Semi-invariants of Binary Forms Pertaining to a Unimodality Theorem of Reiner and Stanton

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Dedicated to the Memory of Professor S. S. Chern

Abstract

The strange symmetric difference of the q-binomial coefficients $F_{n,k}(q) = \left[{n+k \atop k}\right] - q^n \left[{n+k-2 \atop k-2}\right]$ as called by Stanley, was introduced by Reiner and Stanton. They proved that $F_{n,k}(q)$ is symmetric and unimodal for $k \geq 2$ and any even nonnegative integer n by using the representation theory for Lie algebras. Inspired by the connection between the Gaussian coefficients, or the q-binomial coefficients, and semi-invariants of binary forms established by Sylvester in his proof of the unimodality of the Gaussian coefficients as conjectured by Cayley, we find an interpretation of the unimodality of $F_{n,k}(q)$ in terms of semi-invariants. In the spirit of the strict unimodality of the Gaussian coefficients due to Pak and Panova, we prove the strict unimodality of $G_{n,k,r}(q) = \left[{n+k \atop k}\right] - q^{nr/2} \left[{n+k-r \atop k-r}\right]$, where $n, r \geq 8$, $k \geq r$ and at least one of n and r is even.

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1 Introduction

The story begins with the Gaussian coefficients called by Rota with no particular reasons (private conversation, see also [9]), or sometimes the Gaussian polynomials, or often under the name of the q-binomial coefficients, as given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)},$$

where $0 \leq k \leq n$. The Gaussian coefficients are polynomials of q and they enjoy the fundamental property that their coefficients are symmetric and unimodal, as conjectured by Cayley [4] in 1856 and confirmed by Sylvester [30] in 1878, who had even believed that settling the conjecture of Cayley was a task that lay outside the human power, see also, Pak and Panova [15,16]. Ever since a great deal of work has been done in this vein, see, for example, [3,6–8,10,13,14,17–21,24,25,28,31,33], to mention only a few. In particular, O'Hara [14] found a combinatorial proof, Zeilberger [33] came up with an identity, known as the KOH theorem, which serves the purpose of justifying the unimodality.

The unimodality is not only associated with the Gaussian coefficients, it can also be said about certain differences of the Gaussian coefficients. Employing the representation theory for Lie algebras, Reiner and Stanton [22] established the unimodality of

$$\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}, \tag{1.1}$$

where n is an odd positive integer and $2k \leq n+1$. They also showed that

$$F_{n,k}(q) = \begin{bmatrix} n+k \\ k \end{bmatrix} - q^n \begin{bmatrix} n+k-2 \\ k-2 \end{bmatrix} \tag{1.2}$$

is symmetric and unimodal when $k \geq 2$ and n is even. Furthermore, Reiner and Stanton conjectured the unimodality of

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} - q^{n-2rk+1+4(r-1)} \begin{bmatrix} n-1+4(r-1) \\ k-2 \end{bmatrix}, \tag{1.3}$$

where n is odd, and r and k are nonnegative integers with $n \geq 2rk - 4r + 3$. This conjecture is still open.

The above symmetric differences are called the strange symmetric differences by S-tanley and Zanello [27]. They extended the above conjecture involving (1.3) to a broader framework, namely, for each $k \geq 5$, the polynomials

$$f_{k,m,b}(q) = \begin{bmatrix} m \\ k \end{bmatrix} - q^{\frac{k(m-b)}{2}+b-2k+2} \begin{bmatrix} b \\ k-2 \end{bmatrix} \tag{1.4}$$

are nonnegative and unimodal subject to certain conditions. They verified their conjecture for $k \leq 5$ by means of the KOH theorem of Zeilberger [33]. Notice that when $b = m-2$, (1.4) reduces to (1.2).

It is worth mentioning that Bergeron [2] investigated the differences of the Gaussian coefficients of the following form

$$\begin{bmatrix} b+c \\ b \end{bmatrix} - \begin{bmatrix} a+d \\ d \end{bmatrix}, \tag{1.5}$$

where a, b, c, d are positive integers, with a being the smallest and $ad = bc$. It was conjectured that such differences are nonnegative. Zanello [32] further conjectured the unimodality of (1.5), and proved its unimodality for $a \leq 3$ and $b, c \geq 4$, resorting to the KOH theorem of Zeilberger [33].

