Discrete Math.
345(1) (2022) 112661

A Context-free Grammar for the e-Positivity of the Trivariate Second-order Eulerian Polynomials

William Y.C. Chen[1] and Amy M. Fu[2]

[1]Center for Applied Mathematics
Tianjin University
Tianjin 300072, P. R. China

[2]School of Mathematics
Shanghai University of Finance and Economics
Shanghai 200433, P.R. China

emails: chenyc@tju.edu.cn, fu.mei@mail.shufe.edu.cn

Abstract. Ma-Ma-Yeh made a beautiful observation that a transformation of the grammar of Dumont instantly leads to the γ-positivity of the Eulerian polynomials. We notice that the transformed grammar bears a striking resemblance to the grammar for 0-1-2 increasing trees also due to Dumont. The appearance of the factor of two fits perfectly in a grammatical labeling of 0-1-2 increasing plane trees. Furthermore, the grammatical calculus is instrumental to the computation of the generating functions. This approach can be adapted to study the e-positivity of the trivariate second-order Eulerian polynomials first introduced by Dumont in the contexts of ternary trees and Stirling permutations, and independently defined by Janson, in connection with the joint distribution of the numbers of ascents, descents and plateaux over Stirling permutations.

Keywords: Context-free grammars, increasing plane trees, Stirling permutations, Second-order Eulerian polynomials, γ-positivity, e-positivity

AMS Classification: 05A15, 05A19

1 Introduction

The objective of this paper is to present a context-free grammar to derive the e-positivity of the trivariate second-order Eulerian polynomials $C_n(x,y,z)$ defined on Stirling permutations, first introduced by Dumont [9] in terms of ternary trees and Stirling permutations, and rediscovered by Janson [18].

This work was inspired by a beautiful observation of Ma-Ma-Yeh [22] that a transformation of a context-free grammar found by Dumont [10] instantly leads to the γ-positivity of the Eulerian polynomials. We find that the transformed grammar not only implies the γ-positivity, it also provides a combinatorial interpretation of the γ-coefficients in terms of increasing plane trees.

For $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$ and let S_n denote the set of permutations of $[n]$. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, we assume that a zero is patched at the beginning and at the end, that is, $\sigma_0 = \sigma_{n+1} = 0$. An index $1 \leq i \leq n$ is said to be a descent (ascent) of a permutation $\sigma \in S_n$ if $\sigma_i > \sigma_{i+1}$ ($\sigma_{i-1} < \sigma_i$). The number of permutations of $[n]$ with k descents ($1 \leq k \leq n$) is often denoted by $A(n,k)$, or sometimes by $\left\langle {n \atop k} \right\rangle$. The Eulerian polynomials $A_n(x)$ are defined by $A_0(x) = 1$ and for $n \geq 1$,

$$A_n(x) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} = \sum_{k=1}^{n} A(n,k)x^k, \tag{1.1}$$

where $\mathrm{des}(\sigma)$ denotes the number of descents of a permutation σ. A bivariate version of the Eulerian polynomials is given by

$$A_n(x,y) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} = \sum_{k=1}^{n} A(n,k)x^k y^{n+1-k}, \tag{1.2}$$

where $n \geq 1$ and $\mathrm{asc}(\sigma)$ stands for the number of ascents of σ. Bear in mind that for any permutation $\sigma \in S_n$, we have

$$\mathrm{des}(\sigma) + \mathrm{asc}(\sigma) = n + 1. \tag{1.3}$$

The first few values of $A_n(x,y)$ are given below,

$$\begin{aligned}
A_1(x,y) &= xy, \\
A_2(x,y) &= xy^2 + x^2y, \\
A_3(x,y) &= xy^3 + 4x^2y^2 + x^3y, \\
A_4(x,y) &= xy^4 + 11x^2y^3 + 11x^3y^2 + x^4y, \\
A_5(x,y) &= xy^5 + 26x^2y^4 + 66x^3y^3 + 26x^4y^2 + x^5y,
\end{aligned}$$

$$A_6(x,y) \quad = \quad xy^6+57x^2y^5+302x^3y^4+302x^4y^3+57x^5y^2+x^6y.$$

A celebrated theorem of Foata and Schützenberger [12] states that for $n \geq 1$, the Eulerian polynomial $A_n(x)$ can be expanded uniquely in the following form

$$A_n(x) = \sum_{k=1}^{[(n+1)/2]} \gamma_{n,k} x^k (1+x)^{n-2k+1} \tag{1.4}$$

with nonnegative coefficients $\gamma_{n,k}$. The above expression (1.4) is called the γ-expansion of $A_n(x)$, which can be restated as

$$A_n(x,y) = \sum_{k=1}^{[(n+1)/2]} \gamma_{n,k} \, (xy)^k (x+y)^{n-2k+1}. \tag{1.5}$$

