistribution of these three statistics derived by Bóna [1].

The symmetry property of $C_n(x, y, z)$ was further extended to r-Stirling permutations by Janson et al. [19]. For an r-Stirling permutation, they introduced the notion of a j-plateau. For an r-Stirling permutation $\pi = \pi_1\pi_2\ldots\pi_{nr}$ and an integer $1 \leq j \leq r-1$, a number $1 \leq i < nr$ is called a j-plateau of π if $\pi_i = \pi_{i+1}$ and there are $j-1$ indices $l < i$ such that $\pi_l = \pi_i$, i.e., the number π_i appears j times up to the i-th position of π. Let j-$\mathrm{plat}(\pi)$ denote the number of j-plateaux of π. Define a descent and an ascent of π as in the case of ordinary permutations, and let $\mathrm{des}(\pi)$ and $\mathrm{asc}(\pi)$ denote the number of descents and ascents of π. Janson et al. [19] showed that the distribution of $(\mathrm{des}, 1\text{-plat}, 2\text{-plat}, \ldots, (r-1)\text{-plat}, \mathrm{asc})$ is symmetric over the set of r-Stirling permutations.

Based on the theory of stable multivariate polynomials recently developed by Borcea and Brändén [3–5], Haglund and Visontai [16] presented a unified approach to the stability of the generating functions of Stirling permutations and r-Stirling permutations.

A polynomial $f(z_1, z_2, \ldots, z_n) \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ is said to be stable, if whenever the imaginary part $\mathrm{Im}(z_i) > 0$ for all i then $f(z_1, z_2, \ldots, z_n) \neq 0$. Clearly, a univariate polynomial $f(z) \in \mathbb{R}[z]$ has only real roots if and only if it is stable.

For the case of univariate real polynomials, Pólya and Schur [22] characterized all diagonal operators preserving stability or real-rootedness. Recently, Borcea and Brändén [3–5] characterized all linear operators preserving stability of multivariate polynomials, see also the survey by Wagner [26]. This implies a characterization of linear operators preserving stability of univariate polynomials.

A multivariate polynomial is called multiaffine if the degree of each variable is at most 1. Borcea and Brändén [4] showed that each of the operators preserving stability of multiaffine polynomials has a simple form. Using this property, Haglund and Visontai [16] obtained a stable multiaffine refinement of the second-order Eulerian polynomial $C_n(x)$. Similar methods are employed for other related combinatorial structures, see [2, 6, 7, 15, 24, 25] for a few of other instances.

Given a Stirling permutation $\pi = \pi_1\pi_2 \cdots \pi_{2n} \in Q_n$, let

$$\begin{aligned} A(\pi) =& \{i | \pi_{i-1} < \pi_i, 1 \leq i \leq 2n\}, \\ D(\pi) =& \{i | \pi_i > \pi_{i+1}, 1 \leq i \leq 2n\}, \\ P(\pi) =& \{i | \pi_{i-1} = \pi_i, 1 \leq i \leq 2n\} \end{aligned}$$

denote the set of ascents, the set of descents and the set of plateaux of π, respectively. We set $\pi_0 = \pi_{2n+1} = 0$. Let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$ and $Z = (z_1, z_2, \ldots, z_n)$. Define

$$C_n(X, Y, Z) = \sum_{\pi \in Q_n} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

Haglund and Visontai [16] proved the stability of $C_n(X, Y, Z)$.

Theorem 1.2 (Haglund and Visontai [16]) *The polynomial $C_n(X, Y, Z)$ is stable.*

It is worth mentioning that, as observed by Haglund and Visontai [16], the recurrence relation between $C_{n-1}(X, Y, Z)$ and $C_n(X, Y, Z)$ can be used to derive the symmetry of $C_n(X, Y, Z)$, which implies the symmetry of $C_n(x, y, z)$ obtained by Janson et al. [19].

Moreover, Haglund and Visontai [16] extended the stability of $C_n(X, Y, Z)$ to generating polynomials of r-Stirling permutations by taking the j-plateau statistic into consideration. Let $P_j(\pi)$ denote the set of j-plateaux of π. For $i = 1, 2, \ldots, r-1$, let $Z_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,n})$. Haglund and Visontai [16] obtained the following stable multivariate polynomial over r-Stirling permutations

$$E_n(X,Y,Z_1,\ldots,Z_{r-1}) = \sum_{\pi\in Q_n(r)} \prod_{i\in D(\pi)} x_{\pi_i} \prod_{i\in A(\pi)} y_{\pi_i} \prod_{j=1}^{r-1} \prod_{i\in P_j(\pi)} z_{j,\pi_i}.$$

They also obtained a similar stable multivariate polynomial for generalized Stirling permutations.

Motivated by the real-rootedness of $C_n(x)$ and its stable multivariate refinement $C_n(X,Y,Z)$, Haglund and Visontai further considered the problem of finding stable multivariate polynomials as refinements of the generating polynomials of the descent statistic over Legendre-Stirling permutations. The Legendre-Stirling permutations were introduced by Egge [12] as a generalization of Stirling permutations in the study of Legendre-Stirling numbers of the second kind. For any $n \geq 1$, let M_n be the multiset $\{1,1,\bar{1},2,2,\bar{2},\ldots,n,n,\bar{n}\}$. A permutation $\pi = \pi_1\pi_2\ldots\pi_{3n}$ on M_n is called a Legendre-Stirling permutation if whenever $i < j < k$ and $\pi_i = \pi_k$ are both unbarred, then $\pi_j > \pi_i$. For a Legendre-Stirling permutation π on M_n, we say that i is a descent if either $i = 3n$ or $\pi_i > \pi_{i+1}$. Let $B_{n,k}$ denote the number of Legendre-Stirling permutations of M_n with k descents. Define

$$B_n(x) = \sum_{k=1}^{2n-1} B_{n,k}x^k.$$

Egge [12] proved the real-rootedness of $B_n(x)$.

Theorem 1.3 (Egge [12]) *For $n > 0$, $B_n(x)$ has distinct, real, non-positive roots.*

In order to derive a stable multivariate refinement of $B_n(x)$, we introduce an approach of generating stable polynomials by a sequence of grammars. Based on the Stirling grammar given by Chen and Fu [10], we find a sequence $G_1, G_2, \ldots$ of context-free grammars to generate Legendre-Stirling permutations. Let D_n denote the differential operator associated with the grammar G_n, which leads to a stable multivariate refinement $B_n(X,Y,Z,U,V)$ of $B_n(x)$, that is,

$$B_n(X,Y,Z,U,V) = D_{2n}D_{2n-1}\ldots D_2D_1(x_0),$$

where $U = (u_1,u_2,\ldots,u_n)$ and $V = (v_1,v_2,\ldots,v_n)$, respectively. Then by applying Borcea and Brändén's characterization of linear operators and the grammatical interpretation of $B_n(X,Y,Z,U,V)$, we prove the stability of $B_n(X,Y,Z,U,V)$. On the other hand, according to the grammars, we obtain the following combinatorial interpretation

$$B_n(X,Y,Z,U,V) = \sum_{\pi} \prod_{i\in X(\pi)} x_{\pi_i} \prod_{i\in Y(\pi)} y_{\pi_i} \prod_{i\in Z(\pi)} z_{\pi_i} \prod_{i\in U(\pi)} u_{\pi_i} \prod_{i\in V(\pi)} v_{\pi_i},$$

where π runs over all Legendre-Stirling permutations on M_n. Here $X(\pi)$, $Y(\pi)$, $Z(\pi)$, $U(\pi)$ and $V(\pi)$ are defined as follows: For a Legendre-Stirling permutation π on M_n,

$$\begin{aligned}
X(\pi) =&\{i \,|\, \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears for the first time}\},\\
Y(\pi) =&\{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is unbarred}\},\\
Z(\pi) =&\{i \,|\, \pi_{i-1} \leq \pi_i, \pi_i \text{ is unbarred and appears for the second time}\},\\
U(\pi) =&\{i \,|\, \pi_{i-1} \leq \pi_i \text{ and } \pi_i \text{ is barred}\},\\
V(\pi) =&\{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is barred}\}.
\end{aligned}$$

Here we set $\pi_0 = \pi_{3n+1} = 0$. Then the real-rootedness of $B_n(x)$ is a consequence of the stability of $B_n(X,Y,Z,U,V)$ by setting $v_i = y_i = y$ and $x_i = z_i = u_i = 1$ for $0 \leq i \leq n$.