A notable progress on the unimodality of the Gaussian coefficients was achieved by Pak and Panova [15,16]. They showed that the Gaussian coefficients are strictly unimodal except for a few cases. Recall that a polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ with positive coefficients is said to be unimodal if there is an index m such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n. \tag{1.6}$$

We say that $f(x)$ is strictly unimodal if (1.6) is to be replaced by

$$a_0 < a_1 < \cdots < a_m > a_{m+1} > \cdots > a_n. \tag{1.7}$$

The humble goal of this paper is to present an interpretation of the unimodality of $F_{n,k}(q)$ as in (1.2) in terms of semi-invariants of binary forms, following the original aspiration of Sylvester. Once the connection to semi-invariants is at our disposal, we carry on to prove the strict unimodality of

$$G_{n,k,r}(q) = \begin{bmatrix} n+k \\ k \end{bmatrix} - q^{\frac{nr}{2}} \begin{bmatrix} n+k-r \\ k-r \end{bmatrix}, \tag{1.8}$$

when $n, r \geq 8$, $k \geq r$ and at least one of n and r is even.

It is our hope that after a sound rest for more than a century, it might be the time for the once shining invariant theory of binary forms to shed light on the ongoing study of the theory of partitions, presumably under a banner bearing the name of enumerative invariant theory.

2 Semi-invariants

A binary form of degree n, or a binary n-form, is a homogeneous polynomial in x and y,

$$f(x, y) = a_0x^n + \binom{n}{1}a_1x^{n-1}y + \binom{n}{2}a_2x^{n-2}y^2 + \cdots + a_ny^n. \tag{2.1}$$

Roughly speaking, a semi-invariant or a source of a covariant, called by Cayley [5], or a differentiant called by Sylvester [30], of the binary form $f(x, y)$, is a polynomial $I(a_0, a_1, \ldots, a_n)$ in $a_0, a_1, \ldots, a_n$ with rational coefficients that is invariant under the transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{2.2}$$

that is, $x = x' + hy'$ and $y = y'$. The matrix in (2.2) is called a shear matrix. Notice that Hilbert [12] used the Greek letter μ in place of h in the shear matrix in (2.2). Here we prefer the above notation of Sylvester [30] for the reason that the Greek letters λ, μ are now often reserved for integer partitions. Indeed, the transformation (2.2) for binary forms goes back to Lagrange, see [11,23]. More precisely, suppose that the binary form (2.1) becomes

$$f'(x', y') = a'_0x'^n + \binom{n}{1}a'_1x'^{n-1}y' + \binom{n}{2}a'_2x'^{n-2}y'^2 + \cdots + a'_ny'^n \tag{2.3}$$

under the transformation (2.2), where for $0 \leq i \leq n$,

$$a_i' = a_i + \binom{i}{1} a_{i-1} h + \binom{i}{2} a_{i-2} h^2 + \cdots + a_0 h^i. \tag{2.4}$$

Then we say that a polynomial $I(a_0, a_1, \ldots, a_n)$ with rational coefficients is a semi-invariant of the binary form (2.1) if

$$I(a_0, a_1, \ldots, a_n) = I(a_0', a_1', \ldots, a_n') \tag{2.5}$$

for any shear transformation in (2.2).

For example, for the quadratic form

$$f(x, y) = a_0 x^2 + 2a_1 xy + a_2 y^2, \tag{2.6}$$

it can be easily checked that the polynomial

$$I(a_0, a_1, a_2) = a_0 a_2 - a_1^2 \tag{2.7}$$

is a semi-invariant. In fact, it is the discriminant, which is more than a semi-invariant in the sense that it is an invariant with respect to a more general transformation, see, for example, [11, 29]. Below is a semi-invariant but not an invariant:

$$J(a_0, a_1, a_2) = a_0^2 a_2 - a_0 a_1^2. \tag{2.8}$$

As far as this paper is concerned, a semi-invariant can be viewed as nothing but a polynomial $I(a_0, a_1, \ldots, a_n)$ with certain degree conditions that satisfies a partial differential equation. Needless to say, the coefficients $a_0, a_1, \ldots, a_n$ are perceived as variables. First, the degree of a monomial

