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New Asymptotics and Inequalities Related to the Partition Function

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Abstract

This thesis is devoted to a detailed study of asymptotics and inequalities related to the partition function from analytic and combinatorial aspects.
In the first part of this thesis, we present a short and concise discussion on the evolution of respective developments and the state-of-art.
The second part of the thesis focuses on to derive an infinite family of inequalities associated to the partition function and its consequences. We start with deriving an infinite family of inequalities for the logarithm of the partition function that in turn shows its full asymptotic expansion. Applications of these inequalities help us to construct a unified framework to prove multiplicative inequalities for the partition function which in turn reveal combinatorial properties like log-concavity, strong log-concavity, Bessenrodt-Ono type inequality, and its variants, among many others. Furthermore, we also obtain a full asymptotic expansion of $(-1)^{r-1}\Delta^r \log p(n)$ which extends a result of Chen, Wang, and Xie. We also discuss the limitations of such an infinite family of inequalities for $\log p(n)$ in comparison to getting a similar family for $p(n)$. Crossing this barrier, we obtain the full asymptotic expansion for the partition function along with explicit computations on the error term so as to obtain an infinite family of inequalities for $p(n)$ which leads to the completion of works done by Wright, Szekeres, and O'Sullivan. These new inequalities for $p(n)$ and the proof of getting such inequalities have manifold applications. First, we show how one can follow the proof systematically to obtain a similar family of inequalities for the shifted value of the partition function, denoted by $p(n-\ell)$ with ℓ a non-negative integer, which finally settles all the conjectures of Chen on the partition function inequalities arising from the invariants of a quartic binary form. Second, we prove higher order shifted Laguerre-Pólya inequalities for $p(n)$ that confirms a conjecture of Wagner. Third, we get the full asymptotic expansion of the finite differences for $p(n)$ which completes the work of Odlyzko. Consequently, this helps to work out on asymptotic enumeration of ranks and cranks of partitions. Fourth, we partially resolve a conjecture of Andrews and Merca on partition function inequalities arising from truncated theta series. In the final chapter of this part, we obtain the bounds of the error

term after truncation of the asymptotic expansion of the modified Bessel function of non-negative order which extends and simultaneously refines a result of Bringmann, Kane, Rolen, and Tripp. We also discuss the importance and novelty of the result in the context of asymptotic enumeration of Fourier coefficients arising from certain periodic meromorphic functions, which admit a Hardy-Ramanujan-Rademacher type series expansion.
The third part of this thesis concentrates on proving inequalities related to the partition function by applications of elementary combinatorial techniques. Avoiding the analytic approach, we show the positivity of the second-order shifted difference for integer partitions and overpartitions by getting a combinatorial description in the terms of restricted partitions and overpartitions. This in turn provides a combinatorial proof of the monotonicity of ranks and cranks of partitions. Inequalities for the partition function by imposing certain restrictions on its parts and counting the parity of the length of parts is the key topic in the last chapter of this thesis.

Kurzfassung

Diese Arbeit ist einer detaillierten Untersuchung Vou Asymptotik und Ungleichungen im Zusammenhang mit der Partitionsfunktion unter analytischen und kombinatorischen Aspekten gewidmet.
Im ersten Teil dieser Arbeit wird eine kurze Diskussion über die Evolution der entsprechenden Entwicklungen und den aktuellen Stand der Technik gegeben.
Der zweite Teil der Arbeit konzentriert sich auf die Ableitung einer unendlichen Familie von Ungleichungen, die mit der Partitionsfunktion und ihren Konsequenzen verbunden sind. Wir beginnen mit der Ableitung einer unendlichen Familie von Ungleichungen für den Logarithmus der Partitionsfunktion, die ihrerseits ihre vollständige asymptotische Entwicklung zeigt. Anwendungen dieser Ungleichungen helfen uns, einen einheitlichen Rahmen zu konstruieren, um multiplikative Ungleichungen für die Partitionsfunktion zu beweisen, die wiederum kombinatorische Eigenschaften wie log-Konkavität, starke log-Konkavität, Ungleichungen vom Bessenrodt-Ono-Typ und ihre Varianten, neben vielen anderen, aufzeigen. Darüber hinaus erhalten wir auch eine vollständige asymptotische Erweiterung von $(-1)^{r-1}\Delta^r \log p(n)$, die ein Ergebnis von Chen, Wang und Xie erweitert. Wir diskutieren auch die Grenzen einer solchen unendlichen Familie von Ungleichungen für $\log p(n)$ im Vergleich zu einer ähnlichen Familie für $p(n)$. Nach Überwindung dieser Schranke erhalten wir die vollständige asymptotische Erweiterung für die Partitionsfunktion zusammen mit expliziten Berechnungen zum Fehlerterm, um eine unendliche Familie von Ungleichungen für $p(n)$ zu erhalten, die zur Vervollständigung der Arbeiten von Wright, Szekeres und O'Sullivan führt. Diese neuen Ungleichungen für $p(n)$ und der Beweis für den Erhalt solcher Ungleichungen haben vielfältige Anwendungen. Erstens zeigen wir, wie man dem Beweis systematisch folgen kann, um eine ähnliche Familie von Ungleichungen für den verschobenen Wert der Partitionsfunktion zu erhalten, die mit $p(n-\ell)$ bezeichnet wird, wobei ℓ eine nicht-negative ganze Zahl ist, die schließlich alle Vermutungen von Chen über die Ungleichungen der Partitionsfunktion, die sich aus den Invarianten einer quartischen binären Form ergeben, klärt. Zweitens beweisen wir verschobene Laguerre-Pólya-Ungleichungen höherer Ordnung für $p(n)$,

die eine Vermutung von Wagner bestätigen. Drittens erhalten wir die vollständige asymptotische Entwicklung der endlichen Differenzen für $p(n)$, was die Arbeit von Odlyzko vervollständigt. Dies hilft, die asymptotische Aufzählung von Rängen und Verzweigungen von Partitionen zu erarbeiten. Viertens lösen wir teilweise eine Vermutung von Andrews und Merca über Ungleichungen von Partitionsfunktionen, die sich aus abgeschnittenen Thetareihen ergeben. Im letzten Kapitel dieses Teils erhalten wir die Schranken des Fehlerterms nach Abschneiden der asymptotischen Erweiterung der modifizierten Besselfunktion nichtnegativer Ordnung, die ein Ergebnis von Bringmann, Kane, Rolen und Tripp erweitert und gleichzeitig verfeinert. Wir diskutieren auch die Bedeutung und Neuartigkeit des Ergebnisses im Zusammenhang mit der asymptotischen Aufzählung von Fourier-Koeffizienten, die aus bestimmten periodischen meromorphen Funktionen entstehen, die eine Reihenentwicklung vom Typ Hardy-Ramanujan-Rademacher zulassen.
Der dritte Teil dieser Arbeit konzentriert sich auf den Nachweis von Ungleichungen im Zusammenhang mit der Partitionsfunktion durch Anwendung elementarer kombinatorischer Techniken. Unter Vermeidung des analytischen Ansatzes zeigen wir die Positivität der verschobenen Differenz zweiter Ordnung für ganzzahlige Partitionen und Überpartitionen, indem wir eine kombinatorische Beschreibung in Form von beschränkten Partitionen und Überpartitionen erhalten. Dies wiederum liefert einen kombinatorischen Beweis für die Monotonizität von Rängen und Verzweigungen von Partitionen. Ungleichungen für die Partitionsfunktion durch Auferlegung bestimmter Beschränkungen für ihre Teile und das Zählen der Parität der Länge der Teile ist das Hauptthema im letzten Kapitel dieser Arbeit.

To
My family and Manosij

Acknowledgements

I am immensely grateful to my advisor Prof. Peter Paule for his care, and guidance throughout my Ph.D. years from both academic and non-academic aspects. Not being from an affluent family, it would be hard for me to come and stay in Linz without his support. I must admit that words like help and support are too little or shallow to portray the humanity of Prof. Paule. Honestly, I have no idea as to how much effort he has put to secure the extension of the period of funding up to one year so that I could complete my Ph.D. without being distracted by financial distress in the future. I must confess that I am grateful to have Prof. Paule as a mentor in my life. Ever since the day of my Ph.D. interview, whenever I met with Prof. Paule, always I always realized that I am privileged. On the very first meeting, he gave me a piece of deep and beautiful advice by quoting from his advisor Prof. Johann Cigler: "Stick honestly to your own understanding." Before moving on to his mathematical advice, I would like to thank him for sharing his memories of Gian-Carlo Rota at Bayreuth and explaining "phenomenology" in simple terms to me. The topics covered in this thesis were impossible to execute without his encouragement, profound questions, and remarks. It gives me great joy to see how much he cherishes simple and beautiful solutions to difficult mathematical problems. Being inexperienced in writing and speaking about research, I faced a lot of difficulties at the beginning of my Ph.D. years but with immense patience, Prof. Paule taught me these two aspects akin to how a mother teaches her child to walk. Above all, the very personage of Prof. Paule leaves a permanent imprint on my life.

The formation of this thesis from its incubation period was nurtured under the tutelage of my second supervisor Dr. Cristian-Silviu Radu. He planted the seeds of inequalities for the partition function by providing me with a simple problem that expanded slowly like a tree to form my thesis. Whenever I showed him a solution to a problem, he always asked me very pertinent questions which in turn helped me furnish my answers most optimally. We also had a lot of discussions both inside and outside of mathematics. During the pandemic period, staying alone was not all difficult for me because he used to call me twice or thrice a day even when he was on holiday spending time with his family. Among all of his reactions to solutions to any problem, I found the most amusing one to be "disaster" which happened when I sent him a simple and elegant solution.

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Almost two years ago, I attempted to prove an inequality for the partition function arising from one of the invariants of a quartic binary form which was conjectured by Prof. William Y. C. Chen. This was the first time when I had a chance to correspond with Prof. Chen and he explained the current status of his conjecture and stated

> I did not know any reason why they should be valid for the partition function for large n.

This statement intrigued me to demystify all the conjectures of Prof. Chen related to the partition function inequalities, which are documented in his survey paper. I am immensely grateful to Prof. William Y. C. Chen for his kindness to be the second examiner of this thesis.

I am deeply honored that Prof. Kathy Qing Ji and Janet J. W. Dong kindly pointed out several typographical errors in the earlier version of this thesis. Moreover, their invaluable suggestions make a substantial improvement of this draft from both mathematical and grammatical aspects.

My journey in the academic world of mathematics would not be possible at all if I had never met Prof. Atul Dixit. After finishing my master's, I got an opportunity to work under his supervision during my summer project internship at IIT, Gandhinagar. In summer, usually, the temperature is too high (above 50° C!) in Gandhinagar, and therefore, he always called me to his apartment after 18 : 00 PM so that I don't get dehydrated and fall in. This is just one of the examples that reflect how much he cares for his students. During the period of the next two years, I applied for Ph.D. positions in several US universities but got rejected due to my inability in writing and speaking fluently in the English language. The important thing at this time to me was the wonder of how a professor kept on telling me "whenever you need a recommendation for any of your academic purpose, don't hesitate at all to send me an email". He indeed stayed true to that for the last six years.

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It is very hard for me to write about my academic brother Manosij. I feel that this is too little space to write about him. He opened the horizon of mathematics in front of me since the day I first met him about eleven years ago. One of his precious advice to me was “try to create a good problem and solve it beautifully in order to do a justice”. Instead of posing problems, he just asked a few questions from time to time to see my reactions. During that period, whenever I went to his room asking for a solution, his reply was “think for yourself”. Even after showing him the solution, most often I received a cold reception like “okay but you wasted a lot of energy and the solution doesn’t look good, so think again”. More surprisingly, when I came up with a beautiful solution to a problem, his reaction was “why did it take so much time?”. He always kept on telling me “prove theorems on your own so that you can understand the difficulty hidden inside”. Till today, I learn a lot from him and I owe him.

After my arrival in Linz, I found a really good friend from the western world and he is Nicolas Allen Smoot. He took me to all the necessary places so as to complete the essential administrative processes. In order to help me to adapt to this new culture, quite often he took me to cafes and restaurants so that I could feel at ease. In the beginning, we played chess in cafes and he discussed the research he had done in his Ph.D. years. I never felt uneasy for a single moment while spending time with him.

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Contents

Part I

Prologue

Chapter 1

Introduction

1.1 Contributions of this thesis

A major portion of this thesis is devoted to a rigorous and comprehensive study of inequalities related to the partition function from an analytic aspect. We also address inequalities for a certain class of restricted partition functions that are of combinatorial flavor.

Following the usual notation, $p(n)$ denotes the total number of partitions of n. Let us begin with the Hardy-Ramanujan-Rademacher formula [124] for $p(n)$ which reads

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\Big(1 - \frac{k}{\mu(n)}\Big) e^{\mu(n)/k} + \Big(1 + \frac{k}{\mu(n)}\Big) e^{-\mu(n)/k} \right] + R_2(n, N), \tag{1.1}$$

and an error bound due to Lehmer [98] that states

$$\Big| R_2(n, N) \Big| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\Big(\frac{N}{\mu(n)}\Big)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \Big(\frac{N}{\mu(n)}\Big)^2 \right]; \tag{1.2}$$

where $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$. After truncating the series (1.1) at $N = 2$ and applying (1.2) Chen, Jia, and Wang [37, Lemma 2.2] proved that for all $n \geq 1206$,

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1} \left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{10}} \right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1} \left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{10}} \right). \tag{1.3}$$

This is the starting point of our journey to derive the asymptotics and inequalities discussed in Chapters 3-7.

In Chapter 3 we generalize (1.3) as follows: for all $n > N(w)$,

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^w}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^w}\right), \quad (1.4)$$

where $N(w)$ is given explicitly in Theorem 3.4.4. Applying the logarithm[1] on both sides of (1.4), we obtain an infinite family of inequalities for $\log p(n)$ as given below. For $w \in \mathbb{Z}_{>0}$ with $\lceil w/2 \rceil \geq \gamma_0$ and $n > g(w)$, we have

$$P_n(w-1) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil}}\left(\frac{1}{\sqrt{n}}\right)^w < \log p(n) < P_n(w-1) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil}}\left(\frac{1}{\sqrt{n}}\right)^w, \quad (1.5)$$

where

$$P_n(U) := -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{U} \frac{g_u}{\sqrt{n}^u},$$

with $\gamma_0, \gamma_1, \gamma_2, \alpha$ and the sequence $(g_u)_{u\geq 1}$ are computed explicitly in Theorem 3.6.6. As an application of (1.5), we obtain (cf. Theorem 3.7.6) for all $n \geq 120$,

$$\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{1}{n^2}\right) < u_n := \frac{p(n)^2}{p(n-1)p(n+1)} < \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right). \quad (1.6)$$

From the last inequality, we can conclude two facts: (i) $(p(n))_{n\geq 26}$ is log-concave, and (ii) that the rate of decay of the quotient u_n is indeed $\frac{\pi}{\sqrt{24}n^{3/2}}$ which leads to a completion of the study in this direction [39, eq. (1.2)-(1.4), Sec. 2].
Chapter 4 presents a couple of examples on inequalities related to $\log p(n)$ as applications of (1.4). First, we obtain a similar inequality in Theorem 4.3.9 for $\log p(n+s)$ for $s \in \mathbb{Z}_{\geq 0}$ as in (1.5). The inequalities for $\log p(n+s)$ and its generalization for $\sum_i \log p(n+s_i)$ in Theorem 4.3.13 help us to study the following aspects.

1. To prove inequalities[2] of the following form:

$$L(T,\vec{\mathbf{s}}) := \prod_{i=1}^{T} p(n+s_i) \geq \prod_{i=1}^{T} p(n+r_i) := R(T,\vec{\mathbf{r}}), \quad (1.7)$$

   where $T \geq 1$, and $(s_i)_{1\leq i\leq T}$, $(r_i)_{1\leq i\leq T}$ are non-negative integers by an algorithmic approach, see Section 4.5. One can also decide the rate of decay of the quotient $L(T,\vec{\mathbf{s}})/R(T,\vec{\mathbf{r}})$ as shown in (1.6). Moreover, (1.7) provides an unified framework to prove log-concavity, strong log-concavity [53, Sec. 5], and Bessenrodt-Ono type inequalities [26] for $p(n)$, see Remark 4.3.14.

[1]Throughout the thesis, the logarithm is taken with respect to the base e; i.e., natural logarithm.
[2]Which we call multiplicative inequalities

2. To present both an upper and lower bound of $(-1)^{r-1}\Delta^r \log p(n)$ along with a cut-off $n(r)$ for n in Theorem 4.4.6 so that we can compute from which point on the inequality holds for each positive integer r. This extends a result of Chen, Wang, and Xie [39, Thm. 3.1, 4.1]. Moreover, we provide a full asymptotic expansion of $(-1)^{r-1}\Delta^r \log p(n)$ in Theorem 4.4.7.

We conclude Chapter 4 by discussing the limitations of applying inequalities for $\log p(n+s)$ to other partition function inequalities, for example, higher order Turán inequalities. In Theorem 4.6.9, we obtain an infinite family of inequalities for $\prod_i p(n+s_i)$ directly from inequalities for $\sum_i \log p(n+s_i)$ by taking the exponential. But in order to prove an inequality of the following form,

$$\sum_{j=1}^{M}\prod_{i=1}^{T} p(n+s_{i,j}) \geq \sum_{j=1}^{M}\prod_{i=1}^{T} p(n+r_{i,j}), \tag{1.8}$$

a major difficulty is in the computation for bounds of error terms.
To remove this obstacle, we trace back to (1.4). Instead of applying the logarithm on (1.4), we compute the Taylor series expansion of $\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}\Big)$. This in turn gives (cf. Lemma 5.3.19)

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}\Big) = \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}.\sum_{t=0}^{\infty}\frac{g(t)}{\sqrt{n}^{t}}, \tag{1.9}$$

where

$$g(t) = \frac{1}{(-4\sqrt{6})^t}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\Big(\frac{\pi}{6}\Big)^{t-2k}.$$

Recently, O'Sullivan [112] obtained (1.9) but it does not lead to proving inequalities for the partition function because as it seems, no results on bounds for the error term are available in the literature. To obtain both an upper and lower bound of the error term $\sum\limits_{t\geq m} g(t)/\sqrt{n}^{t}$, we need to estimate $g(t)$ in the following way:

$$f(t)-\ell(t) \leq g(t) \leq f(t)+u(t), \text{ with } \lim_{t\to\infty}\frac{g(t)}{f(t)}=1, \lim_{t\to\infty}\frac{\ell(t)}{f(t)}=0, \text{ and } \lim_{t\to\infty}\frac{u(t)}{f(t)}=0.$$

In Chapter 5, we resolve this problem by simplifying the deceptively simple looking sum $g(t)$ using the symbolic summation package `Sigma` due to Schneider [128].

Finally, we obtain an infinite family of inequalities for $p(n)$ that reads

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t+\frac{L(w)}{\sqrt{n}^w}\right)<p(n)<\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t+\frac{U(w)}{\sqrt{n}^w}\right), \tag{1.10}$$

given in Theorem 5.7.5.
Applying a similar proof methodology as described in the last paragraph, in Chapter 6, we derive inequalities for $p(n-\ell)$ with $\ell\in\mathbb{Z}_{\geq 0}$ of the following form:

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t+\frac{L(w,\ell)}{\sqrt{n}^w}\right)<p(n-\ell)$$
$$<\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t+\frac{U(w,\ell)}{\sqrt{n}^w}\right). \tag{1.11}$$

For the definitions of $g(t,\ell), L(w,\ell)$, and $U(w,\ell)$, we refer to Theorem 6.4.5). Using this infinite family of inequalities, we obtain the following results.

1. In Section 6.5, we confirm all the conjectures of Chen [36, eq. (6.17)-(6.18), Conj. 6.15-6.16] on partition function inequalities arising from invariants of a quartic binary form.

2. $(p(n))_{n\geq 95}$ satisfies the higher order Turán inequality.

3. $(p(n))_{n\geq 222}$ is 2-log-concave.

Further applications of Theorem 6.4.5 are discussed in Section 6.7.
In Chapter 7, we study the asymptotic growth of coefficients of truncations of theta series by applying Theorem 6.4.5 and this leads to settling (partially!) Andrews and Merca's [10, Sec. 7] conjecture. Moreover, we also show that the conjecture of Andrews and Merca is even true for the excluded case; i.e., n even and k odd.
The Chapter 8 focuses on deriving a family of inequalities for the modified Bessel function of the first kind of non-negative order, denoted by $I_\nu(x)$ with ν a non-negative integer or a half-integer, and x a real number ≥ 1. A natural question is why do we need such an infinite family of inequalities for $I_\nu(x)$? Among many others, our attention is on the appearance of $I_\nu(x)$ in Hardy-Ramanujan-Rademacher type series expansions for Fourier coefficients of certain classes of Dedekind eta quotients[3];

[3]For a definition, see [117, Definition 1.63]

see for example [41, Thm. 1.1] or [138, Thm. 1.1]. These coefficients are quite often entangled with combinatorial features that emerge from the question of whether a real polynomial associated with such sequences has roots all real. For example, consider the Jensen polynomial of degree d and shift n for a sequence $(\alpha(n))_{n\geq 0}$ of real numbers, defined as

$$J_{\alpha}^{d,n}(x) = \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j)x^j.$$

To solve problems like log-concavity, higher order Turán inequalities for a sequence, say $a_f(n)$, arising from the Fourier expansion of a periodic meromorphic function, say a Dedekind eta quotient $f(q)$, we would like to estimate $a_f(n)$ by computing a precise estimation of the associated Hardy-Ramanujan-Rademacher type series, say S_f. Now, in order to provide such a precise estimate for the main term obtained after truncating the series S_f to a finite number of terms, inequalities for $I_{\nu(f)}(x)$ are needed, where the index $\nu(f)$ is depending on f. For example, in order to prove log-concavity of the colored partition function $p_k(n)$[4], Bringmann et. al. [32, Lemma 2.2 (4)] estimated the error term by truncation of the asymptotic expansion of $I_\nu(x)$ at $N = 3$, which plays a key role in their proof of the conjecture [32, Conjecture 1]. Recently, Dong, Ji, and Jia [57] proved log-concavity and other associated inequalities for broken k-diamond partitions $\Delta_k(n)$[5] (with $k = \{1, 2\}$). But in order to prove the higher order Turán inequalities for $\Delta_k(n)$, Jia [81, Thm 2.1] estimated the error term obtained after truncation at $N = 5$ and obtain a similar inequality. The above examples indicate that to prove real rootedness of $J_{p_k}^{d,n}(x)$ $(k \geq 2)$ and $J_{\Delta_k}^{d,n}(x)$ $(k \in \{1, 2\})$ for $d \geq 4$, we need the truncation point N higher than 5; i.e., as d increases, the choice of truncation $N(d)$ also increases. This is one of the key motivations to compute bounds for the absolute value of the error term for truncation of the asymptotic expansion of $I_\nu(x)$ at any point N. Section 8.3 presents the case when ν is non-negative integer, Section 8.4 treats the case when ν being a half-integer. For both cases, our Theorems 8.3.9 and 8.4.6 extend the results of Bringmann et. al. and Jia.

In Chapter 9, we provide a very simple combinatorial proof of a result due to Gomez, Males, and Rolen which states that

$$\Delta_j^2(p(n)) := p(n) - 2p(n-j) + p(n-2j) > 0. \tag{1.12}$$

We prove the analogous inequality (1.12) for the overpartition function $\overline{p}(n)$[6] also

[4] Defined in [32, p. 2]

[5] For definition, see [57, p. 1]

[6] See the definition [46, eq. (1.1)]

(see Theorems 9.1.5 and 9.1.9).
In Chapter 10, we develop a coherent and systematic procedure to construct a map from its domain set to the range by decomposing the map into successive injective maps so that we can almost exhaust the domain set but on the range, we have a bigger portion left which is not being mapped. This fundamental principle helps us prove the results of parity bias in parts of partitions and settle a conjecture due to Kim, Kim, and Lovejoy. We further show a couple of similar phenomena for a certain class of restricted partition functions.

Motivations and the history of all the inequalities related to $p(n)$ discussed in the previous paragraphs are given in detail in Sections 2.4-2.7.

1.2 How to read the thesis

This thesis is comprised of several chapters which are connected but, to relate the interest of readers, we provide a road map to read the thesis.
A more detailed version of the abstract of the thesis including its purposes and novelty is given in Chapter 1. For readers interested in the history of asymptotics and inequalities related to the partition function, we refer to Chapter 2. Inequalities and asymptotics related to the logarithm for the partition function can be found in Chapter 3 and its applications in Chapter 4. The reader can easily follow up on the materials covered in Chapter 5 without having a look at Chapters 3 and 4. Chapter 5 is essential to follow Chapter 6. Chapter 7 can be read independently after Chapter 6. Chapters 8-10 can be followed without looking at its previous chapters.

1.3 Connection to (joint) articles by the author

Chapters in this thesis are excerpted from the papers written by the author with his collaborators or by himself alone. We associate all the chapters with their corresponding articles by using the format "Chapter → Article listed the bibliography:"

- Chapter 3 is essentially (up to minor modifications, cf. [22]):

  K. Banerjee, P. Paule, C. S. Radu, and W. H. Zeng. New inequalities for $p(n)$ and $\log p(n)$. *Ramanujan Journal*, (2022). https://doi.org/10.1007/s11139-022-00653-6.

- Chapter 4 is essentially (up to minor modifications, cf. [13]):

K. Banerjee. A unified framework to prove multiplicative inequalities for the partition function. *Submitted for publication*, (2022). https://www3.risc.jku.at/publications/download/risc_6614/inequalityproductform.pdf

- Chapter 5 is essentially (up to minor modifications, cf. [21]):

  K. Banerjee, P. Paule, C. S. Radu, and C. Schneider. Error bounds for the asymptotic expansion of the partition function. *Submitted for publication*, (2022). https://arxiv.org/abs/2209.07887

- Chapter 6 is essentially (up to minor modifications, cf. [16, 14]):

  K. Banerjee. Invariants of the quartic binary form and proofs of Chen's conjectures for partition function inequalities. *Submitted for publication*, (2022). https://www3.risc.jku.at/publications/download/risc_6615/Chen.pdf

  K. Banerjee. Higher order Laguerre inequalities for the partition function and its consequences. *In preparation*, (2022).

- Chapter 7 is essentially (up to minor modifications, cf. [19]):

  K. Banerjee and M. G. Dastidar. Inequalities for the partition function arising from truncated theta series. *Submitted for publication*, (2022). https://www3.risc.jku.at/publications/download/risc_6622/AndMer.pdf

- Chapter 8 is essentially (up to minor modifications, cf. [15]):

  K. Banerjee. Inequalities for the modified Bessel function of first kind of non-negative order. *Journal of Mathematical Analysis and Applications*, **524** (2023), p. 127082.

- Chapter 9 is essentially (up to minor modifications, cf. [17]):

  K. Banerjee. Positivity of second shifted difference of partitions and overpartitions: a combinatorial approach. *Enumerative Combinatorics and Applications* **3:2** (2023).