Haglund and Visontai [16] also raised the question of finding stable multivariate refinements of the polynomials $T_n(x)$, which are given by

$$T_n(x) = 2^n C_n\left(\frac{x}{2}\right) = \sum_k 2^{n-k} C(n,k) x^k, \tag{1.1}$$

where $C(n,k)$ and $C_n(x)$, as before, denote the second-order Eulerian numbers and the second-order Eulerian polynomials respectively. The polynomials $T_n(x)$ were introduced by Riordan [23].

In view of the relation (1.1) between $T_n(x)$ and $C_n(x)$, we mark the Stirling permutations by some rule. We consider the following multivariate polynomials

$$T_n(X,Y,Z) = \sum_{\pi} \prod_{i\in D(\pi)} x_{\pi_i} \prod_{i\in A(\pi)} y_{\pi_i} \prod_{i\in P(\pi)} z_{\pi_i},$$

where π ranges over marked Stirling permutations of $[n]_2$. We shall show that the polynomials $T_n(X,Y,Z)$ are stable. The polynomial $T_n(x)$ becomes the specialization of $T_n(X,Y,Z)$ by setting $x_i = z_i = 1$ and $y_i = x$ for $0 \leq i \leq n$. This implies that $T_n(x)$ is real-rooted.

This paper is organized as follows. In Section 2, we give an overview of differential operators associated with context-free grammars and find context-free grammars to generate the polynomials $C_n(X,Y,Z)$. In Section 3, we give context-free grammars to generate the multivariate polynomials $T_n(X,Y,Z)$. In Section 4, we obtain context-free grammars that lead to the multivariate generating polynomials $B_n(X,Y,Z,U,V)$. In Section 5, based on Borcea and Brändén's characterization of linear operators preserving stability, we prove that the formal derivative with respect to the grammar that generates $T_n(X,Y,Z)$ preserves stability of multiaffine polynomials. This leads to the stability of $T_n(X,Y,Z)$. In Section 6, we provide an approach to find a new stability preserving operator when a grammar is not suitable to prove the stability of polynomials. In particular, we prove the stability of the multivariate polynomials $B_n(X,Y,Z,U,V)$.

2 Context-free grammars

In this section, we give an overview of the idea of using context-free grammars to generate combinatorial polynomials and combinatorial structures as developed in [9]. A context-free grammar G over an alphabet A is defined to be a set of production rules. A production rule means to substitute a letter in the alphabet A by a polynomial in A over a field. Given a context-free grammar, one may define a formal derivative D as a linear operator on polynomials in A, where the action of D on a letter is defined by the substitution rule of the grammar, the action of D on a sum of two polynomials u and v is defined by linear extension:

$$D(u+v) = D(u) + D(v),$$

and the action of D on the product of u and v is defined by the Leibniz rule, that is,

$$D(uv) = D(u)v + uD(v).$$

Many combinatorial polynomials can be generated by context-free grammars. Context-free grammars can also be used to generate combinatorial structures. More precisely, one may use a word on an alphabet to label a combinatorial structure such that the context-free grammar serves as the procedure to recursively generate the combinatorial structures. Such a labeling of a combinatorial structure is called a grammatical labeling in [10].

For example, we consider the Eulerian grammar

$$G = \{x \to xy,\, y \to xy\}$$

introduced by Dumont [11].

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n]$, let

$$\begin{aligned} A(\pi) =& \{i \,|\, \pi_{i-1} < \pi_i\}, \\ D(\pi) =& \{i \,|\, \pi_i > \pi_{i+1}\} \end{aligned}$$

denote the set of ascents and the set of descents of π, respectively. Here, as usual, we set $\pi_0 = \pi_{n+1} = 0$. Let $A(n,k)$ denote the Eulerian number, that is, the number of permutations on $[n]$ with k descents.

In order to show how to use the Eulerian grammar to generate permutations, Chen and Fu [10] introduced a grammatical labeling of a permutation π on $[n]$: If i is an ascent of π, then π_{i-1} is labeled by x; if i is a descent, then π_i is labeled by y. The weight of π is defined as the product of labels of elements in π, that is,

$$w(\pi) = x^{|A(\pi)|}y^{|D(\pi)|}.$$

For example, the grammatical labeling of the permutation $\pi = 325641$ is as follows:

$$\begin{array}{ccccccc} & 3 & 2 & 5 & 6 & 4 & 1 \\ x & y & x & x & y & y & y \end{array}.$$

Thus the weight of π equals $w(\pi) = x^3y^4$. This grammatical labeling leads to the following expression of the Eulerian polynomials. Dumont [11] obtained an equivalent form in terms of cyclic permutations and gave an inductive proof.

Theorem 2.1 (Dumont [11]) *Let D denote the formal derivative with respect to the Eulerian grammar. For $n \geq 1$, we have*

$$D^n(x) = \sum_{m=1}^{n} A(n,m)y^m x^{n+1-m}.$$

Let us now consider the grammar to generate Stirling permutations. Chen and Fu [10] introduced the grammar

$$G = \{x \to x^2y, y \to x^2y\}.$$

They defined a grammatical labeling of a Stirling permutation π in Q_n as follows: Let $1 \leq i \leq 2n$. If $i \in A(\pi)$ or $i \in P(\pi)$, the element π_{i-1} is labeled by x; if $i \in D(\pi)$, the element π_i is labeled by y. The weight of π, denoted by $w(\pi)$, is defined as the product of labels of elements in π. For example, the Stirling permutation $\pi = 233211$ has the following grammatical labeling

$$\begin{array}{ccccccc} & 2 & 3 & 3 & 2 & 1 & 1 \\ x & x & x & y & y & x & y \end{array}.$$

Then the weight of π is $w(\pi) = x^4y^3$.

Theorem 2.2 (Chen and Fu [10]) *Let D denote the formal derivative with respect to the above grammar G. For $n \geq 1$, we have*

$$D^n(x) = \sum_{m=1}^{n} C(n,m)x^{2n+1-m}y^m.$$

We shall give two sequences of grammars based on the Eulerian grammar and the Stirling grammar to solve the problems of Haglund and Visontai [16]. On one hand, we use these grammars to construct multivariate polynomials over Legendre-Stirling permutations and marked Stirling permutations. On the other hand, we use the grammars to construct stability preserving operators leading to the stability of the multivariate polynomials.

3 Marked Stirling permutations

In this section, we obtain a stable multivariate refinement of the polynomial $T_n(x)$, denoted by $T_n(X,Y,Z)$, which is defined as the generating function of marked Stirling permutations on $[n]_2$. This provides a solution to the problem of Haglund and Visontai.

In order to prove the stability of $T_n(X,Y,Z)$, we find grammars $G_1, G_2, \ldots$ that can be used to generate $T_n(X,Y,Z)$. More precisely, define

$$G_n = \{x_i, z_i \to x_n y_n z_n, y_i \to 2x_n y_n z_n \mid 0 \le i \le n-1\}.$$

Let D_n denote the formal derivative with respect to G_n. Using a grammatical labeling of marked Stirling permutations, we shall show that the polynomial $T_n(X,Y,Z)$ can be generated by $D_1, D_2, \ldots, D_n$. The stability of $T_n(X,Y,Z)$ can be established in Section 6 by using the operators $D_1, D_2, \ldots, D_n$.