The coefficients $\gamma_{n,k}$ are called the γ-coefficients of the Eulerian polynomials. Remarkably, Foata and Schützenberger discovered a combinatorial interpretation of the coefficients $\gamma_{n,k}$, that is, for $n \geq 1$ and $1 \leq k \leq [(n+1)/2]$, $\gamma_{n,k}$ equals the number of permutations of $[n]$ with k descents, but no double descents. Here a double descent of a permutation $\sigma \in S_n$ is defined to be an index $1 \leq i \leq n-1$ such that $\sigma_i > \sigma_{i+1} > \sigma_{i+2}$.

The nonnegativity of the coefficients $\gamma_{n,k}$ has been referred to as the γ-positivity. This property of the Eulerian polynomials and other polynomials along with the q-analogues has been extensively studied ever since, see, for example, [1, 3, 8, 12, 13, 14, 17, 20, 21, 22, 23, 24, 26, 27, 28, 29, 32, 33].

A context-free grammar is a set of substitution rules on a set of variables X. A variable can be substituted with a polynomial (or a Laurent polynomials) in X. Our starting point is the grammar of Dumont for the Eulerian polynomials, namely,

$$G = \{x \to xy, \quad y \to xy\},$$

which can be expressed as a differential operator

$$\Delta = xy\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{1.6}$$

Ma-Ma-Yeh [22] realized that by a change of variables

$$u = xy, \quad v = x+y,$$

the above grammar is transformed into a new grammar

$$H = \{u \to uv, \quad v \to 2u\}, \tag{1.7}$$

ensuring the γ-positivity of the Eulerian polynomials.

Without the setting of a grammar, the above argument can be recast in terms of the differential operator Δ in (1.6). Clearly, we have $\Delta(x+y) = 2xy$ and $\Delta(xy) = xy(x+y)$. Since Δ is a derivative, we see that for $n \geq 1$, $A_n(x,y) = \Delta^{n-1}(xy)$ is a polynomial in $x+y$ and xy with nonnegative coefficients.

It turns out that the grammar H plays an essential role in the combinatorial understanding of the γ-coefficients of the Eulerian polynomials. First, we notice that the grammar H bears a striking resemblance to the following grammar for 0-1-2 increasing trees and the André polynomials, namely,

$$G = \{x \to xy, \quad y \to x\}.$$

Recall that for $n \geq 1$, a 0-1-2 increasing tree on $[n]$ is a rooted increasing tree on $[n]$ for which every vertex has at most two children. For $n \geq 1$, the André polynomials are defined by

$$E_n(x,y) = \sum_T x^{l(T)} y^{u(T)},$$

where the sum ranges over 0-1-2 increasing trees T on $[n]$, $l(T)$ denotes the number of leaves of T and $u(T)$ denotes the number of vertices of T having an only child.

Examining the factor of two in the grammar H, we are guided precisely to the structure of 0-1-2 increasing plane trees. This formulation is in agreement with the known interpretation in reference to binary increasing trees on $[n]$ with exactly k leaves and no vertices with left children only. It is also in agreement with the formula of Han-Ma [17] in terms of 0-1-2 increasing trees on $[n]$ with k leaves. Nevertheless, it seems to be convenient to work with 0-1-2 increasing plane trees in order to describe the labeling consistent with the grammar H.

The grammatical approach associated with a grammatical labeling of 0-1-2 increasing plane trees offers a test ground for the main result of this paper, which is concerned with the trivariate second-order Eulerian polynomials on Stirling permutations, introduced by Gessel and Stanley [15], see also Elizalde [11]. For $n \geq 1$, let $[n]_2$ denote the multiset $\{1,1,2,2,\ldots,n,n\}$. A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ of $[n]_2$ is said to be a Stirling permutation if for any $1 \leq j \leq n$ the elements between the two j's in σ, if any, are greater than j. For $n \geq 1$, the set of Stirling permutations of $[n]_2$ is denoted by Q_n. A descent and an ascent of $\sigma \in Q_n$ can be defined analogously to the case of an ordinary permutation. For a Stirling permutation σ, we adopt the convention that σ is patched with a zero both at the beginning and at the end, that is, $\sigma_0 = \sigma_{2n+1} = 0$. The number of Stirling permutations of $[n]_2$ with k descents is called the second-order Stirling number, denoted by $C(n,k)$, or $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$.