- Chapter 10 is essentially (up to minor modifications, cf. [18]):

  K. Banerjee, S. Bhattacharjee, M. G. Dastidar, P. J. Mahanta, and M. P. Saikia. Parity biases in partitions and restricted partitions. *European Journal of Combinatorics*, **103** (2022), p. 103522.

Chapter 2

Preliminaries

2.1 Approximations, asymptotics, and inequalities

"All exact science is dominated by the idea of approximation."

Bertrand Russell

The problem of approximation of a given number or a sequence or a function is one of the oldest challenges of mathematics. Since the discovery of irrationality, considerations for the theory of approximation had become necessary. The formula for approximating the square root of a number, often attributed to the Babylonians, is a case in point. Mathematical formulae were developed to assist in approximating mainly transcendental functions and initially, the representations primarily relied on Taylor's formula and some interpolation formulae developed by Newton. Probably the first application of this subject outside the regime of mathematics is due to Euler who tried to solve the problem of drawing a map of the Russian empire with exact latitudes. After Euler, Gauß, Laplace, Fourier, Cauchy, Chebyshev, Lagrange, Poisson, Fejér, Weierstraß, Runge among many others expanded the theory of approximations while working on several problems in different directions.
Asymptotic analysis is a branch of mathematical analysis that provides a rigorous foundation to understand the language of approximation. Let us start with a well-known asymptotic result, Stirling's formula:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}n^n e^{-n}} = 1. \tag{2.1}$$

This formula was first discovered by de Moivre in the following form [45]:

$$\lim_{n\to\infty} \frac{n!}{Cn^{n+1/2}e^{-n}} = 1, \tag{2.2}$$

and he gave a rational approximation of $\log C$ and Stirling's contribution was to derive explicitly $C = \sqrt{2\pi}$. Now, the question naturally arises what is the significance of (2.1) when one can easily compute $n!$ theoretically? The point is, as n became larger, we do not know how the function $n!$ really behaves. Thanks to Stirling's approximation, we have now the information that $n!$ has exponential growth; i.e., in other words, we perceive the "unknown" function $n!$ in terms of our known and familiar functions. Before moving forward, we state another deep and famous asymptotic formula. Let x be a positive real number and $\pi(x)$ denotes the number of primes not exceeding x. Based on the tables by Felkel and Vega, Legendre conjectured in 1797-1798 that

$$\lim_{x\to\infty} \frac{\pi(x)}{\left(\frac{x}{A\log x+B}\right)} = 1, \tag{2.3}$$

where A and B are unspecified constants and later in 1808, he proposed that $A = 1$ and $B = -1.08366$. The prime number theorem, originally conjectured by Gauß, and independently proved by Hadamard [74] and de la Vallée Poussin [52], states that

$$\lim_{x\to\infty} \frac{\pi(x)}{\left(x/\log x\right)} = 1. \tag{2.4}$$

For a proof of the prime number theorem without using the Riemann zeta function, we refer to Selberg's proof [131].
We observe that (2.1)-(2.4) are all limit formulas and therefore, they have little value for numerical purposes. For example, we can not draw any concrete numerical conclusion about $n!$ from (2.1) as it merely says for large n, $n!$ behaves like $\sqrt{2\pi n}n^n e^{-n}$. The limit

$$f(n) \to 1 \quad \text{as} \quad n \to \infty$$

tells that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n > N(\epsilon)$, $|f(n) - 1| < \epsilon$. Here two key factors are suppressed; (i) the order of magnitude of ϵ and (ii) exact information about $N(\epsilon)$. We can remove the factor (i) by allowing the Bachman-Landau O-notation[1] [94]. The big-O notation is defined in the usual way: $f(n) = O(g(n))$ means that there exists an absolute constant C such that $|f(n)| \leq Cg(n)$

[1]Here O stands for "Ordnung", which means "order of" in German.

as $n \to \infty$. For example, if we say $f(n) - 1 = O(1/n)$, this implies that

$$|f(n) - 1| \leq \frac{C}{n} \quad \text{as} \quad n \to \infty.$$

Now we have at least the growth of ϵ, but still factor (ii) remains open because we have no control over the constant C so as to comment explicitly on the cut-off value $N(\epsilon)$. So, whenever we have an inequality of the following form, say

$$|f(n) - 1| \leq \frac{3}{n},$$

then we can study the cut-off value for n in a systematic way. We say that $f(n)$ is asymptotically equivalent to $g(n)$ as $n \to \infty$, if the quotient $f(n)/g(n)$ for $n \to \infty$ tends to unity, denoted by $f(n) \underset{n\to\infty}{\sim} g(n)$. Our next goal is to understand the growth of $f(n)/g(n) - 1$ as $n \to \infty$ in the following manner:

$$\begin{aligned} \frac{f(n)}{g(n)} - 1 &= O\Big(\frac{1}{\chi(n)}\Big), \\ \frac{f(n)}{g(n)} - 1 - \frac{c_1}{\chi(n)} &= O\Big(\frac{1}{\chi(n)^2}\Big), \\ &\vdots \\ \frac{f(n)}{g(n)} - 1 - \sum_{i=1}^{k} \frac{c_i}{\chi(n)^i} &= O\Big(\frac{1}{\chi(n)^{k+1}}\Big), \end{aligned} \tag{2.5}$$

where $\chi(n)$ is an increasing function in n with $\chi(n) \to \infty$ as $n \to \infty$, the c_i are constants, and the truncation point k chosen at will. Equation (2.5) gives the asymptotic expansion of the sequence $f(n)$ of order k; i.e.,

$$f(n) \underset{n\to\infty}{\sim} g(n) \sum_{i=0}^{k} \frac{c_i}{\chi(n)^i}, \tag{2.6}$$

with $c_0 = 1$. One of the earliest instances to get a full asymptotic expansion for a smooth function is the Euler-Maclaurin summation formula, independently discovered by Euler [61] and Maclaurin [102]. It states that if m and n are natural numbers and $f(x)$ is a real or complex-valued smooth function over the interval $[m, n]$, then

$$\sum_{i=m}^{n} f(i) = \int_{m}^{n} f(x)\, dx + \frac{f(n) + f(m)}{2} + \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{B_{2k}}{(2k)!}\Big(f^{(2k-1)}(n) - f^{(2k-1)}(m)\Big) + R_N, \tag{2.7}$$

where B_k denotes the Bernoulli numbers defined by $\sum_{k=1}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}$, $f^{(j)}(x)$ denotes the j-th derivative of f with respect to x, and

$$R_N = (-1)^{N+1} \int_m^n f^{(N)}(x) \frac{P_N(x)}{N!} \, dx, \tag{2.8}$$

with $P_N(x) = B_N\big(x - \lfloor x \rfloor\big)$ is the periodic Bernoulli function defined by the Bernoulli polynomial $B_N(x)$ recursively defined by $B_0(x) = 1$, for all $N \geq 1$, $B'_N(x) = NB_{N-1}(x)$ and $\int_0^1 B_N(x) \, dx = 0$. Here we quote three famous examples as applications of (2.7).

$$\sum_{k=1}^{n} \frac{1}{k} \underset{n\to\infty}{\sim} \gamma + \log n + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}, \tag{E1}$$

where γ is the Euler-Mascheroni constant.

$$\sum_{k=1}^{n} \frac{1}{k^2} \underset{n\to\infty}{\sim} \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{30n^5} - \dots. \tag{E2}$$

$$\log n! \underset{n\to\infty}{\sim} n \log\Big(\frac{n}{e}\Big) + \frac{1}{2}\log n + \log\sqrt{2\pi} - \sum_{k=1}^{\infty} \frac{(-1)^k B_{k+1}}{k(k+1)n^k}. \tag{S1}$$

We note that from none of the above three examples, we can get information about the order of growth of the respective error terms because until now, we have not provided any information about the size of the error term R_N (2.8). Using Lehmer's estimate [97] on maxima of Bernoulli polynomials, it readily follows that

$$|R_N| \leq \frac{2\zeta(N)}{(2\pi)^N} \int_m^n |f^{(N)}(x)| \, dx,$$

where ζ denotes the Riemann zeta function.
In the literature on asymptotic analysis, it seems that computations of the error bound have been largely ignored. To illustrate this statement, we go back to (2.6). We can retrieve (2.6), each step given in the scheme (2.5) along with an estimation of the cut-off , if we consider the following procedure from the inequality aspect:

$$\left(1 + \frac{c'_1}{\chi(n)}\right) < \frac{f(n)}{g(n)} < \left(1 + \frac{c_1}{\chi(n)}\right) \text{ for all } n > N_1,$$

$$\left(1+\frac{c_1}{\chi(n)}+\frac{c_2'}{\chi(n)^2}\right)<\frac{f(n)}{g(n)}<\left(1+\frac{c_1}{\chi(n)}+\frac{c_2}{\chi(n)^2}\right) \text{ for all } n>N_2,$$

$$\vdots$$

$$\left(\sum_{k=0}^{M}\frac{c_k}{\chi(n)^k}+\frac{E^{\ell}_{M+1}}{\chi(n)^{M+1}}\right)<\frac{f(n)}{g(n)}<\left(\sum_{k=0}^{M}\frac{c_k}{\chi(n)^k}+\frac{E^{u}_{M+1}}{\chi(n)^{M+1}}\right) \text{ for all } n>N_{M+1}. \tag{2.9}$$

As an example, we refer to Nemes' [109] error bound computation for the asymptotic expansion of $n!$. All the results on inequalities in Part II follow the theme presented in (2.9). For a more detailed study on asymptotic analysis, we refer the reader to [51].

2.2 The partition function

In the history of the literature on partitions, Leibniz seems to be the first person to talk about integer partitions. In a 1674 letter [100, p. 37] he asked J. Bernoulli about the number of "divulsions" of an integer. In modern terminology, "divulsion" is rephrased as the number of partitions of a positive integer. Leibniz observed that there are three partitions of 3 $(3, 2+1, 1+1+1)$, five partitions of 4 $(4, 3+1, 2+2, 2+1+1, 1+1+1+1)$, seven partitions of 5 and eleven partitions of 6 and consequently it leads to a problem which is still open: are there infinitely many integers n for which the total number of partitions of n is a prime? Keeping this question aside, let us define the partition function.

Definition 2.2.1. *A partition of a positive integer n is a finite non-increasing sequence of positive integers $\pi_1, \pi_2, \ldots, \pi_r$ such that $\sum_{i=1}^{r} \pi_i = n$. The π_i are called the parts of the partition. The partition $(\pi_1, \pi_2, \ldots, \pi_r)$ will be denoted by π, and we shall write $\pi \vdash n$ to denote that π is a partition of n. The partition function $p(n)$ is the number of partitions of n. The set of all partitions of n is denoted by $P(n)$. Following the standard convention, we define $P(0) = \{\}$ and $p(0) = 1$.*

Euler undertook a rigorous and systematic investigation of the theory of partitions. Ph. Naudé [62] wrote a letter to Euler asking about the number of partitions of n with the total number of parts in each partition being m. Precisely the question of Naudé was: what is the total number of partitions of 50 into seven distinct parts? It is quite unlikely to get the total number by writing down all the partitions of 50 into seven distinct parts. To avoid this, Euler introduced the generating functions. Let $p_m(n)$ denote the number of partitions of n into m parts. Then following Euler's observation, we get

$$\sum_{m,n\geq 0} p_m(n) z^m q^n = \prod_{k=1}^{\infty}(1+zq^k) = (1+zq)\prod_{k=1}^{\infty}(1+(zq)q^k) = (1+zq)\sum_{m,n\geq 0} p_m(n) z^m q^{m+n}.$$

Comparing the coefficients of $z^m q^n$ on both sides of the above identity, we find a beautiful recursive formula

$$p_m(n) = p_m(n-m) + p_{m-1}(n-m),$$

which gives $p_7(50) = 522$. Euler proceeded further to obtain a generating function for $p(n)$. Euler's computation can be put in the following way:

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n =& (1 + q^1 + q^{1+1} + q^{1+1+1} + \dots) \\ & \times (1 + q^2 + q^{2+2} + q^{2+2+2} + \dots) \\ & \times (1 + q^3 + q^{3+3} + q^{3+3+3} + \dots) \\ & \vdots \\ =& \prod_{n=1}^{\infty} (1 + q^n + q^{n+n} + q^{n+n+n} + \dots) \\ =& \prod_{n=1}^{\infty} \frac{1}{1-q^n}.\end{aligned} \tag{2.10}$$

In order to simplifying the computations for $p(n)$, Euler realized that a power series expansion for $\prod_{n=1}^{\infty}(1-q^n)$ is essential. His empirical discovery leads to the following identity which is now known as Euler's Pentagonal Number Theorem.

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{2.11}$$

This identity was proved by Euler himself many years after the discovery. A modern exposition of Euler's proof and the importance of the theorem is beautifully described in Andrews' article [7]. Putting (2.10) and (2.11) together, we see that

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}\right) = 1, \tag{2.12}$$

and comparing the coefficients of q^n on both sides of the last identity, Euler found the following recurrence for $p(n)$: $p(0) = 1$, and for all $n \geq 1$,

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \dots = 0.$$

After Euler, the theory of partitions propagates through the works of great partition theorists like Sylvester, Cayley, Jacobi, MacMahon, Hardy, Ramanujan, Rademacher, Gordon, and Andrews among many others. The reader can consult Andrews' monumental treatise [8]. In addition to that, the entire history of partitions up to 1918 is documented in [55], and for a survey article, we refer to [72].

The counting problem for $p(n)$ (for large values of n) has been one of the most predominant themes in the literature on integer partitions. The question is how fast does $p(n)$ grow. First of all, we point out a simple fact that states: $(p(n))_{n\geq 1}$ is a strictly increasing sequence. For a partition $\pi \vdash n-1$, define a map $\phi : P(n-1) \to P(n)$ by $\phi(\pi) = (\pi, 1)$; i.e., insertion of 1 as part in π that yields a partitions of n and it is clear that ϕ is an injective map and $P(n) \setminus \phi(P(n-1))$ is the set of all partitions of n where 1 is not a part (also known as non-unitary partitions of n). One can also prove this by using Euler's generating function for $p(n)$ (2.10). Let us formulate the problem of counting $p(n)$ in terms of counting partitions of n subject to the condition that each partition has at most k parts. Let $p_{\leq k}(n)$ denotes total number of such partitions of n. Observe that for $k = n$, $p_{\leq k}(n) = p(n)$. Cayley [34] and Sylvester [139] gave a number of formulas for $p_{\leq k}(n)$ with small values of k, which was anticipated by Herschel [77]. For example, $p_{\leq 2}(n) = \lfloor (n+1)/2 \rfloor$. But still, the question on growths of $p(n)$ remains unanswered. We will come back to this question and a surprising answer in Section 2.4.

2.3 A brief introduction to modular forms

Elementary trigonometric functions play a great role in mathematics and mathematical physics since antiquity. Among many discoveries, one of the more remarkable ones was due to Euler which states that $e^{ix} = \cos x + i \sin x$, where i is an imaginary number (termed by Descartes) satisfying $i^2 = -1$, and consequently, we get $e^{i\pi} + 1 = 0$. Fourier made important contributions to the study of trigonometric series, after preliminary investigations by Euler, d'Alembert, and D. Bernoulli. In 1807, Fourier introduced a series, what is nowadays known as the "Fourier series", for the purpose of solving the heat equation in a metal plate. Roughly we can say that a Fourier series is an infinite sum that represents a periodic function as a sum of sine and cosine functions. Both sine and cosine function are periodic with period 2π; i.e., $\sin(x+2\pi) = \sin x$ and trivially, $\sin(x+2\pi k) = \sin x$ for all $k \in \mathbb{Z}$. In group theoretic language, this translates to the fact: $\sin 2\pi x$ is invariant under the action of the abelian group $(\mathbb{Z}, +)$. Therefore, the question arises which class of functions are invariant under the action of a non-abelian group? Before giving an example of such a class of functions, let us introduce a few preliminary definitions.

Define the set of complex numbers by $\mathbb{C}$ and the complex upper half plane by $\mathbb{H} := \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}$. Let $M_2(\mathbb{Z})$ be the set of 2×2 matrices with integer entries and define the general linear group of order 2 with positive determinant by

$$\mathrm{GL}_2^+(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc > 0 \right\}.$$

The non-abelian group $\left(\mathrm{GL}_2^+(\mathbb{Z}), \cdot\right)$ acts on $\mathbb{H}$ in the usual manner; i.e., for $\gamma \in \mathrm{GL}_2^+(\mathbb{Z})$ and $\tau \in \mathbb{H}$, $\gamma\tau := \dfrac{a\tau + b}{c\tau + d}$. The full modular group is defined as

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Z}) : ad - bc = 1 \right\}.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices S and T [88, Prop. 4, Chap. III] with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Definition 2.3.1. *Let k be an integer and a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called weakly modular of weight k over $SL_2(\mathbb{Z})$ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and for all $\tau \in \mathbb{H}$,*

$$f(\gamma\tau) := (c\tau + d)^k f(\tau). \tag{2.13}$$

Note that for $\gamma = T$, we have $f(\tau+1) = f(\tau)$. From the theory of complex analysis, we know that such a periodic meromorphic function admits a Fourier expansion $f(\tau) = \sum_{n\in\mathbb{Z}} a_f(n)q^n$, where $q = e^{2\pi i\tau}$ and $a_f(n)$ are called Fourier coefficients. Moreover, if f satisfies (2.13) and has a pole at $i\infty$ of order $m \in \mathbb{Z}_{\geq 0}$, then the Fourier expansion of f is of the form:

$$f(\tau) = \sum_{n=-m}^{\infty} a_f(n)q^n \quad \text{and} \quad a_f(-m) \neq 0. \tag{2.14}$$

Definition 2.3.2. *Let k be an integer and f be a meromorphic function on $\mathbb{H}$. Then f is called a weakly holomorphic modular form of weight k over $SL_2(\mathbb{Z})$ if the following hold:*

1. *f satisfies (2.13) for all $\gamma \in SL_2(\mathbb{Z})$ and all $\tau \in \mathbb{H}$.*

2. *f has a Fourier expansion of the form (2.14).*

If $k = 0$, then f is called a modular function over $SL_2(\mathbb{Z})$. We say that f is a modular form if f is holomorphic on $\widehat{\mathbb{H}} := \mathbb{H} \cup \{i\infty\}$.

Now it is clear that the modular functions are invariant under the non-abelian group $\mathrm{SL}_2(\mathbb{Z})$. For a more detailed study on modular forms, we refer to [54, 117]. For the results and definitions assumed here from complex analysis, we refer the reader to [136]. We only need Definition 2.3.2 (which will be useful in Section 2.7) so that the thesis would be self-contained.

2.4 Hardy-Ramanujan-Rademacher formula for $p(n)$

We return back to the question *what is the growth of* $p(n)$? In 1918, Hardy and Ramanujan [76] found an asymptotic series for $p(n)$. The simplest special form of their result is the assertion that,

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}, \quad \text{as} \quad n \to \infty. \tag{2.15}$$

A few years later, Uspensky [144] independently arrived at (2.15). Before jumping over directly to the main result of their paper, it is worth pointing out how Hardy and Ramanujan came up systematically with the asymptotic series for $p(n)$. In [76, eq. (2.11)], they proved that for some positive constants H and K,

$$\frac{H}{n} e^{2\sqrt{n}} < p(n) < \frac{K}{n} e^{2\sqrt{2n}} \tag{2.16}$$

holds for all $n \geq 1$. From (2.16), it became clear that $p(n)$ has exponential growth. So the obvious question arises *what is the order of the exponential growth?* In other words, determine the constant C such that

$$C = \lim_{n\to\infty} \frac{\log p(n)}{\sqrt{n}}.$$

The computation of $C = \pi\sqrt{2/3}$ is given with details in [76, Sec. 3]. Recall the generating function of $p(n)$ (2.10) that states

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} := f(q).$$

Applying the Cauchy integral formula, we have

$$p(n) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(q)}{q^{n+1}} \, dq, \tag{2.17}$$

where the path Γ encloses the origin and lies entirely inside the unit circle. Truncating the generating function (2.4) of $p(n)$, we observe that each partial porduct $f_N(q) := \prod_{n=1}^{N} \frac{1}{1-q^n}$ has a pole at $q = 1$ of order N, a pole at $q = -1$ of order $\lfloor N/2 \rfloor$, poles at $q = e^{2\pi i/3}$ and $q = e^{4\pi i/3}$ of order $\lfloor N/3 \rfloor$, and so on. Hardy and Ramanujan defined the following auxiliary function

$$F(q) := \frac{1}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \Psi(n) q^n,$$

where

$$\Psi(n) := \frac{d}{dn}\left(\frac{\cosh C\lambda_n - 1}{\lambda_n}\right),$$

$C = \pi\sqrt{2/3}$, and $\lambda_n = \sqrt{n - 1/24}$. Now the behaviour of f and F is similar inside the unit circle and in the neighbourhood of $q = 1$. Applying Cauchy's integral formula for $f - F$, they obtain the following error bound, namely:

$$p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}), \tag{2.18}$$

where $D > C/2$. Taking $n \to \infty$, (2.18) gives (2.15). But how close the formula (2.18) with real values of $p(n)$? For example, taking $n = 61, 62, 63$ (2.18) gives 1121538.672, 1300121.359, 1505535.606, whereas the the correct values are 1121505, 1300156, 1505499. So the errors alternate in sign. To explain this factor, the same principle is applied near the point -1 on the unit circle which contributes to the next dominant term in the asymptotic series expansion for $p(n)$; i.e.,

$$p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + \frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}), \tag{2.19}$$

where $D > C/3$. This process can be continued further by taking into consideration the points on the unit circle where f has singularities. For example, the singularities which are important after $q = -1$ are $q = e^{2\pi i/3}$ and $q = e^{4\pi i/3}$, and so on. The major obstacle to proceeding systematically is to construct the auxiliary functions associated with the points $q = e^{2\pi i h/k}$ of singularity lying on the unit circle. The construction is as follows:

$$F_{h,k}(q) := \omega_{h,k}\frac{\sqrt{k}}{\pi\sqrt{2}}F_{C/k}(q_{h,k}),$$

where $\omega_{h,k}$ is a 24th root of unity, $q_{h,k} = qe^{-2\pi i h/k}$; and for α being positive and independent of n,

$$\Phi(q) = f(q) - \sum_{k=1}^{\alpha\sqrt{n}} \sum_{\substack{1\le h\le k\\(h,k)=1}} F_{h,k}(q).$$

If then $F_{h,k}(q) = \sum c_{h,k,n}q^n$, we obtain from Cauchy integral formula,

$$p(n) - \sum_{k=1}^{\alpha\sqrt{n}} \sum_{\substack{1\le h\le k-1\\(h,k)=1}} c_{h,k,n} = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(q)}{q^{n+1}}\,dq, \tag{2.20}$$

where Γ is a circle of radius $R < 1$ and its center is the origin. By dissecting the circle Γ by means of Farey series[2] and computing the bounds of the integral on the right-hand side of (2.20), Hardy and Ramanujan finally proved that the error term is of order $O(1/n^4)$. The final form of their formula for $p(n)$ can be stated as follows.

Theorem 2.4.1. *For all sufficiently large values of n,*

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{N} \sqrt{k} A_k(n) \frac{d}{dn}\Big(\frac{e^{C\lambda_n/k}}{\lambda_n}\Big) + O(n^{-1/4}), \tag{2.21}$$

where

$$N = \alpha\sqrt{n} \quad \text{and} \quad A_k(n) = \sum_{\substack{1 \le h < k \\ (h,k)=1}} e^{-2\pi inh/k + \pi i s(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1} \Big(\frac{\mu}{k} - \Big\lfloor \frac{\mu}{k} \Big\rfloor - \frac{1}{2}\Big)\Big(\frac{h\mu}{k} - \Big\lfloor \frac{h\mu}{k} \Big\rfloor - \frac{1}{2}\Big).$$

To know in detail about the contributions of both Hardy and Ramanujan, we refer to [101, 132] where the reader might be enhanced to get different opinions from the two eminent number theorists. We end this discussion by quoting further two instances for verifying (2.21) with the actual values of n. MacMahon[3] computed values of $p(n)$ for $1 \le n \le 200$. The actual values for

$$p(100) = 190569292 \quad \text{and} \quad p(200) = 3972999029388,$$

whereas if taking the first six terms of (2.21) for $n = 100$ and first eight terms of (2.21) for $n = 200$ gives

$$p(100) \approx 190569291.996 \quad \text{and} \quad p(200) \approx 3972999029388.004.$$

This proves the accuracy of the formula which is, needless to say, astounding and beautiful. In [76, Sec. 6, 6.22], they remarked that it remains unanswered whether the infinite series (by extending $n \to \infty$ in (2.21)) is convergent or divergent and if it is convergent, then whether it represents $p(n)$. Lehmer [96] proved that (2.21) is divergent when $N \to \infty$.

[2] This gives birth of the celebrated “Circle Method”

[3] See the table [76, p. 377-378]

In the fall of 1936, Rademacher [124] improved the formula (2.21) so that a convergent infinite series was found for $p(n)$, namely,

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}\right)}{\left(x - \frac{1}{24}\right)^{1/2}} \right]_{x=n}. \tag{2.22}$$

Rademacher [124] also proved that if the series (2.22) is truncated after N terms, the absolute value of the error is bounded by

$$\frac{2\pi^2}{9\sqrt{3N}} e^{\frac{\pi}{N+1}\sqrt{2n/3}},$$

which tends to 0 as $N \to \infty$. If we truncate the series (2.22) at N and compare it with (2.21), it clearly shows two significant differences between them:

1. In (2.21), the parameters n and N are entangled whereas in (2.22), we have the complete freedom over n and N.

2. The exponential function in (2.21) is replaced by the sine hyperbolic function in (2.22) which made the series convergent.

Selberg [132, p. 705] came up with the same formula (2.22) for $p(n)$ around the same time but never published his result when he came to know that Rademacher already had it. Lehmer [99] made a significant improvement on the bounds of the absolute value of the error term obtained by Rademacher before. After Rademacher's contribution in deriving convergent series for $p(n)$, Hagis, Lehner, Zuckerman among many others adapted his refinements using Ford circles [123] to get similar convergent series for Fourier coefficients of certain weakly holomorphic modular forms and related functions.

2.5 Real rootedness of polynomials

> "The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics can be viewed as problems of location of the zeros of certain polynomials and in giving these zeros a combinatorial interpretation"
>
> Gian-Carlo Rota

The study of the roots of polynomials dates back to the Sumerians (third millennium B.C.) and has deeply influenced the development of mathematics throughout the centuries. This study has motivated the introduction of some fundamental concepts of mathematics such as irrational and complex numbers, and Galois theory to name a few, which has substantially influenced the earlier development of numerical computing. In this section, we restrict ourselves to study on polynomials with non-negative coefficients which have only real roots.
Let $f(x) = \sum_{k=1}^{n} a_k x^k$ be a polynomial of degree n with non-negative coefficients. We say that $f(x)$ is real-rooted if all its zeros are real. Real-rooted polynomials have attracted much attention during the past decades. One of the earliest instances of studying relations between the real-rootedness of a polynomial with its coefficients dates back to Newton's theorem stated below.

