A Simple Proof of Dixon’s Identity

Victor J. W. Guo

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, People’s Republic of China
Email: jwguo@eyou.com

**Keywords**: Dixon, identity, polynomial

Dixon [?] established the following famous identity

\[
\sum_{k=-a}^{a} (-1)^k \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!},
\]

(1)

where \(a, b, c\) are nonnegative integers (for short proofs, cf. [?, ?]).

In this note, we give another simple proof of (1) in its polynomial version.

**Theorem 1** Let \(m, r\) be nonnegative integers, and \(x\) an indeterminate. Then

\[
\sum_{k=0}^{2r} (-1)^k \binom{m+2r}{m+k} \binom{x}{k} \binom{x+m}{m+2r-k} = (-1)^r \binom{x}{r} \binom{x+m+r}{m+r}.
\]

(2)

Here and in what follows,

\[
\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!}.
\]

**Proof.** Denote the left-hand side of (2) by \(P(x)\). We want to show that

\[P(x) = 0, \quad \text{for } -m - r \leq x < r.\]

(i) \(x = 0, 1, \ldots, r - 1\), we have 0 \(\leq x < k\) or 0 \(\leq x < 2r - k\). Hence, \(\binom{x}{k} = 0\) or \(\binom{x+m}{m+2r-k} = 0\).

(ii) \(x = -m, -m + 1, \ldots, -1\), we have 0 \(\leq x + m < m \leq m + 2r - k\). So,

\[\binom{x+m}{m+2r-k} = 0.\]
(iii) $x = -m - r, -m - r + 1, \ldots, -m - 1$. Set $x = -p - 1$, where $p = m, m + 1, \ldots, m + r - 1$. Then,

$$P(-p - 1) = \sum_{k=0}^{2r} (-1)^{m-k} \binom{m+2r}{m+k} \binom{p+k}{k} \binom{p+2r-k}{m+2r-k} = 0.$$ 

The last identity holds because $\binom{p+k}{k}$ is a polynomial in $k$ of degree $2p - m < m + 2r$, and we have

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} k^i = 0, \quad 0 \leq i < n,$$

which is well-known.

Moreover, $P(r)$ has only one nonzero term $(-1)^r \binom{m+2r}{m+r}$. Thus, $P(x)$ coincides with $(-1)^r \binom{p+k}{k} \binom{p+2r-k}{p-m}$ at $m + 2r + 1$ values of $x$. Hence they must be identical. This completes the proof.

Set $m + 2r = a + b$, $x = b + c$, $x + m = c + a$ in Theorem 33. Then multiplying (??) by $(-1)^b$, and changing $k$ to $b + k$, we obtain the form (??).

Acknowledgment. This work was done under the auspices of the National “973” Project on Mathematical Mechanization, and the National Science Foundation of China. The author thanks a referee for helpful suggestions to improve the clarity of the manuscript.

References