Bóna [2] introduced the notion of a plateau of $\sigma \in Q_n$, which is defined to be a pair of two adjacent elements (σ_i, σ_{i+1}) such that $\sigma_i = \sigma_{i+1}$. More precisely, for $\sigma \in Q_n$, the number plateaux, denoted $\mathrm{plat}(\sigma)$, is defined to be the number of indices $1 \le i \le 2n$ such that $\sigma_i = \sigma_{i+1}$. Bóna showed that for $n \ge 1$, the statistics $\mathrm{asc}(\sigma)$, $\mathrm{des}(\sigma)$ and $\mathrm{plat}(\sigma)$ have the same distribution over Q_n. Janson [18] constructed an urn model to prove the symmetry of the joint distribution of the three statistics.

It should be noted that a plateau of a Stirling permutation was defined earlier by Dumont [9] in the name of a repetition. For $n \ge 1$, Dumont defined the polynomials $C_n(x,y,z)$ as

$$C_n(x,y,z) = \sum_{\sigma \in Q_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} z^{\mathrm{plat}(\sigma)}.$$

Note that for $n \ge 1$ and any $\sigma \in Q_n$,

$$\mathrm{des}(\sigma) + \mathrm{asc}(\sigma) + \mathrm{plat}(\sigma) = 2n + 1. \tag{1.8}$$

Dumont [9] obtained the following recurrence relation. For $n \ge 1$,

$$C_{n+1}(x,y,z) = xyz \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) C_n(x,y,z) \tag{1.9}$$

with $C_1(x,y,z) = xyz$. A refinement of the above recurrence relation (1.9) was established by Haglund-Visontai [16].

The differential operator in the recurrence relation (1.9) can be prescribed as a grammar

$$G = \{x \to xyz, \quad y \to xyz, \quad z \to xyz\}. \tag{1.10}$$

Indeed, the grammatical labeling as described in the arXiv version of [7] or in [22] is essentially the same argument as that given in [16], which in turn is in the same vein as the recursive construction of Janson [18]. So the above grammar G in (1.10) should be attributed to Dumont [9].

The symmetry of $C_n(x,y,z)$ suggests that we may consider the expansion into the elementary symmetric functions, as denoted by

$$u = x + y + z, \quad v = xy + xz + yz, \quad w = xyz.$$

Thanks to the idea of Ma-Ma-Yeh [22], we come to the following grammar

$$H = \{u \to 3w, \quad v \to 2uw, \quad w \to vw\}. \tag{1.11}$$

Utilizing this grammar, we realize that the argument for the γ-expansion of the Eulerian polynomials can be carried over to the expansion of $C_n(x,y,z)$ into the elementary symmetric functions. To be more specific, we prove that for $n \geq 1$, $C_n(x,y,z)$ is a polynomial in u,v,w whose coefficients can be interpreted in terms of 0-1-2-3 increasing plane trees.

Very recently, Ma-Ma-Yeh-Yeh [25] have extended the grammatical approach to the e-positivity of the symmetric polynomials in $k+1$ variables defined on k-Stirling permutations. For $k \geq 1$, they defined the polynomials $C_n(x_1,x_2,\ldots,x_{k+1})$ by means of the grammar $G = \{x_i \to x_1x_2\cdots x_{k+1} \,|\, i = 1,2,\ldots,k+1\}$ and provided a grammatical labeling for k-Stirling permutations, which makes the symmetry property transparent. It turns out that the polynomials $C_n(x_1,x_2,\ldots,x_{k+1})$ are associated with the generating function of the joint distribution of the numbers of ascents, j-plateaux and descents of k-Stirling permutations of $\{1^k,2^k,\ldots,n^k\}$, introduced by Janson-Kuba-Panholzer [19], where i^k represents k occurrences of i. The symmetry of $C_n(x_1,x_2,\ldots,x_{k+1})$ was discovered in [19]. The coefficients in the e-expansion of $C_n(x_1,x_2,\ldots,x_{k+1})$ can be interpreted in terms of increasing plane trees for which no vertex has more than $k+1$ children, see [25].

The background on the use of context-free grammars for combinatorial enumeration including the notion of a grammatical labeling can be found in [5, 6]. In the next section, we shall give a glimpse of how to compute a generating function based on a context-free grammar. In a certain sense, this approach can be thought of as a formal calculus in the spirit of the symbolic method, while we may enjoy the advantage that there is no fear of the lack of rigor.