Theorem 2.5.1. *[75, p. 52] Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree n with $a_k \in \mathbb{R}$ for all $0 \leq k \leq n$ and all roots of $p(x)$ are real. Then $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k < n$.*

A positive sequence $(a_k)_{0\leq k\leq n}$ satisfying

$$a_k^2 \geq a_{k-1}a_{k+1} \tag{2.23}$$

for all $1 \leq k < n$ is called logarithmically concave or log-concave[4] in short. For example, the binomial coefficient $\left(\binom{n}{k}\right)_{1\leq k<n}$ is log-concave because

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.$$

[4] Also called "Turán inequalities", named after Pál Turán [143] who found certain inequalities for Legendre polynomials which was first published by Gábor Szegö [140] in 1948

Stanley [135] and Brenti [31] have written extensive surveys of various techniques that can be used to prove the real–rootedness of polynomials and log-concavity. Now the question is why all of a sudden we use the terminology "log-concave" for a positive sequence $(a_k)_{0\le k<n}$ satisfying (2.23)? To answer this question, let us define the notion of log-concavity of a function. A strictly positive function $f:\mathbb{R}\to\mathbb{R}$ is called log-concave if it satisfies the inequality

$$f\big(tx+(1-t)y\big)\ge f(x)^t f(y)^{1-t} \tag{2.24}$$

for all $x,y\in\mathbb{R}$ and $t\in[0,1]$. Observe that the sums and products of log-concave functions are also log-concave. Now we show that a polynomial $p(x)\in\mathbb{R}[x]$ whose all roots are real is a log-concave function. A polynomial $p(x)$ having all roots real can be expressed as

$$p(x)=a_n\prod_{i=1}^{d}(x-\alpha_i)^{m_i}:=a_n\prod_{i=1}^{d}p_i(x)^{m_i},$$

where $p_i(x)=x-\alpha_i$, $\sum_{i=1}^{d}m_i=n$, and α_i are the roots with multiplicity m_i for all $1\le i\le d$. Define $\alpha:=\max\limits_{1\le i\le d}\{\alpha_i\}$ and so for all $x>\alpha$, $p_i(x)>0$ for all $1\le i\le d$ and consequently, $p(x)>0$. To show that $p(x)$ is log-concave function for $x\in\mathbb{R}\setminus\alpha$, it remains to show that $p_i(x)$ is log-concave. By the weighted AM-GM inequality, we have

$$tx+(1-t)y-\alpha_i=t(x-\alpha_i)+(1-t)(y-\alpha_i)\ge(x-\alpha_i)^t(y-\alpha_i)^{1-t},$$

and therefore each $p_i(x)$ is log-concave. So, the polynomial $p(x)$ is log-concave. According to the assumption in Theorem 2.5.1, the polynomial $p(x)=\sum_{k=0}^{n}a_kx^k$ has only real roots and we have already shown that such a polynomial is a log-concave function. Now Theorem 2.5.1 tells that the coefficient sequence $(a_k)_{0\le k\le n}$ of $p(x)$ satisfies the inequality $a_k^2\ge a_{k-1}a_{k+1}$ for all $1\le k<n$ and such a property is called log-concave because the polynomial $p(x)$ is a log-concave function.

The notion of log-concavity can be further generalized in the following way. Consider the operator $\mathcal{L}$ defined on a sequence $\mathcal{A}:=(a_n)_{n\ge0}\subset\mathbb{R}_{>0}$ by $\mathcal{L}(\mathcal{A}):=\mathcal{A}_1:=(b_n)_{n\ge0}$ with

$$b_0=a_0^2\ \text{ and }\ b_n=a_n^2-a_{n-1}a_{n+1},\ \text{for } n\ge1.$$

Hence a sequence $\mathcal{A}$ is log-concave if and only if $\mathcal{L}(\mathcal{A})$ is a non-negative sequence. A sequence is k-log-concave if j-fold applications of $\mathcal{L}$ on $\mathcal{A}$, denoted by $\mathcal{L}^j(\mathcal{A})$, is a non-negative sequence for all $0\le j\le k$. A sequence is called infinitely log-concave if it is k-log-concave for all $k\ge1$. Brändén [29] proved that the sequence of binomial coefficients $\left(\binom{n}{k}\right)_{0\leq k\leq n}$ is infinitely log-concave for all $n \geq 0$ which was conjectured by Boros and Moll [28]. Considering the application of $\mathcal{L}$ on a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ in the following way

$$\mathcal{L}(p) := \sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1})x^k,$$

Brändén [29] proved the following generalized version of Theorem 2.5.1.

Theorem 2.5.2. *If $p(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial with real and non-positive zeros only, then so is $\mathcal{L}(p)$. In particular, the sequence $(a_k)_{0\leq k\leq n}$ is infinitely* log-*concave.*

This theorem was independently conjectured by Fisk [63] and MacNamara-Sagan [106]. Similar notions for higher order log-concavity were given by Craven and Csordas [48, 47].

Now let us look back at the premises of Theorems 2.5.1 and 2.5.2. To be precise, in both Theorems 2.5.1 and 2.5.2, we assumed that all those polynomials are real-rooted. But here two questions turn up. (1) How do we know a priori whether a $p(x) \in \mathbb{R}[x]$ of degree n has all roots real or not? (2) What is the importance of studying the previous question in mathematics? First, we proceed with the second question. Let $p_1(x) := a_{n+1}x + a_n$ be a polynomial of degree 1 with its coefficient sequence $(a_n)_{n\geq 0}$ of real numbers. Then $p_1(x)$ has always one real root. Consider the polynomial $p_2(x) := a_{n+2}x^2 + 2a_{n+1}x + a_n$ of degree 2 with coefficients $(a_n)_{n\geq 0}$ of real numbers and observe that $p_2(x)$ has two real roots if and only if the sequence $(a_n)_{n\geq 0}$ is log-concave. The necessary and sufficient condition for real-rootedness of the polynomial $p_3(x) := a_{n+3}x^3 + a_{n+2}x^2 + a_{n+1}x + a_n$ is that the coefficient sequence $(a_n)_{n\geq 0}$ satisfies the following inequality

$$4(a_{n+1}^2 - a_n a_{n+2})(a_{n+2}^2 - a_{n+1}a_{n+3}) \geq (a_{n+1}a_{n+2} - a_n a_{n+3})^2, \tag{2.25}$$

and we say $(a_n)_{n\geq 0}$ satisfies the higher order Turán inequalities[5]. The inequalities (2.23) and (2.25) share a deep connection with a certain class of functions, known as Laguerre-Pólya class. A real entire function[6] $\psi(x) = \sum_{n\geq 0}^{\infty} \frac{a_n}{n!} x^k$ is said to be in the Laguerre-Pólya class, denoted by $\psi(x) \in \mathcal{LP}$, if it has the following form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{x_n}\right) e^{-x/x_n}, \tag{2.26}$$

[5]Sometimes it is also stated as "Turán inequality of order 2

[6]An entire function (analytic on $\mathbb{C}$) is said to be real if it takes real values on the real axis

where c, β, x_n are real numbers, $\alpha \geq 0$, m is a non-negative integer, and the series $\sum_{n=1}^{\infty} x_n^{-2}$ is convergent. For example, consider the real entire function $\sin x$. We know that the sine function over $\mathbb{C}$ has the following infinite product representation[7]:

$$\sin x = x \prod_{n=1}^{\infty} \Big(1 - \frac{x^2}{\pi^2 n^2}\Big).$$

Taking $c = m = 1$, $\alpha = \beta = 0$, and $x_n = -\pi^2 n^2$, we have

$$\sin x = x \lim_{N \to \infty} \prod_{n=1}^{N} \Big(1 - \frac{x^2}{\pi^2 n^2}\Big) e^{-x/\pi^2 n^2},$$

with $\sum_{n \geq 1}^{\infty} (\pi n)^{-4} < \infty$. This proves that $\sin x \in \mathcal{LP}$. For a more detailed study on $\mathcal{LP}$ class of functions, we refer to [125]. Pólya and Schur [130] proved that the Maclaurin coefficient sequence $(a_n)_{n \geq 0}$ of $\psi(x) \in \mathcal{LP}$ is log-concave. Dimitrov [56] showed that for $\psi(x) \in \mathcal{LP}$, its coefficient sequence $(a_n)_{n \geq 0}$ satisfies the higher order Turán inequalities, which was first observed by Pólya and Schur [130]. Based on notes of Jensen [80], Pólya and Schur [130] showed that the Riemann hypothesis[8] is equivalent to say that $(-1+4z^2)\Lambda\Big(\frac{1}{2}+z\Big) \in \mathcal{LP}$, where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s)$, and ζ denotes the Riemann zeta function. In the language of roots of polynomials, Pólya [121] proved that the Riemann hypothesis is equivalent to the real-rootedness of the Jensen polynomial associated with the Taylor coefficients $(\gamma_n)_{n \geq 0}$ defined by

$$J_{\gamma}^{d,n}(x) := \sum_{j=0}^{d} \gamma_{n+j} x^j \tag{2.27}$$

for all positive integers n, and d, where $(\gamma_n)_{n \geq 0}$ is defined by

$$\xi\Big(\frac{1}{2} + z\Big) := \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^{2n}.$$

The Riemann ξ-function is defined as $\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Recently, Griffin, Ono, Rolen, and Zagier [69] proved that $J_{\gamma}^{d,n}(x)$ has only real roots for all positive integers d and for all sufficiently large n and based on this work, Griffin et. al. [70] provided an estimate for the cut-off $N(d) = ce^d$ (for some positive real

[7]See [136, P. 142, eq. (3)]

[8]For a detailed and rudimentary discussion on the Riemann hypothesis, we refer to [136, Chap. 6-7]

number c) so that for all $n \geq N(d)$, $J_\gamma^{d,n}(x)$ has all real roots. We put an end to this discussion by introducing another family of inequalities called Laguerre inequalities. A polynomial $p(x)$ is said to satisfy the Laguerre inequality if

$$p'(x)^2 - p(x)p''(x) \geq 0. \tag{2.28}$$

Laguerre [93] proved that if $p(x)$ is a polynomial with only real zeroes, then $p(x)$ satisfies (2.28). In 1913, Jensen [80] found an nth generalization of the Laguerre inequality

$$L_n(p(x)) := \frac{1}{2}\sum_{k=0}^{2n}(-1)^{n+k}\binom{2n}{k}p^{(k)}(x)p^{(2n-k)}(x) \geq 0, \tag{2.29}$$

where $p^{(k))}(x)$ denotes the kth derivative of $p(x)$. For $n = 1$, we get back (2.28). Equivalence of the Riemann hypothesis in the context of higher-order Laguerre inequalities can be found in [49].

Finally, we conclude this section by discussing necessary and sufficient conditions for the real-rootedness of a polynomial $p(x)$. If $p(x)$ is a polynomial of degree 2 or 3, then non-negativity of the discriminant of $p(x)$ (denoted by $\mathrm{Disc}_x(p)$) is enough to show that $p(x)$ has two real roots or three real roots respectively. But for a polynomial, say $p(x)$ of degree $d \geq 4$, $\mathrm{Disc}_x(p) \geq 0$ is not sufficient enough to prove that $p(x)$ has only real roots. On the other hand, if $p(x)$ is real rooted, then $\mathrm{Disc}_x(p) \geq 0$. Let $p(x) := a_d x^d + \cdots + a_0$ be a polynomial of degree d and $\alpha_1, \ldots, \alpha_d$ are the roots of $p(x)$. Hermite [113] proved that $p(x)$ has all roots real if and only if

$$\det\begin{pmatrix} S_0 & S_1 & \ldots & S_{m-1} \\ S_1 & S_2 & \ldots & S_m \\ \vdots & \vdots & & \vdots \\ S_{m-1} & S_m & \ldots & S_{2m-2} \end{pmatrix} \geq 0$$

for all $2 \leq m \leq d$, where $S_k = \alpha_1^k + \cdots + \alpha_d^k$.

2.6 State of the art: inequalities for $p(n)$

In this section, we present only those inequalities for the partition function which fit into the discussions in Section 2.5.

Independently, Nicolas [111] and DeSalvo-Pak [53] proved that $(p(n))_{n\geq 26}$ is log-concave. DeSalvo and Pak [53, Thm. 4.1-4.2] proved the following two companion inequalities

$$\frac{p(n-1)}{p(n)}\Big(1+\frac{1}{n}\Big) > \frac{p(n)}{p(n+1)} \quad \text{for all } n \geq 2,$$

and

$$\frac{p(n-1)}{p(n)}\Big(1+\frac{240}{(24n)^{3/2}}\Big) > \frac{p(n)}{p(n+1)} \quad \text{for all } n \geq 7.$$

Chen, Wang, and Xie [39, Sec. 2] proved that for all $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}\Big) > \frac{p(n)}{p(n+1)},$$

conjectured by DeSalvo and Pak [53, Conj. 4.3]. DeSalvo and Pak [53, Thm. 5.1] also proved the strong log-concavity of $(p(n))_{n\geq 1}$ which states that for all $n > m > 1$, we have

$$p(n)^2 > p(n-m)p(n+m).$$

By extending the partition function to a multiplicative function on partitions, Bessenrodt and Ono [26, Thm. 2.1] obtained for any two positive integers a and b with $a+b > 8$, then

$$p(a)p(b) \geq p(a+b),$$

where equality holds only for $\{a,b\} = \{2,7\}$. Hou and Zhang [79, Thm. 4.1] proved that for any positive integer r, there exists an integer $N(r) > 0$ such that $(p(n))_{n\geq N(r)}$ is r-log-concave and consequently, proved that $(p(n))_{n\geq 221}$ is 2-log-concave. A determinantal approach for proving 2-log-concavity for $(p(n))_{n\geq 221}$ was due to Jia and Wang [82].

In 2019, Chen, Jia, and Wang [37, Thm. 1.3-1.4] showed that $(p(n))_{n\geq 95}$ satisfies the higher order Turán inequality and proposed that the Jensen polynomial associated with $p(n)$

$$J_p^{d,n}(x) = \sum_{j=0}^{d} \binom{n}{j} p(n+j)x^j$$

has only real roots for all $d \geq 4$ and $n > N(d)$, where $N(d)$ is a positive integer depending on the parameter d. For fixed degree d and for sufficiently large n, Griffin, Ono, Rolen, and Zagier [69, Thm. 5] proved that $J_p^{d,n}(x)$ has only distinct and real roots and moreover, conjectured the minimal values for $N(d)$ with $N(d) \approx 10d^2 \log d$. Later Larson and Wagner [95, Thm. 1.1 and 1.3] computed that $N(3) = 94, N(4) = 206, N(5) = 381$, and proved that for all $d \geq 1$, $N(d) \leq (3d)^{24d}(50d)^{3d^2}$. In [95, Thm 1.2], they proved a companion inequality related to the higher order Turán inequality for $p(n)$, which states that for $u_n := p(n+1)p(n-1)/p(n)^2$ and for all $n \geq 2$,

$$4(1-u_n)(1-u_{n+1}) < \Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\Big)(1-u_nu_{n+1})^2,$$

conjectured by Chen-Jia-Wang [37]. Wang and Yang [149, Thm. 2.1] initiated the study on Laguerre inequalities for $p(n)$. They proved that $(p(n))_{n\geq 184}$ satisfies the Laguerre inequality of order 2. Wagner [145] proved that $p(n)$ satisfies all the higher order Laguerre inequalities of order m as $n \to \infty$ and proposed a conjecture on the $(N(m))_{3\leq m\leq 10}$ such that $(p(n))_{n\geq N(m)}$ satisfies the Laguerre inequality of order m. Dou and Wang [58] gave an explicit bound $(N(m))_{3\leq m\leq 10}$ and consequently, confirmed the case $m = 3$ and 4 of Wanger's conjecture.
Let us conclude with the real rootedness property of Jensen polynomials associated with a broader class of sequences. The Hermite polynomial $H_d(x)$ is defined by the generating function [69, eq. 3]

$$\sum_{d=0}^{\infty} H_d(x)\frac{t^d}{d!} = e^{-t^2+xt} = 1 + xt + (x^2-2)\frac{t^2}{2!} + (x^3-6x)\frac{t^3}{3!} + \dots.$$

Due to Griffin, Ono, Rolen, and Zagier [69], we have the following result.

Theorem 2.6.1. *Let $(\alpha_n)_{n\geq 0}$, $(A_n)_{n\geq 0}$, and $(\delta_n)_{n\geq 0}$ be three sequences of positive real numbers with $\delta_n \underset{n\to\infty}{\to} 0$ and satisfying*

$$\log\Big(\frac{\alpha_{n+j}}{\alpha_n}\Big) = A_nj - \delta_n^2 j^2 + o\big(\delta_n^d\big) \quad \text{as } n \to \infty, \tag{2.30}$$

for some integer $d \geq 1$ and all $0 \leq j \leq d$. Then we have

$$\lim_{n\to\infty}\left(\frac{\delta_n^{-d}}{\alpha_n}J_\alpha^{d,n}\left(\frac{\delta_n x - 1}{e^{A_n}}\right)\right) = H_d(x), \tag{2.31}$$

uniformly for x in any compact subset of $\mathbb{R}$.

Since the Hermite polynomials $H_d(x)$ has only distinct and real roots [141], the normalized Jensen polynomial $J_\alpha^{d,n}\left(\frac{\delta_n x - 1}{e^{A_n}}\right)$ associated with the sequence $(\alpha_n)_{n\geq 0}$

has also only distinct and real roots. Following [69, Thm. 7], while having a Hardy-Ramanujan-Rademacher type series expansion for a weakly holomorphic modular from, say f, we can choose $\alpha_n = a_f(n)$, where $(a_f(n))_{n\geq 0}$ are Fourier coefficients of f and consequently, determine an asymptotic expansion of the form

$$\log\Big(\frac{a_f(n+j)}{a_f(n)}\Big) = A_f(n)j - \delta_f^2(n)j^2 + o\big(\delta_f^d(n)\big) \ \text{ as } n \to \infty.$$

For example, we can choose $(a_f(n))_{n\geq 0}$ arising from the Dedekind eta-quotients f considered by Sussman [138] and Chern [41].

2.7 Modified Bessel function of first kind

The theory of ordinary differential equations is one of the frequently celebrated topics in the history of mathematics. Among many others, one such equation is Bessel's equation over the real domain, defined by

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{2.32}$$

where ν is an arbitrary parameter. The solutions of this equation are termed the Bessel functions. The theory of Bessel functions traces its root by sharing an intimate connection with Riccati's equation, named after Jacopo Riccati. Riccati investigated certain curves whose radii of curvature are functions only of their ordinates and subsequently came up with a non-linear first-order differential equation of the form

$$y' + ay^2 = bx^\alpha, \tag{2.33}$$

where a and b are constants and α is not necessarily an integer. In 1694 John Bernoulli came up with

$$y' = x^2 + y^2, \tag{2.34}$$

which is a particular form of (2.33), but was unable to solve it. James Bernoulli, the older brother of John found a solution of (2.34) in the form of an infinite power series in x. This solution appears to be the earliest example in the mathematical paradigm of a result reducible to Bessel functions. These functions sailed off its journey along the line of physical problems primarily related to mechanics, astronomy, and the conduction of heat appeared in works of D. Bernoulli, Euler, Lagrange, Fourier, and Poisson among many others. Bessel functions of the first kind, denoted as $J_\nu(x)$, are solutions of (2.32). For integer or positive real ν, $J_\nu(x)$ is finite at $x = 0$ and for negative non-integer ν, $J_\nu(x)$ diverges as $x \to 0$. The series expansion of $J_\nu(x)$ around $x = 0$ is given by:

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}, \tag{2.35}$$

where Γ denotes the gamma function defined by:

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}\,dt \quad \text{for } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0.$$

Extending the function $J_\nu(x)$ for the complex arguments z introduces the theory of modified Bessel functions denoted by $I_\nu(z)$ and $K_\nu(z)$ which are two linearly independent solutions of

$$z^2y'' + zy' - (z^2 + \nu^2)y = 0, \tag{2.36}$$

the so-called modified version of (2.32). $I_\nu(z)$ is called the modified Bessel function of the first kind and its series representation is

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{2.37}$$

By ratio test, one can easily show that the radius of convergence of the series on the right hand side of (2.37) is infinite. In 1854 Kirchhoff [86] established an asymptotic expansion of $I_\nu(z)$: for fixed $\nu \in \mathbb{C}$,

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 - \frac{4\nu^2-1}{8z} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8z)^2} - \dots\right), \; |\arg z| < \frac{\pi}{2}. \tag{2.38}$$

Inequalities related to $I_\nu(x)$ with $x \in \mathbb{R}_{>0}$ have been studied to prove its several properties like monotonicity, Turán inequalities, etc. But estimation of error bounds for the asymptotic expansion of $I_\nu(x)$ (2.38) has not been found until the work of Bringmann, Kane, Rolen, and Tripp [32] appeared recently. Bringmann et. al. [32, Lemma 2.2 (4)] proved that for $\nu \geq 2$ and $x \geq \frac{1}{120}(\nu+\frac{7}{2})^6$,

$$\left|\frac{I_\nu(x)\sqrt{2\pi x}}{e^x} - 1 + \frac{4\nu^2-1}{8x} - \frac{(4\nu^2-1)(4\nu^2-9)}{128x^2} + \frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3072x^3}\right| < \frac{31\nu^8}{6x^4}. \tag{2.39}$$

Jia [81, Thm. 2.1] considered the truncation point $N = 5$ and proved that for all $\nu \geq 2$ and $x \geq \frac{1}{120}(\nu+\frac{7}{2})^6$,

$$\begin{aligned}
&\left|\frac{I_\nu(x)\sqrt{2\pi x}}{e^x} - \sum_{i=0}^{5}(-1)^i \frac{\prod_{j=0}^{i}\left(-j+\nu-\frac{1}{2}+1\right)\left(j+\nu+\frac{1}{2}-1\right)}{(2x)^i}\right| \\
&\leq \frac{52e^{-x}}{17\Gamma(\nu+\frac{1}{2})}\sum_{i=0}^{5}\left|\binom{\nu-\frac{1}{2}}{i}\right|\frac{x^{\nu-\frac{1}{2}}}{2^i} + \frac{e^{-x}x^{\nu+\frac{1}{2}}}{2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})} + \\
&\qquad \left|\frac{\prod_{i=0}^{5}\left(\nu^2 - \frac{(2i+1)^2}{4}\right)}{6!2^{\nu-\frac{1}{2}}x^6}\right| \max\left\{2^{\nu-\frac{13}{2}}, 1\right\}.
\end{aligned} \tag{2.40}$$

For a more detailed study of the theory of modified Bessel functions, we refer the reader to Watson's monumental treatise [150].

Part II

Asymptotic inequalities

Chapter 3

New inequalitites for the partition function and logarithm of the partition function

Let $p(n)$ denote the number of partitions of n. A new infinite family of inequalities for $p(n)$ is presented. This generalizes a result by William Chen et al. From this infinite family, another infinite family of inequalities for $\log p(n)$ is derived. As an application of the latter family one, for instance obtains that for $n \geq 120$,

$$p(n)^2 > \Big(1 + \frac{\pi}{\sqrt{24} n^{3/2}} - \frac{1}{n^2}\Big) p(n-1) p(n+1).$$

3.1 On the asymptotic growth of $p(n)$

We denote by $p(n)$ the number of partitions of n. The first 50 values of $p(n)$ starting from $n = 0$ read as follows,

```
1,1,2,3,5,7,11,15,22,30,42,56,77,101,135,176,231,297,385,490,
627,792,1002,1255,1575,1958,2436,3010,3718,4565,5604,6842,
8349,10143,12310,14883,17977,21637,26015,31185,37338,44583,
53174,63261,75175,89134,105558,124754,147273,173525.
```

A well-known asymptotic formula for $p(n)$ was found by G.H. Hardy and Srinivasa Ramanujan [76] in 1918 and independently by James Victor Uspensky in 1920 [144]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}}. \tag{3.1}$$

An elementary proof of (3.1) was given by Paul Erdős [60] in 1942. At MICA 2016 (Milestones in Computer Algebra) held in Waterloo in July 2016, Zhenbing Zeng et al. [133] reported that using numerical analysis they found a better asymptotic formula[1] for $p(n)$ by searching for constants $C_{i,j}$ to fit the following formula,

$$\log p(n) = \pi\sqrt{\frac{2}{3}}\sqrt{n} - \log n - \log(4\sqrt{3}) + \frac{C_{0,-1}}{\log n} + \frac{C_{1,0}}{\sqrt{n}} + \frac{C_{1,-1}}{\sqrt{n}\log(n)} + \frac{C_{2,1}\log n}{n} + \frac{C_{2,0}}{n} + \cdots . \tag{3.2}$$

By substituting for $n = 2^{10}, 2^{11}, \ldots, 2^{20}$ into (3.2) they obtained,

$$C_{0,-1} = 0, \quad C_{1,0} = -0.4432\ldots, \quad C_{1,-1} = 0, \quad C_{2,1} = 0, \quad C_{2,0} = -0.0343\ldots.$$

The OEIS [134] for A000041 shows that a similarly refined asymptotic formula for $p(n)$ was discovered by Jon E. Schoenfield in 2014, this reads

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\cdot\left(\frac{2n}{3} + c_0 + \frac{c_1}{\sqrt{n}} + \frac{c_2}{n} + \frac{c_3}{n\sqrt{n}} + \frac{c_4}{n^2} + \ldots\right)^{\frac{1}{2}}}, \tag{3.3}$$

where the coefficients are approximately

$$c_0 = -0.230420\ldots, \quad c_1 = -0.017841\ldots, \quad c_2 = 0.005132\ldots,$$

$$c_3 = -0.001112\ldots, \quad c_4 = 0.000957\ldots,$$

Later Vaclav Kotesovec according to OEIS [134] for A000041 got the precise value of $c_0, c_1, \ldots, c_4$ as follows:

$$c_0 = -\frac{1}{36} - \frac{2}{\pi^2}, \quad c_1 = \frac{1}{6\sqrt{6}\pi} - \frac{\sqrt{6}}{2\pi^3}, \quad c_2 = \frac{1}{2\pi^4},$$

$$c_3 = -\frac{5}{16\sqrt{6}\pi^3} + \frac{3\sqrt{6}}{8\pi^5}, \quad c_4 = \frac{1}{576\pi^2} - \frac{1}{24\pi^4} + \frac{93}{80\pi^6}.$$

To the best of our knowledge, the details of the methods of Schoenfield and Kotesovec have not yet been published.

[1] In the literature, the Hardy-Ramanujan-Rademacher is also called an asymptotic formula/approximation. However, it is built by an expression of substantially more complicated type. For example, the log concavity of $p(n)$ follows nontrivially from it, as shown in the work of DeSalvo and Pak [53].