A marked Stirling permutation is defined as follows. Given a Stirling permutation $\pi = \pi_1\pi_2\cdots\pi_{2n}$, if π_i is an element of π such that π_i occurs the second time in π and $\pi_i < \pi_{i+1}$, then we may mark the element π_i. We denote a marked element i by $\bar{i}$. A marked Stirling permutation is a Stirling permutation with some elements marked according to the above rule. Let $\bar{Q}_n$ denote the set of marked Stirling permutations on $[n]_2$. For example, there is only one marked Stirling permutation on $[1]_2$: 11, whereas there are four marked Stirling permutations on $[2]_2$:

$$2211, 1221, 1122, 1\bar{1}22.$$

Let $T(n,k)$ be the number of marked Stirling permutations on $[n]_2$ with k descents. Clearly,

$$T(n,k) = 2^{n-k} \cdot C(n,k),$$

where $C(n,k)$ denotes the second-order Eulerian number. Recall that $T_n(x)$ is defined by

$$T_n(x) = 2^n \cdot C_n\left(\frac{x}{2}\right) = \sum_{k=0}^{n} 2^{n-k} C(n,k) x^k.$$

Hence $T_n(x)$ is the generating function of marked Stirling permutations on $[n]_2$, that is,

$$T_n(x) = \sum_{k=0}^{n} T(n,k) x^k = \sum_{\pi \in \bar{Q}_n} x^{|D(\pi)|}.$$

In fact, Riordan [23] introduced the polynomials $T_n(x)$ and proved that $T_n(1)$ equals the Schröder number, namely, the number of series-reduced rooted trees with $n+1$ labeled leaves.

We shall prove that the polynomials $T_n(x)$ can be generated by the grammar

$$G = \{x \to x^2 y, y \to 2x^2 y\}.$$

The proof relies on the following grammatical labeling of a marked Stirling permutation. Let π be a marked Stirling permutation on $[n]_2$. If $i \in D(\pi)$, we label π_i by y. If $i \in A(\pi)$ or $i \in P(\pi)$, we label π_{i-1} by x. The weight of a marked Stirling permutation π on $[n]_2$ with m descents is given by

$$w(\pi) = x^{2n+1-m} y^m.$$

Theorem 3.1 *Let G be the grammar $G = \{x \to x^2y, y \to 2x^2y\}$ and D be the formal derivative associated with G. For $n \geq 1$,*

$$D^n(x) = \sum_{k=1}^{n} T(n,k) x^{2n-k+1} y^k.$$

Setting $x = 1$, we have

$$D^n(x)|_{x=1} = T_n(y).$$

Proof. We aim to show that $D^n(x)$ equals the sum of the weights of marked Stirling permutations of $[n]_2$ by induction on n, that is,

$$D^n(x) = \sum_{\pi \in \bar{Q}_n} w(\pi). \tag{3.1}$$

For $n = 1$, (3.1) follows from the fact that the weight of 11, the only marked Stirling permutation on $[1]_2$, is x^2y. Assume that (3.1) holds for $n-1$, that is,

$$D^{n-1}(x) = \sum_{\pi \in \bar{Q}_{n-1}} w(\pi).$$

We now use an example to demonstrate the action of D on a marked Stirling permutation of $[n-1]_2$. Let $\pi = 12\bar{2}331$ with the following grammatical labeling

$$\begin{array}{ccccccc} & 1 & 2 & \bar{2} & 3 & 3 & 1 \\ x & x & x & x & x & y & y \end{array}.$$

If we apply the substitution rule $x \to x^2y$ to the fourth letter x, then we insert the two elements 44 after $\bar{2}$. We keep all the old labels and assign the labels x and y to the two new letters 44 from left to right. It is not difficult to see that the generated marked Stirling permutation has a consistent grammatical labeling

$$\begin{array}{ccccccccc} & 1 & 2 & \bar{2} & 4 & 4 & 3 & 3 & 1 \\ x & x & x & x & x & y & x & y & y \end{array}.$$

If we apply the substitution rule $y \to 2x^2y$ to the first letter y, then we insert 44 after the second element 3. We change the label of the second element 3 from y to x

and assign x and y to the two new elements 44 from left to right. According to the marking rule, the second element 3 may be marked or unmarked. These two choices correspond the coefficient 2 in the substitution rule $y \to 2x^2y$. So we are led to the following two marked Stirling permutations with consistent grammatical labelings,

$$\begin{array}{cccccccccc} & 1 & 2 & \bar{2} & 3 & 3 & 4 & 4 & 1 & \\ x & x & x & x & x & x & x & y & y \end{array},$$

and

$$\begin{array}{cccccccccc} & 1 & 2 & \bar{2} & 3 & \bar{3} & 4 & 4 & 1 & \\ x & x & x & x & x & x & x & y & y \end{array}.$$

In general, it can be verified that the action of D on the weights of marked Stirling permutations in $\bar{Q}_{n-1}$ generates the weights of marked Stirling permutations in $\bar{Q}_n$. So we deduce that (3.1) holds for n, that is,

$$D^n(x) = D(D^{n-1}(x)) = D\left(\sum_{\pi\in\bar{Q}_{n-1}} w(\pi)\right) = \sum_{\sigma\in\bar{Q}_n} w(\sigma).$$

Hence the proof is complete by induction. ∎

As a multivariate refinement of $T_n(x)$, we define the following generating function of marked Stirling permutations on $[n]_2$,

$$T_n(X,Y,Z) = \sum_{\pi\in\bar{Q}_n} \prod_{i\in A(\pi)} x_{\pi_i} \prod_{i\in D(\pi)} y_{\pi_i} \prod_{i\in P(\pi)} z_{\pi_i}.$$

Let

$$G_n = \{x_i \to x_ny_nz_n, z_i \to x_ny_nz_n, y_i \to 2x_ny_nz_n \mid 0 \le i \le n-1\}.$$

We give a grammatical labeling of a marked Stirling permutation. For a marked Stirling permutation π on $[n]_2$, if $i \in A(\pi)$, we label π_{i-1} by x_{π_i}; if $i \in D(\pi)$, we label π_i by y_{π_i}; and if $i \in P(\pi)$, we label π_{i-1} by z_{π_i}. Then the weight of π equals

$$w(\pi) = \prod_{i\in A(\pi)} x_{\pi_i} \prod_{i\in D(\pi)} y_{\pi_i} \prod_{i\in P(\pi)} z_{\pi_i}.$$

The following theorem shows that the polynomials $T_n(X,Y,Z)$ can be generated by the grammars $G_1, G_2, \ldots, G_n$.