2 A grammatical calculus for $A_n(x,y)$

A context-free grammar can also be understood as a formal differential operator. For the purpose of combinatorial enumeration, the variables are attached to combinatorial structures, whereas the rules reflect the recursive construction of combinatorial objects. Computationally speaking, a grammar is a derivative which is often informative for deriving the generating functions.

Let us take the Eulerian polynomials $A_n(x,y)$ to demonstrate the efficiency of the grammatical calculus. Dumont [10] discovered the following grammar for $A_n(x,y)$:

$$G = \{x \to xy, \quad y \to xy.\} \tag{2.1}$$

Let D denote the formal derivative with respect to the above grammar G. Dumont showed that $A_n(x,y)$ can be generated by the grammar G, that is, for $n \geq 1$,

$$A_n(x,y) = D^n(x). \tag{2.2}$$

Chen and Fu [6] introduced the notion of a grammatical labeling in the sense that the grammar G preserves information of significance along with the generation of permutations in S_{n+1} from permutations in S_n.

If we express the formal derivative in terms of a differential operator, the above relation (2.2) can be written as $A_1(x,y) = xy$ and for $n \geq 1$,

$$A_{n+1}(x,y) = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)A_n(x,y).$$

The above recurrence relation also appeared in Haglund-Visontai [16]. It is apparent that for $n \geq 1$,

$$A_n(x) = A_n(x,y)|_{y=1}. \tag{2.3}$$

Let us proceed to present a derivation of the well-known generating function of $A_n(x)$:

$$\sum_{n\geq 0} A_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{(1-x)t}}. \tag{2.4}$$

As noted by Carlitz and Scoville [4], it is not so easy to recover the above generating function from the recurrence relation for the Eulerian numbers. However, by employing the grammatical calculus one can perform this task with ease. In fact, we find it more convenient to deal with the following generating function of the bivariate version $A_n(x,y)$.

Theorem 2.1. *Set $A_0(x,y) = y$. Then we have*

$$\sum_{n=0}^{\infty} A_n(x,y)\frac{t^n}{n!} = \frac{y-x}{1-xy^{-1}e^{(y-x)t}}. \tag{2.5}$$

It is evident that setting $y = 1$ in (2.5) yields (2.4). To present a grammatical proof of (2.5), recall that for a Laurent polynomial f in x and y, the generating function of f with respect to the grammar G is defined by

$$\mathrm{Gen}(f,t) = \sum_{n\geq 0} D^n(f)\frac{t^n}{n!}. \tag{2.6}$$

Assume that g is also a Laurent polynomial in x and y. The first and foremost property of D is that it is a derivative, that is,

$$D(fg) = D(f)g + fD(g), \tag{2.7}$$

and hence it obeys the Leibniz rule

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^k(f) D^{n-k}(g), \tag{2.8}$$

for any $n \geq 0$. This implies the multiplicative property

$$\mathrm{Gen}(fg,t) = \mathrm{Gen}(f,t)\,\mathrm{Gen}(g,t). \tag{2.9}$$

Proof of Theorem 2.1 by Using the Grammar of Dumont. Under the assumption $A_0(x,y) = y$, we have $A_n(x,y) = D^n(y)$. So our goal is to compute the generating function $\mathrm{Gen}(y,t)$.

For the formal derivative D with respect to the grammar G in (2.1), we have

$$D(y^{-1}) = -y^{-2}D(y) = -xy^{-1} \tag{2.10}$$

and

$$D(xy^{-1}) = xy^{-1}(y-x). \tag{2.11}$$

As noted in [6], since $x-y$ is a constant with respect to D, we deduce that for $n \geq 0$,

$$D^n(xy^{-1}) = xy^{-1}(y-x)^n. \tag{2.12}$$

In light of the property (2.9) and the fact $D(c) = 0$ when c is a constant, it suffices to consider $\mathrm{Gen}(y^{-1},t)$, since

$$\mathrm{Gen}(y,t) = \frac{1}{\mathrm{Gen}(y^{-1},t)}. \tag{2.13}$$

Using (2.10), we obtain that

$$\mathrm{Gen}(y^{-1},t) = \sum_{n\geq 0} D^n(y^{-1})\frac{t^n}{n!} = y^{-1} - \sum_{n\geq 1} D^{n-1}(xy^{-1})\frac{t^n}{n!}. \tag{2.14}$$