$$a^\nu = a_0^{\nu_0} a_1^{\nu_1} \cdots a_n^{\nu_n} \tag{2.9}$$

is meant to be the total degree $\nu_0 + \nu_1 + \cdots + \nu_n$. Besides, the weight of the above monomial a^ν is defined by

$$\nu_1 + 2\nu_2 + \cdots + n\nu_n. \tag{2.10}$$

We shall be concerned with semi-invariants that are homogeneous not only in the degree but also in the weight. For example, the semi-invariant $I(a_0, a_1, a_2)$ in (2.7) is homogeneous with degree two and weight two, whereas the semi-invariant $J(a_0, a_1, a_2)$ in (2.8) has degree three and weight two.

We are now led to define $Q_n(k, m)$ as the vector space of polynomials of $a_0, a_1, \ldots, a_n$ over the rational numbers that are homogeneous of degree k and weight m. Taking the dimension of $Q_n(k, m)$ into account, the notion of partitions immediately comes into play. The monomials in $Q_n(k, m)$ are in one-to-one correspondence with partitions of m with k parts. More precisely, a partition of an integer m with k parts, is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 +$

$\cdots + \lambda_k = m$. It should be stressed that zero is allowed to be a part. Recall that a partition of m with k parts such that no parts exceed n can be represented as

$$\lambda = 0^{\nu_0} 1^{\nu_1} \cdots n^{\nu_n}, \tag{2.11}$$

where ν_i signifies the number of occurrences of the part i in λ and $\nu_0 + \nu_1 + \cdots + \nu_n = k$. We say that such a partition is contained in a $k \times n$ rectangle. Clearly, λ is a partition of $\nu_1 + 2\nu_2 + \cdots + n\nu_n$, which turns out to be the weight of the monomial a^ν. This basic fact brings us right to the stage of the Gaussian coefficients which admit the following partition interpretation.

Let $p(k,n,m)$ denote the number of partitions of m contained in a $k \times n$ rectangle, then we have

$$\begin{bmatrix} n+k \\ k \end{bmatrix} = \sum_{m=0}^{nk} p(k,n,m) q^m, \tag{2.12}$$

see [1, 26]. Sylvester found an interpretation of the coefficients $p(k,n,m)$ in terms of semi-invariants, which demystifies the unimodality of the Gaussian coefficients.

The following characterization is crucial for the route from the vector space $Q_n(k,m)$ to semi-invariants, see Cayley [4] or Hilbert [12].

Theorem 2.1. *Let*

$$D = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \cdots + n a_{n-1} \frac{\partial}{\partial a_n}. \tag{2.13}$$

A polynomial $I(a_0, a_1, \ldots, a_n)$ in $Q_n(k,m)$ is a semi-invariant of the binary form (2.1) if and only if it satisfies the differential equation $DI(a_0, a_1, \ldots, a_n) = 0$.

Taking the semi-invariant $J(a_0, a_1, a_2)$ in (2.8) as an example, it can be seen that

$$DJ(a_0, a_1, a_2) = \left(a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} \right) (a_0^2 a_2 - a_0 a_1^2) \tag{2.14}$$

is identically zero.

Let $S_n(k,m)$ denote the set of semi-invariants of degree k and weight m, that is,

$$S_n(k,m) = \{ I \in Q_n(k,m) \mid D(I) = 0 \}, \tag{2.15}$$

where D is the linear operator defined in (2.13). Bear in mind that $S_n(k,m)$ forms a vector space over $\mathbb{Q}$. By the number of semi-invariants of degree k and weight m of a binary n-form, we really mean the dimension of the vector space $S_n(k,m)$. For example, $\dim S_4(4,6) = 2$. Below are the two semi-invariants of degree 4 and weight 6 of a binary 4-form:

$$I_1 = 3a_1^2 a_2^2 - 4a_1^3 a_3 - 2a_0 a_1 a_2 a_3 + 3a_0^2 a_3^2 + 4a_0 a_1^2 a_4 - 4a_0^2 a_2 a_4, \tag{2.16}$$

$$I_2 = a_0 a_2^3 - 2a_0 a_1 a_2 a_3 + a_0^2 a_3^2 + a_0 a_1^2 a_4 - a_0^2 a_2 a_4. \tag{2.17}$$

We now come to the remarkable discovery of Sylvester [30].