Theorem 3.2 *Let D_n denote the formal derivative associated with the grammar G_n. For $n \ge 1$,*

$$T_n(X,Y,Z) = D_nD_{n-1}\cdots D_1(z_0).$$

The proof of the above theorem is analogous to that of Theorem 3.1. Hence the details are omitted. Here we use an example to illustrate the action of D_4 on the above marked Stirling permutation $\pi = 12\bar{2}331$ with the grammatical labeling

$$\begin{array}{cccccc} 1 & 2 & \bar{2} & 3 & 3 & 1 \\ x_1 & x_2 & z_2 & x_3 & z_3 & y_3 & y_1 \end{array}.$$

Applying the substitution rule $x_3 \to x_4y_4z_4$ to π, we get a marked Stirling permutation by inserting the two elements 44 after $\bar{2}$ and the consistent grammatical labeling is given below:

$$\begin{array}{cccccccc} 1 & 2 & \bar{2} & 4 & 4 & 3 & 3 & 1 \\ x_1 & x_2 & z_2 & x_4 & z_4 & y_4 & z_3 & y_3 & y_1 \end{array}.$$

Similarly, applying the substitution rule $y_3 \to 2x_4y_4z_4$ leads to two marked Stirling permutations by inserting 44 after the second element 3, since the second element 3 can be marked. The consistent grammatical labelings are

$$\begin{array}{cccccccc} 1 & 2 & \bar{2} & 3 & 3 & 4 & 4 & 1 \\ x_1 & x_2 & z_2 & x_3 & z_3 & x_4 & z_4 & y_4 & y_1 \end{array},$$

and

$$\begin{array}{cccccccc} 1 & 2 & \bar{2} & 3 & \bar{3} & 4 & 4 & 1 \\ x_1 & x_2 & z_2 & x_3 & z_3 & x_4 & z_4 & y_4 & y_1 \end{array}.$$

For $n = 0$, the empty permutation is labeled by z_0. We have $T_0(X, Y, Z) = z_0$. For $n = 1, 2$, we have

$$\begin{aligned} T_1(X,Y,Z) = D_1(z_0) &= x_1 \overset{1}{z_1} \overset{1}{y_1}, \\ T_2(X,Y,Z) = D_2D_1(z_0) &= D_2(x_1 \overset{1}{z_1} \overset{1}{y_1}) \\ &= x_2 \overset{2}{z_2} \overset{2}{y_2} \overset{1}{z_1} \overset{1}{y_1} + x_1 \overset{1}{x_2} \overset{2}{z_2} \overset{2}{y_2} \overset{1}{y_1} + x_1 \overset{1}{z_1} \overset{1}{x_2} \overset{2}{z_2} \overset{2}{y_2} \\ &\quad + x_1 \overset{1}{z_1} \overset{\bar{1}}{x_2} \overset{2}{z_2} \overset{2}{y_2} \\ &= y_1z_1x_2y_2z_2 + x_1y_1x_2y_2z_2 + 2x_1z_1x_2y_2z_2. \end{aligned}$$

4 Legendre-Stirling permutations

In this section, we give refinements of the Stirling grammar and the Eulerian grammar, and we show that these refined grammars can be used to generate stable multivariate polynomials. For $n \geq 1$, let

$$G_{2n-1} = \{x_i, y_i, z_i, u_i, v_i \to u_nv_n \mid 0 \leq i < n\},$$

and let

$$G_{2n} = \{u_n \to x_n z_n u_n, v_n \to x_n y_n z_n,$$
$$x_i, y_i, z_i, u_i, v_i \to x_n y_n z_n \mid 0 \le i < n\}.$$

Clearly, G_{2n-1} is a refinement of the Eulerian grammar, and G_{2n} is a refinement of the Stirling grammar.

Let D_n denote the formal derivative with respect to the grammar G_n. We give a grammatical labeling of Legendre-Stirling permutations, which leads to a combinatorial interpretation of the multivariate polynomial $D_{2n}D_{2n-1}\cdots D_1(x_0)$. To this end, we introduce several statistics of a Legendre-Stirling permutation. In terms of these statistics, we obtain a multivariate polynomial $B_n(X, Y, Z, U, V)$ as a refinement of $B_n(x)$, which can be generated by the operators $D_1, D_2, \ldots, D_{2n}$.

Recall that M_n denotes the multiset $\{1, 1, \bar{1}, 2, 2, \bar{2}, \ldots, n, n, \bar{n}\}$. Let L_n denote the set of Legendre-Stirling permutations on M_n. For a Legendre-Stirling permutation $\pi = \pi_1\pi_2 \ldots \pi_{3n} \in L_n$, define

$$\begin{aligned}
X(\pi) =& \{i \,|\, \pi_{i-1} \le \pi_i, \pi_i \text{ is unbarred and appears for the first time}\},\\
Y(\pi) =& \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is unbarred}\},\\
Z(\pi) =& \{i \,|\, \pi_{i-1} \le \pi_i, \pi_i \text{ is unbarred and appears for the second time}\},\\
U(\pi) =& \{i \,|\, \pi_{i-1} \le \pi_i \text{ and } \pi_i \text{ is barred}\},\\
V(\pi) =& \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is barred}\}.
\end{aligned}$$

As usual, we set $\pi_0 = \pi_{3n+1} = 0$.

For example, let $\pi = \bar{1}1\bar{2}233 2\bar{3}1$. Then we have $X(\pi) = \{2, 4, 5\}$, $Y(\pi) = \{6, 9\}$, $Z(\pi) = \{6\}$, $U(\pi) = \{1, 3, 8\}$ and $V(\pi) = \{8\}$.

Define

$$B_n(X, Y, Z, U, V) = \sum_{\pi \in L_n} \prod_{i \in X(\pi)} x_{\pi_i} \prod_{i \in Y(\pi)} y_{\pi_i} \prod_{i \in Z(\pi)} z_{\pi_i} \prod_{i \in U(\pi)} u_{\pi_i} \prod_{i \in V(\pi)} v_{\pi_i}. \tag{4.1}$$

For example, there are only two Legendre-Stirling permutations on M_1: $11\bar{1}$ and $\bar{1}11$. So we have

$$B_1(X, Y, Z, U, V) = x_1y_1z_1u_1 + x_1z_1u_1v_1.$$

For $n = 2$, there are 40 Legendre-Stirling permutations on M_2 and we have

$$\begin{aligned} B_2(X,Y,Z,U,V) = {} & 2x_2y_2z_2u_2x_1z_1u_1 + x_2y_2z_2u_2x_1y_1z_1 + x_2y_2z_2u_2x_1y_1u_1 \\ & + x_2y_2z_2u_2y_1z_1u_1 + x_2y_2z_2u_2x_1u_1v_1 + x_2y_2z_2u_2z_1u_1v_1 \\ & + x_2y_2z_2u_2x_1z_1v_1 + 2x_2z_2u_2v_2x_1z_1u_1 + x_2z_2u_2v_2x_1y_1z_1 \\ & + x_2z_2u_2v_2x_1y_1u_1 + x_2z_2u_2v_2y_1z_1u_1 + x_2z_2u_2v_2x_1z_1v_1 \\ & + x_2z_2u_2v_2x_1u_1v_1 + x_2z_2u_2v_2z_1u_1v_1 + 4x_2y_2z_2u_2v_2x_1z_1 \\ & + 4x_2y_2z_2u_2v_2x_1u_1 + 4x_2y_2z_2u_2v_2u_1z_1 + 2x_2y_2z_2u_2v_2x_1y_1 \\ & + 2x_2y_2z_2u_2v_2y_1z_1 + 2x_2y_2z_2u_2v_2y_1u_1 + 2x_2y_2z_2u_2v_2x_1v_1 \\ & + 2x_2y_2z_2u_2v_2u_1v_1 + 2x_2y_2z_2u_2v_2z_1v_1. \end{aligned}$$

We now give a grammatical labeling of a Legendre-Stirling permutation. Let π be a Legendre-Stirling permutation in L_n. For $i \in X(\pi)$, $i \in Z(\pi)$ or $i \in U(\pi)$, we label π_{i-1} by x_{π_i}, z_{π_i} or u_{π_i}, respectively; for $i \in Y(\pi)$ or $i \in V(\pi)$, we label π_i by y_{π_i} or v_{π_i}, respectively. The weight of π is defined as the product of these letters labeled on entries of π and denoted by $w(\pi)$. For example, the grammatical labeling of the aforementioned Legendre-Stirling permutation $\pi = 1\bar{2}\bar{1}233 2\bar{3}1$ is given below:

$$\begin{array}{cccccccccc} & 1 & \bar{2} & \bar{1} & 2 & 3 & 3 & 2 & \bar{3} & 1 \\ x_1 & u_2 & v_2 & x_2 & x_3 & z_3 & y_3 & u_3 & v_3 & y_1 \end{array}.$$

Theorem 4.1 *For $n \geq 1$, let D_n denote the differential operator with respect to the grammar G_n, then we have*

$$D_{2n}D_{2n-1}\cdots D_1(x_0) = B_n(X,Y,Z,U,V). \tag{4.2}$$

Proof. We proceed by induction on n to show that

$$D_{2n}D_{2n-1}\cdots D_1(x_0) = \sum_{\pi\in L_n} w(\pi). \tag{4.3}$$

It can be checked that (5) holds for $n = 1$. For $n \geq 2$, we assume that (4.3) holds for $n-1$, that is,

$$D_{2n-2}D_{2n-3}\cdots D_1(x_0) = \sum_{\pi\in L_{n-1}} w(\pi).$$

Note that any Legendre-Stirling permutation on M_n can be obtained from a Legendre-Stirling permutation on M_{n-1} by inserting nn and $\bar{n}$. We use examples to illustrate that the application of the operator $D_{2n}D_{2n-1}$ reflects the insertions of nn and $\bar{n}$.