Invoking (2.12), we deduce that

$$\begin{aligned}
\mathrm{Gen}(y^{-1},t) &= y^{-1} - \sum_{n\geq 1} xy^{-1}(y-x)^{n-1}\frac{t^n}{n!} \\
&= y^{-1} - \frac{xy^{-1}}{y-x}\left(e^{(y-x)t} - 1\right) \\
&= \frac{1 - xy^{-1}e^{(y-x)t}}{y-x},
\end{aligned}$$

which completes the proof by utilizing (2.13). ∎

A variation of the generating function of $A_n(x,y)$ was considered by Carlitz-Scoville [4], which equals

$$F(t)=\frac{e^{xt}-e^{yt}}{xe^{yt}-ye^{xt}}. \tag{2.15}$$

Recall that $A_0(x,y)$ is defined to be y and $A_n(x,y)$ is symmetric in x and y for $n\geq 1$. We are led to consider the generating function of $A_n(x,y)$ for $n\geq 1$. Using the above generating function, we see that

$$\mathrm{Gen}(y,t)-y=xy\,\frac{e^{xt}-e^{yt}}{xe^{yt}-ye^{xt}}. \tag{2.16}$$

The reason for the appearance of the factor xy in the above expression becomes evident once we take a close look at the definition of $F(t)$ given by Carlitz-Scoville [4].

3 The γ-positivity of $A_n(x,y)$

The γ-coefficients $\gamma_{n,k}$ of the Eulerian polynomials $A_n(x,y)$ as given in (1.4) and (1.5) have a number of combinatorial interpretations. For the purpose of this paper, we shall single out the one in connection with 0-1-2 increasing plane trees.

A 0-1-2 increasing plane tree on $[n]$ is an increasing plane tree for which each vertex has degree at most two, where the degree of a vertex is referred to the number its children. For a 0-1-2 increasing plane tree T on $[n]$, assume that it has f_0 leaves and $f_2$vertices of degree two, then it is easily seen that

$$f_2=f_0-1. \tag{3.1}$$

Let $s(n,k)$ be the number of 0-1-2 increasing trees on $[n]$ with k leaves, and let $t(n,k)$ be the number of 0-1-2 increasing plane trees on $[n]$ with k leaves. Then we have

$$t(n,k)=2^{k-1}s(n,k). \tag{3.2}$$

We now turn to the observation of Ma-Ma-Yeh [22] on a grammatical explanation of the γ-positivity of $A_n(x,y)$. Observe that

$$D(xy)=(x+y)xy,\quad D(x+y)=2xy.$$

If we set

$$u=xy,\quad v=x+y,$$

then we get $D(u) = uv$ and $D(v) = 2u$. In other words, we have a new grammar

$$H = \{u \to uv, \quad v \to 2u\}. \tag{3.3}$$

Let D denote the formal derivative with respect to the grammar G as well as the grammar H. It is safe to do so since G and H have distinct variables. Since for $n \geq 1$,

$$A_n(x,y) = D^n(x) = D^{n-1}(u),$$

we infer that $A_n(x,y)$ is a polynomial in u and v with nonnegative coefficients. That is to say, the polynomials $A_n(x,y)$ are γ-positive.

Let T be a 0-1-2 increasing plane tree on $[n]$, where $n \geq 1$. We define a grammatical labeling of T as follows. A leaf is labeled by u, a degree one vertex is labeled by v and a degree two vertex is labeled by 1. The weight of T is defined to be the product of the labels associated with the vertices of T.

For example, Figure 1 is a 0-1-2 increasing plane on [6] with weight u^3v, where the grammatical labels are in parentheses.

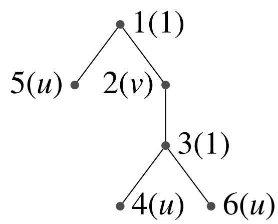

Figure 1: A 0-1-2 increasing plane tree.

Increasing plane trees are also called plane recursive trees, see Janson [18]. The above grammatical labeling of 0-1-2 increasing plane trees shows that the γ-coefficients for the Eulerian polynomials can be interpreted based on 0-1-2 increasing plane trees.

Theorem 3.1. *For $n \geq 1$ and $1 \leq k \leq [(n+1)/2]$, the number $\gamma_{n,k}$ equals the number of 0-1-2 increasing plane trees on $[n]$ with k leaves.*

It is not hard to transform a 0-1-2 increasing plane tree into a permutation without double descents. There is a one-to-one correspondence ϕ between the set of permutations on $[n]$ with k descents and no double descents and the set of 0-1-2 increasing plane trees on $[n]$ with k leaves. This is exactly the classical bijection between permutations and increasing binary trees, restricted to binary trees without vertices having only left children, see Stanley [30].