In this chapter, using symbolic-numeric computation, we present our method to derive (3.2) together with a closed form formula for the $C_{i,j}$ in (3.2). Namely we show that

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log n - \log 4\sqrt{3} + \sum_{u=1}^{\infty} \frac{g_u}{\sqrt{n}^u},$$

where the g_u are as in Definition 3.5.1. By $\sim$ in the above expression we mean that for each $N \geq 1$

$$\log p(n) = \pi\sqrt{\frac{2n}{3}} - \log n - \log 4\sqrt{3} + \sum_{u=1}^{N-1} \frac{g_u}{\sqrt{n}^u} + O_N(n^{-N/2}).$$

In particular $C_{i,j} = 0$, if $j \neq 0$, and $C_{i,0} = g_i$, otherwise. This result is obtained as a consequence of an infinite family of inequalities for $\log p(n)$, Theorem 3.6.6 (main theorem). We also apply our method to conjecture an analogous formula to (3.2) for $a(n)$, the cubic partitions of n, with $a(n)$ given by

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{2n})}. \tag{3.4}$$

In the OEIS, this sequence is registered as A002513. The first 50 values of $a(n)$, $n \geq 0$, are

```
1,1,3,4,9,12,23,31,54,73,118,159,246,329,489,651,940,
1242,1751,2298,3177,4142,5630,7293,9776,12584,16659,
21320,27922,35532,46092,58342,75039,94503,120615,
151173,191611,239060,301086,374026,468342,579408.
```

This sequence appears in a letter from Richard Guy to Morris Newman [73]. In [38], William Chen and Bernard Lin proved that the sequence $a(n)$ satisfies several congruence properties. For example, $a(3n+2) \equiv 0 \pmod 3$, $a(25n+22) \equiv 0 \pmod 5$. An asymptotic formula for $a(n)$ was obtained by Kotesovec [91] in 2015 as follows:

$$a(n) \sim \frac{e^{\pi\sqrt{n}}}{8n^{5/4}}. \tag{3.5}$$

In [155] the fourth author investigated the combinatorial properties of the sequence $a(n)$ by using Maple.

We summarize some of our main results:

Theorem 3.1.1. *For the usual partition function $p(n)$ we have*

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log n - \log 4\sqrt{3} - \frac{0.44\ldots}{\sqrt{n}}, \ n \to \infty. \tag{3.6}$$

The proof of this theorem will be given in Section 3.6.

Conjecture 3.1.2. *For the cubic partitions $a(n)$ we have*

$$\log a(n) \sim \pi\sqrt{n} - \frac{5}{4}\log n - \log 8 - \frac{0.79\ldots}{\sqrt{n}}, \ n \to \infty. \tag{3.7}$$

Theorem 3.1.3. *For the partition numbers $p(n)$ we have the inequalities*

$$\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1-\frac{1}{2\sqrt{n}}\Big) < p(n) < \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1-\frac{1}{3\sqrt{n}}\Big), \ n \geq 1.$$

The proof of this is given in Section 3.3.

This chapter is organized as follows. In Section 3.2 we present the methods used in the mathematical experiments that led us Theorem 3.1.1 and Conjecture 3.1.2. In Section 3.3 we prove Theorem 3.1.3 by adapting methods used by Chen et al. to fit our purpose. In Section 3.4 we generalize an inequality by Chen et al. by extending it to an infinite family of inequalities for $p(n)$. In Section 3.5 we introduce preparatory results required to prove Theorem 3.6.6. In Section 3.6 we prove our main result, Theorem 3.6.6, by using the main result from Section 3.4, Theorem 3.4.4. This gives an infinite family of inequalities for $\log p(n)$. Finally in Section 3.7 we give an application of the results in Section 3.5 which extends DeSalvo's and Pak's log concavity theorem for $p(n)$. In Section 3.8 (the Appendix) we give additional information on the method used to discover the asymptotic formulas. We remark explicitly that to finalize the proof of Theorem 3.6.6, we use the Cylindrical Algebraic Decomposition in Mathematica; the details of this are also put to Section 3.8.

3.2 Mathematical experiments for better asymptotics for $a(n)$ and $p(n)$

Before proving our theorems, in this section we briefly describe the experimental mathematics which led us to their discovery. Our strategy is as follows. If we have sufficiently many instances of a given sequence, how can we find an asymptotic formula for this sequence? Take the cubic partitions $a(n)$ and the partition numbers $p(n)$ as examples.

We have

$$p(10) = 42, \ldots, p(100) = 190569292, \ldots, p(1000) = 24061467864032622473692149727991,$$
$$a(10) = 118, \ldots, a(100) = 16088094127, \ldots,$$
$$a(1000) = 30297813107652163371961415787616527927 6.$$

A plot of the two curves through the points $(n, a(n))$, resp. $(n, p(n))$, for $n \in \{1, \ldots, 1000\}$ are shown in the Fig. 3.1(a) and 3.1(b). According to the Hardy-Ramanujan Theorem 3.1 and the asymptotic formula of Kotesovec (3.5), the curves are increasing with "sub-exponential" speeds. Thus, we may plot two curves using data points $(n, \log a(n))$ and $(n, \log p(n))$ as in Fig. 3.1(c). One observes that the new curves look like parabolas $y = \sqrt{x}$. This is also very natural in view of,

$$\log p(n) \sim \sqrt{\frac{2}{3}}\pi \cdot \sqrt{n} - \log n - \log 4\sqrt{3},$$

$$\log a(n) \sim \pi \cdot \sqrt{n} - \frac{5}{4} \cdot \log n - \log 8. \tag{3.8}$$

So if we modify further with $(\sqrt{n}, \log a(n))$ and $(\sqrt{n}, \log p(n))$ to plot the curves, we get two almost-straight lines as shown in the Fig. 3.1(d).

This provides the starting point for finding the improved asymptotic formulas (3.6) for $p(n)$ and (3.7) for $a(n)$ from their data sets. We restrict our description to the latter case. Motivated by (3.8), we compute the differences of $\log a(n)$ with the estimation values $a_e(n) := \frac{e^{\pi\sqrt{n}}}{8n^{5/4}}$:

$$\Delta(n) := \log a_e(n) - \log a(n) = \pi\sqrt{n} - \frac{5}{4}\log n - \log 8 - \log a(n).$$

Then we can plot curves from the data points $(n, \Delta(n))$ in Fig. 3.2(a) and 3.2(b), and $(n, n \cdot \Delta(n))$ and $(n, \sqrt{n} \cdot \Delta(n))$ in Fig. 3.2(c) and 3.2(d), in order to confirm the next dominant term approximately. We can see in Fig. 3.2(d) that after multiplying $\Delta(n)$ by $\sqrt{n}$ the curve is almost constant, confirming that the next term is $\frac{C}{\sqrt{n}}$. Also multiplying $\Delta(n)$ by n, in Fig. 3.2(c) we see that the behaviour is like $\sqrt{n}$ as expected. By using least square regression on the original data set $(n, a(n))$ for $1 \leq n \leq 10000$, we aimed at finding the best constant C that minimizes [2]

$$-\log a(n) + \alpha \cdot \sqrt{n} - \beta \cdot \log n - \log \gamma + \frac{C}{\sqrt{n}},$$

where we fixed $\alpha = \pi, \beta = 5/4, \gamma = 8$ according to (3.5). As a result, we obtained that $C \approx 0.7925$.

In the Appendix, Section 3.8, we explain that the constants α, β, γ can also be found via regression analysis with MAPLE instead of getting them from (3.5) directly.

[2] The fourth author of the paper [22] told the result to V. Kotesovec in May 2016 and got a reply in January 2017 that the precise value of C could be `Pi/16+15/(8*Pi)=0.7931...`

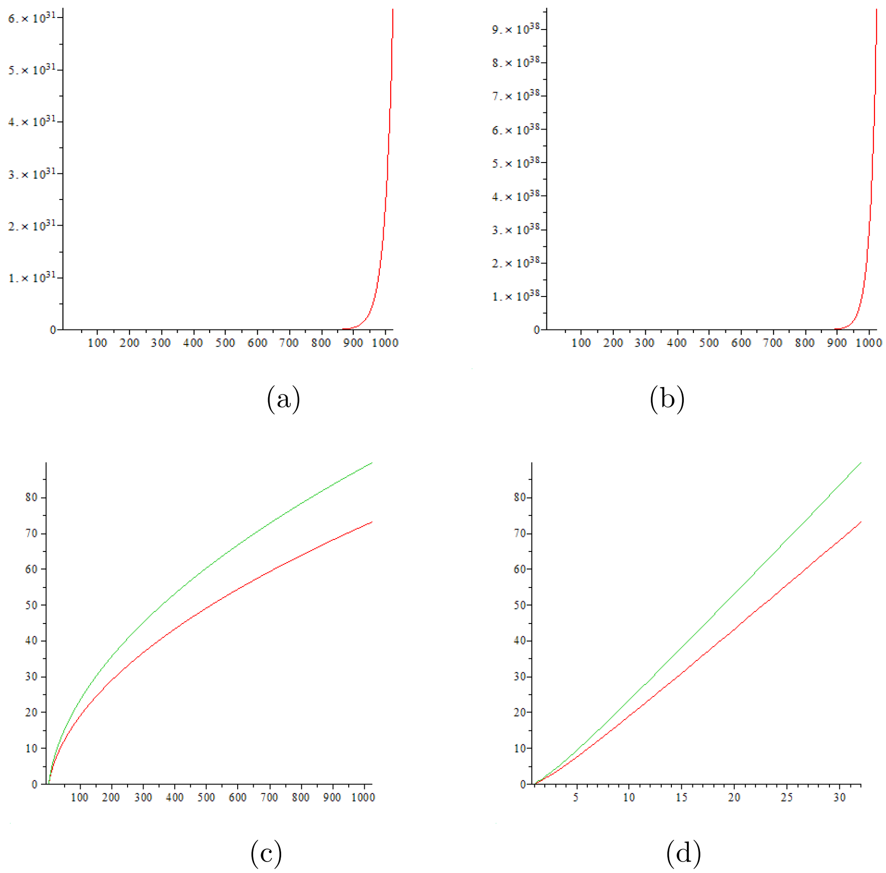

Figure 3.1: In (a) $p(n)$ is plotted and in (b) $a(n)$ is plotted. In (c) the upper curve is $\{(n, \log a(n))|1 \leq n \leq 1000\}$, and the lower curve is $\{(n, \log p(n))|1 \leq n \leq 1000\}$. The two curves are like the parabola $y = \sqrt{x}$. In (d) the two lines are for $\{(\sqrt{n}, \log a(n))|1 \leq n \leq 1000\}$ (upper) and $\{(\sqrt{n}, \log p(n))|1 \leq n \leq 1000\}$ (lower).

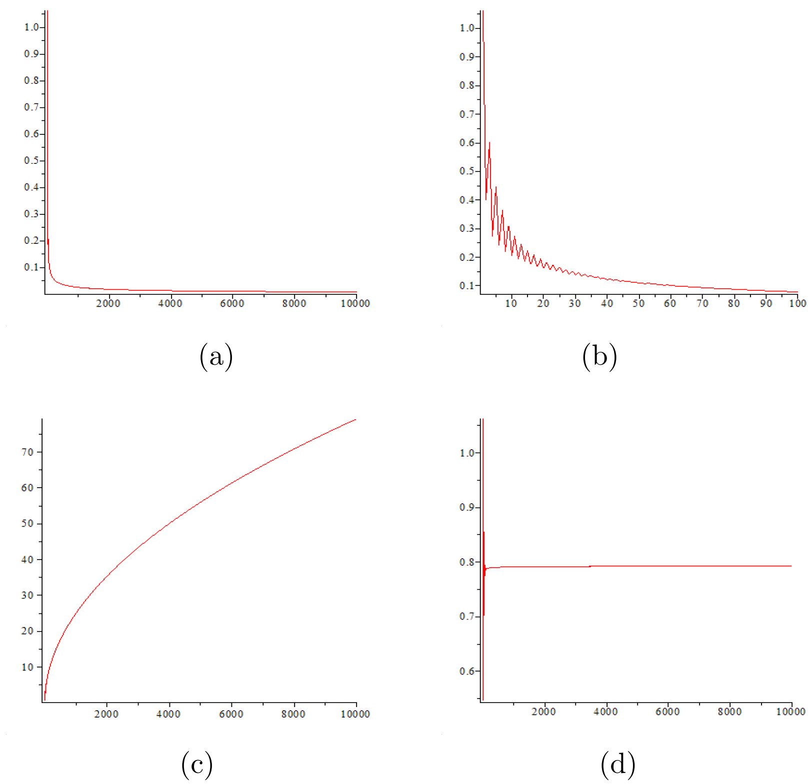

Figure 3.2: The curve in (a) is for $(n, \Delta(n))$ where $1 \leq n \leq 10000$, (b) is for $(n, \Delta(n))$ where $1 \leq n \leq 100$. The curve in (c) is for $(n, n \cdot \Delta(n))$, and (d) is for $(n, \sqrt{n} \cdot \Delta(n))$ where $1 \leq n \leq 10000$.

3.3 An inequality for $p(n)$

We separate the proof into two lemmas. The first lemma is the upper bound for $p(n)$ and second lemma is the lower bound. In order to prove these lemmas we will state several facts which are routine to prove.

Lemma 3.3.1. *For all $n \geq 1$, we have*

$$p(n) < \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1 - \frac{1}{3\sqrt{n}}\Big).$$

Proof. By [37, (2.7)-(2.8)] and with $A_k(n)$ and $R(n, N)$[3] as defined there, we have,

$$p(n) = \frac{\sqrt{12}}{24n-1}\sum_{k=1}^{N}\frac{A_k(n)}{\sqrt{k}}\Big[\Big(1 - \frac{k}{\mu(n)}\Big)e^{\frac{\mu(n)}{k}} + \Big(1 + \frac{k}{\mu(n)}\Big)e^{-\frac{\mu(n)}{k}}\Big] + R(n, N),\ n \geq 1,$$

where

$$\mu(n) := \frac{\pi}{6}\sqrt{24n-1}.$$

We will exploit the case $N = 2$ together with $A_1(n) = 1$ and $A_2(n) = (-1)^n$ for any positive integer n. For $N \geq 1$, Lehmer [98, (4.14), p. 294] gave the following error bound:

$$|R(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}}\Big[\Big(\frac{N}{\mu(n)}\Big)^3 \sinh\frac{\mu(n)}{N} + \frac{1}{6} - \Big(\frac{N}{\mu(n)}\Big)^2\Big],\ n \geq 1, \tag{3.9}$$

and for $N = 2$ (cf. [37, (2.9)-(2.10)]):

$$p(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)} + T_1(n)\Big),\ n \geq 1, \tag{3.10}$$

where

$$T_1(n) := \frac{(-1)^n}{\sqrt{2}}\Big(\Big(1 - \frac{2}{\mu(n)}\Big)e^{-\frac{\mu(n)}{2}} + \Big(1 + \frac{2}{\mu(n)}\Big)e^{-\frac{3\mu(n)}{2}}\Big) + \Big(1 + \frac{1}{\mu(n)}\Big)e^{-2\mu(n)} + \frac{(24n-1)R(n,2)}{\sqrt{12}e^{\mu(n)}}.$$

[3]Note that in [37] $R(n, N)$ is denoted by $R_2(n, N)$.

We first estimate the absolute value of $T_1(n)$; for convenience we denote subexpressions by a_1, b_1, c_1 and d_1:

$$|T_1(n)| \leq \underbrace{\frac{1}{\sqrt{2}}\Big(1-\frac{2}{\mu(n)}\Big)e^{-\frac{\mu(n)}{2}}}_{=:a_1} + \underbrace{\frac{1}{\sqrt{2}}\Big(1+\frac{2}{\mu(n)}\Big)e^{-\frac{3\mu(n)}{2}}}_{=:b_1} + \underbrace{\Big(1+\frac{1}{\mu(n)}\Big)e^{-2\mu(n)}}_{=:c_1} + \underbrace{\Big|\frac{(24n-1)R(n,2)}{\sqrt{12}e^{\mu(n)}}\Big|}_{=:d_1}.$$

The following facts are easily verified.

Fact A. *For all* $n \geq 1$, $a_1 < e^{-\frac{\mu(n)}{2}}$.

Fact B. *For all* $n \geq 1$, $b_1 < e^{-\frac{\mu(n)}{2}}$.

Fact C. *For all* $n \geq 1$, $c_1 < e^{-\frac{\mu(n)}{2}}$.

Now,

$$\begin{aligned} d_1 =& \frac{36}{\pi^2\sqrt{12}}\frac{\mu(n)^2}{e^{\mu(n)}}|R(n,2)| \\ <& \frac{\mu(n)^2 e^{-\mu(n)}}{2^{2/3}} + \frac{12\sqrt[3]{2}e^{-\frac{\mu(n)}{2}}}{\mu(n)} - \frac{12\sqrt[3]{2}e^{-\frac{3\mu(n)}{2}}}{\mu(n)} - 12\sqrt[3]{2}e^{-\mu(n)} \quad \text{(by (3.9))} \\ <& \underbrace{\frac{\mu(n)^2 e^{-\mu(n)}}{2^{2/3}}}_{=:d_1^*} + \underbrace{\frac{12\sqrt[3]{2}e^{-\frac{\mu(n)}{2}}}{\mu(n)}}_{=:d_2^*}. \end{aligned}$$

Fact D. *For all* $n \geq 7$, $d_1^* < e^{-\frac{\mu(n)}{2}}$.

Fact E. *For all* $n \geq 35$, $d_2^* < e^{-\frac{\mu(n)}{2}}$.

By Fact D and Fact E, we have

Fact F. $d_1 = d_1^* + d_2^* < 2e^{-\frac{\mu(n)}{2}}$ *for all* $n \geq 35$.

Now, by Facts A, B, C and Fact F we conclude that for all $n \geq 35$,

$$|T_1(n)| \leq a_1 + b_1 + c_1 + d_1 < 5e^{-\frac{\mu(n)}{2}}. \tag{3.11}$$

By (3.11), we have for all $n \geq 35$ that

$$1 - \frac{1}{\mu(n)} - 5e^{-\frac{\mu(n)}{2}} < 1 - \frac{1}{\mu(n)} + T_1(n) < 1 - \frac{1}{\mu(n)} + 5e^{-\frac{\mu(n)}{2}}. \tag{3.12}$$

Fact G. *For all* $n \geq 3$, $1 - \frac{1}{\mu(n)} - 5e^{-\frac{\mu(n)}{2}} > 0$.

Therefore from (3.10), (3.12) and Fact G, we have for all $n \geq 35$,

$$p(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)} + T_1(n)\Big) < \underbrace{\frac{\sqrt{12}e^{\mu(n)}}{24n-1}}_{=:e_1}\underbrace{\Big(1 - \frac{1}{\mu(n)} + 5e^{-\frac{\mu(n)}{2}}\Big)}_{=:f_1}. \quad (3.13)$$

Fact H. $f_1 < 1 - \frac{1}{3\sqrt{n}}$ *for all* $n \geq 23$.

Fact I. $e_1 < \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$ *for all* $n \geq 1$.

Therefore by Facts H, I and (3.13) we have for all $n \geq 35$,

$$p(n) < \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}\Big(1 - \frac{1}{3\sqrt{n}}\Big).$$

This completes the proof of the stated upper bound in Lemma 3.3.1. □

Lemma 3.3.2. *For all* $n \geq 1$,

$$\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}\Big(1 - \frac{1}{2\sqrt{n}}\Big) < p(n). \quad (3.14)$$

Proof. In the proof of [53, Prop 2.4], it is noted that for all $n \geq 1$,

$$p(n) > T_2(n)\Big(1 - \frac{|R(n)|}{T_2(n)}\Big),$$

where

$$T_2(n) := \frac{\sqrt{12}}{24n-1}\Big[\Big(1 - \frac{1}{\mu(n)}\Big)e^{\mu(n)} + \frac{(-1)^n}{\sqrt{2}}e^{\frac{\mu(n)}{2}}\Big]$$

and $R(n)$ is as in [53, (7)].

From the definition of $T_2(n)$ one verifies:

Fact J. $T_2(n) > 0$ *for all* $n \geq 1$.

The following bound holds for $|R(n)|$ (see [53, (13)]),

$$0 < \frac{|R(n)|}{T_2(n)} < e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}, \ n \geq 2.$$

Hence by Fact J,

$$T_2(n)\Big(1-\frac{|R(n)|}{T_2(n)}\Big) > T_2(n)\Big(1-e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}\Big),\ n\geq 2. \tag{3.15}$$

Plugging the definition of $T_2(n)$ into (3.15) gives for $n\geq 2$,

$$\begin{aligned} p(n) >& \frac{\sqrt{12}}{24n-1}\Big[\underbrace{\Big(1-\frac{1}{\mu(n)}\Big)}_{=:a_2}e^{\mu(n)}+\frac{(-1)^n}{\sqrt{2}}e^{\frac{\mu(n)}{2}}\Big]\underbrace{(1-e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}})}_{=:d_2} \\ >& \frac{\sqrt{12}}{24n}e^{\pi\sqrt{\frac{2n}{3}}}\Big[a_2\times\underbrace{e^{\mu(n)-\frac{\pi}{6}\sqrt{24n}}}_{=:b_2}+\underbrace{\frac{(-1)^n}{\sqrt{2}}e^{\frac{\mu(n)}{2}-\frac{\pi}{6}\sqrt{24n}}}_{=:c_2}\Big]\times d_2 \\ =& \frac{1}{4\sqrt{3}n}e^{\pi\sqrt{\frac{2n}{3}}}(a_2b_2+c_2)d_2. \end{aligned}$$

We now bound a_2, b_2, c_2, and d_2:

Fact K. $a_2 > 1-\frac{2}{5\sqrt{n}} > 0$ *for all* $n\geq 1$.

Fact L. $b_2 > 1-\frac{2}{37\sqrt{n}} > 0$ *for all* $n\geq 1$.

Fact M. $c_2 > -\frac{1}{225\sqrt{n}}$ *for all* $n\geq 29$.

Fact N. $d_2 > 1-\frac{1}{25\sqrt{n}} > 0$ *for all* $n\geq 631$.

By Facts K, L and M we have,

Fact O. $a_2b_2+c_2 > (1-\frac{2}{5\sqrt{n}})(1-\frac{2}{37\sqrt{n}})-\frac{1}{225\sqrt{n}} > 0$ *for all* $n\geq 1$.

From Facts O and N we have for all $n\geq 631$,

$$(a_2b_2+c_2)d_2 > \underbrace{\Big[\Big(1-\frac{2}{5\sqrt{n}}\Big)\Big(1-\frac{2}{37\sqrt{n}}\Big)-\frac{1}{225\sqrt{n}}\Big]\Big(1-\frac{1}{25\sqrt{n}}\Big)}_{=:I(n)}.$$

Fact P. $I(n) > 1-\frac{1}{2\sqrt{n}} > 0,\quad$ *for all* $n\geq 1$.

From all the above facts we can conclude that (3.14) holds for all $n\geq 631$. Using Mathematica we checked that (3.14) also holds for all $1\leq n\leq 630$. This concludes the proof of Lemma 3.3.2. □

Finally, combining Lemma 3.3.1 and Lemma 3.3.2, we have Theorem 3.1.3.

3.4 A generalization of a result by Chen, Jia, and Wang

In this section we have again that $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$; this should not be confused with the real variable μ which we will use below. Eventually, we will set the real variable μ equal to $\mu(n)$. The main goal of this section is to generalize [37, Lem. 2.2] which says that for $n \geq 1206$, we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{10}}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{10}}\Big).$$

Our improvement is Theorem 3.4.4 where we replace the 10 in this formula by k and the 1206 by a parametrized bound $g(k)$. In order to achieve this, for a fixed k one needs to find an explicit constant $\nu(k) \in \mathbb{R}$ such that $\frac{1}{6}e^{\mu/2} > \mu^k$ for all $\mu \in \mathbb{R}$ with $\mu > \nu(k)$. One can show that

$$\tilde{\nu}(k) := \min\Big\{h \in \mathbb{R}\Big|\forall_{\mu\in\mathbb{R}}\Big(\mu > h \Rightarrow \frac{1}{6}e^{\mu/2} > \mu^k\Big)\Big\}$$

satisfies $\frac{1}{6}e^{\tilde{\nu}(k)/2} = \tilde{\nu}(k)^k$. Theorem 3.4.4 is crucial for proving our main result, Theorem 3.6.6, presented in the next section. In Lemma 3.4.1 we find such a constant $\nu(k)$ for all $k \geq 7$. In Lemma 3.4.2 we find a lower bound on $\tilde{\nu}(k)$. In this way, we see that what is delivered by Lemma 3.4.1, is best possible in the sense that our $\nu(k)$ from Lemma 3.4.1 and the minimal possible $\tilde{\nu}(k)$ satisfies $|\nu(k) - \tilde{\nu}(k)| < \frac{3k\log\log k}{\log k}$ for all $k \geq 7$.

Lemma 3.4.1. *For $k \geq 7$ let*

$$\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \frac{5k\log\log k}{\log k},$$

then

$$\frac{1}{6}\cdot e^{\nu(k)/2} > \nu(k)^k, \qquad k \geq 7. \tag{3.16}$$

Moreover, if $\mu > \nu(k)$ for some $k \geq 7$, then

$$\frac{1}{6}\cdot e^{\mu/2} > \mu^k, \qquad k \geq 7. \tag{3.17}$$

Proof. Let $f(\mu) := -\log 6 + \mu/2 - k\log\mu$. By $f'(\mu) = 1/2 - k/\mu$, f is increasing for $\mu > 2k$. Hence the fact $\nu(k) > 2k$ gives $f(\mu) > f(\nu(k))$, and (3.17) follows from (3.16) which is equivalent to $f(\nu(k)) > 0$, $k \geq 7$. We set

$$\overline{\nu}(k) := -1 + \frac{\nu(k)}{2k\log k} = \frac{\log 6}{k\log k} + \frac{\log 2}{\log k} + \frac{\log\log k}{\log k} + \frac{5\log\log k}{2(\log k)^2}.$$

The positivity of $f(\nu(k))$ is shown as follows:

$$\begin{aligned}
f(\nu(k)) &= -\log 6 + \nu(k)/2 - k\log(2k\log k) - k\log(1+\overline{\nu}(k)) \\
&= \frac{5k\log\log k}{2\log k} - k\log(1+\overline{\nu}(k)) \\
&> k\Big(\frac{5\log\log k}{2\log k} - \overline{\nu}(k)\Big) \qquad (\text{by } \log(1+x) < x \text{ for } 0 < x) \\
&= \frac{k}{2\log k}\Big(3\log\log k - \frac{2\log 6}{k} - 2\log 2 - \frac{5\log\log k}{\log k}\Big) \\
&> \frac{k}{2\log k}\Big(3\log\log k - \frac{1}{5} - \frac{7}{5} - 2\Big) = \frac{k}{2\log k}\Big(3\log\log k - \frac{18}{5}\Big).
\end{aligned}$$

The last inequality holds for all $k \geq 18$, because for such k:

$$\frac{2\log 6}{k} < \frac{1}{5}, \frac{5\log\log k}{\log k} < 2, \text{ and } 2\log 2 < \frac{7}{5}.$$

It is also straight-forward to prove $\log\log k > 6/5$ for all $k \geq 28$. For the the remaining cases $7 \leq k \leq 27$ the inequality (3.16) is verified by numerical computation, which completes the proof of Lemma 3.4.1. □

Lemma 3.4.2. *For $k \geq 7$ let*

$$\kappa(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \frac{2k\log\log k}{\log k}.$$

Then we have

$$\frac{1}{6}e^{\kappa(k)/2} < \kappa(k)^k.$$

Proof. Let f defined as in Lemma 3.4.1, then the statement is equivalent to proving that

$$f(\kappa(k)) = -\log 6 + \frac{\kappa(k)}{2} - k\log\kappa(k) < 0.$$

Setting

$$\tilde{\kappa}(k) := -1 + \frac{\kappa(k)}{2k\log k} = \frac{\log(6)}{k\log k} + \frac{\log 2}{\log k} + \frac{\log\log k}{\log k} + \frac{\log\log k}{(\log k)^2}$$

we observe that

$$\begin{aligned}
f(\kappa(k)) &= -\log 6 + \kappa(k)/2 - k\log(2k\log k) - k\log(1+\tilde{\kappa}(k)) \\
&= \frac{2k\log\log k}{2\log k} - k\log(1+\tilde{\kappa}(k)) \\
&< \frac{k\log\log k}{\log k} - k(\tilde{\kappa}(k) - \tilde{\kappa}(k)^2/2),
\end{aligned}$$

because of $\log(1+x) > x - x^2/2$ for $x \in \mathbb{R}_{>0}$.