Consider the Legendre-Stirling permutation $\pi = \bar{1}1\bar{2}233 2\bar{3}1$ with the following grammatical labeling:

$$\begin{array}{cccccccccc} & \bar{1} & 1 & \bar{2} & 2 & 3 & 3 & 2 & \bar{3} & 1 \\ u_1 & x_1 & u_2 & x_2 & x_3 & z_3 & y_3 & u_3 & v_3 & y_1 \end{array}.$$

Let w be the weight of the above grammatical labeling, that is,

$$w = u_1 x_1 u_2 x_2 x_3 z_3 y_3 u_3 v_3 y_1.$$

Let us consider the action of D_7 on w. Recall that

$$G_7 = \{x_i, y_i, z_i, u_i, v_i \to u_4 v_4 \mid i = 1, 2, 3\}.$$

Consider a substitution rule that replaces a letter s by $u_4 v_4$. Assume that π_k is labeled by s, where $0 \leq k \leq 9$. This rule corresponds to an insertion of $\bar{4}$ after the entry π_k in π. Then the element π_k is relabeled by u_4, and the element $\bar{4}$ is labeled by v_4.

For example, the substitution rule $z_3 \to u_4 v_4$ corresponds to the insertion of $\bar{4}$ after the first element 3 in π. After the insertion, we obtain a Legendre-Stirling permutation with a consistent grammatical labeling:

$$\begin{array}{ccccccccccc} & \bar{1} & 1 & \bar{2} & 2 & 3 & \bar{4} & 3 & 2 & \bar{3} & 1 \\ u_1 & x_1 & u_2 & x_2 & x_3 & u_4 & v_4 & y_3 & u_3 & v_3 & y_1 \end{array}.$$

As for the action of D_8, consider the above permutation $\sigma = \bar{1}1\bar{2}23\bar{4}32\bar{3}1$. Let w' denote the weight of σ, that is,

$$w' = u_1 x_1 u_2 x_2 x_3 u_4 v_4 y_3 u_3 v_3 y_1.$$

The two substitution rules $u_4 \to x_4 z_4 u_4$ and $v_4 \to x_4 y_4 z_4$ of G_8 correspond to the insertions of the element 44 into σ before $\bar{4}$ or after $\bar{4}$, respectively, resulting in two Legendre-Stirling permutations: $\bar{1}1\bar{2}2344\bar{4}32\bar{3}1$ or $\bar{1}1\bar{2}23\bar{4}4432\bar{3}1$.

It remains to consider the substitution rules of G_8 that are of the form $s \to x_4 y_4 z_4$, where $s \in \{x_i, y_i, z_i, u_i, v_i \mid i = 1, 2, 3\}$. Suppose that σ_i is the element in σ that is labeled by s. The substitution rule $s \to x_4 y_4 z_4$ corresponds to the insertion of 44 into σ after σ_i. Let τ denote the resulting permutation obtained from σ after the insertion. Then one can obtain a consistent grammatical labeling of τ by relabeling σ_i by x_4 and assigning the two labels z_4 and y_4 to the inserted two elements 44 from left to right. For example, by applying the substitution rule $u_2 \to x_4 y_4 z_4$, we obtain the Legendre-Stirling permutation by inserting 44 after the first element 1 with a consistent grammatical labeling:

$$\begin{array}{ccccccccccccc} & \bar{1} & 1 & 4 & 4 & \bar{2} & 2 & 3 & \bar{4} & 3 & 2 & \bar{3} & 1 \\ u_1 & x_1 & x_4 & z_4 & y_4 & x_2 & x_3 & u_4 & v_4 & y_3 & u_3 & v_3 & y_1 \end{array}.$$

In general, it can be verified that the action of $D_{2n} D_{2n-1}$ on the weights of the Legendre-Stirling permutations in L_{n-1} generates the weights of Legendre-Stirling permutations in L_n. So we conclude that (4.3) holds for n, that is,

$$D_{2n} D_{2n-1} \cdots D_1(x_0) = \sum_{\pi \in L_n} w(\pi).$$

Thus (4.3) holds for all n. This completes the proof. ∎

We note that the grammars $G_2, G_4, \ldots$ are related to the polynomials $C_n(X,Y,Z)$ introduced by Haglund and Vistonai [16], as defined by

$$C_n(X,Y,Z) = \sum_{\pi \in Q_n} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

Clearly, $C_1(X,Y,Z) = x_1y_1z_1$. Based on the combinatorial interpretation of $C_n(X,Y,Z)$, Haglund and Visontai [16] established the following recurrence relation for $n \geq 1$:

$$C_{n+1}(X,Y,Z) = x_{n+1}y_{n+1}z_{n+1}\partial C_n(X,Y,Z), \tag{4.4}$$

where

$$\partial = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial}{\partial y_i} + \sum_{i=1}^{n} \frac{\partial}{\partial z_i}. \tag{4.5}$$

The following theorem shows that the grammar D_{2n} has the same effect as the operator $x_ny_nz_n\partial$ when acting on $C_{n-1}(X,Y,Z)$.

Theorem 4.2 *For $n \geq 0$,*

$$D_{2n+2}(C_n(X,Y,Z)) = x_{n+1}y_{n+1}z_{n+1}\partial C_n(X,Y,Z). \tag{4.6}$$

The relation (4.6) implies that

$$D_{2n+2}D_{2n}\cdots D_4D_2(z_0) = C_{n+1}(X,Y,Z).$$

To prove Theorem 4.2, we observe the following property of the formal derivative D with respect to a grammar G. The verification is straightforward.

Proposition 4.3 *Let X denote the set of variables of a grammar G. For a polynomial f in X, we have*

$$D(f) = \sum_{x \in X} D(x)\frac{\partial f}{\partial x}.$$

5 The stability of $T_n(X,Y,Z)$

In this section, we prove the stability of the multivariate polynomials $T_n(X,Y,Z)$ by showing that the related formal derivatives with respect to the generating grammars are stability preserving operators. The proof relies on the characterization of stability preserving linear operators on multiaffine polynomials due to Borcea and Brändén [4].

Recall that a multivariate polynomial $f(z_1, z_2, \ldots, z_n)$ is called multiaffine if the degree of any variable in f is at most 1. An operator T is called a stability preserver of multiaffine polynomials if $T(f)$ is either stable or identically 0 for any stable multiaffine polynomial $f \in \mathbb{C}[z_1, z_2, \ldots, z_n]$.

Theorem 5.1 (Borcea and Brändén) *Let T denote a linear operator acting on the polynomials in $\mathbb{C}[z_1, z_2, \ldots, z_n]$. If*

$$T\left(\prod_{i=1}^{n}(z_i + w_i)\right) \in \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n]$$

is stable, then T is a stability preserver of multiaffine polynomials.