Theorem 2.2. *For $n, k \geq 0$ and $0 \leq m \leq nk/2$, the number of semi-invariants of a binary n-form of degree k and weight m equals $p(k,n,m) - p(k,n,m-1)$, that is,*

$$\delta(k,n,m) = p(k,n,m) - p(k,n,m-1), \tag{2.18}$$

with the convention that $p(k,n,-1) = 0$.

For example, for the quadratic form

$$f(x,y) = a_0x^2 + 2a_1xy + a_2y^2, \tag{2.19}$$

it is easily checked that there is exactly one semi-invariant of degree 3 and weight 2, that is,

$$I(a_0,a_1,a_2) = a_0^2a_2 - a_0a_1^2. \tag{2.20}$$

On the other hand, we have $p(3,2,2) = 2$ and $p(3,2,1) = 1$, which is consistent with the above theorem.

Note that the condition $0 \leq m \leq nk/2$ is obviously needed in the above theorem because the coefficients $p(k,n,m)$ are symmetric, namely, for $0 \leq m \leq nk/2$,

$$p(k,n,m) = p(k,n,nk-m). \tag{2.21}$$

Observe that this symmetry can also be understood from the perspective of semi-invariants. Assume that $I(a_0,a_1,\ldots,a_n)$ is a semi-invariant of degree k and weight m. It is not hard to see that $I(a_n,a_{n-1},\ldots,a_0)$ is a semi-invariant of degree k and weight $nk-m$, which leads to the symmetry (2.21). For example, there are two semi-invariants I_1 and I_2 of degree 4 and weight 6 of a binary 4-form as given in (2.16) and (2.17). Substituting a_i with a_{4-i} for $0 \leq i \leq 4$, we get the following two semi-invariants of a binary 4-form of degree 4 and weight 10,

$$\begin{aligned} J_1 &= 3a_2^2a_3^2 - 4a_1a_3^3 - 2a_1a_2a_3a_4 + 3a_1^2a_4^2 + 4a_0a_3^2a_4 - 4a_0a_2a_4^2, \\ J_2 &= a_2^3a_4 - 2a_1a_2a_3a_4 + a_1^2a_4^2 + a_0a_3^2a_4 - a_0a_2a_4^2. \end{aligned} \tag{2.22}$$

Up to now, we are sufficiently equipped to move on to the next section to explore the symmetric differences of the Gaussian coefficients by means of semi-invariants.

3 Symmetric Differences of the Gaussian Coefficients

The first objective of this section is to give a semi-invariant interpretation of the unimodality theorem of Reiner and Stanton on $F_{n,k}(q)$ as given in (1.2). Then we proceed to prove the strict unimodality of $G_{n,k,r}(q)$ as given in (1.8).

Lemma 3.1. *If n is even, then there is exactly one semi-invariant of a binary n-form of degree 2 and weight n.*

Proof. It is not hard to see that for $m \leq n$,

$$p(2, n, m) = \left\lfloor \frac{m+2}{2} \right\rfloor, \tag{3.1}$$

which can be found in Stanley and Zanello [27, 32]. Since n is even, by Theorem 2.2 we obtain that the number of semi-invariants of degree 2 and weight n equals one, as claimed. ∎

For example, the only semi-invariant of a 4-from having degree 2 and weight 4 is

$$J = 3a_2^2 - 4a_1a_3 + a_0a_4. \tag{3.2}$$

The argument in the proof of Lemma 3.1 indicates that when n is odd, there are no semi-invariants of a binary n-form of degree 2 and weight n. The following relation is essentially a consequence of the fact that the set of semi-invariants forms a ring. To be precise, if I and J are two semi-invariants of a binary n-form, then so are $I + J$ and IJ. In fact, the ring property of semi-invariants is an easy corollary of the operator characterization as stated in Theorem 2.1.