The structure of 0-1-2 increasing plane trees can be employed to partition the set of permutations of $[n]$ into classes similar to the classification according to the Foata-Strehl action on permutations defined based on the notion of x-factorization [13], see also Brändén [3]. Foata and Strehl gave a proof of (1.4) using the group action on S_n, by which S_n can be partitioned into equivalence classes.

Let T be a 0-1-2 increasing plane tree on $[n]$. We now consider a labeling of T by assigning a label x or y to a degree one vertex, and a label xy to a leaf, and 1 to a degree two vertex. Let $\alpha(T)$ denote the set of labeled trees obtained from T, and let $w(T)$ denote the sum of weights of all trees in $\alpha(T)$. Assume that T has k leaves. As has already been mentioned, T has $k-1$ degree two vertices. Thus it contains $n+1-2k$ degree one vertices. It follows that the total weight of the trees in $\alpha(T)$ amounts to

$$w(T) = (xy)^k(x+y)^{n+1-2k}. \tag{3.4}$$

The above relation reveals that the set of permutations of $[n]$ can be partitioned into classes with each class corresponding to a 0-1-2 increasing plane tree T. More precisely, for a 0-1-2 increasing plane tree T, we get a set $\alpha(T)$ of labeled trees with each corresponding to a permutation. Now, the sum of weights of these labeled trees is given by (3.4). It is readily seen that the sum of weights of trees can be translated into a weighted sum of permutation statistics.

We notice that 0-1-2 increasing plane trees with the above labeling scheme can be represented as increasing binary trees. For each labeled tree in $\alpha(T)$, we can represent it by an increasing binary tree on $[n]$. For a degree one vertex v, if it is labeled by x, then we turn its child into a left child as in a binary tree, otherwise, we turn it into a right child as in a binary tree. By the classical bijection between permutations and increasing binary trees, we find that the labeling of a 0-1-2 increasing plane tree is suitable for keeping track of the number of descents of a permutation. As can be seen, the grammatical labeling of 0-1-2 increasing plane trees provides a combinatorial justification of the relation of Foata-Schützenberger back to the original form.

4 The Second-order Eulerian Polynomials

Gessel and Stanley [15] introduced the notion of Stirling permutations and defined the second-order Eulerian polynomials $C_n(x)$ by $C_0(x) = 1$ and for $n \geq 1$,

$$C_n(x) = \sum_{k=1}^{n} C(n,k)x^k, \tag{4.1}$$

where $C(n,k)$ is the number of Stirling permutations on $[n]_2$ with k descents. A homogeneous version of $C_n(x)$ is given by

$$C_n(x,y) = \sum_{k=1}^{n} C(n,k) x^k y^{2n+1-k}. \tag{4.2}$$

Let $C_n(x,y,z)$ be the trivariate polynomials first defined by Dumont [9] and rediscovered by Janson [18]. As a symmetric function in x,y,z, $C_n(x,y,z)$ can be expressed as a polynomial in the elementary symmetric functions in x,y,z. If the coefficients are all nonnegative, we say that the symmetric function is e-positive, see Stanley [31]. We shall show that for $n \geq 1$, $C_n(x,y,z)$ is e-positive along with a combinatorial interpretation of the coefficients.

Let G be the following grammar

$$G = \{x \to xyz, \quad y \to xyz, \quad z \to xyz\}. \tag{4.3}$$

Let D denote the formal derivative with respect to G. It has been shown by Dumont [9] that for $n \geq 1$,

$$C_n(x,y,z) = D^n(x). \tag{4.4}$$

For $n \geq 1$, assume that

$$C_n(x,y,z) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} (x+y+z)^i (xy+xz+yz)^j (xyz)^k. \tag{4.5}$$

Let

$$u = x+y+z, \quad v = xy+xz+yz, \quad w = xyz.$$

Then we have

$$D(u) = 3w, \quad D(v) = 2uw, \quad D(w) = vw. \tag{4.6}$$

For $n \geq 1$, we have $C_n(x,y,z) = D^{n-1}(xyz)$, where D is the formal derivative with respect to the grammar G in (4.3). Now, we might as well use the same symbol D for the formal derivative with respect to the grammar H. For $n \geq 1$, we have

$$D^n(x) = D^{n-1}(w),$$

which is clearly a polynomial in u,v,w with nonnegative coefficients. In other words, $C_n(x,y,z)$ is e-positive.