To prove the stability of $T_n(X, Y, Z)$, we use the grammatical expression

$$T_n(X, Y, Z) = D_n D_{n-1} \cdots D_1(z_0)$$

in Theorem 3.2, where D_n is the formal derivative with respect to the grammar

$$G_n = \{x_i, z_i \to x_n y_n z_n, y_i \to 2x_n y_n z_n \mid 0 \leq i < n\}.$$

We shall show that D_n is a stability preserver, and this proves the stability of $T_n(X, Y, Z)$.

Theorem 5.2 *For $n \geq 1$, $T_n(X, Y, Z)$ is stable.*

Proof. Let

$$F = \prod_{i=0}^{n}(x_i + u_i)(y_i + v_i)(z_i + w_i), \tag{5.1}$$

and let

$$\xi = \sum_{i=0}^{n-1}\left(\frac{1}{x_i + u_i} + \frac{2}{y_i + v_i} + \frac{1}{z_i + w_i}\right). \tag{5.2}$$

We have

$$\begin{aligned}
D_n(F) &= \sum_{i=0}^{n-1} D(x_i)\frac{\partial F}{\partial x_i} + \sum_{j=0}^{n-1} D(y_j)\frac{\partial F}{\partial y_j} + \sum_{k=0}^{n-1} D(z_k)\frac{\partial F}{\partial z_k} \\
&= \sum_{i=0}^{n-1} x_n y_n z_n \frac{F}{x_i + u_i} + \sum_{j=0}^{n-1} 2x_n y_n z_n \frac{F}{y_i + v_i} + \sum_{k=0}^{n-1} x_n y_n z_n \frac{F}{z_k + w_k} \\
&= x_n y_n z_n \xi F.
\end{aligned}$$

To prove that D_n preserves stability of multiaffine polynomials, we assume that x_i, y_i, z_i, u_i, v_i and w_i have positive imaginary parts for all $0 \leq i \leq n$. We proceed to show that $D_n(F) \neq 0$.

Under the above assumptions, for $0 \leq i \leq n$, $x_i + u_i$, $y_i + v_i$ and $z_i + w_i$ also have positive imaginary parts. It follows that $\frac{1}{x_i+u_i}$, $\frac{2}{y_i+v_i}$ and $\frac{1}{z_i+w_i}$ have negative imaginary parts. By the definition (5.1), we see that $F \neq 0$. By (5.2), we find that $\xi \neq 0$. Hence $D_n(F) \neq 0$. Thus D_n is a stability preserver. This completes the proof. ∎

6 The stability of $B_n(X, Y, Z, U, V)$

In this section, we prove the stability of the multivariate polynomials $B_n(X, Y, Z, U, V)$. Unlike the proof for $T_n(X, Y, Z)$, the formal derivatives with respect to the grammars do not preserve stability. Fortunately, as for the multiaffine polynomials that we are concerned with, the formal derivatives in our case are equivalent to linear operators which turn out to be stability preserving.

More specifically, the idea goes as follows: Let $G_1, G_2, \ldots$ be context-free grammars, and $D_1, D_2, \ldots$ be the formal derivatives with respect to $G_1, G_2, \ldots$. Suppose that we wish to prove the stability of the multivariate polynomials

$$f_n = D_n D_{n-1} \cdots D_1(x),$$

for $n \geq 1$, where $D_1, D_2, \ldots$ may not be stability preserving. We aim to construct stability preservers $T_1, T_2, \ldots$ such that

$$T_n T_{n-1} \cdots T_1(x) = D_n D_{n-1} \cdots D_1(x).$$

Once such stability preservers $T_1, T_2, \ldots$ are found, it can be asserted that the multivariate polynomials f_n are stable. The following lemma provides a way to find such operators T_n.

Lemma 6.1 *Let G be a context-free grammar over the alphabet $X \cup Y$, where*

$$X = \{x_1, x_2, \ldots, x_r\}$$

and

$$Y = \{y_1, y_2, \ldots, y_s\}.$$

Let D denote the formal derivative with respect to G. Assume that $D(x_i)$ contains a factor x_i for $i = 1, 2, \ldots, r$, namely, $x_i \to x_i h_i(X, Y)$ is a substitution rule in G. Let T denote the following operator

$$T = \sum_{i=1}^{r} h_i(X, Y) I + \sum_{j=1}^{s} D(y_j) \frac{\partial}{\partial y_j},$$

where I denotes the identity operator. Let $g(Y)$ be any polynomial in Y and let $f(X,Y) = x_1x_2\ldots x_r g(Y)$. Then we have

$$D(f(X,Y)) = T(f(X,Y)).$$

Proof. By Proposition 4.3, we find that

$$\begin{aligned}
D(f(X,Y)) &= \sum_{i=1}^{r} D(x_i)\frac{\partial f(X,Y)}{\partial x_i} + \sum_{j=1}^{s} D(y_j)\frac{\partial f(X,Y)}{\partial y_j} \\
&= \sum_{i=1}^{r} x_i h_i(X,Y)\cdot\frac{f(X,Y)}{x_i} + \sum_{j=1}^{s} D(y_j)\frac{\partial f(X,Y)}{\partial y_j} \\
&= \sum_{i=1}^{r} h_i(X,Y)f(X,Y) + \sum_{j=1}^{s} D(y_j)\frac{\partial f(X,Y)}{\partial y_j},
\end{aligned}$$

which equals $T(f(X,Y))$. This completes the proof. ∎

For example, the grammar

$$G = \{a \to ax,\ x \to x\}$$

is used in [9] to generate the set of partitions of $[n]$ and the Stirling polynomials

$$S_n(x) = \sum_{i=0}^{n} S(n,k)x^k,$$

where $S(n,k)$ denotes the Stirling number of the second kind.

For $n \geq 1$, we have

$$D^n(a) = \sum_{k=1}^{n} S(n,k)ax^k = aS_n(x). \tag{6.1}$$

Many properties of the Stirling polynomials follow from the above expression in terms of the differential operator D with respect to the grammar G.

Let $X = \{a\}$ and $Y = \{x\}$. Then D satisfies the conditions in Lemma 6.1. Thus $D(af(x)) = T(af(x))$ for any polynomial $f(x)$, where the operator T is given by

$$T = x\left(I + \frac{\partial}{\partial x}\right).$$

In particular, we have

$$T(aS_n(x)) = D(aS_n(x)).$$

In fact, the above operator T corresponds to the following recurrence relation for $S_n(x)$:

$$S_n(x) = T(S_{n-1}(x)),$$

which is equivalent to the recurrence relation of $S(n,k)$:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{6.2}$$

where $n \geq k > 1$. Harper [17] proved that $S_n(x)$ has only real roots for $n \geq 1$. Liu and Wang [21] showed that T preserves the real-rootedness of polynomials in x.

As a generalization of the real-rootedness of $S_n(x)$, we consider the stability of the multivariate polynomials $S_n(a, x_1, x_2, \ldots, x_n)$, which can be viewed as a refinement of the Stirling polynomial $S_n(x)$. Let

$$G_n = \{a \to ax_n, x_i \to x_n \mid 1 \leq i < n\},$$

and let D_n denote the formal derivative associated with G_n. It will be shown that for $n \geq 1$, $S_n(a, x_1, x_2, \ldots, x_n)$ can be generated by $G_1, G_2, \ldots, G_n$.

The polynomial $S_n(a, x_1, x_2, \ldots, x_n)$ is defined by using the following grammatical labeling of a partition $P = \{P_1, P_2, \ldots, P_k\}$ of $[n]$. The partition itself is labeled by the letter a and a block P_i is labeled by the letter x_{m_i}, where m_i is the maximal element in P_i. The weight of P is given by the product of all labelings in P, that is,

$$w(P) = a \prod_{i=1}^{k} x_{m_i}.$$

Denote by $S_n(a, x_1, x_2, \ldots, x_n)$ the sum of weights of partitions of $[n]$. Clearly, $S_n(a, x_1, x_2, \ldots, x_n)$ is the generating function of partitions of $[n]$ involving not only the number of blocks, but also the maximal elements of the blocks.