Theorem 3.2. *If $k \geq 2$, $m \geq n$ and n is even, then the number of semi-invariants of a binary n-form of degree k and weight m is at least the number of semi-invariants of degree $k-2$ and weight $m-n$, that is,*

$$\delta(k, n, m) \geq \delta(k-2, n, m-n). \tag{3.3}$$

Proof. Assume that there are t linearly independent semi-invariants of degree $k-2$ and weight $m-n$, say, $I_1, I_2, \ldots, I_t$, where the variables $a_0, a_1, \ldots, a_n$ are suppressed. We wish to show that there are at least t semi-invariants of degree k and weight m. By the above Lemma, we may assume that J is a semi-invariant of an n-form of degree 2 and weight n. Thanks to the ring structure of semi-invariants, we see that $JI_1, JI_2, \ldots, JI_t$ are semi-invariants of degree k and weight m. To complete the proof, one only needs to realize that JI_1, JI_2, $\ldots$, JI_t are linearly independent, which is by any means a plain fact. ∎

Next we demonstrate that Reiner and Stanton's unimodality theorem for $F_{n,k}(q)$ is immediate from Theorem 3.2. Recall that

$$F_{n,k}(q) = \begin{bmatrix} n+k \\ k \end{bmatrix} - q^n \begin{bmatrix} n+k-2 \\ k-2 \end{bmatrix}, \tag{3.4}$$

where $k \geq 2$ and n is even.

While keeping the symmetry of $F_{n,k}(q)$ in mind, to confirm the unimodality, let

$$F_{n,k}(q) = \sum_{m=0}^{nk} f_m q^m, \tag{3.5}$$

so that

$$f_m = p(k, n, m) - p(k-2, n, m-n), \tag{3.6}$$

with the convention that $p(k-2,n,j)=0$ if j is negative. For $0 \le m \le nk/2$, in light of Theorem 2.2, the inequality (3.3) yields

$$p(k,n,m)-p(k,n,m-1) \ge p(k-2,n,m-n)-p(k-2,n,m-n-1), \quad (3.7)$$

which can be recast as

$$p(k,n,m)-p(k-2,n,m-n) \ge p(k,n,m-1)-p(k-2,n,m-n-1). \quad (3.8)$$

But this is exactly $f_m \ge f_{m-1}$.

Notice that $F_{n,k}(q)$ is not always strictly unimodal. For example, for $F_{5,9}(q)$ and $F_{14,5}(q)$, the maximal coefficient occurs more than twice in the middle. The following conjecture is supported by numerical evidence.

Conjecture 3.3. *For $n \ge 8$ and $k \ge 15$, $F_{n,k}(q)$ is strictly unimodal.*

One might expect to push forward along this direction to tackle the strict unimodality of $F_{n,k}(q)$. But it is not clear as to whether or how this can be pursued. Nevertheless, we are given a chance to establish the strict unimodality of $G_{n,k,r}(q)$, with the two terms at the beginning and at the end being excluded, to be precise. To this end, we will rely on two special semi-invariants, as assured by the following lemma.

Lemma 3.4. *If $n, r \ge 8$ and at least one of n and r is even, then there are at least two linearly independent semi-invariants of a binary n-form of degree r and weight $nr/2$, that is,*

$$\delta(r,n,nr/2) \ge 2. \quad (3.9)$$

In a more general setting, Pak and Panova [15, 16] obtained the strict unimodality as stated below in the language of semi-invariants.

Theorem 3.5. *For all $n, k \ge 8$ and $2 \le m \le nk/2$, we have*

$$\delta(k,n,m) \ge 1. \quad (3.10)$$

It should be pointed out that the following proof of Lemma 3.4 is reminiscent of the reduction argument of Pak and Panova.

Proof of Lemma 3.4. For $8 \le n, r < 16$, it is easily verified (3.9) holds. We now turn to the case that $8 \le n < 16$ and $r \ge 16$. Write $r = 8s + t$, where $s \ge 1$ and $8 \le t < 16$. As is known for the first case, we have $\delta(8,n,4n) \ge 2$. This means that there exists a semi-invariant I, with the variables being suppressed, of a binary n-form of degree 8 and weight $4n$. Again, the evidence of the first case implies $\delta(t,n,nt/2) \ge 2$. This enables us to find two linearly independent semi-invariants J_1, J_2 of a binary n-form of degree t and weight $nt/2$. Observe that $I^s J_1, I^s J_2$ are linearly independent semi-invariants of a binary n-form of degree r and weight $nr/2$, from which we find that $\delta(r,n,nr/2) \ge 2$ for $8 \le n < 16$ and $r \ge 16$. Together with the first case $8 \le n, r < 16$, we see that $\delta(r,n,nr/2) \ge 2$ for $8 \le n < 16$ and $r \ge 8$.