The main objective of this paper is to give a combinatorial interpretation of the coefficients $\gamma_{i,j,k}$ in (4.5). The relations in (4.6) prompt us to define the grammar

$$H = \{u \to 3w, \quad v \to 2uw, \quad w \to vw\}. \tag{4.7}$$

Theorem 4.1. *For $n \geq 1$ and $i+2j+3k=2n+1$, the coefficient $\gamma_{i,j,k}$ in the expansion (4.5) of $C_n(x,y,z)$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices.*

Proof. Let T be a 0-1-2-3 increasing plane tree on $[n]$. We first give a labeling of T as follows. Label a leaf by w, a degree one vertex by v, a degree two vertex by u and a degree three vertex by 1. Given any 0-1-2-3 increasing plane tree T on $[n]$ with k leaves, j degree one vertices and i degree two vertices, it has $n-i-j-k$ vertices of degree three. Taking the number of edges into consideration, we get

$$3(n-i-j-k)+2i+j=n-1.$$

Thus we have verified that

$$i+2j+3k=2n+1.$$

Let us examine how to generate a 0-1-2-3 increasing plane tree T' on $[n+1]$ by adding $n+1$ to T as a leaf. We can add $n+1$ to T only as a child of a vertex r that is not of degree three. Thus we have the following three possibilities.

Case 1: The vertex r is a leaf with label w. In the resulting tree T', r becomes a degree one vertex with label v and $n+1$ becomes a leaf with label w. This operation corresponds to the substitution $w \to vw$.

Case 2: The vertex r is a degree one vertex with label v. In this case, $n+1$ can be attached to r either as the first child, or the second child. In either case, in the resulting tree T', r becomes a degree two vertex with label u and $n+1$ becomes a leaf with label w. This operation corresponds to the substitution $v \to 2uw$.

Case 3: The vertex r is a degree two vertex with label u. In this case, $n+1$ can be attached to r either as the first child, or the second child, or the third child. In either case, in the resulting tree T', r becomes a degree three vertex with label 1 and $n+1$ becomes a leaf with label w. This operation corresponds to the substitution $u \to 3w$.

The aforementioned three cases exhaust all the possibilities to construct a 0-1-2-3 increasing plane tree T' on $[n+1]$ from a 0-1-2-3 increasing plane tree T on $[n]$ by adding $n+1$ as a leaf. Since each case corresponds to an application of a substitution rule in H, we see that for $n \geq 1$, $D^n(x)$ equals the sum of the weights of 0-1-2-3 increasing plane trees on $[n]$, that is,

$$D^n(w) = \sum_{i+2j+3k=2n+1} \gamma_{i,j,k} u^i v^j w^k. \tag{4.8}$$

Therefore, $\gamma_{i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices. ∎

Figure 2 is an illustration of a 0-1-2-3 increasing plane tree on $[10]$, where the grammatical labels are in parentheses.

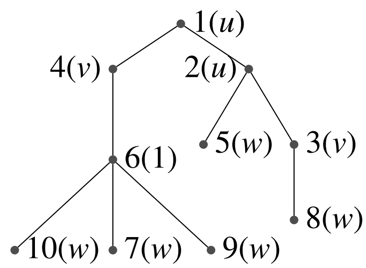

Figure 2: A 0-1-2-3 increasing plane tree.

In view of the above combinatorial interpretation or the relation $D^n(u) = D(D^{n-1}(u))$, we conclude with the following recurrence relation:

$$\gamma_{n,i,j,k} = 3(i+1)\gamma_{n-1,i+1,j,k-1} + 2(j+1)\gamma_{n-1,i-1,j+1,k-1} + k\gamma_{n-1,i,j-1,k} \quad (4.9)$$

with $\gamma_{1,0,0,1} = 1$ and $\gamma_{1,i,j,k} = 0$ if $k \neq 1$.

Acknowledgments. We wish to thank M. Bóna, D. Foata, S.-M. Ma, P. Paule and the referee for invaluable comments and suggestions. This work was supported by the National Science Foundation and the Ministry of Science and Technology of China.

References

[1] C.A. Athanasiadis, Gamma-positivity in combinatorics and geometry, Sém. Lothar. Combin., 77 (2018), Article B77i.

[2] M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, SIAM J. Discrete Math., 23 (2009), 401–406.

[3] P. Brändén, Actions on permutations and unimodality of descent polynomials, European J. Combin., 29 (2008), 514–531.

[4] L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, J. Reine Angew. Math., 265 (1974), 110–137.