For example, for $n = 1, 2, 3$, we have

$$\begin{aligned}
S_1(a, x_1) &= ax_1, \\
S_2(a, x_1, x_2) &= ax_1x_2 + ax_2, \\
S_3(a, x_1, x_2, x_3) &= ax_1x_2x_3 + 2ax_2x_3 + ax_1x_3 + ax_3.
\end{aligned}$$

The following theorem gives a grammatical expression of $S_n(a, x_1, x_2, \ldots, x_n)$.

Theorem 6.2 *For $n \geq 1$,*

$$S_n(a, x_1, x_2, \ldots, x_n) = D_nD_{n-1} \cdots D_1(a). \tag{6.3}$$

Let us give an example to demonstrate the action of the differential operator D_7 on a partition of $\{1, 2, 3, 4, 5, 6\}$. Recall that

$$G_7 = \{a \to ax_7, x_i \to x_7 \mid 1 \le i \le 6\}.$$

Consider the following partition along with its grammatical labeling:

$$\begin{array}{cccc} \{1,3,6\} & \{2,5\} & \{4\} & \\ x_6 & x_5 & x_4 & a \end{array}.$$

Applying the substitution rule $a \to ax_7$ to the above partition leads to a partition with a consistent grammatical labeling:

$$\begin{array}{ccccc} \{1,3,6\} & \{2,5\} & \{4\} & \{7\} & \\ x_6 & x_5 & x_4 & x_7 & a \end{array}.$$

Similarly, applying the substitution rule $x_5 \to x_7$ to the partition, we get the following partition with a consistent grammatical labeling

$$\begin{array}{cccc} \{1,3,6\} & \{2,5,7\} & \{4\} & \\ x_6 & x_7 & x_4 & a \end{array}.$$

In fact, the above arguments are sufficient to justify the expression (6.3).

It should be noticed that the relation (6.3) cannot be directly used to prove the stability of $S_n(a, x_1, x_2, \ldots, x_n)$, since the operator D_n does not preserve stability in general. Take D_2 as an example. Consider the polynomial $(a + 1)(x_1 + 1)$, which is clearly stable. But

$$D_2((a + 1)(x_1 + 1)) = x_2(ax_1 + 2a + 1)$$

is not stable since it vanishes when $a = i$ and $x_1 = i - 2$. It follows that D_2 is not stability preserving.

Fortunately, we can find a stability preserving operator T_n for the purpose of justifying the stability of $S_n(a, x_1, x_2, \ldots, x_n)$. It is easy to see that $S_n(a, x_1, x_2, \ldots, x_n)$ can be written as $ah(X)$, where $h(X)$ is a multivariate polynomial in $x_1, x_2, \ldots, x_n$ that is independent of the variable a. Let

$$T_n = x_n I + x_n \sum_{i=1}^{n} \frac{\partial}{\partial x_i}. \tag{6.4}$$

According to Lemma 6.1, for each $n \ge 1$, we have

$$T_n(S_n(a, x_1, x_2, \ldots, x_n)) = D_n(S_n(a, x_1, x_2, \ldots, x_n)).$$

It turns out that $S_n(a, x_1, x_2, \ldots, x_n)$ can be obtained by using $T_1, T_2, \ldots, T_n$.

Theorem 6.3 *For $n \geq 1$, we have*

$$S_n(a, x_1, x_2, \ldots, x_n) = T_n T_{n-1} \cdots T_1(a). \tag{6.5}$$

The following theorem establishes the stability of $S_n(a, x_1, x_2, \ldots, x_n)$.

Theorem 6.4 *For $n \geq 1$, the multivariate polynomial $S_n(a, x_1, x_2, \ldots, x_n)$ is stable.*

Proof. It suffices to show that the linear operator T_n preserves stability of multiaffine polynomials. By Theorem 5.1, it is enough to prove that $T_n(F)$ is stable, where

$$F = (a+u) \prod_{i=1}^{n} (x_i + v_i).$$

Let

$$\xi = 1 + \sum_{i=1}^{n-1} \frac{1}{x_i + v_i}.$$

Then

$$\begin{aligned} T_n(F) =& x_n F + x_n \sum_{i=1}^{n-1} \frac{\partial F}{\partial x_i} \\ =& x_n F + x_n F \sum_{i=1}^{n-1} \frac{1}{x_i + v_i} \\ =& x_n \xi F. \end{aligned}$$

To prove that $T_n(F)$ is stable, we assume that $a, u, x_1, x_2, \ldots, x_n$ and $v_1, v_2, \ldots, v_n$ have positive imaginary parts. It remains to show that $T_n(F) \neq 0$.

Under the above assumptions, for $1 \leq i \leq n$, $x_i + v_i$ has a positive imaginary part. It follows that $\frac{1}{x_i+v_i}$ has a negative imaginary part. Furthermore, the imaginary part of ξ is also negative. Thus we have $F \neq 0$ and $\xi \neq 0$. Consequently, $T_n(F) \neq 0$. This completes the proof. ∎

Next we prove the stability of $B_n(X, Y, Z, U, V)$, where $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, $Z = (z_1, z_2, \ldots, z_n)$, $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$. We shall show that D_{2n-1} is stability preserving for $n \geq 1$. It should be noticed that D_{2n} is not always stability preserving for $n \geq 1$. For example, the polynomial $(u_n+1)(v_n+1)$ is clearly stable, but

$$D_{2n}((u_n+1)(v_n+1)) = x_n z_n (u_n(v_n+1) + y_n(u_n+1))$$

is not stable since it vanishes for $y_n = i+2, u_n = i$ and $v_n = i-4$. Nevertheless, when restricted to polynomials $u_n g$, where g is a polynomial in X, Y, Z, U and V that is independent of the variable u_n, there is a stability preserving operator T_n that is equivalent to D_{2n}.

Theorem 6.5 *For $n \geq 1$, the multivariate polynomial $B_n(X,Y,Z,U,V)$ is stable.*

Proof. For $1 \leq k \leq 2n$, let

$$f_k = D_k D_{k-1} \cdots D_1(x_0),$$

which is a polynomial in

$$A_k = \{x_i, y_i, z_i, u_i, v_i \mid 1 \leq i \leq \lfloor (k+1)/2 \rfloor\}.$$

So $f_{2n} = B_n(X,Y,Z,U,V)$. For $1 \leq k \leq 2n$, it can be seen that f_k is multiaffine. We proceed to prove the stability of f_{2n} by induction on n. The stability of z_0 is evident. For $n \geq 1$, assume that f_{2n-2} is stable. Let us consider the actions of D_{2n-1} and D_{2n}.