In order to show $f(\kappa(k)) < 0$, it would be enough therefore to show that $2\Big(\tilde{\kappa}(k) - \frac{\log\log k}{\log k}\Big) > \tilde{\kappa}^2$ below. We have

$$2\frac{\log 6 \log k + (\log 2)k \log k + k \log\log k}{k(\log k)^2} > \Big(\frac{\log 6 \log k + (\log 2)k \log k + k(\log\log k)\log k + k \log\log k}{k(\log k)^2}\Big)^2,$$

which is equivalent to the inequality

$$2\log k\Big(\frac{\log 6}{k} + \log 2 + \frac{\log\log k}{\log k}\Big) > (\log\log k)^2\Big(\frac{\log 6}{k\log\log k} + \frac{\log 2}{\log\log k} + 1 + \frac{1}{\log k}\Big)^2.$$

Since

$$2\log k\Big(\frac{\log 6}{k} + \log 2 + \frac{\log\log k}{\log k}\Big) > (2\log 2)\log k > \frac{5}{4}\log k, \;\; k \geq 3,$$

it suffices to show

$$\frac{5}{4}\log k > (\log\log k)^2\Big(\frac{\log 6}{k\log\log k} + \frac{\log 2}{\log\log k} + 1 + \frac{1}{\log k}\Big)^2,$$

which after division by $(\log\log k)^2$ gives the equivalent inequality

$$\frac{5}{4}\frac{\log k}{(\log\log k)^2} > \Big(\frac{\log 6}{k\log\log k} + \frac{\log 2}{\log\log k} + 1 + \frac{1}{\log k}\Big)^2.$$

Now note that $\frac{\log k}{(\log\log k)^2}$ is increasing and the right-hand side of the above inequality is decreasing for $k \geq \lceil e^{e^2} \rceil = 1619$. Evaluating both sides at $k = e^{e^2}$ gives $\frac{5}{4}\frac{e^2}{4} > \frac{23}{10}$ for the left, and $\Big(1 + \frac{1}{e^2} + \frac{\log 2}{2} + \frac{\log 6}{2e^{e^2}}\Big)^2 < \frac{22}{10}$ for the right side. This proves the inequality for $k \geq 1619$. For $7 \leq k \leq 1618$ the result follows by numerical evaluation. $\square$

Definition 3.4.3. *For $k \geq 2$ define*

$$g(k) := \frac{3}{2\pi^2}(\nu(k)^2 + 1),$$

where $\nu(k)$ is as in Lemma 3.4.1.

Theorem 3.4.4. *For all $k \geq 2$ and $n > g(k)$ such that $(n,k) \neq (6,2)$ we have*

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^k}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^k}\Big). \tag{3.18}$$

Proof. From [37, p. 8, (2.9)] we find that

$$p(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+T(n)\Big) \text{ for } n \geq 1,$$

where $T(n)$ is defined in [37, (2.10)]. In [37, (2.22)] it is proven that

$$|T(n)| < 6e^{-\frac{\mu(n)}{2}} \text{ for } n > 350. \tag{3.19}$$

By Lemma 3.4.1 we have that $\mu(n)^k < \frac{1}{6}e^{\frac{\mu(n)}{2}}$ for $k \geq 7$ and $\mu(n) > \nu(k)$, which is equivalent to

$$6e^{-\frac{\mu(n)}{2}} < \frac{1}{\mu(n)^k}, \text{ for } \mu(n) > \nu(k). \tag{3.20}$$

Since $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$, it follows that $\mu(n) > \nu(k)$ if and only if $n > g(k)$. Furthermore for $k \geq 7$, we have $g(k) > 350$, this means that (3.19) is satisfied for $n > g(k)$.

By (3.19) and (3.20) we obtain that $|T(n)| < \frac{1}{\mu(n)^k}$ for $n > g(k)$ which proves that statement for $k \geq 7$. To prove the statement for $k \in \{2,\ldots,6\}$ we use the statement for $k = 7$ which says that for all $n \geq \lceil g(7) \rceil = 581$ we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^7}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^7}\Big).$$

However

$$p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^7}\Big) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^k}\Big) \tag{3.21}$$

for $k \in \{2,\ldots,6\}$ and $n \geq 581$. To prove (3.21) for $g(k) < n < 581$ it is enough to do a numerical evaluation of (3.21) for these values of n with the exception $n = 6$ when $k = 2$. We did this using computer algebra. Analogously, we see that for $k \in \{2,\ldots,6\}$ and $n \geq 581$ we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^k}\Big) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^7}\Big) < p(n). \tag{3.22}$$

In the same way we prove (3.22) for $g(k) < n < 581$.

□

3.5 Set up

In this section we prepare for the proof of our main theorem, Theorem 3.6.6, which is presented in Section 3.6. To this end, we need to introduce a variety of lemmas.

Definition 3.5.1. *For* $y \in \mathbb{R}$, $0 < y^2 < 24$, *we define*

$$G(y) := -\log\left(1 - \frac{y^2}{24}\right) + \frac{\pi}{6y}\sqrt{24}\left(\sqrt{1 - \frac{y^2}{24}} - 1\right) + \log\left(1 - \frac{y}{\frac{\pi}{6}\sqrt{24 - y^2}}\right),$$

and its sequence of Taylor coefficients by

$$\sum_{u=1}^{\infty} g_u y^u := G(y).$$

Definition 3.5.2. *For* $0 < y^2 < 24$ *and* $i \in \{-1, 1\}$, *define*

$$G_{i,k}(y) := G(y) + \log\left(1 + \frac{i\left(\frac{y}{\frac{\pi}{6}\sqrt{24-y^2}}\right)^k}{1 - \frac{y}{\frac{\pi}{6}\sqrt{24-y^2}}}\right).$$

Lemma 3.5.3. *Let* $g(k)$ *be as in Definition 3.4.3. Then for all* $k \geq 2$ *and* $n > g(k)$ *with* $(k, n) \neq (2, 6)$ *we have*

$$-\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + G_{-1,k}(1/\sqrt{n}) < \log p(n) <$$
$$-\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + G_{1,k}(1/\sqrt{n}).$$

Proof. Taking log of both sides of (3.18) gives

$$\log E_{-1,k}(n) < \log p(n) < \log E_{1,k}(n)$$

where

$$E_{i,k}(n) := \log\sqrt{12} - \log(24n - 1) + \mu(n) + \log\left(1 - \frac{1}{\mu(n)} + \frac{i}{\mu(n)^k}\right).$$

Now

$$
\begin{aligned}
E_{i,k}(n) = &\log \frac{\sqrt{12}}{24} - \log n - \log\left(1 - \frac{1}{24n}\right) + \pi\sqrt{\frac{2n}{3}} + \mu(n) \\
&- \frac{\pi}{6}\sqrt{24n} + \log\left(1 - \frac{1}{\mu(n)} + \frac{i}{\mu(n)^k}\right) \\
= &- \log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + R_{i,k}(n),
\end{aligned}
$$

where

$$
R_{i,k}(x) := -\log\left(1 - \frac{1}{24x}\right) + \mu(x) - \frac{\pi}{6}\sqrt{24x} + \log\left(1 - \frac{1}{\mu(x)} + \frac{i}{\mu(x)^k}\right).
$$

Finally one verifies that $R_{i,k}(x) = G_{i,k}(1/\sqrt{x})$. □

The quantity

$$
\alpha := \frac{\pi^2}{36 + \pi^2}
$$

will play an important role in this and the next section.

Lemma 3.5.4. *Let $G(y) = \sum_{u=1}^{\infty} g_u y^u$ be the Taylor series expansion of $G(y)$ as in Definition 3.5.1. Then*

$$
g_{2n} = \frac{1}{3^n 2^{3n} n} - \frac{1}{2^{3n+1} 3^n n}\left(-1 + \frac{1}{\alpha^n}\right), \ n \geq 1, \tag{3.23}
$$

and for $n \geq 0$,

$$
g_{2n+1} = \sqrt{6}\left[(-1)^{n+1}\binom{1/2}{n+1}\frac{\pi}{2^{3n+3}3^{n+2}} - \frac{1}{2^{3n+1}3^n\alpha^n(1+2n)\pi}\sum_{j=0}^{n}\alpha^j\binom{-\frac{1}{2}+j}{j}\right]. \tag{3.24}
$$

Proof. By using

$$
\log\left(1 - \frac{y}{\frac{\pi}{6}\sqrt{24 - y^2}}\right) = -\sum_{k=1}^{\infty} y^k k^{-1}\pi^{-k}6^k 24^{-k/2}\left(1 - \left(\frac{y}{\sqrt{24}}\right)^2\right)^{-k/2},
$$

together with

$$
\left(1 - \left(\frac{y}{\sqrt{24}}\right)^2\right)^{-k/2} = \sum_{n=0}^{\infty}(-1)^n\binom{-k/2}{n}\left(\frac{y}{\sqrt{24}}\right)^{2n},
$$

we obtain

$$g_{2n} = \frac{1}{3^n 2^{3n} n} - \sum_{u=0}^{n-1} \frac{1}{3^{2u-n} 2^{n+2u} \pi^{2n-2u} (2n-2u)} (-1)^u \binom{u-n}{u}, \ n \geq 1.$$

For $n \geq 0$,

$$\begin{aligned} g_{2n+1} =& \sqrt{6} \Big[(-1)^{n+1} \binom{1/2}{n+1} \frac{\pi}{2^{3n+3} 3^{n+2}} \\ & - \sum_{u=0}^{n} \frac{1}{3^{2u-n} 2^{n+1+2u} \pi^{2n+1-2u} (2n+1-2u)} (-1)^u \binom{u-n-1/2}{u} \Big]. \end{aligned}$$

Inputting this into the package Sigma developed by Carsten Schneider [128], we obtain (3.23) and (3.24). □

We need various additional facts about the Taylor coefficients g_u of $G(y)$.

Lemma 3.5.5. *For* $0 \leq a < 1$,

$$\frac{a}{2} \leq \sum_{j=1}^{n} a^j \binom{j-1/2}{j} \leq \frac{a}{2(1-a)}.$$

Proof. First we note that $\binom{j-1/2}{j} = (-1)^j \binom{-\frac{1}{2}}{j} > 0$. Hence

$$\begin{aligned} \sum_{j=1}^{n} a^j \binom{j-\frac{1}{2}}{j} &= \sum_{j=1}^{n} (-a)^j \binom{-\frac{1}{2}}{j} = \sum_{j=0}^{n} (-a)^j \binom{-\frac{1}{2}}{j} - 1 \\ &< \sum_{j=0}^{\infty} (-a)^j \binom{-\frac{1}{2}}{j} - 1 = \frac{1}{\sqrt{1-a}} - 1 \leq \frac{a}{2(1-a)}. \end{aligned}$$

This proves the upper bound. To prove the lower bound note that the first term of the sum is $\frac{a}{2}$ and the other terms are all positive. □

Lemma 3.5.6. *Let* $s_n := (-1)^n \binom{1/2}{n+1}$. *For* $n \geq 0$ *we have* $s_n \geq 0$ *and* s_n *is a decreasing sequence, that is* $s_n > s_{n+1}$ *for all* $n \geq 0$.

Lemma 3.5.7. *For* $n \geq 0$ *we have*

$$-\frac{\sqrt{6}}{2\pi 2^{3n} 3^n \alpha^n (1+2n)} \left(1 + \frac{\alpha}{2}\right) \geq g_{2n+1} \geq -\frac{\sqrt{6}}{2\pi 2^{3n} 3^n \alpha^n (1+2n)} \left(\frac{\pi^2}{72} + 1 + \frac{\alpha}{2(1-\alpha)}\right).$$

Proof. From Lemma 3.5.4, Lemma 3.5.5 and Lemma 3.5.6 we obtain

$$-\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}\Big(1+\frac{\alpha}{2}\Big) \geq g_{2n+1}.$$

Again by Lemma 3.5.4, Lemma 3.5.5 and Lemma 3.5.6 we have:

$$\begin{aligned} g_{2n+1} \geq &- \frac{\sqrt{6}}{2^{3n}3^n}\Big(\frac{\pi}{72}(-1)^{0+1}\binom{1/2}{0+1} + \frac{1+\frac{\alpha}{2(1-\alpha)}}{2\pi\alpha^n(1+2n)}\Big) \\ =& - \frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}\Big(\frac{\pi^2\alpha^n(1+2n)}{72} + 1 + \frac{\alpha}{2(1-\alpha)}\Big) \\ \geq& - \frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(1+2n)}\Big(\frac{\pi^2\alpha^0(1+2\cdot 0)}{72} + 1 + \frac{\alpha}{2(1-\alpha)}\Big). \end{aligned}$$

The last line is because $\alpha^n(1+2n)$ is a decreasing sequence of n for $n \geq 0$. □

Lemma 3.5.8. *For $n \geq 1$ we have*

$$-\frac{1}{3^n 2^{3n+1} n\alpha^n} \leq g_{2n} \leq \frac{1}{3^n 2^{3n} n\alpha^n}\Big(\frac{3\alpha}{2} - \frac{1}{2}\Big).$$

Proof. By Lemma 3.5.4 the statement follows from

$$g_{2n} = \frac{1}{3^n 2^{3n} n} - \frac{1}{2^{3n+1}3^n n}\Big(-1+\frac{1}{\alpha^n}\Big) = \frac{1}{3^n 2^{3n}\alpha^n n}\Big(\frac{3\alpha^n}{2} - \frac{1}{2}\Big).$$

□

Lemma 3.5.9. *Define*

$$\mu_1 := \frac{\sqrt{6}}{2\pi}\Big(\frac{\pi^2}{72} + 1 + \frac{\alpha}{2(1-\alpha)}\Big) \quad \textit{and} \quad \mu_2 := \frac{\sqrt{6}}{2\pi}\Big(1+\frac{\alpha}{2}\Big).$$

Then for $m \geq 0$ and $0 < y \leq \epsilon < 2\sqrt{6\alpha}$,

$$-\frac{\mu_2}{2^{3m}3^m\alpha^m(1+2m)}y^{2m+1} \geq \sum_{n=m}^{\infty} g_{2n+1}y^{2n+1} \geq -\frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}}y^{2m+1}.$$

Proof. By Lemma 3.5.7 we have

$$\begin{aligned} \sum_{n=m}^{\infty} g_{2n+1}y^{2n+1} \geq &- \mu_1\sum_{n=m}^{\infty}\frac{1}{2^{3n}3^n\alpha^n(1+2n)}y^{2n+1} \geq -\frac{\mu_1 y^{2m+1}}{1+2m}\sum_{n=0}^{\infty}\frac{1}{2^{3(n+m)}3^{n+m}\alpha^{n+m}}y^{2n} \\ =& - \frac{\mu_1 y^{2m+1}}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{y^2}{3\alpha\cdot 2^3}} \geq -\frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}}y^{2m+1}, \end{aligned}$$

and again by Lemma 3.5.7 we have

$$\sum_{n=m}^{\infty} g_{2n+1}y^{2n+1} \leq -\mu_2 \sum_{n=m}^{\infty} \frac{y^{2n+1}}{2^{3n}3^n\alpha^n(1+2n)} \leq -\mu_2 \frac{y^{2m+1}}{2^{3m}3^m\alpha^m(1+2m)}.$$

□

Lemma 3.5.10. *For $m \geq 1$ and $0 < y \leq \epsilon < 2\sqrt{6\alpha}$,*

$$\frac{3\alpha - 1}{3^m 2^{3m+1} m\alpha^m} y^{2m} \geq \sum_{n=m}^{\infty} g_{2n}y^{2n} \geq -y^{2m} \frac{1}{3^m 2^{3m+1} m\alpha^m} \frac{1}{1 - \frac{\epsilon^2}{3\cdot 2^3\cdot\alpha}}.$$

Proof. By Lemma 3.5.8,

$$\begin{aligned}\sum_{n=m}^{\infty} g_{2n}y^{2n} &\geq -\frac{1}{2}\sum_{n=m}^{\infty} \frac{1}{3^n 2^{3n} n\alpha^n} y^{2n} \geq -y^{2m}\frac{1}{2}\sum_{n=m}^{\infty} \frac{1}{3^n 2^{3n} m\alpha^n} y^{2n-2m} \\ &= -y^{2m} \frac{1}{3^m 2^{3m+1} m\alpha^m} \frac{1}{1 - \frac{y^2}{3\cdot 2^3\cdot\alpha}} \geq -y^{2m} \frac{1}{3^m 2^{3m+1} m\alpha^m} \frac{1}{1 - \frac{\epsilon^2}{3\cdot 2^3\cdot\alpha}}.\end{aligned}$$

Again by Lemma 3.5.8,

$$\sum_{n=m}^{\infty} g_{2n}y^{2n} \leq \frac{3\alpha-1}{2}\sum_{n=m}^{\infty} \frac{1}{3^n 2^{3n} n\alpha^n} y^{2n} \leq \frac{3\alpha-1}{2} \frac{1}{3^m 2^{3m} m\alpha^m} y^{2m}.$$

□

Definition 3.5.11. *For $0 < y \leq \epsilon < 1$ define*

$$B(y) := \frac{y}{\frac{\pi}{6}\sqrt{24 - y^2}} \text{ and } B_{\epsilon,k} := \epsilon^{-k}\frac{B(\epsilon)^k}{1 - B(\epsilon)}. \tag{3.25}$$

Lemma 3.5.12. *If $0 < y \leq \epsilon < 1$, then*

$$\log\Big(1 + \frac{B(y)^k}{1 - B(y)}\Big) \leq \frac{B_{\epsilon,k}}{1 - (B_{\epsilon,k}\epsilon^k)^2} y^k, \ \ k \geq 1.$$

Proof. First note that for $0 < y < \sqrt{24}$ the function $B(y)$ is increasing and also that $\frac{B(y)^k}{1-B(y)} \leq \frac{B(y)^k}{1-B(\epsilon)}$ and $B(y) < \frac{y}{\frac{\pi}{6}\sqrt{24-\epsilon^2}} = \epsilon^{-1}yB(\epsilon)$. Hence

$$\frac{B(y)^k}{1 - B(\epsilon)} < \frac{\epsilon^{-k}y^k B(\epsilon)^k}{1 - B(\epsilon)} = B_{\epsilon,k}y^k.$$

Consequently,

$$\begin{aligned}
\log\left(1+\frac{B(y)^k}{1-B(y)}\right) &\le \log\left(1+B_{\epsilon,k}y^k\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}B_{\epsilon,k}^n y^{kn} \\
&= -\sum_{n=1}^{\infty}\frac{1}{2n}B_{\epsilon,k}^{2n}y^{2kn} + \sum_{n=0}^{\infty}\frac{1}{2n+1}B_{\epsilon,k}^{2n+1}y^{k(2n+1)} \\
&\le \sum_{n=0}^{\infty}\frac{1}{2n+1}B_{\epsilon,k}^{2n+1}y^{k(2n+1)} \le \sum_{n=0}^{\infty}B_{\epsilon,k}^{2n+1}y^{k(2n+1)} \\
&= \frac{B_{\epsilon,k}y^k}{1-(B_{\epsilon,k}y^k)^2} \le \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}y^k.
\end{aligned}$$

□

Lemma 3.5.13. *If $0 < y \le \epsilon < 1$, then*

$$\log\Big(1-\frac{B(y)^k}{1-B(y)}\Big) \ge -\frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}y^k,\ \ k \ge 1.$$

Proof.

$$\begin{aligned}
\log\Big(1-\frac{B(y)^k}{1-B(y)}\Big) &\ge \log\Big(1-B_{\epsilon,k}y^k\Big) = -\sum_{n=1}^{\infty}\frac{1}{n}B_{\epsilon,k}^n y^{kn} \ge -\sum_{n=1}^{\infty}B_{\epsilon,k}^n y^{kn} \\
&= -\frac{B_{\epsilon,k}y^k}{1-B_{\epsilon,k}y^k} \ge -\frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}y^k.
\end{aligned}$$

□

Lemma 3.5.14. *For all $k \ge 2$ and $0 < \epsilon \le \frac{1}{\sqrt{7}}$ we have*

$$\frac{6^k}{5^k\pi^k} < B_{\epsilon,k} \le \frac{b_0\cdot 6^k}{\pi^k(\sqrt{24-\frac{1}{7}})^k},$$

where $b_0 := \frac{1}{1-\frac{6}{\sqrt{7}\pi\sqrt{24-\frac{1}{7}}}}$ and again $B_{\epsilon,k}$ as in (3.25).

Proof. Define

$$s := \sqrt{24-\epsilon^2},\ l_s := \sqrt{24-\frac{1}{7}},\ u_s := 4.9,\ l_\epsilon := 0,\ \text{and}\ u_\epsilon := \frac{1}{\sqrt{7}}.$$

For all $k \geq 2$ and $0 < \epsilon \leq \frac{1}{\sqrt{7}}$, we have

$$l_s \leq s < u_s \text{ and } l_\epsilon < \epsilon \leq u_\epsilon.$$

The following conventions for the letters l and u will be useful: l_a denotes a lower bound for the quantity a, and u_a will denote an upper bound for the quantity a. And again we use $B(y)$ as defined in Definition 3.5.11.

Then

$$0 = \frac{l_\epsilon}{\frac{\pi}{6}u_s} < B(\epsilon) = \frac{\epsilon}{\frac{\pi}{6}s} \leq \frac{u_\epsilon}{\frac{\pi}{6}l_s}.$$

Let us define $l_B := 0$ and $u_B := \frac{u_\epsilon}{\frac{\pi}{6}l_s}$. Then

$$l_B < B(\epsilon) \leq u_B \Rightarrow 1 - u_B \leq 1 - B(\epsilon) < 1 - l_B = 1 \Rightarrow \frac{1}{1-l_B} = 1 < \frac{1}{1-B(\epsilon)} \leq \frac{1}{1-u_B},$$

and $\frac{1}{(\frac{\pi}{6}u_s)^k} < \frac{1}{(\frac{\pi}{6}s)^k} \leq \frac{1}{(\frac{\pi}{6}l_s)^k}$. Hence

$$\frac{6^k}{5^k\pi^k} < \frac{6^k}{(4.9)^k\pi^k} = \frac{1}{(1-l_B)(\frac{\pi}{6}u_s)^k} < B_{\epsilon,k} \leq \frac{1}{(1-u_B)(\frac{\pi}{6}l_s)^k}$$
$$= \frac{1}{\left(1 - \frac{\frac{1}{\sqrt{7}}}{\frac{\pi}{6}\sqrt{24-\frac{1}{7}}}\right)\left(\frac{\pi^k}{6^k}\left(\sqrt{24-\frac{1}{7}}\right)^k\right)} = \frac{b_0}{\frac{\pi^k}{6^k}\left(\sqrt{24-\frac{1}{7}}\right)^k}.$$

□

Definition 3.5.15. *Define*

$$\beta := \sqrt{24 - \frac{1}{7}}$$

and for $k \geq 0$*,*

$$C_k := \frac{6^k}{(\pi\beta)^k}.$$

Lemma 3.5.16. *Let* $0 < \epsilon \leq \frac{1}{\sqrt{7}}$ *and* $B_{\epsilon,k}$ *be as in* (3.25)*. Then for* $k \geq 2$*,*

$$\frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2} \leq b_1 B_{\epsilon,k} \text{ and } \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} \leq b_2 B_{\epsilon,k},$$

with

$$b_1 := \frac{1}{1-\frac{1}{49}b_0^2C_4},\ b_2 := \frac{1}{1-\frac{1}{7}b_0C_2},$$

and b_0 *as in Lemma 3.5.14.*

Proof. We obtain, using Lemma 3.5.14,

$$\frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{7}B_{\epsilon,k}} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{7}b_0C_2} = b_2B_{\epsilon,k},$$

and

$$\frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{49}B_{\epsilon,k}^2} \leq \frac{B_{\epsilon,k}}{1-\frac{1}{49}b_0^2C_4} = b_1B_{\epsilon,k}.$$

□

Lemma 3.5.17. *Let C_k be as in Definition 3.5.15, then*

$$C_{2m} < \frac{1}{3^m2^{3m}\alpha^m m}, \ m \geq 10, \ \textit{and } C_{2m-1} < \frac{69}{25}\frac{1}{2^{3m}3^m\alpha^m(2m-1)}, \ m \geq 14.$$

Proof. We start with the first inequality:

$$C_{2m} = \Big(\frac{252}{167\pi^2}\Big)^m < \frac{(36+\pi^2)^m}{3^m2^{3m}m\pi^{2m}} \Leftrightarrow \Big(\frac{6048}{6012+167\pi^2}\Big)^m m < 1.$$

To prove the inequality in the rewritten form, define $\ell := \frac{6048}{6012+167\pi^2}$ and note that $\ell < 1$. Moreover, for $m \geq 10$,

$$m\ell^m < 1 \Leftrightarrow \log m + m\log\ell < 0.$$

Define $f(m) := m\log\ell + \log m$. We have to show $f(m) < 0$ for all $m \geq 10$. We first show that $f(m)$ is decreasing for $m \geq 10$. This is equivalent to $f'(m) = \log\ell + \frac{1}{m} < 0$ for $m \geq 10$. This is equivalent to showing $\ell e^{1/m} < 1$ for $m \geq 10$. Now for $m \geq 10$ we have $\ell e^{1/m} \leq \ell e^{1/10}$. By numerics, $\ell e^{1/10} < 1$ and $f(10) < 0$. Since $f(m)$ is decreasing and $f(m) \leq f(10) < 0$ for $m \geq 10$, the first inequality is proven. Now for the second inequality, first note that

$$C_{2m-1} = \Big(\frac{6}{\pi\beta}\Big)^{2m-1} = \Big(\frac{252}{167\pi^2}\Big)^m\Big(\frac{\pi}{6}\sqrt{\frac{167}{7}}\Big).$$

Hence we have to show

$$\Big(\frac{252}{167\pi^2}\Big)^m\Big(\frac{\pi}{6}\sqrt{\frac{167}{7}}\Big) < \frac{69}{25}\frac{1}{2^{3m}3^m\alpha^m(2m-1)},$$

which is equivalent to

$$\Big(\frac{6048}{6012+167\pi^2}\Big)^m(2m-1) < \frac{414}{25\pi}\sqrt{\frac{7}{167}} \Leftrightarrow (2m-1)\ell^m < \frac{414}{25\pi}\sqrt{\frac{7}{167}}$$
$$\Leftrightarrow \underbrace{m\log\ell + \log(2m-1) - \log\Big(\frac{414}{25\pi}\sqrt{\frac{7}{167}}\Big)}_{=:g(m)} < 0.$$