Due to the symmetry of the Gaussian coefficients, we have

$$p(r,n,nr/2) = p(n,r,nr/2). \tag{3.11}$$

Thus $\delta(r,n,nr/2) \geq 2$ also holds for $n \geq 8$ and $8 \leq r < 16$.

So we are left with only the case $n \geq 8$ and $r \geq 16$. For $n \geq 8$, we write $r = 8s + t$, where $s \geq 1$ and $8 \leq t < 16$. Mimicking the above reasoning for $8 \leq n < 16$ and $r \geq 16$, we may reduce this case ($n \geq 8$ and $r \geq 16$) back to the case $8 \leq n < 16$ and $r \geq 16$. In summary, we conclude that $\delta(r,n,nr/2) \geq 2$ holds for all $n, r \geq 8$. ∎

Theorem 3.6. *If $n, r \geq 8$, $k \geq r$, $m \geq nr/2$, and at least one of n and r is even, then the number of semi-invariants of a binary n-form of degree k and weight m is greater than the number of semi-invariants of degree $k - r$ and weight $m - nr/2$, that is,*

$$\delta(k,n,m) > \delta(k-r,n,m-nr/2). \tag{3.12}$$

Let us see how the above theorem gives rise to the strict unimodality of $G_{n,k,r}(q)$.

Theorem 3.7. *For $n, r \geq 8$, $k \geq r$ and at least one of n and r is even,*

$$G_{n,k,r}(q) = \begin{bmatrix} n+k \\ k \end{bmatrix} - q^{\frac{nr}{2}} \begin{bmatrix} n+k-r \\ k-r \end{bmatrix} \tag{3.13}$$

is symmetric and strictly unimodal, except for the two terms at both ends.

Proof. The symmetry of $G_{n,k,r}(q)$ can be easily verified. Let

$$G_{n,k,r}(q) = \sum_{m=0}^{nk} g_m q^m, \tag{3.14}$$

and so

$$g_m = p(k,n,m) - p(k-r,n,m-nr/2), \tag{3.15}$$

where we assume that $p(k-r,n,j) = 0$ whenever j is negative. Notice that $g_0 = g_1 = 1$ and $g_{nk-1} = g_{nk} = 1$. For $2 \leq m < nr/2$, we have $g_m = p(k,n,m)$. By the strict unimodality obtained by Pak and Panova [15, 16] as stated in Theorem 3.5, we find that

$$g_m - g_{m-1} = \delta(k,n,m) \geq 1. \tag{3.16}$$

For $nr/2 \leq m \leq nk/2$, in virtue of Theorem 2.2, the inequality (3.12) takes the form

$$p(k,n,m) - p(k,n,m-1) > p(k-r,n,m-nr/2) - p(k-r,n,m-nr/2-1), \tag{3.17}$$

which can be reformulated as $g_m > g_{m-1}$. Thus we have shown that $G_{n,k,r}(q)$ is symmetric and strictly unidomal. ∎

For example, we have

$$G_{8,14,10}(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7$$

$$+\cdots+8310q^{53}+8408q^{54}+8450q^{55}+8479q^{56}$$
$$+8450q^{57}+8408q^{58}+8310q^{59}+\cdots+11q^{106}$$
$$+7q^{107}+5q^{108}+3q^{109}+2q^{110}+q^{111}+q^{112}.$$

To present the proof Theorem 3.6, it is necessary to define the leading term of a polynomial $f(a_0,a_1,\ldots,a_n)$. First, we write a monomial in the form $a^\nu = a_0^{\nu_0}a_1^{\nu_1}\cdots a_n^{\nu_n}$. Then we order the monomials of degree k and weight m according to the anti-lexicographic order of their exponents. Clearly, this order extends to the set of all monomials of $a_0,a_1,\ldots,a_n$. For example, for $k=4$, $n=4$ and $m=6$, we have

$$a_1^2a_2^2 > a_0a_2^3 > a_1^3a_3 > a_0a_1a_2a_3 > a_0^2a_3^2 > a_0a_1^2a_4 > a_0^2a_2a_4.$$