First, we show that D_{2n-1} preserves stability of multiaffine polynomials. Let

$$A'_k = \{x'_i, y'_i, z'_i, u'_i, v'_i \mid 1 \leq i \leq \lfloor (k+1)/2 \rfloor\}.$$

According to Theorem 5.1, it suffices to show that the polynomial $D_{2n-1}(F)$ is stable, where

$$F = \prod_{i=1}^{n}(x_i + x'_i) \prod_{i=1}^{n}(y_i + y'_i) \prod_{i=1}^{n}(z_i + z'_i) \prod_{i=1}^{n}(u_i + u'_i) \prod_{i=1}^{n}(v_i + v'_i).$$

Let

$$\xi = \sum_{i=1}^{n-1}\left(\frac{1}{x_i + x'_i} + \frac{1}{y_i + y'_i} + \frac{1}{z_i + z'_i} + \frac{1}{u_i + u'_i} + \frac{1}{v_i + v'_i}\right).$$

By Proposition 4.3,

$$\begin{aligned}
D_{2n-1}(F) &= \sum_{i=1}^{n-1} D_{2n-1}(x_i)\frac{\partial F}{\partial x_i} + \sum_{i=1}^{n-1} D_{2n-1}(y_i)\frac{\partial F}{\partial y_i} + \sum_{i=1}^{n-1} D_{2n-1}(z_i)\frac{\partial F}{\partial z_i} \\
&\quad + \sum_{i=1}^{n-1} D_{2n-1}(u_i)\frac{\partial F}{\partial u_i} + \sum_{i=1}^{n-1} D_{2n-1}(v_i)\frac{\partial F}{\partial v_i} \\
&= u_n v_n \sum_{i=1}^{n-1}\left(\frac{F}{x_i + x'_i} + \frac{F}{y_i + y'_i} + \frac{F}{z_i + z'_i} + \frac{F}{u_i + u'_i} + \frac{F}{v_i + v'_i}\right) \\
&= u_n v_n \xi F.
\end{aligned}$$

Assume that all the variables in A_{2n} and A'_{2n} have positive imaginary parts. Then each factor in F is nonzero, and so $F \neq 0$. Similarly, each term in ξ has a negative imaginary part, which implies that $\xi \neq 0$. Hence $D_{2n-1}(F) \neq 0$. This proves that D_{2n-1} is stability preserving. By the induction hypothesis, we deduce that f_{2n-1} is stable.

Next we turn to the operator D_{2n}. Define

$$T_n = x_n z_n I + x_n y_n z_n \sum_{i=1}^{n-1} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} + \frac{\partial}{\partial z_i} + \frac{\partial}{\partial u_i} \right) + x_n y_n z_n \sum_{i=1}^{n} \frac{\partial}{\partial v_i}.$$

Since f_{2n-1} can be written in the form $u_n g$, where g is a polynomial in X, Y, Z, U and V that is independent of u_n, using Lemma 6.1, we find that

$$f_{2n} = D_{2n}(f_{2n-1}) = T_n(f_{2n-1}).$$

To prove that T_n preserves stability of multiaffine polynomials, let

$$F = \prod_{i=1}^{n}(x_i + x_i') \prod_{i=1}^{n}(y_i + y_i') \prod_{i=1}^{n}(z_i + z_i') \prod_{i=1}^{n}(u_i + u_i') \prod_{i=1}^{n}(v_i + v_i').$$

Then

$$\begin{aligned} T_n\left(F\right) &= x_n y_n z_n F \sum_{i=1}^{n-1} \left(\frac{1}{x_i + x_i'} + \frac{1}{y_i + y_i'} + \frac{1}{z_i + z_i'} + \frac{1}{u_i + u_i'} \right) \\ &\quad + x_n y_n z_n F \sum_{i=1}^{n} \frac{1}{v_i + v_i'} + x_n z_n F \\ &= x_n y_n z_n \xi F, \end{aligned}$$

where

$$\xi = \frac{1}{y_n} + \sum_{i=1}^{n-1} \left(\frac{1}{x_i + x_i'} + \frac{1}{y_i + y_i'} + \frac{1}{z_i + z_i'} + \frac{1}{u_i + u_i'} \right) + \sum_{i=1}^{n} \frac{1}{v_i + v_i'}.$$

Assume that all the variables in A_{2n} and A'_{2n} have positive imaginary parts. By Theorem 5.1, it suffices to verify that $T_n(F) \neq 0$. For $1 \leq i \leq n$, since $x_i + x_i', y_i + y_i', z_i + z_i', u_i + u_i'$, and $v_i + v_i'$ all have positive imaginary parts, we see that

$$\frac{1}{x_i + x_i'}, \frac{1}{y_i + y_i'}, \frac{1}{z_i + z_i'}, \frac{1}{u_i + u_i'}, \quad \text{and} \quad \frac{1}{v_i + v_i'}$$

all have negative imaginary parts. Similarly, under the assumption that y_n has a positive imaginary part, it can be seen that $\frac{1}{y_n}$ has a negative imaginary part. Thus we find that $\xi \neq 0$ and $F \neq 0$. Consequently, $T_n\left(F\right) \neq 0$. This leads to the stability of $T_n(F)$. Finally, in light of Theorem 5.1, we conclude that f_{2n} is stable. This completes the proof. ∎

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References

[1] M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, SIAM J. Discrete Math. 23 (2009) 401–406.

[2] J. Borcea, P. Brändén, Applications of stable polynomials to mixed determinants: Johnsons conjectures, unimodality and symmetrized Fischer products, Duke Math. J. 143 (2) (2008) 205–223.

[3] J. Borcea and P. Brändén, Pólya–Schur master theorems for circular domains and their boundaries, Ann. Math. 170 (2009) 465–492.

[4] J. Borcea and P. Brändén, The Lee–Yang and Pólya–Schur programs I: linear operators preserving stability, Invent. Math. 177 (3) (2009) 541–569.

[5] J. Borcea and P. Brändén, The Lee–Yang and Pólya–Schur programs II: theory of stable polynomials and applications, Comm. Pure Appl. Math. 62 (12) (2009) 1595–1631.

[6] P. Brändén and M. Leander, Multivariate P-Eulerian polynomials. arXiv preprint arXiv:1604.04140, 2016.

[7] P. Brändén, M. Leander and M. Visontai, Multivariate Eulerian polynomials and exclusion processes. Combin. Probab. Comput. 25 (2016) 486–499.

[8] F. Brenti, Unimodal, Log-concave and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc. 413 (1989).

[9] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1) (1993) 113–129.

[10] W.Y.C. Chen and A.M. Fu, Context-free grammars for permutations and increasing trees, Adv. in Appl. Math. 82 (2017) 58–82.

[11] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, Sém. Lothar. Combin. 37 (1996) 1–21.

[12] E.S. Egge, Legendre-Stirling permutations, European J. Combin. 31 (7) (2010) 1735–1750.

[13] I. Gessel and R.P. Stanley, Stirling polynomials, J. Combin. Theory Ser. A 24 (1) (1978) 24–33.

[14] R.L. Graham, D.E. Knuth and O. Patashnik, Concrete Mathematics. A Foundation for Computer Science, Addison-Wesley, Reading, MA, 1994.

[15] J. Haglund and M. Visontai, On the monotone column permanent conjecture, Proceedings of FPSAC 2009, Disc. Math. and Theor. Comp. Sci. (2009) 443–454.

[16] J. Haglund and M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, European J. Combin. 33 (4) (2012) 477–487.

[17] L.H. Harper, Stirling behaviour is asymptotically normal, Ann. Math. Statist. 38 (2) (1967) 410–414.

[18] S. Janson, Plane recursive trees, Stirling permutations and an urn model, Proceedings of Fifth Colloquium on Mathematics and Computer Science, Disc. Math. and Theor. Comp. Sci., AI. (2008) 541–547.

[19] S. Janson, M. Kuba and A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, J. Combin. Theory Ser. A 118 (1) (2011) 94–114.

[20] P. Levande, Two new interpretations of the Fishburn numbers and their refined generating functions, arXiv:1006.3013v1, 2010.

[21] L.L. Liu and Y. Wang, A unified approach to polynomial sequences with only real zeros, Adv. in Appl. Math. 38 (4) (2007) 542–560.

[22] G. Pólya and J. Schur, Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen, J. Reine Angew. Math. 144 (1914) 89–113.

[23] J. Riordan, The blossoming of Schröder's fourth problem, Acta Math. 137 (1) (1976) 1–16.

[24] A. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, Surveys in Combinatorics 2005, Cambridge Univ. Press, Cambridge, (2005) 173–226.

[25] M. Visontai and N. Williams, Stable multivariate W-Eulerian polynomials, J. Combin. Theory Ser. A 120 (7) (2013) 1929–1945.

[26] D.G. Wagner, Multivariate stable polynomials: theory and applications, Bull. Amer. Math. Soc. (N.S.) 48 (1) (2011) 53–84.