Now anolgously to the proof of the first case one observes that $g(m)$ is decreasing for $m \geq 14$ and that $g(14) < 0$, hence $g(m) \leq g(14) < 0$. □

3.6 An infinite family of inequalities for $\log p(n)$ and its growth

After the preparations made in Section 3.5, in this section we prove our Main Theorem, Theorem 3.6.6, which implies Theorem 3.1.1 as a corollary. Again we let

$$\alpha = \frac{\pi^2}{36+\pi^2}.$$

Definition 3.6.1. *Let $B_{\epsilon,k}$ be as in Definition 3.5.11 and μ_1, μ_2 as in Lemma 3.5.9 and $\nu := \frac{3\alpha-1}{2}$. Moreover, let $0 < \epsilon \leq \frac{1}{\sqrt{7}}$. For $m, k \geq 1$ we define*

$$\begin{aligned}
A_{1,k}(2m) &:= \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}\epsilon^{k-2m} + \nu\frac{1}{3^m 2^{3m} m\alpha^m},\\
A_{-1,k}(2m) &:= \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}\epsilon^{k-2m} + \frac{1}{3^m 2^{3m+1} m\alpha^m}\frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\alpha}} +\\
&\qquad \frac{\mu_1}{2^{3m}3^m\alpha^m(1+2m)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}},\\
A_{1,k}(2m-1) &:= \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}\epsilon^{k-2m+1} - \frac{\mu_2}{2^{3m-3}3^{m-1}\alpha^{m-1}(2m-1)},\\
A_{-1,k}(2m-1) &:= \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}\epsilon^{k-2m+1} + \frac{1}{3^m 2^{3m+1} m\alpha^m}\frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3\alpha}} +\\
&\qquad \frac{\mu_1}{2^{3m-3}3^{m-1}\alpha^{m-1}(2m-1)}\frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}}.
\end{aligned}$$

Lemma 3.6.2. *Let $\sum_{n=1}^{\infty} g_n y^n$ as in Definition 3.5.1 and $G_{i,k}(y)$ as in Definition 3.5.2. Moreover let $0 < y \le \epsilon \le \frac{1}{\sqrt{7}}$. Then for $k \ge 2m \ge 2$, we have*

$$\sum_{n=1}^{2m-1} g_n y^n - A_{-1,k}(2m) y^{2m} \le G_{-1,k}(y) \text{ and } G_{1,k}(y) \le \sum_{n=1}^{2m-1} g_n y^n + A_{1,k}(2m) y^{2m},$$

and for $k \ge 2m-1 \ge 1$,

$$\sum_{n=1}^{2m-2} g_n y^n - A_{-1,k}(2m-1) y^{2m-1} \le G_{-1,k}(y)$$

and

$$G_{1,k}(y) \le \sum_{n=1}^{2m-2} g_n y^n + A_{1,k}(2m-1) y^{2m-1}.$$

Proof. For $k \ge 2m \ge 2$, by using the Lemmas 3.5.9 to 3.5.12, we obtain

$$\begin{aligned} G_{1,k}(y) &\le \sum_{n=1}^{2m-1} g_n y^n + \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2} y^k + \nu \frac{1}{3^m 2^{3m} m \alpha^m} y^{2m} - \frac{\mu_2}{2^{3m} 3^m \alpha^m (1+2m)} y^{2m+1} \\ &\le \sum_{n=1}^{2m-1} g_n y^n + \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2} \epsilon^{k-2m} y^{2m} + \nu \frac{1}{3^m 2^{3m} m \alpha^m} y^{2m} \\ &= \sum_{n=1}^{2m-1} g_n y^n + A_{1,k}(2m) y^{2m}. \end{aligned}$$

By using the Lemmas 3.5.9 to 3.5.10 together with Lemma 3.5.13 we obtain

$$\begin{aligned} G_{-1,k}(y) \ge \sum_{n=1}^{2m-1} g_n y^n &- \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} y^k - \frac{1}{3^m 2^{3m+1} m \alpha^m} \frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3 \alpha}} y^{2m} \\ &- \frac{\mu_1}{2^{3m} 3^m \alpha^m (1+2m)} \frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}} y^{2m+1} \\ \ge \sum_{n=1}^{2m-1} g_n y^n &- \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k} \epsilon^{k-2m} y^{2m} - \frac{1}{3^m 2^{3m+1} m \alpha^m} \frac{1}{1-\frac{\epsilon^2}{3\cdot 2^3 \alpha}} y^{2m} \\ &- \frac{\mu_1}{2^{3m} 3^m \alpha^m (1+2m)} \frac{1}{1-\frac{\epsilon^2}{3\alpha\cdot 2^3}} y^{2m} \\ = \sum_{n=1}^{2m-1} g_n y^n &- A_{-1,k}(2m) y^{2m}. \end{aligned}$$

The statement for $A_{-1,k}(2m-1)$ is proven analogously. $\square$

Lemma 3.6.3. *We have for $m \geq 10$ that*

$$A_{1,k}(2m) < \frac{1}{3^m 2^{3m} m \alpha^m}, \ A_{-1,k}(2m) < \frac{2}{3^m 2^{3m} m \alpha^m}$$

and for $m \geq 14$

$$A_{1,k}(2m-1) < \frac{2}{3^m 2^{3m}(2m-1)\alpha^m}, \ A_{-1,k}(2m-1) < \frac{7}{3^m 2^{3m}(2m-1)\alpha^m}.$$

Proof. For $m \geq 10$ we have,

$$\begin{aligned}
A_{1,k}(2m) &= \frac{B_{\epsilon,k}}{1-(B_{\epsilon,k}\epsilon^k)^2}\epsilon^{k-2m} + \nu\frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Definition 3.6.1)} \\
&< b_1 B_{\epsilon,k}\epsilon^{k-2m} + \nu\frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Lemma 3.5.16)} \\
&< b_1 b_0 \frac{6^k}{(\pi\beta)^k}\epsilon^{k-2m} + \nu\frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Lemma 3.5.14)} \\
&= b_0 b_1 C_k \epsilon^{k-2m} + \nu\frac{1}{3^m 2^{3m}\alpha^m m} \text{ (using Definition 3.5.15)} \\
&\leq b_0 b_1 C_{2m} + \nu\frac{1}{3^m 2^{3m}\alpha^m m} \\
&\qquad \text{(because } f(k) := C_k\epsilon^{k-2m} \text{ is decreasing for all } k \geq 2m) \\
&< b_0 b_1 \frac{1}{3^m 2^{3m}\alpha^m m} + \nu\frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by Lemma 3.5.17)} \\
&= \left(b_0 b_1 + \nu\right)\frac{1}{3^m 2^{3m}\alpha^m m} \\
&< \frac{1}{3^m 2^{3m}\alpha^m m} \text{ (by evaluating } b_0 b_1 + \nu \text{ numerically).}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& A_{-1,k}(2m) \\
= \ & \frac{B_{\epsilon,k}}{1-B_{\epsilon,k}\epsilon^k}\epsilon^{k-2m} + \frac{1}{3^m 2^{3m+1}\alpha^m m}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} + \\
& \qquad \frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} \text{ (by Definition 3.6.1)} \\
< \ & b_2 B_{\epsilon,k}\epsilon^{k-2m} + \frac{1}{2}\frac{1}{3^m 2^{3m}\alpha^m m}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} +
\end{aligned}$$

$$
\begin{aligned}
& \qquad\qquad\qquad \frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} \text{ (by Lemma 3.5.16)} \\
< \quad & b_2 b_0 \frac{6^k}{(\pi\beta)^k}\epsilon^{k-2m} + \frac{1}{2}\frac{1}{3^m 2^{3m}\alpha^m m}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} + \\
& \qquad\qquad\qquad \frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{\epsilon^2}{24\alpha}} \text{ (by Lemma 3.5.14)} \\
\leq \quad & b_0 b_2 \cdot C_{2m} + \frac{1}{2}\frac{1}{3^m 2^{3m}\alpha^m m}\frac{1}{1-\frac{1}{168\alpha}} + \frac{\mu_1}{2^{3m}3^m\alpha^m(2m+1)}\frac{1}{1-\frac{1}{168\alpha}} \\
< \quad & b_0 b_2 \frac{1}{3^m 2^{3m}\alpha^m m} + \frac{1}{2}\frac{1}{3^m 2^{3m}\alpha^m m}\frac{1}{1-\frac{1}{168\alpha}} + \\
& \qquad\qquad\qquad \frac{1}{2}\frac{\mu_1}{2^{3m}3^m\alpha^m m}\frac{1}{1-\frac{1}{168\alpha}} \text{ (by Lemma 3.5.17)} \\
= \quad & \left(b_0 b_2 + \frac{1}{2}\frac{1}{1-\frac{1}{168\alpha}}(1+\mu_1)\right)\frac{1}{3^m 2^{3m}\alpha^m m} \\
< \quad & \frac{2}{3^m 2^{3m}\alpha^m m} \text{ (by evaluating } b_0 b_2 + \frac{1}{2}\frac{1}{1-\frac{1}{168\alpha}}(1+\mu_1) \text{ numerically).}
\end{aligned}
$$

The statements for $A_{1,k}(2m-1)$ and $A_{-1,k}(2m-1)$ are proven analogously.

□

Definition 3.6.4. *For $n, U \geq 1$ we define*

$$P_n(U) := -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{U} g_u(1/\sqrt{n})^u.$$

Lemma 3.6.5. *Let $g(k)$ be as in Definition 3.4.3 and $P_n(U)$ as in Definition 3.6.4. If $m \geq 1$, $k \geq 2m$ and*

$$n > \begin{cases} 6 & \text{if } m = 1, \\ g(k), & \text{if } m \geq 2, \end{cases}$$

then

$$-A_{-1,k}(2m)\frac{1}{n^m} < \log p(n) - P_n(2m-1) < A_{1,k}(2m)\frac{1}{n^m}. \tag{3.26}$$

If $m \geq 2$, $k \geq 2m-1$, and $n > g(k)$, then

$$-A_{-1,k}(2m-1)\frac{1}{n^{m-\frac{1}{2}}} < \log p(n) - P_n(2m-2) < A_{1,k}(2m-1)\frac{1}{n^{m-\frac{1}{2}}}. \tag{3.27}$$

Proof. We start with the inequality from Lemma 3.5.3. Next we use Lemma 3.6.2 to bound $G_{1,k}(y)$. Finally we set $y = \frac{1}{\sqrt{n}}$ and obtain the desired result. □

Theorem 3.6.6. *Let $G(y) = \sum_{n=1}^{\infty} g_n y^n$ be as in Definition 3.5.1. Let $g(k)$ be as in Definition 3.4.3 and $P_n(U)$ as in Definition 3.6.4. If $m \geq 1$ and $n > g(2m)$, then*

$$P_n(2m-1) - \frac{2}{3^m 2^{3m} \alpha^m m n^m} < \log p(n) < P_n(2m-1) + \frac{1}{3^m 2^{3m} \alpha^m m n^m}; \tag{3.28}$$

if $m \geq 2$ and $n > g(2m-1)$, then

$$P_n(2m-2) - \frac{7}{3^m 2^{3m} \alpha^m (2m-1) n^{m-1/2}} < \log p(n) < P_n(2m-2) + \frac{2}{3^m 2^{3m} \alpha^m (2m-1) n^{m-1/2}}. \tag{3.29}$$

Proof. We start by setting $k = 2m$ in (3.26) of Lemma 3.6.5, and $k = 2m-1$ in (3.27). In this inequality we bound $A_{1,k}(m)$ resp $A_{-1,k}(m)$ by using Lemma 3.6.3. This gives (3.29) for all $m \geq 14$ and $n > g(2m-1)$, and (3.28) for $m \geq 10$ and $n > g(2m)$.

In order to prove (3.28) and (3.29) for the remaining values of m, firstly we will prove that

$$\text{if (3.28) holds for } m \geq 2 \text{ and all } n \geq y \geq 1, \text{ then (3.28) holds for } m-1 \text{ and all } n \geq y. \tag{3.30}$$

In particular, if we subtract from the lower bound on $\log p(n)$ with parameter m in (3.28) the lower bound on $\log p(n)$ with parameter $m-1$, we obtain $f(2m,-4) - g(2m-2,-4)$, where

$$f(w,x) := \sum_{u=w-2}^{w-1} g_u \left(\frac{1}{\sqrt{n}}\right)^u + \frac{x}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w} \left(\frac{1}{\sqrt{n}}\right)^w$$

and

$$g(w,x) := \frac{x}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w} \left(\frac{1}{\sqrt{n}}\right)^w.$$

Similarly, if we subtract from the upper bound for $m \to m-1$ in (3.28) the upper bound for m, we obtain $g(2m-2,2) - f(2m,2)$. Hence in order to prove (3.30), it suffices to prove

$$f(2m,-4) > g(2m-2,-4) \text{ and } f(2m,2) < g(2m-2,2). \tag{3.31}$$

Analogously, in order to prove that if (3.29) holds for all $m \geq 3$ and all $n \geq y \geq 1$, then (3.29) holds for $m-1$ and all $n \geq y$, it suffices to prove

$$f(2m-1,-7) > g(2m-3,-7) \text{ and } f(2m-1,2) < g(2m-3,2). \tag{3.32}$$

For proving (3.31) and (3.32), we shall prove

$$f(w,x_0(w)) > g(w-2,x_0(w)) \text{ with } x_0(w) := \begin{cases} -4, & \text{if } w \text{ is even} \\ -7, & \text{if } w \text{ is odd} \end{cases} \tag{3.33}$$

and

$$f(w,y_0) < g(w-2,y_0) \text{ with } y_0 > 0. \tag{3.34}$$

From Lemma 3.5.7 and Lemma 3.5.8 we have

$$\frac{\ell_w}{(24\alpha)^{\lfloor \frac{w}{2} \rfloor} w} \leq g_w \leq \frac{u_w}{(24\alpha)^{\lfloor \frac{w}{2} \rfloor} w}$$

with

$$\ell_w := \begin{cases} -\mu_1, & \text{if } w \text{ is odd} \\ -1, & \text{if } w \text{ is even} \end{cases} \text{ and } u_w := \begin{cases} -\mu_2, & \text{if } w \text{ is odd} \\ 2\nu, & \text{if } w \text{ is even} \end{cases},$$

where μ_1 and μ_2 are as in Lemma 3.5.9 and ν as in Definition 3.6.1. Consequently,

$$\begin{aligned} f(w,x_0) &= \sum_{u=w-2}^{w-1} g_u \Big(\frac{1}{\sqrt{n}}\Big)^u + \frac{x_0}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w} \Big(\frac{1}{\sqrt{n}}\Big)^w \\ &\geq \frac{\ell_{w-2}}{(24\alpha)^{\lfloor \frac{w-2}{2} \rfloor}(w-2)} \Big(\frac{1}{\sqrt{n}}\Big)^{w-2} + \frac{\ell_{w-1}}{(24\alpha)^{\lfloor \frac{w-1}{2} \rfloor}(w-1)} \Big(\frac{1}{\sqrt{n}}\Big)^{w-1} \\ &\qquad + \frac{x_0}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w} \Big(\frac{1}{\sqrt{n}}\Big)^w. \end{aligned}$$

In order to prove (3.33), it is enough to prove

$$\frac{\ell_{w-2}}{w-2} + \frac{\ell_{w-1}}{(24\alpha)^{\alpha_w}(w-1)} \frac{1}{\sqrt{n}} + \frac{x_0}{(24\alpha)^{\beta_w} w} \frac{1}{n} > \frac{x_0}{(24\alpha)^{\delta_w}(w-2)}, \tag{3.35}$$

where

$$\alpha_w = \Big\lfloor \frac{w-1}{2} \Big\rfloor - \Big\lfloor \frac{w-2}{2} \Big\rfloor = \begin{cases} 0, & \text{if } w \text{ is even} \\ 1, & \text{if } w \text{ is odd} \end{cases},$$

$$\beta_w = \Big\lceil \frac{w}{2} \Big\rceil - \Big\lfloor \frac{w-2}{2} \Big\rfloor = \begin{cases} 1, & \text{if } w \text{ is even} \\ 2, & \text{if } w \text{ is odd} \end{cases},$$

and

$$\delta_w = \left\lceil \frac{w-2}{2} \right\rceil - \left\lfloor \frac{w-2}{2} \right\rfloor = \begin{cases} 0, & \text{if } w \text{ is even} \\ 1, & \text{if } w \text{ is odd} \end{cases}.$$

Inequality (3.35) is equivalent to

$$\Big(\ell_{w-2} - \frac{x_0}{(24\alpha)^{\alpha_w}}\Big)\frac{1}{w-2} > -\frac{\ell_{w-1}}{(24\alpha)^{\delta_w}(w-1)}\frac{1}{\sqrt{n}} + \frac{x_0}{(24\alpha)^{\beta_w} w}\frac{1}{n},$$

which is implied by

$$\Big(\ell_{w-2} - \frac{x_0}{(24\alpha)^{\alpha_w}}\Big)\frac{1}{w-2} > -\Big(\frac{\ell_{w-1}}{(24\alpha)^{\alpha_w}(w-1)} + \frac{x_0}{(24\alpha)^{\beta_w} w}\Big)\frac{1}{\sqrt{n}} \tag{3.36}$$

since $\delta_w = \alpha_w$, $x_0 < 0$ and $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ for all $n \geq 1$. Inequality (3.36) is equivalent to

$$n \geq \left\lceil \frac{(w-2)^2\Big(\frac{\ell_{w-1}}{(24\alpha)^{\alpha_w}(w-1)} + \frac{x_0}{(24\alpha)^{\beta_w} w}\Big)^2}{\Big(\ell_{w-2} - \frac{x_0}{(24\alpha)^{\alpha_w}}\Big)^2} \right\rceil =: N_1(w, x_0).$$

We checked with Mathematica that $N_1(w, x_0(w)) \leq 1$; see the Appendix, Section 3.8.3.

Similarly to above, for $y_0 > 0$ one has,

$$\begin{aligned} f(w, y_0) = &\sum_{u=w-2}^{w-1} g_u\Big(\frac{1}{\sqrt{n}}\Big)^u + \frac{y_0}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w \\ \leq &\frac{u_{w-2}}{(24\alpha)^{\lceil \frac{w-2}{2} \rceil}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2} + \frac{u_{w-1}}{(24\alpha)^{\lceil \frac{w-1}{2} \rceil}(w-1)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-1} \\ &+ \frac{y_0}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w. \end{aligned}$$

In order to prove (3.34), it is enough to show

$$\begin{aligned} &\frac{u_{w-2}}{(24\alpha)^{\lceil \frac{w-2}{2} \rceil}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2} + \frac{u_{w-1}}{(24\alpha)^{\lceil \frac{w-1}{2} \rceil}(w-1)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-1} + \frac{y_0}{(24\alpha)^{\lceil \frac{w}{2} \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w \\ < &\frac{y_0}{(24\alpha)^{\lceil \frac{w-2}{2} \rceil}(w-2)}\Big(\frac{1}{\sqrt{n}}\Big)^{w-2}. \end{aligned}$$

This last inequality can be rewritten as the following equivalent inequality,

$$\frac{u_{w-2}}{w-2} + \frac{u_{w-1}}{(24\alpha)^{\alpha_w}(w-1)}\frac{1}{\sqrt{n}} + \frac{y_0}{(24\alpha)^{\beta_w} w}\frac{1}{n} < \frac{y_0}{(24\alpha)^{\alpha_w}(w-2)},$$

which is implied by

$$\left(\frac{y_0}{(24\alpha)^{\alpha_w}} - u_{w-2}\right)\frac{1}{w-2} > \left(\frac{u_{w-1}}{(24\alpha)^{\alpha_w}(w-1)} + \frac{y_0}{(24\alpha)^{\beta_w} w}\right)\frac{1}{\sqrt{n}} \tag{3.37}$$

since $y_0 > 0$ and $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$. Inequality (3.37) is equivalent to

$$n \geq \left\lceil \frac{(w-2)^2\left(\frac{u_{w-1}}{(24\alpha)^{\alpha_w}(w-1)} + \frac{y_0}{(24\alpha)^{\beta_w} w}\right)^2}{\left(\frac{y_0}{(24\alpha)^{\alpha_w}} - u_{w-2}\right)^2} \right\rceil =: N_2(w, y_0).$$

We checked using Mathematica that $N_2(w, y_0) \leq 1$ for all $y_0 \geq 1$; see the Appendix, Section 3.8.3.

We have checked with Mathematica that (3.28) holds for $m \in \{2, \ldots, 10\}$ and $n \in \mathbb{N}$ such that

$$g(2m-2) < n \leq g(2m). \tag{3.38}$$

Now (3.28) is true for $m = 10$ and $n > g(2m)$. Next, assume that (3.28) is true for $m = N$ with $2 \leq N \leq 10$ and $n > g(2N)$. Then, as shown above, (3.28) is true for $m = N-1$ if $n > g(2N)$. By (3.38), (3.28) is true for $m = N-1$ if $g(2N-2) < n \leq g(2N)$. This implies that (3.28) is true for $m = N-1$ and $n > g(2N-2)$. Hence the result follows inductively. The proof of (3.29) is done analogously.

□

Finally, we are put into the position to prove Theorem 3.1.1.

Proof of Theorem 3.1.1: We apply (3.28) in Theorem 3.6.6, with $m = 1$. Then for $n \geq 1$, we have

$$\begin{aligned}
&-\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} - \sqrt{6}\left(\frac{\pi}{144} + \frac{1}{2\pi}\right)\frac{1}{\sqrt{n}} - \frac{2}{24\alpha}\frac{1}{n} \\
&< \log p(n) < -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} - \sqrt{6}\left(\frac{\pi}{144} + \frac{1}{2\pi}\right)\frac{1}{\sqrt{n}} + \frac{1}{24\alpha}\frac{1}{n}.
\end{aligned}$$

Noting that $\sqrt{6}\left(\frac{\pi}{144} + \frac{1}{2\pi}\right) = 0.44\ldots$ finishes the proof. □

3.7 An application to Chen-DeSalvo-Pak log-concavity result

In 2010 at FPSAC [35], William Chen conjectured that $\{p(n)\}_{n\geq 26}$ is log-concave and that for $n \geq 1$,

$$p(n)^2 < \Big(1 + \frac{1}{n}\Big) p(n-1)p(n+1). \tag{3.39}$$

DeSalvo and Pak [53] proved these two conjectures. Moreover, they refined (3.39) by proposing the following conjecture

$$p(n)^2 < \Big(1 + \frac{\pi}{\sqrt{24} n^{3/2}}\Big) p(n-1)p(n+1), \ n \geq 45. \tag{3.40}$$

Chen, Wang and Xie [39] gave an affirmative answer to (3.40). In this section, using Theorem 3.6.6, we continue this research by obtaining the following inequality,

$$\Big(1 + \frac{\pi}{\sqrt{24} n^{3/2}} - \frac{1}{n^2}\Big) p(n-1)p(n+1) < p(n)^2 < \Big(1 + \frac{\pi}{\sqrt{24} n^{3/2}}\Big) p(n-1)p(n+1);$$

for a more precise statement see Theorem 3.7.6. Note that the right inequality is just (3.40), but we give here our proof in order to show that, alternatively, one can obtain this from Theorem 3.6.6. In order to achieve our goal we also need to prove the Lemmas 3.7.3 to 3.7.5 in this section. These lemmas deal with estimating the tail of an infinite series involving standard binomial coefficients.

Proposition 3.7.1. *For $s \geq 1$ and $k \geq 0$ we have*

$$\binom{-\frac{2s-1}{2}}{k} = \frac{(-1)^k}{4^k} \frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}}$$

and

$$\binom{-s}{k} = (-1)^k \binom{s+k-1}{s-1}.$$

Proof. By simplifying quotients formed by taking each expression in $k+1$ divided by the original expression in k. $\square$

Lemma 3.7.2. *For $k, m \geq 0$ and $s \geq 1$,*

$$\binom{s-1+m+k}{s-1} \leq \binom{s-1+m}{s-1} s^k. \tag{3.41}$$

Proof. From

$$\binom{s-1+m+k}{s-1} = \frac{(s-1+m+k)!}{(s-1)!(m+k)!} = \binom{s-1+m}{s-1}\frac{(s+m)\cdots(s+m+k-1)}{(m+1)\cdots(m+k)}$$

we have $\frac{s+m+j}{m+j+1} \leq s$ for each $0 \leq j \leq k-1$; this is because

$$s+m+j \leq s(m+j+1) \Leftrightarrow m(s-1)+j(s-1) \geq 0.$$

This proves (3.41). ☐

Lemma 3.7.3. *For $n, s \geq 1$, $m \geq 0$, and $n > 2s$ let*

$$b_{m,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+m-1}}\binom{s+m-1}{s-1}\frac{1}{n^m},$$

then

$$-b_{m,n}(s) < \sum_{k=m}^{\infty}\binom{-\frac{2s-1}{2}}{k}\frac{1}{n^k} < b_{m,n}(s) \tag{3.42}$$

and

$$0 < \sum_{k=m}^{\infty}\binom{-\frac{2s-1}{2}}{k}\frac{(-1)^k}{n^k} < b_{m,n}(s). \tag{3.43}$$

Proof. For $s \geq 1$:

$$\begin{aligned}
\left|\sum_{k=m}^{\infty}\binom{-\frac{2s-1}{2}}{k}\frac{1}{n^k}\right| &= \left|\sum_{k=m}^{\infty}\frac{(-1)^k}{4^k}\frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}}\frac{1}{n^k}\right| \text{ (by Proposition 3.7.1)}\\
&\leq \sum_{k=m}^{\infty}\frac{1}{4^k}\frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}}\frac{1}{n^k}\\
&\leq \sum_{k=m}^{\infty}\frac{2\sqrt{s-1}}{\sqrt{\pi(s+k-1)}}\binom{s+k-1}{s-1}\frac{1}{n^k} \text{ (using } \frac{4^n}{2\sqrt{n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}})\\
&< \frac{2\sqrt{s-1}}{\sqrt{s+m-1}}\sum_{k=m}^{\infty}\binom{s-1+k}{s-1}\frac{1}{n^k}\\
&\quad\left(\text{using } \frac{1}{\sqrt{\pi}} < 1 \text{ and } \frac{1}{\sqrt{s+k-1}} \leq \frac{1}{\sqrt{s+m-1}} \text{ for all } k \geq m\right)\\
&= \frac{2\sqrt{s-1}}{\sqrt{s+m-1}}\sum_{k=0}^{\infty}\binom{s-1+m+k}{s-1}\frac{1}{n^{m+k}}\\
&= \frac{2\sqrt{s-1}}{\sqrt{s+m-1}}\frac{1}{n^m}\sum_{k=0}^{\infty}\binom{s-1+m+k}{s-1}\frac{1}{n^k}.
\end{aligned}$$

Now we apply Lemma 3.7.2 to obtain,

$$\begin{aligned}\Big|\sum_{k=m}^{\infty}\binom{-\frac{2s-1}{2}}{k}\frac{1}{n^k}\Big| \leq& \frac{2\sqrt{s-1}}{\sqrt{s-1+m}}\frac{1}{n^m}\binom{s-1+m}{s-1}\sum_{k=0}^{\infty}\frac{s^k}{n^k} \\ =&\frac{2\sqrt{s-1}}{\sqrt{s+m-1}}\binom{s-1+m}{s-1}\frac{1}{n^m}\frac{n}{n-s} < b_{m,n}(s),\end{aligned}$$

where the latter inequality is by $n > 2s$. This proves (3.42). Moreover, the bound we obtained also works for

$$\sum_{k=m}^{\infty}\frac{1}{4^k}\frac{\binom{2s+2k-2}{s+k-1}\binom{s+k-1}{s-1}}{\binom{2s-2}{s-1}}\frac{1}{n^k},$$

because this term showed up along the way in the proof of the previous case. Hence applying Proposition 3.7.1 implies (3.43). □

Lemma 3.7.4. *For $n, s \geq 1$, $m \geq 0$, and $n > 2s$ let*

$$\beta_{m,n}(s) := \frac{2}{n^m}\binom{s+m-1}{s-1},$$

then

$$-\beta_{m,n}(s) < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{1}{n^k} < \beta_{m,n}(s) \tag{3.44}$$

and

$$0 < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{3.45}$$

Proof.

$$\begin{aligned}\Big|\sum_{k=m}^{\infty}\binom{-s}{k}\frac{1}{n^k}\Big| =&\Big|\sum_{k=m}^{\infty}(-1)^k\binom{s+k-1}{s-1}\frac{1}{n^k}\Big| \quad \text{(by Proposition 3.7.1)} \\ \leq&\sum_{k=m}^{\infty}\binom{s+k-1}{s-1}\frac{1}{n^k} \\ =&\frac{1}{n^m}\sum_{k=0}^{\infty}\binom{s+k-1+m}{s-1}\frac{1}{n^k} \\ <&\frac{1}{n^m}\binom{s-1+m}{s-1}\sum_{k=0}^{\infty}\frac{s^k}{n^k} \quad \text{(by Lemma 3.7.2)},\end{aligned}$$

and geometric series summation implies (3.44). The proof of (3.45) is analogous. □

Finally, we need another similar lemma which is easy to prove.