The leading term of a semi-invariant $I(a_0,a_1,\ldots,a_n)$, denoted by $\alpha(I(a_0,a_1,\ldots,a_n))$ or $\alpha(I)$ for short, is defined to be the largest monomial with a nonzero coefficient with respect to the above order. For example, for the semi-invariants I_1 and I_2 in (2.16) and (2.17), we have

$$\alpha(I_1)=a_1^2a_2^2, \quad \text{and} \quad \alpha(I_2)=a_0a_2^3.$$

Since the set of semi-invariants of a binary n-form of degree k and weight m forms a vector space, by a triangulation process or the Gauss elimination we may always find a set of semi-invariants whose leading terms are strictly decreasing with respect to the anti-lexicographic order.

We are now in a position to complete the proof of Theorem 3.6.

Proof of Theorem 3.6. Let $t=\delta(k-r,n,m-nr/2)$. If $t=0$, then by the strict unimodality of the Gaussian coefficients established by Pak and Panova [15, 16] as stated in Theorem 3.5, we have $\delta(k,n,m)\geq 1$. If $t>0$, we assume that $I_1,I_2,\ldots,I_t$, with the variables $a_0,a_1,\ldots,a_n$ being suppressed, are t linearly independent semi-invariants of degree $k-r$ and weight $m-nr/2$. By the triangulation process, we may further assume that the leading terms of $I_1,I_2,\ldots,I_t$ are strictly decreasing with respect to the anti-lexicographic order, that is,

$$\alpha(I_1)>\alpha(I_2)>\cdots>\alpha(I_t). \tag{3.18}$$

Next we attempt to construct $t+1$ linearly independent semi-invariants of degree k and weight m. By Lemma 3.4, there exist two linearly independent semi-invariants J_1 and J_2 of a binary n-form of degree r and weight $nr/2$. Without loss of generality, let us assume that

$$\alpha(J_1)>\alpha(J_2). \tag{3.19}$$

We claim that $J_1I_1, J_1I_2,\ldots,J_1I_t, J_2I_t$ are the desired linearly independent semi-invariants of degree k and weight m. The degree and weight conditions are easily satisfied. It remains to verify that $J_1I_1, J_1I_2,\ldots,J_1I_t, J_2I_t$ are linearly independent. To this end, it suffices to show that the leading terms of $J_1I_1, J_1I_2,\ldots,J_1I_t, J_2I_t$ are distinct. For two semi-invariants K_1 and K_2, it is evident that

$$\alpha(K_1K_2)=\alpha(K_1)\alpha(K_2). \tag{3.20}$$

Under the assumption (3.18), we see that

$$\alpha(J_1I_1) > \alpha(J_1I_2) > \cdots > \alpha(J_1I_t). \tag{3.21}$$

Invoking the assumption (3.19), we get

$$\alpha(J_1I_t) > \alpha(J_2I_t). \tag{3.22}$$

Combining (3.21) and (3.22), we reach the conclusion that $J_1I_1, J_1I_2, \ldots, J_1I_t, J_2I_t$ have distinct leading terms, and hence they must be linearly independent. This completes the proof. ∎

With the same construction as above, the following lemma provides a more general construction of semi-invariants of higher degrees and higher weights from those of lower degrees and lower weights. The proof is omitted.

Lemma 3.8. *Let $k_1, k_2, n \geq 0$, $0 \leq m_1 \leq nk_1/2$, $0 \leq m_2 \leq nk_2/2$. For a binary n-form, assume that there are t_1 linearly independent semi-invariants of degree k_1 and weight m_1, and t_2 linearly independent semi-invariants of degree k_2 and weight m_2, where $t_1, t_2 \geq 1$. Then there exist $t_1 + t_2 - 1$ linearly independent semi-invariants of degree $k_1 + k_2$ and weight $m_1 + m_2$.*

We conclude with a remark of Professor S. S. Chern. He once commented that Gian-Carlo Rota was fond of invariant theory, which he considered as a subject of old mathematics, in his own words, but not in a negative tone, to avoid any misunderstanding. In the same context, Professor Chern also mentioned that some papers of Gauss remain to be explored, but perhaps not easy to comprehend. As time passes, it may be witnessed that Rota had his reasons. At least it is our belief that this is something that should not completely slip our minds, or "there is something to it", as Rota would have had put it.

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