[5] W.Y.C. Chen, Context-free grammars, differential operator and formal power series, Theoret. Comput. Sci., 117 (1993), 113–129.

[6] W.Y.C. Chen and A.M. Fu, Context-free grammars for permutations and increasing trees, Adv. in Appl. Math., 82 (2017), 58–82.

[7] W.Y.C. Chen, R.X.J. Hao and H.R.L.Yang, Context-free grammars and multivariate stable polynomials over Stirling permutations, In: V. Pillwein and C. Schneider (eds.), Algorithmic Combinatorics: Enumerative Combinatorics, Special Functions and Computer Algebra, pp. 109–135, Springer, 2021

[8] C.-O. Chow, On certain combinatorial expansions of the Eulerian polynomials, Adv. in Appl. Math., 41 (2008), 133–157.

[9] D. Dumont, Une généralisation trivariée symétrique des nombres eulériens, J. Combin. Theory, Ser. A, 28 (1980), 307-320.

[10] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, Sém. Lothar. Combin., 37, Art. B37a (1996), 1–21.

[11] S. Elizalde, Descents on quasi-Stirling permutations, J. Combin. Theory, Ser. A 180 (2021) 105429.

[12] D. Foata and M.P. Schützenberger, Théorie géométrique des polynômes eulériens, Lecture Notes in Math., Vol. 138, Springer, Berlin, 1970.

[13] D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, Math. Z., 137 (1974), 257–264.

[14] V. Gasharov, On the Neggers-Stanley conjecture and the Eulerian polynomials, J. Combin. Theory, Ser. A, 82 (1998), 134–146.

[15] I.M. Gessel and R.P. Stanley, Stirling permutations, J. Combin. Theory, Ser. A, 24 (1978), 25–33.

[16] J. Haglund and M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, European J. Combin., 33 (2012), 477–487.

[17] G.-N. Han and S.M. Ma, Derivatives, Eulerian polynomials and the g-indexes of Young tableaux, arXiv:2006.14064.

[18] S. Janson, Plane recursive trees, Stirling permutations and an urn model, Proceedings of Fifth Colloquium on Mathematics and Computer Science, Discrete Math. Theor. Comput. Sci. Proc., vol. AI, 2008, pp. 541–547.

[19] S. Janson, M. Kuba, A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, J. Combin. Theory, Ser. A, 118 (2011), 94–114.

[20] Z. Lin, On γ-positive polynomials arising in pattern avoidance, Adv. in Appl. Math., 82 (2017), 1–22.

[21] Z. Lin and J. Zeng, The γ-positivity of basic Eulerian polynomials via group actions, J. Combin. Theory, Ser. A, 135 (2015), 112–129.

[22] S.-M. Ma, J. Ma and Y.-N. Yeh, γ-Positivity and partial γ-positivity of descent-type polynomials, J. Combin. Theory, Ser. A, 167 (2019), 257–293.

[23] S.-M. Ma, J. Ma and Y.-N. Yeh, The ascent-plateau statistics on Stirling permutations, Electron. J. Combin. 26(2) (2019), P2.5.

[24] S.-M. Ma, J. Ma, J. Yeh and Y.-N. Yeh, Excedance-type polynomials and gamma-positivity, arXiv:2102.00899.

[25] S.-M. Ma, J. Ma, Jean Yeh and Y.-N. Yeh, The e-positivity of multivariate the k-th order Eulerian polynomials, arXiv:2106.16189

[26] S.-M. Ma and Y.-N. Yeh, Eulerian polynomials, Stirling permutations of the second kind and perfect matchings, Electron. J. Combin., 24(4) (2017), #P4.27

[27] T.K. Petersen, Eulerian Numbers. Birkhäuser/Springer, New York, 2015.

[28] L.W. Shapiro, W.-J. Woan, S. Getu, Runs, slides and moments, SIAM J. Algebraic Discrete Methods, 4 (1983), 459–466.

[29] H. Shin and J. Zeng, The symmetric and unimodal expansion of Eulerian polynomials via continued fractions, European J. Combin., 33 (2012), 111–127.

[30] R.P. Stanley, Enumerative Combinatorics, Vol. I, second ed., Cambridge University Press, Cambridge, 2012.

[31] R.P. Stanley, Enumerative Combinatorics, Vol. II, Cambridge University Press, Cambridge, 1999.

[32] J.R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, Discrete Math., 99 (1992), 307–320.

[33] S.H.F. Yan, H. Zhou and Z. Lin, A new encoding of permutations by Laguerre histories, Electron. J. Combin., 26(3) (2019), #P3.47.