Lemma 3.7.5. *For $m, n, s \geq 1$ and $n > 2s$ let*

$$c_{m,n}(s) := \frac{2}{m}\frac{s^m}{n^m}.$$

Then

$$-c_{m,n}(s) < \sum_{k=m}^{\infty} \frac{(-1)^{k+1}}{k}\frac{s^k}{n^k} < c_{m,n}(s) \text{ and } -c_{m,n}(s) < -\sum_{k=m}^{\infty} \frac{1}{k}\frac{s^k}{n^k} < 0$$

and

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty} \binom{1/2}{k}\frac{s^k}{n^k} < \frac{c_{m,n}(s)}{\sqrt{m}} \text{ and } -\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty} \binom{1/2}{k}\frac{(-1)^k s^k}{n^k} < 0. \tag{3.46}$$

The following theorem was announced in the abstract; its proof is the goal of this section. To arrive at the intermediate inequality (3.51), we need our main result, Theorem 3.6.6. For the remainder of the proof, one spends some time on simplifying (3.51) in order to arrive at the desired form. In order to do, one needs the Lemmas 3.7.3 to 3.7.5 which we have proven above in this section.

Theorem 3.7.6. *For $n \geq 45$,*

$$p(n)^2 < \Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\Big)p(n-1)p(n+1),$$

and for $n \geq 120$

$$p(n)^2 > \Big(1 + \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\Big)p(n-1)p(n+1).$$

Proof. We set $m = 3$ in the first equation of Theorem 3.6.6, which gives for all $n \geq \lceil g(6) \rceil$ that

$$\underbrace{P_n(5) - \frac{2}{3(24\alpha)^3}\frac{1}{n^3}}_{=:l(n)} < \log p(n) < \underbrace{P_n(5) + \frac{1}{3(24\alpha)^3}\frac{1}{n^3}}_{=:u(n)},$$

using the notation from Definition 3.6.4. This inequality has the form

$$l(n) < \log p(n) < u(n). \tag{3.47}$$

By substituting n by $n+1$ and multiplying by -1 into (3.47) we obtain

$$-u(n+1) < -\log p(n+1) < -l(n+1), \tag{3.48}$$

and by substituting n by $n-1$ and multiplying by -1 again into (3.47) gives

$$-u(n-1) < -\log p(n-1) < -l(n-1). \tag{3.49}$$

Multiplying (3.47) by 2, and by adding (3.48) and (3.49), results in

$$\begin{aligned}2l(n) - u(n-1) - u(n+1) < 2\log p(n) - \log p(n-1) - \log p(n+1) <\\ 2u(n) - l(n-1) - l(n+1).\end{aligned} \tag{3.50}$$

We define

$$A_1(n) := \log\left(1 + \frac{1}{n}\right) + \log\left(1 - \frac{1}{n}\right),$$

$$A_2(n) := -\pi\sqrt{\frac{2n}{3}}\left(\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{(-1)^k}{n^k} + \sum_{k=1}^{\infty}\binom{1/2}{k}\frac{1}{n^k}\right),$$

and for $t \geq 3$

$$A_t(n) := -\frac{g_{t-2}}{(\sqrt{n})^{t-2}}\left(\sum_{k=1}^{\infty}\binom{-\frac{t-2}{2}}{k}\frac{(-1)^k}{n^k} + \sum_{k=1}^{\infty}\binom{-\frac{t-2}{2}}{k}\frac{1}{n^k}\right),$$

where g_n is as in Definition 3.5.1. Then from (3.50), by substituting $l(n)$ and $u(n)$ according to their definitions, we obtain

$$\begin{aligned}-\frac{7}{(24\alpha)^3 \cdot 3}\frac{1}{n^3} + \sum_{t=1}^{7} A_t(n) < 2\log p(n) - \log p(n-1) - \log p(n+1)\\ < \sum_{t=1}^{7} A_t(n) + \frac{8}{(24\alpha)^3}\frac{1}{n^3},\end{aligned}$$

which implies

$$\begin{aligned}-\frac{3}{(24\alpha)^3}\frac{1}{n^3} + \sum_{t=1}^{7} A_t(n) < 2\log p(n) - \log p(n-1) - \log p(n+1)\\ < \sum_{t=1}^{7} A_t(n) + \frac{3}{(24\alpha)^3}\frac{1}{n^3}.\end{aligned} \tag{3.51}$$

Finally, we establish bounds for the $A_t(n)$. For $t=1$,

$$A_1(n) = \log(1+\frac{1}{n}) + \log(1-\frac{1}{n}) = -\frac{1}{n^2} - \frac{1}{2n^4} + \sum_{k=5}^{\infty} \frac{(-1)^{k+1}}{kn^k} - \sum_{k=5}^{\infty} \frac{1}{kn^k}.$$

Taking $s=1$ and $m=5$ in Lemma 3.7.5 we have

$$-\frac{1}{n^2} - \frac{1}{2n^4} - \frac{4}{5n^5} < A_1(n) < -\frac{1}{n^2} - \frac{1}{2n^4} + \frac{2}{5n^5}$$

which implies

$$-\frac{1}{n^2} - \frac{2}{n^3} < A_1(n) < -\frac{1}{n^2}. \tag{3.52}$$

For $t=2$, note that

$$A_2(n) = -\pi\sqrt{\frac{2n}{3}}\Big(-\frac{5}{64n^4} - \frac{1}{4n^3} + \sum_{k=5}^{\infty}\binom{1/2}{k}\frac{(-1)^k}{n^k} + \sum_{k=5}^{\infty}\binom{1/2}{k}\frac{1}{n^k}\Big).$$

Applying Lemma 3.7.5, (3.46), with $s=1$ and $m=5$ gives

$$-\pi\sqrt{\frac{2n}{3}}\Big(-\frac{1}{4n^2} - \frac{5}{64n^4} - \frac{4}{5\sqrt{5}}\frac{1}{n^5}\Big) < A_2(n) < -\pi\sqrt{\frac{2n}{3}}\Big(-\frac{1}{4n^2} - \frac{5}{64n^4} + \frac{2}{5\sqrt{5}}\frac{1}{n^5}\Big),$$

which implies,

$$\frac{\pi}{\sqrt{24}n^{3/2}} < A_2(n) < \frac{\pi}{\sqrt{24}n^{3/2}} + \frac{2}{n^{5/2}}. \tag{3.53}$$

Next we consider odd indices; i.e., for $1 \le t \le 3$,

$$A_{2t+1}(n) = -\frac{g_{2t-1}}{(\sqrt{n})^{2t-1}}\Big(\frac{\left(\frac{2t-1}{2}\right)_2}{n^2} + \frac{\left(\frac{2t-1}{2}\right)_4}{12n^4} + \sum_{k=5}^{\infty}\binom{-\frac{2t-1}{2}}{k}\frac{(-1)^k}{n^k} + \sum_{k=5}^{\infty}\binom{-\frac{2t-1}{2}}{k}\frac{1}{n^k}\Big),$$

where $(a)_k := a(a-1)\dots(a-k+1)$. Applying Lemma 3.7.3 with $s=t$ and $m=5$ gives

$$-\frac{g_{2t-1}}{(\sqrt{n})^{2t-1}}\Big(\frac{\left(\frac{2t-1}{2}\right)_2}{n^2} + \frac{\left(\frac{2t-1}{2}\right)_4}{12n^4} - \frac{4\sqrt{t}}{\sqrt{t+4}}\binom{t+4}{t-1}\frac{1}{n^5}\Big) < A_{2t+1}(n)$$
$$< -\frac{g_{2t-1}}{(\sqrt{n})^{2t-1}}\Big(\frac{\left(\frac{2t-1}{2}\right)_2}{n^2} + \frac{\left(\frac{2t-1}{2}\right)_4}{12n^4} + \frac{8\sqrt{t}}{\sqrt{t+4}}\binom{t+4}{t-1}\frac{1}{n^5}\Big),$$

which implies

$$-\frac{3g_1}{4n^{5/2}}+\frac{4g_1}{\sqrt{5}}\frac{1}{n^3}<A_3(n)<-\frac{5g_1}{n^{5/2}}, \tag{3.54}$$

$$\frac{4\sqrt{6}g_3}{n^3}<A_5(n)<-\frac{29g_3}{n^{5/2}}, \tag{3.55}$$

$$\frac{4\sqrt{2}}{\sqrt{7}}\binom{7}{2}\frac{g_5}{n^3}<A_7(n)<-\frac{117g_5}{n^{5/2}}. \tag{3.56}$$

Finally, we consider even indices; i.e., for $1\le t\le 2$,

$$A_{2t+2}(n)=-\frac{g_{2t}}{(\sqrt{n})^{2t}}\Big(\frac{\left(-\frac{2t}{2}\right)_2}{n^2}+\frac{\left(-\frac{2t}{2}\right)_4}{12n^4}+\sum_{k=5}^{\infty}\binom{-\frac{2t}{2}}{k}\frac{(-1)^k}{n^k}+\sum_{k=5}^{\infty}\binom{-\frac{2t}{2}}{k}\frac{1}{n^k}\Big).$$

Applying Lemma 3.7.4 with $s=t$ and $m=5$, we obtain

$$-\Big(\frac{(-t)_2}{n^2}+\frac{(-t)_4}{12n^4}-\frac{2}{n^5}\binom{t+4}{t-1}\Big)\frac{g_{2t}}{(\sqrt{n})^{2t}}<A_{2t+2}(n)$$
$$<-\Big(\frac{(-t)_2}{n^2}+\frac{(-t)_4}{12n^4}+\frac{4}{n^5}\binom{t+4}{t-1}\Big)\frac{g_{2t}}{(\sqrt{n})^{2t}}.$$

From this,

$$\frac{2g_2}{n^3}<A_4(n)<-\frac{8g_2}{n^{5/2}}, \tag{3.57}$$

$$\frac{12g_4}{n^3}<A_6(n)<-\frac{40g_4}{n^{5/2}}. \tag{3.58}$$

Now, substituting (3.52) to (3.58) into (3.51) gives,

$$L(n)<2\log p(n)-\log p(n-1)-\log p(n+1)<U(n),$$

where

$$L(n):=\frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}}-\frac{1}{n^2}-\frac{3g_1}{4}\frac{1}{n^{5/2}}+$$
$$\left(-2+\frac{4g_1}{\sqrt{5}}+2g_2+4\sqrt{6}g_3+12g_4+\frac{4\sqrt{2}}{\sqrt{7}}\binom{7}{2}g_5-\frac{3}{(24\alpha)^3}\right)\frac{1}{n^3}$$

and

$$U(n):=\frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}}-\frac{1}{n^2}+\left(2-5g_1-8g_2-29g_3-40g_4-117g_5+\frac{3}{(24\alpha)^3}\right)\frac{1}{n^{5/2}}.$$

By using numerical estimations of the coefficient of $1/n^{5/2}$ and of the coefficient of $1/n^3$ in the lower bound, and of the coefficient of $1/n^{5/2}$ in the upper bound above, we are led to

$$L_1(n) < 2\log p(n) - \log p(n-1) - \log p(n+1) < U_1(n),$$

with

$$L_1(n) := \frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} - \frac{1}{n^2} + \frac{1}{4}\frac{1}{n^{5/2}} - \frac{4}{n^3} \quad \text{and} \quad U_1(n) := \frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} - \frac{1}{n^2} + \frac{7}{n^{5/2}}.$$

Next we observe that

$$-\frac{1}{n^2} + \frac{7}{n^{5/2}} < -\frac{\pi^2}{48n^3} \text{ for all } n \geq 50$$

and

$$-\frac{1}{n^2} + \frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} + \frac{1}{4}\frac{1}{n^{5/2}} - \frac{4}{n^3} > -\frac{1}{n^2} + \frac{\pi}{\sqrt{24}}\frac{1}{n^{3/2}} \text{ for all } n \geq 257.$$

Therefore, for $n \geq 257$,

$$\frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2} < 2\log p(n) - \log p(n-1) - \log p(n+1) < \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{\pi^2}{48n^3}. \quad (3.59)$$

Because of $\log(1+x) < x$ for $x > 0$, we have

$$\log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\right) < \frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}, \quad (3.60)$$

and because of $x - \frac{x^2}{2} < \log(1+x)$ for all $x > 0$, we have

$$\frac{\pi}{\sqrt{24}n^{3/2}} - \frac{\pi^2}{48n^3} < \log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right). \quad (3.61)$$

Applying (3.60) and (3.61) to (3.59) gives

$$\log\left(1+\frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\right) < 2\log p(n) - \log p(n-1) - \log p(n+1) < \log\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right),$$

which after exponentiation gives the desired result for $n \geq 257$. To extend the proofs of the statements for $n \geq 45$, resp. $n \geq 120$, is done by straight forward numerics. □

3.8 Appendix

3.8.1 Methods to discover the results

We will describe very briefly the mathematical experiments used in this research. We want to point out that without these experiments, the theoretical results of this chapter would never have been found. For this reason we feel that it is important to give at least a brief sketch of what led us to the final formulas and how we were led to conjecture special cases of related asymptotics. The final asymptotic formulas can easily be derived from our main result, Theorem 3.6.6 presented in Section 3.6.

In Section 3.3 we proved the inequality

$$\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1-\frac{1}{2\sqrt{n}}\Big) < p(n) < \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}\Big(1-\frac{1}{3\sqrt{n}}\Big), \tag{3.62}$$

which was found by mathematical experiments. Our proof uses methods similar to those used in [53] and [37]. In our attempt to prove the following formula for the asymptotics of $\log p(n)$,

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log n - \log(4\sqrt{3}) - \frac{0.44\ldots}{\sqrt{n}}, \tag{3.63}$$

we first tried to prove the log-version of (3.62). However, we soon realised that this inequality is not sharp enough in order to prove (3.63). We noted that the inequality for $p(n)$ in [37, Lemma 2.2] can be used instead. This formula says that for $n \geq 1206$,

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{10}}\Big) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{10}}\Big), \tag{3.64}$$

where $\mu(n) := \frac{\pi}{6}\sqrt{24n-1}$. We observed that after taking the log of both sides, with some extra work, (3.63) can be proven. When we saw the asymptotics (3.3), discovered by Schoenfield and Kotesovec, we naturally wondered whether these asymptotics can also be proven by taking the log of an appropriate inequality. We observed that (3.64) is enough also to prove these asymptotics, and we observed that (3.64) can be used to prove an even more refined asymptotic formula that takes the form

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log n - \log 4\sqrt{3} + b_1\Big(\frac{1}{\sqrt{n}}\Big) + b_2\Big(\frac{1}{\sqrt{n}}\Big)^2 + \cdots + b_9\Big(\frac{1}{\sqrt{n}}\Big)^9,$$

where

$$
\begin{aligned}
b_1 &= -\frac{\pi\sqrt{6}}{2^4 3^2} - \frac{\sqrt{6}}{2\pi} \approx -0.44328...,\\
b_2 &= \frac{1}{3\cdot 2^3} - \frac{3}{2^2\pi^2} \approx -0.034324...,\\
b_3 &= -\frac{\pi\sqrt{6}}{2^9 3^3} - \frac{\sqrt{6}}{2^5 3\pi} - \frac{\sqrt{6}}{2^2\pi^3} \approx -0.028428...,\\
b_4 &= \frac{1}{2^7 3^2} - \frac{1}{2^5\pi^2} - \frac{9}{2^4\pi^4} \approx -0.0080728...,\\
b_5 &= -\frac{\pi\sqrt{6}}{2^{13}3^4} - \frac{\sqrt{6}}{2^{10}3\pi} - \frac{\sqrt{6}}{2^6\pi^3} - \frac{9\sqrt{6}}{5\cdot 8\pi^5} \approx -0.0033007...,\\
b_6 &= \frac{1}{2^9 3^4} - \frac{1}{2^8 3\pi^2} - \frac{3}{2^6\pi^4} - \frac{9}{2^4\pi^6} \approx -0.001174124716...,\\
b_7 &= -\frac{5\pi\sqrt{6}}{2^{19}3^5} - \frac{5\sqrt{6}}{2^{14}3^3\pi} - \frac{5\sqrt{6}}{2^{11}3\pi^3} - \frac{3\sqrt{6}}{2^7\pi^5} - \frac{3^3\sqrt{6}}{2^4 7\pi^7} \approx -0.00045651...,\\
b_8 &= \frac{1}{2^{14}3^4} - \frac{1}{2^{11}3^2\pi^2} - \frac{3}{2^{10}\pi^4} - \frac{3^2 2}{2^7\pi^6} - \frac{3^4}{2^7\pi^8} \approx -0.00017464...,\\
b_9 &= -\frac{7\pi\sqrt{6}}{2^{23}3^6} - \frac{35\sqrt{6}}{2^{20}3^4\pi} - \frac{35\sqrt{6}}{2^{15}3^3\pi^3} - \frac{7\sqrt{6}}{2^{12}\pi^5} - \frac{9\sqrt{6}}{2^8\pi^7} - \frac{9\sqrt{6}}{2^5\pi^9} \approx -0.000068757...,\\
&\vdots
\end{aligned}
$$

Of course we wondered whether one can get an even better formula. The only obstacle that seemed to limit us was the 10 in the formula (3.64) above. This led us to look into the details of the proof of (3.64), and we observed that the 10 can be replaced by a k. This then led us to the discovery of the complete asymptotics. That is, we also got $b_{10}, b_{11}, \ldots$, etc. At this point we still were not fully satisfied. Even though we observed that the formula (3.64) could be generalised, it was not a proper generalization because we could not say explicitly for which precise range of n the generalized inequality (3.18) for $p(n)$ holds. We only could say that there is some sufficiently big constant $C(k)$ such that (3.18) for all $n > C(k)$.

We felt that this is not a proper generalization because (3.64) gives $C(10)$ explicitly, namely $C(10) = 1206$. After some work, we realized that we can obtain an explicit expression for $C(k)$, which is very close to the optimal value, according to mathematical experiments. This $C(k)$ is our $g(k)$ of Section 3.4 where we gave a generalization of (3.64).

Because (3.64) could be generalized, we suspected that also (3.62) could be generalized. The difference between the two inequalities is that (3.62) is in terms of

```
with(combinat);
rt := proc (n) local rtn, k;
rtn := combinat:-numbpart(n);
for k to (1/2)*n do
rtn := rtn+combinat:-numbpart(k)*combinat:-numbpart(n-2*k)
end do;
rtn
end proc
```

Figure 3.3: Procedure for computing the number of cubic partitions of n.

$\sqrt{n}$, while (3.64) is in terms of $\mu(n)$. We again took the log of both sides of the generalized version of (3.64) and aimed not only at getting a refined asymptotic but rather a new type of inequality. This was achieved in Section 3.6. However, even after we found a preliminary version of Theorem 3.6.6, still something was missing. We wondered whether we can guarantee that this inequality is optimal in some sense, and not overestimated. After various experiments, we got control in the form (3.28) and (3.29), where the error term in the inequality cannot be improved to a smaller integer in the numerator—the same time keeping the statement unaltered. This is the point where we stopped.

3.8.2 Discovery of Kotesovec's formula (3.5) by regression analysis

We used the procedure shown in Figure 3.3 to compute the sequence $a(n)$ defined in (3.4). This procedure works fine for computing $a(n)$ in the range $1 \leq n \leq 2^{15}$. The computation took 24 hours on a notebook computer with Intel Core i7 CPU.

To find the approximate relation between $\log a(n)$, $\sqrt{n}$ and $\log(n)$, substitute the values $n = 2^k,\ 2^{k+1},\ 2^{k+2}$ into the target expression,

$$\log a(n) \sim \alpha \cdot \sqrt{n} - \beta \cdot \log(n) - \log(\gamma),$$

to obtain a system with three equations:

$$\begin{cases} \log_2 a(2^k) = a_k \log_2(e) \cdot \sqrt{2^k} - b_k \cdot k - c_k + \varepsilon_k, \\ \log_2 a(2^{k+1}) = a_k \log_2(e) \cdot \sqrt{2^{k+1}} - b_k \cdot (k+1) - c_k + \varepsilon_{k+1}, \\ \log_2 a(2^{k+2}) = a_k \log_2(e) \cdot \sqrt{2^{k+2}} - b_k \cdot (k+2) - c_k + \varepsilon_{k+2}, \end{cases}$$

and solve it successively for k from 1 to 13. Let (a_k, b_k, c_k) be the solution of the above equation system under the assumption $\varepsilon_k = \varepsilon_{k+1} = \varepsilon_{k+2} = 0$ for all $k \in \{1, \ldots, 13\}$. The numerical values of the (a_k, b_k, c_k) are presented in Figure 3.4. In the limit $k \to \infty$,

$$\begin{aligned}
a_k &= \tfrac{\log_2 a(2^k)+\log_2 a(2^{k+2})-2\log_2 a(2^{k+1})}{(3-2\sqrt{2})\sqrt{2^k}\log_2(e)} \to \alpha, \\
b_k &= \tfrac{\log_2 a(2^k)+\log_2 a(2^{k+2})-2\log_2 a(2^{k+1})}{\sqrt{2}-1} - \{\log_2 a(2^{k+1}) - \log_2 a(2^k)\} \to \beta, \\
c_k &= \tfrac{2^{a_k \log_2(e)\sqrt{2^k} - k\, b_k}}{a(2^k)} \to \log_2(\gamma).
\end{aligned}$$

$a_1 = 2.856681587,$	$b_1 = 0.829251071,$	$c_1 = 3.414213592,$
$a_2 = 3.104034810,$	$b_2 = 1.124879663,$	$c_2 = 3.536666941,$
$a_3 = 3.138359816,$	$b_3 = 1.182896591,$	$c_3 = 3.502681525,$
$a_4 = 3.134634608,$	$b_4 = 1.173992098,$	$c_4 = 3.516802205,$
$a_5 = 3.133095462,$	$b_5 = 1.168789090,$	$c_5 = 3.530255390,$
$a_6 = 3.135881324,$	$b_6 = 1.182107364,$	$c_6 = 3.482499147,$
$a_7 = 3.138560309,$	$b_7 = 1.200219526,$	$c_7 = 3.399441064,$
$a_8 = 3.140063351,$	$b_8 = 1.214590204,$	$c_8 = 3.319170509,$
$a_9 = 3.140825268,$	$b_9 = 1.224893620,$	$c_9 = 3.251316705,$
$a_{10} = 3.141207944,$	$b_{10} = 1.232211776,$	$c_{10} = 3.195805627,$
$a_{11} = 3.141399944,$	$b_{11} = 1.237403601,$	$c_{11} = 3.151230912,$
$a_{12} = 3.141496152,$	$b_{12} = 1.241082894,$	$c_{12} = 3.115957155,$
$a_{13} = 3.141544378,$	$b_{13} = 1.243690699,$	$c_{13} = 3.088371824.$

Figure 3.4: Numerical values of the (a_k, b_k, c_k).

The numerical values in Figure 3.4 clearly support the precise values

$$\alpha = \pi, \quad \beta = \frac{5}{4}, \quad \gamma = 2^3 = 8.$$

Note that we have used a sub-sequence $a(2^k)$, $k = 1, 2, \ldots, 15$. The regression analysis to obtain the numerical data for Fig. 3.1 and Fig. 3.2 are rather routine, so we will not list any further details here.

3.8.3 Mathematica computations

We present Mathematica computations needed in the proof of Theorem 3.6.6. Note that in order to complete the proof of Theorem 3.6.6 we needed to bound four terms by 1; however, in each inequality proven with Mathematica as shown below, we checked that each inequality holds in fact for bounds smaller than 1, namely $\frac{1}{5}$, $\frac{1}{3}$, $\frac{1}{26}$ and $\frac{1}{26}$. The Mathematica computations are based on Cylindrical Algebraic Decomposition [44].

In[1]:= $\mathbf{a := \frac{\pi^2}{36+\pi^2}}$

In[2]:= $\mathbf{(mu1, mu2, nu) := \left(\frac{\sqrt{6}}{2\pi}\left(\frac{\pi^2}{72}+1+\frac{a}{2(1-a)}\right), \frac{\sqrt{6}}{2\pi}\left(1+\frac{a}{2}\right), 3\frac{a}{2}-\frac{1}{2}\right)}$

In[3]:= $\mathbf{CylindricalDecomposition[\{(2w-2)^2\frac{\left(\frac{-\mu 1}{2w-1}+\frac{-x}{(24a)2w}\right)^2}{(-1+x)^2} < \frac{1}{5}, w \geq 1, x \geq 4\}, \{w, x\}]}$

Out[3]= $w \geq 1$ && $x \geq 4$

In[4]:= $\mathbf{CylindricalDecomposition[\{\left(\left(\frac{2w-3}{x-24\ a\ mu1}\right)\left(\frac{1}{2w-2}+\frac{x}{24a(2w-1)}\right)\right)^2 < \frac{1}{3}, w \geq 2, x \geq 7\}, \{w, x\}]}$

Out[4]= $w \geq 2$ && $x \geq 7$

In[5]:= $\mathbf{CylindricalDecomposition[\{\left((2w-2)\frac{\left(\frac{-mu2}{2\ w-1}+\frac{y}{24a(2w)}\right)}{y-2nu}\right)^2 < \frac{1}{26}, w \geq 1, y \geq 1\}, \{w, y\}]}$

Out[5]= $w \geq 1$ && $y \geq 1$

In[6]:= $\mathbf{CylindricalDecomposition[\{\left(\left(\frac{2w-3}{y+24\ a\ mu2}\right)\left(\frac{2\ nu}{2w-2}+\frac{y}{24a(2w-1)}\right)\right)^2 < \frac{1}{26}, w \geq 2, y \geq 1\}, \{w, y\}]}$

Out[6]= $w \geq 2$ && $y \geq 1$

Chapter 4

A unified framework to prove multiplicative inequalities for the partition function

In this chapter, we consider a certain class of inequalities for the partition function of the following form:

$$\prod_{i=1}^{T} p(n+s_i) \geq \prod_{i=1}^{T} p(n+r_i),$$

which we call multiplicative inequalities. Given a multiplicative inequality with the condition that $\sum_{i=1}^{T} s_i^m \neq \sum_{i=1}^{T} r_i^m$ for at least one $m \geq 1$, we shall construct an unified framework so as to decide whether such a inequality holds or not. As a consequence, we will see that study of such inequalities has manifold applications. For example, one can retrieve the log-concavity property, strong log-concavity, and the inequalities for $p(n)$ considered by Bessenrodt and Ono, to name a few. Furthermore, we obtain the full asymptotic expansion for the finite difference of the logarithm of $p(n)$, denoted by $(-1)^{r-1}\Delta^r \log p(n)$, which extends a result by Chen, Wang, and Xie.

4.1 Multiplicative inequalities for $p(n)$

A partition of a positive integer n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. Hardy and Ramanujan [76] studied the asymptotic growth of $p(n)$

as follows:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty. \tag{4.1}$$

Rademacher [122, 124, 123] improved the work of Hardy and Ramanujan and found a convergent series for $p(n)$ and Lehmer's [99, 98] study was on estimation for the remainder term of the series for $p(n)$. The Hardy-Ramanujan-Rademacher formula reads

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] + R_2(n, N), \tag{4.2}$$

where

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \quad A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i s(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \left\lfloor \frac{\mu}{k} \right\rfloor - \frac{1}{2} \right) \left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2} \right),$$

and

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)} \right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)} \right)^2 \right]. \tag{4.3}$$

A sequence $\{a_n\}_{n \geq 0}$ is said to satisfy the Turán inequlaities or to be log-concave, if

$$a_n^2 - a_{n-1}a_{n+1} \geq 0 \quad \text{for all} \quad n \geq 1. \tag{4.4}$$

Independently Nicolas [111] and DeSalvo and Pak [53, Theorem 1.1] proved that the partition function $p(n)$ is log-concave for all $n \geq 26$, conjectured by Chen [35]. DeSalvo and Pak [53, Theorem 4.1] also proved that for all $n \geq 2$,

$$\frac{p(n-1)}{p(n)} \left(1 + \frac{1}{n}\right) > \frac{p(n)}{p(n+1)}, \tag{4.5}$$

conjectured by Chen [35]. Further, they improved the rate of decay in (4.5) and proved that for all $n \geq 7$,

$$\frac{p(n-1)}{p(n)} \left(1 + \frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \tag{4.6}$$

see [53, p. 4.2]. DeSalvo and Pak [53] finally came up with the conjecture that the coefficient of $1/n^{3/2}$ in (4.6) can be improved to $\pi/\sqrt{24}$; i.e., for all $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \tag{4.7}$$

which was proved by Chen, Wang and Xie [39, Sec. 2]. Recently, the author along with Paule, Radu, and Zeng [22, Theorem 7.6] confirmed that the coefficient of $1/n^{3/2}$ is $\pi/\sqrt{24}$, which is optimal; i.e., they proved that for all $n \geq 120$,

$$p(n)^2 > \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{1}{n^2}\right)p(n-1)p(n+1). \tag{4.8}$$

DeSalvo and Pak [53, Theorem 5.1] also established that $p(n)$ satisfies the strong log-concavity property; i.e., for all $n > m > 1$,

$$p(n)^2 - p(n-m)p(n+m) > 0. \tag{4.9}$$

Ono and Bessenrodt [26] extended (4.9) by considering the border case $m = n$. This leads to the unveiling of multiplicative properties of the partition function encoded in the following theorem.

Theorem 4.1.1. *[26, Theorem 2.1] If a and b are integers with $a, b > 1$ and $a+b > 8$, then*

$$p(a)p(b) \geq p(a+b), \tag{4.10}$$

with equality holding only for $\{a, b\} = \{2, 7\}$.

Let Δ be the forward difference operator define by $\Delta a(n) := a(n+1) - a(n)$ for a sequence $(a(n))_{n\geq 0}$. It is clear that the log-concavity property for $p(n)$ is equivalent to say that $-\Delta^2 \log p(n-1) > 0$ for all $n \geq 26$. Equations (4.7) and (4.8) show the asymptotic growth of $-\Delta^2 \log p(n-1)$. Chen, Wang, and Xie proved the positivity of $(-1)^{r-1}\Delta^r \log p(n)$ along with the estimation of an upper bound.

Theorem 4.1.2. *[39, Thm. 3.1 and 4.1] For each $r \geq 1$, there exists a positive integer $n(r)$ such that for all $n \geq n(r)$,*

$$0 < (-1)^{r-1}\Delta^r \log p(n) < \log\left(1+\frac{\pi}{\sqrt{6}}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right). \tag{4.11}$$

The above inequalities can be rephrased in the following form:

$$\prod_{i=1}^{T} p(n+s_i) \geq \prod_{i=1}^{T} p(n+r_i), \tag{4.12}$$

which we call multiplicative inequalities for the partition function. Instead of applying the Hardy-Ramanujan-Rademacher formula (4.2) and Lehmer's error bound (4.3) but with different methodology for different inequalities for $p(n)$ as done in [26, 53, 111, 39], we will see how one can prove all such multiplicative inequalities under a unified framework so as to decide explicitly $N(T)$, such that for all $n \geq N(T)$, (4.12) holds. To prove (4.12), it is equivalent to show

$$\sum_{i=1}^{T} \log p(n+s_i) \geq \sum_{i=1}^{T} \log p(n+r_i), \tag{4.13}$$

and therefore, an infinite family of inequalities for logarithm of the shifted version of the partition function is a prerequisite, see Theorems 4.3.9 and 4.3.13. As an application of Theorem 4.3.9, we shall complete Theorem 4.1.2 (see Theorems 4.4.6 and 4.4.7 below) in the following aspects:

1. by improving the lower bound in (4.11) to show that the rate of decay given in the upper bound is the optimal one,

2. for each $r \geq 1$, computation of $n(r)$ by estimation of error bound based on the minimal choice of the truncation point w in Theorem 4.3.9,

3. and a full asymptotic expansion for $(-1)^{r-1}\Delta^r \log p(n)$. This seems to be inaccessible from Theorem 4.1.2 because a key tool in the proof was on the relations between the higher order differences and derivatives (cf. Prop. 3.5, [39]) due to Odlyzko [114] which only contributes to the main term in the expansion; i.e., $\dfrac{\pi}{\sqrt{6}}\left(\dfrac{1}{2}\right)_{r-1}\dfrac{1}{(n+1)^{r-\frac{1}{2}}}$.

Even having Theorem 4.3.13 in hand, in order to decide whether (4.12) holds or not, there are two key factors that remain unexplained. First, an explanation of the following assumption

$$\sum_{i=1}^{T} s_i^m \neq \sum_{i=1}^{T} r_i^m \ \text{ for at least one } \ m \geq \mathbb{Z}_{\geq 1}. \tag{4.14}$$

and an appropriate choice of w, i.e., the truncation point as in Theorem 4.3.13. Now we move on to see how these two factors are intricately connected through a classical problem in Diophantine equations known as the Prouhet-Tarry-Escott problem [55, Chapter XXIV]. The problem originated in a different guise from a letter of Goldbach [65] to Euler that dates back to 18 July 1750. The Prouhet–Tarry–Escott problem asks for two distinct tuples of integers $(s_1, s_2, \dots, s_T)$ and $(r_1, r_2, \dots, r_T)$ such that

$$\sum_{i=1}^{T} s_i^k = \sum_{i=1}^{T} r_i^k, \quad \text{for all } 0 \le k \le m-1 \quad \text{and} \quad \sum_{i=1}^{T} s_i^m \neq \sum_{i=1}^{T} r_i^m.$$

We write $(s_1, \dots, s_T) \overset{m}{=} (r_1, \dots, r_T)$ to denote a solution of the Prouhet–Tarry–Escott problem. Recently, Merca and Katriel [108] connect the non-trivial linear homogeneous partition inequalities with the Prouhet–Tarry–Escott problem. In brevity, we shall explain why the optimal choice of truncation point $w = m + 1$, with $(s_1, \dots, s_T) \overset{m}{=} (r_1, \dots, r_T)$ for a given (4.12) in Section 4.5.

The rest of the chapter is organized as follows. In Section 4.2, we state preliminary lemmas and theorems from the work of Paule, Radu, Zeng, and the author [22]. Section 4.3 presents a detailed synthesis on derivation of inequalities for $\log p(n+s)$ for any non-negative integer s that leads to the main result of this chapter, see Theorem 4.3.13. As an application of Theorem 4.3.13, we provide a full asymptotic expansion of $(-1)^{r-1}\Delta^r \log p(n)$ in Section 4.4. In Section 4.5, we work out the steps to verify multiplicative inequalities for the partition function. Section 4.6 is devoted to derive an infinite families of inequalities for $\prod_{i=1}^{T} p(n+s_i)$, given in Theorem 4.6.9. Finally we conclude this chapter with a short discussion on the applications of Theorems 4.3.13 and 4.6.9.

4.2 Set up

Throughout this section, we follow the notations as in [22].

Definition 4.2.1 (Def. 5.1, [22]). *For $y \in \mathbb{R}$, $0 < y^2 < 24$, we define*

$$G(y) := -\log\Big(1 - \frac{y^2}{24}\Big) + \frac{\pi\sqrt{24}}{6y}\left(\sqrt{1 - \frac{y^2}{24}} - 1\right) + \log\left(1 - \frac{y}{\frac{\pi}{6}\sqrt{24 - y^2}}\right), \quad (4.15)$$

and its sequence of Taylor coefficients by

$$G(y) = \sum_{u=1}^{\infty} g_u y^u. \quad (4.16)$$

Define $\alpha := \dfrac{\pi^2}{36+\pi^2}$.

Lemma 4.2.2 (Lem. 5.4, [22]). *Let $G(y) = \sum_{u=1}^{\infty} g_u y^u$ be the Taylor expansion of $G(y)$ as in Definition 4.2.1. Then for $n \geq 1$,*

$$g_{2n} = \frac{1}{3^n 2^{3n} n} - \frac{1}{3^n 2^{3n+1} n}\Big(-1 + \frac{1}{\alpha^n}\Big), \tag{4.17}$$

and for $n \geq 0$,

$$g_{2n+1} = \sqrt{6}\left[(-1)^{n+1}\binom{1/2}{n+1}\frac{\pi}{2^{3n+3}3^{n+2}} - \frac{1}{2^{3n+1}3^n\alpha^n(2n+1)\pi}\sum_{j=0}^{n}\alpha^j\binom{-\frac{1}{2}+j}{j}\right]. \tag{4.18}$$

Lemma 4.2.3 (Lem. 5.8, [22]). *For $n \geq 0$, we have*

$$-\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(2n+1)}\Big(\frac{\pi^2}{72} + 1 + \frac{\alpha}{2(1-\alpha)}\Big) \leq g_{2n+1} \leq -\frac{\sqrt{6}}{2\pi 2^{3n}3^n\alpha^n(2n+1)}\Big(1 + \frac{\alpha}{2}\Big). \tag{4.19}$$

Lemma 4.2.4 (Lem. 5.9, [22]). *For $n \geq 1$, we have*

$$-\frac{1}{3^n 2^{3n+1}\alpha^n n} \leq g_{2n} \leq \frac{1}{3^n 2^{3n}\alpha^n n}\Big(\frac{3\alpha}{2} - \frac{1}{2}\Big). \tag{4.20}$$

Definition 4.2.5 (Def. 4.3, [22]). *For $k \in \mathbb{Z}_{\geq 2}$, define*

$$g(k) := \frac{1}{24}\left(\frac{6^2}{\pi^2}\cdot\nu(k)^2 + 1\right),$$

where $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}$.

Definition 4.2.6 (Def. 6.4, [22]). *For $n, U \in \mathbb{Z}_{\geq 1}$, we define*

$$P_n(U) := -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{U} g_u(1/\sqrt{n})^u.$$

Theorem 4.2.7 (Thm 6.6, [22]). *Let $G(y) = \sum_{u=1}^{\infty} g_u y^u$ as in Definition 4.2.1. Let $g(k)$ be as in Definition 4.2.5 and $P_n(U)$ as in Definition 4.2.6. If $m \geq 1$ and $n > g(2m)$, then*

$$P_n(2m-1) - \frac{2}{3^m 2^{3m}\alpha^m n^m m} < \log p(n) < P_n(2m-1) + \frac{1}{3^m 2^{3m}\alpha^m n^m m}; \tag{4.21}$$

if $m \geq 2$ and $n > g(2m-1)$, then

$$P_n(2m-2) - \frac{7}{3^m 2^{3m} \alpha^m n^{m-1/2}(2m-1)} < \log p(n) < P_n(2m-2) + \frac{2}{3^m 2^{3m} \alpha^m n^{m-1/2}(2m-1)}. \tag{4.22}$$

In other words, for $w \in \mathbb{Z}_{>0}$ with $\lceil w/2 \rceil \geq \gamma_0$ and $n > g(w)$, we have

$$P_n(w-1) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil}} \left(\frac{1}{\sqrt{n}}\right)^w < \log p(n) < P_n(w-1) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil}} \left(\frac{1}{\sqrt{n}}\right)^w, \tag{4.23}$$

where

$$(\gamma_0, \gamma_1, \gamma_2) = \begin{cases} (1, 4, 2), & \text{if } w \text{ is even} \\ (2, 7, 2), & \text{if } w \text{ is odd} \end{cases}. \tag{4.24}$$

Lemma 4.2.8 (Lem 7.3, [22]). *For $n, s \in \mathbb{Z}_{\geq 1}, m \in \mathbb{N}$ and $n > 2s$, let*

$$b_{m,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+m-1}} \binom{s+m-1}{s-1} \frac{1}{n^m},$$

then

$$-b_{m,n}(s) < \sum_{k=m}^{\infty} \binom{-\frac{2s-1}{2}}{k} \frac{1}{n^k} < b_{m,n}(s) \tag{4.25}$$

and

$$0 < \sum_{k=m}^{\infty} \binom{-\frac{2s-1}{2}}{k} \frac{(-1)^k}{n^k} < b_{m,n}(s). \tag{4.26}$$

Lemma 4.2.9 (Lem 7.4, [22]). *For $n, s \in \mathbb{Z}_{\geq 1}, m \in \mathbb{N}$ and $n > 2s$, let*

$$\beta_{m,n}(s) := \frac{2}{n^m} \binom{s+m-1}{s-1},$$

then

$$-\beta_{m,n}(s) < \sum_{k=m}^{\infty} \binom{-s}{k} \frac{1}{n^k} < \beta_{m,n}(s) \tag{4.27}$$

and

$$0 < \sum_{k=m}^{\infty} \binom{-s}{k} \frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{4.28}$$

Lemma 4.2.10 (Lem 7.5, [22]). *For $m, n, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$, let*

$$c_{m,n}(s) := \frac{2}{m}\frac{s^m}{n^m},$$

then

$$-c_{m,n}(s) < \sum_{k=m}^{\infty} \frac{(-1)^{k+1}}{k}\frac{s^k}{n^k} < c_{m,n}(s) \quad \text{and} \quad -c_{m,n}(s) < -\sum_{k=m}^{\infty} \frac{1}{k}\frac{s^k}{n^k} < 0 \tag{4.29}$$

and

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty} \binom{1/2}{k}\frac{s^k}{n^k} < \frac{c_{m,n}(s)}{\sqrt{m}} \quad \text{and} \quad -\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty} \binom{1/2}{k}\frac{(-1)^k s^k}{n^k} < 0. \tag{4.30}$$

4.3 Inequalities for $\log p(n; \vec{\mathbf{s}})$

In this section, first we prove an infinite family of inequalities for $\log p(n+s)$ with s being a non-negative integer, see Theorem 4.3.9. Starting from Theorem 4.2.7, we will estimate $P_{n+s}(U)$ and the error terms given in (4.21) and (4.22), stated in Lemmas 4.3.3-4.3.6. Finally, generalizing Theorem 4.3.9 by taking into consideration $\sum_{i=1}^{T} \log p(n+s_i)$ for $(s_1, s_2, \ldots, s_T) \in \mathbb{Z}_{\geq 0}^T$, we obtain Theorem 4.3.13.

Lemma 4.3.1. *Let the coefficient sequence $(g_n)_{n\geq 1}$ be as in Lemma 4.2.2. Then for all $n \geq 1$, we have*

$$|g_n| \leq \frac{1}{n}\frac{1}{(24\alpha)^{\lfloor n/2 \rfloor}}. \tag{4.31}$$

Proof. Observe that for all $n \geq 0$, $\frac{\sqrt{6}}{2\pi}\left(1 + \frac{\alpha}{2}\right)\frac{1}{(24\alpha)^n(2n+1)} > 0$ and $0 < \frac{\sqrt{6}}{2\pi}\left(\frac{\pi^2}{72} + 1 + \frac{\alpha}{2(1-\alpha)}\right) < 1$. Using (4.19), we obtain for all $n \geq 0$,

$$-\frac{1}{(24\alpha)^n(2n+1)} < g_{2n+1} < 0. \tag{4.32}$$

Since $\frac{3\alpha}{2} - \frac{1}{2} < 0$, from (4.20), it follows that for all $n \geq 1$,

$$-\frac{1}{(24\alpha)^n(2n)} \leq g_{2n} < 0. \tag{4.33}$$

From (4.32) and (4.33), we conclude that for all $n \geq 1$,

$$|g_n| \leq \frac{1}{(24\alpha)^{\lfloor n/2 \rfloor} n}.$$

$\square$

Definition 4.3.2. *For $s \in \mathbb{Z}_{\geq 0}$, define*

$$\delta_s := \begin{cases} 1, & \text{if } s \geq 1 \\ 0, & \text{if } s = 0 \end{cases}.$$

Lemma 4.3.3. *For $(n, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s$, let*

$$P^1_{n,s}(w) := -\log n + \sum_{k=1}^{\lfloor \frac{w-1}{2} \rfloor} \frac{(-1)^k s^k}{k} \left(\frac{1}{\sqrt{n}}\right)^{2k} \text{ and } E^1_{n,s}(w) := \frac{2s^{\lceil \frac{w+1}{2} \rceil}}{\lceil w/2 \rceil} \left(\frac{1}{\sqrt{n}}\right)^{w} \delta_s,$$

then

$$P^1_{n,s}(w) - E^1_{n,s}(w) \leq -\log(n+s) \leq P^1_{n,s}(w) + E^1_{n,s}(w). \tag{4.34}$$

Proof. For all $n, s \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s$, we split $\log(n + s)$ as follows

$$\log(n+s) = \log n + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k} = \log n + \sum_{k=1}^{\lfloor \frac{w-1}{2} \rfloor} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k} + \sum_{k=\lceil \frac{w}{2} \rceil}^{\infty} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k}. \tag{4.35}$$

Applying (4.29) with $m \mapsto \lceil \frac{w}{2} \rceil$, it follows that for all $n > 2s$,

$$-\frac{2}{\lceil w/2 \rceil} \left(\frac{s}{n}\right)^{\lceil w/2 \rceil} < \sum_{k=\lceil \frac{w}{2} \rceil}^{\infty} \frac{(-1)^{k+1}}{k} \frac{s^k}{n^k} < \frac{2}{\lceil w/2 \rceil} \left(\frac{s}{n}\right)^{\lceil w/2 \rceil}. \tag{4.36}$$

Since for all $s \in \mathbb{Z}_{\geq 0}$, $s^{\lceil w/2 \rceil} \leq s^{\lceil \frac{w+1}{2} \rceil}$, from (4.35) and (4.36), it follows that

$$P^1_{n,s}(w) - E^1_{n,s}(w) \leq -\log(n+s) \leq P^1_{n,s}(w) + E^1_{n,s}(w). \tag{4.37}$$

Observe that equality holds when $s = 0$. $\square$

Lemma 4.3.4. *For* $(n,s)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$*,* $w\in\mathbb{Z}_{\geq 2}$*, and* $n>2s$*, let*

$$P^2_{n,s}(w):=\pi\sqrt{\frac{2n}{3}}+\pi\sqrt{\frac{2}{3}}\sum_{k=1}^{\lfloor\frac{w}{2}\rfloor}\binom{1/2}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k-1}\ \text{and}\ E^2_{n,s}(w):=\frac{6s^{\lceil\frac{w+1}{2}\rceil}}{\lceil w/2\rceil}\Big(\frac{1}{\sqrt{n}}\Big)^{w}\delta_s,$$

then

$$P^2_{n,s}(w)-E^2_{n,s}(w)\leq\pi\sqrt{\frac{2n+2s}{3}}\leq P^2_{n,s}(w)+E^2_{n,s}(w). \tag{4.38}$$

Proof. For all $n,s\in\mathbb{Z}_{\geq 1}$, $w\in\mathbb{Z}_{\geq 2}$, and $n>2s$, we split $\pi\sqrt{\frac{2n+2s}{3}}$ as follows

$$\pi\sqrt{\frac{2n+2s}{3}}=\pi\sqrt{\frac{2n}{3}}+\pi\sqrt{\frac{2}{3}}\sum_{k=1}^{\lfloor\frac{w}{2}\rfloor}\binom{1/2}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k-1}+\pi\sqrt{\frac{2n}{3}}\sum_{k=\lfloor\frac{w+2}{2}\rfloor}^{\infty}\binom{1/2}{k}\frac{s^k}{n^k}. \tag{4.39}$$

Applying (4.30) with $m\mapsto\lfloor\frac{w+2}{2}\rfloor$, it follows that for all $n>2s$,

$$-\frac{2}{\big(\lfloor\frac{w+2}{2}\rfloor\big)^{3/2}}\Big(\frac{s}{n}\Big)^{\lfloor\frac{w+2}{2}\rfloor}<\sum_{k=\lfloor\frac{w+2}{2}\rfloor}^{\infty}\binom{1/2}{k}\frac{s^k}{n^k}<\frac{2}{\big(\lfloor\frac{w+2}{2}\rfloor\big)^{3/2}}\Big(\frac{s}{n}\Big)^{\lfloor\frac{w+2}{2}\rfloor}. \tag{4.40}$$

Therefore,

$$\begin{aligned}-2\pi\sqrt{\frac{2}{3}}\frac{s^{\lfloor\frac{w+2}{2}\rfloor}}{\big(\lfloor\frac{w+2}{2}\rfloor\big)^{3/2}}\Big(\frac{1}{\sqrt{n}}\Big)^{2\lfloor\frac{w+2}{2}\rfloor-1}&<\pi\sqrt{\frac{2n}{3}}\sum_{k=\lfloor\frac{w+2}{2}\rfloor}^{\infty}\binom{1/2}{k}\frac{s^k}{n^k}\\&<2\pi\sqrt{\frac{2}{3}}\frac{s^{\lfloor\frac{w+2}{2}\rfloor}}{\big(\lfloor\frac{w+2}{2}\rfloor\big)^{3/2}}\Big(\frac{1}{\sqrt{n}}\Big)^{2\lfloor\frac{w+2}{2}\rfloor-1}.\end{aligned} \tag{4.41}$$

Now for all $s\in\mathbb{Z}_{\geq 0}$,

$$\pi\sqrt{\frac{2}{3}}\frac{s^{\lfloor\frac{w+2}{2}\rfloor}}{\big(\lfloor\frac{w+2}{2}\rfloor\big)^{3/2}}\Big(\frac{1}{\sqrt{n}}\Big)^{2\lfloor\frac{w+2}{2}\rfloor-1}<\frac{6s^{\lceil\frac{w+1}{2}\rceil}}{\lceil w/2\rceil}\Big(\frac{1}{\sqrt{n}}\Big)^{w}.$$

From (4.39) and (4.41), it follows that

$$P^2_{n,s}(w)-E^2_{n,s}(w)\leq\pi\sqrt{\frac{2n+2s}{3}}\leq P^2_{n,s}(w)+E^2_{n,s}(w), \tag{4.42}$$

with equality holds for $s=0$. ∎

Lemma 4.3.5. *For* $(n,s)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$*,* $w\in\mathbb{Z}_{\geq 2}$*, and* $n>2s$*, let*

$$\overline{g}_{\ell}(s;t):=g_{\ell}\binom{-\ell/2}{t-\lfloor \ell/2\rfloor}s^{t-\lfloor\frac{\ell}{2}\rfloor}\quad \textit{for all}\ \ \ell\in\mathbb{Z}_{\geq 1},$$

$$P^{3}_{n,s}(w):=\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^{u}+\sum_{t=1}^{\lfloor\frac{w-2}{2}\rfloor}\sum_{u=0}^{t-1}\overline{g}_{2u+1}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}+\sum_{t=2}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{u=1}^{t-1}\overline{g}_{2u}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t},$$

and

$$E^{3}_{n,s}(w):=\frac{29}{w}\Big(s+\frac{1}{24\alpha}\Big)^{\lceil\frac{w-1}{2}\rceil+1}\Big(\frac{1}{\sqrt{n}}\Big)^{w}\delta_s,$$

then

$$P^{3}_{n,s}(w)-E^{3}_{n,s}(w)\leq\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n+s}}\Big)^{u}\leq P^{2}_{n,s}(w)+E^{3}_{n,s}(w). \tag{4.43}$$

Proof. For all $n,s\in\mathbb{Z}_{\geq 1}$, $w\in\mathbb{Z}_{\geq 2}$, and $n>2s$, we split $\sum_{u=1}^{w-1}g_u(1/\sqrt{n+s})^u$ as

$$\begin{aligned}\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n+s}}\Big)^{u} &= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^{u}\sum_{k=0}^{\infty}\binom{-u/2}{k}\frac{s^k}{n^k}\\ &= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^{u}+\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^{u}\sum_{k=1}^{\infty}\binom{-u/2}{k}\frac{s^k}{n^k}\\ &= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^{u}+\sum_{u=1}^{w-1}g_u\sum_{k=1}^{\infty}\binom{-u/2}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+u}\\ &= \sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^{u}+\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}g_{2u+1}\sum_{k=1}^{\infty}\binom{-\frac{2u+1}{2}}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+2u+1}\\ &\qquad+\sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}g_{2u}\sum_{k=1}^{\infty}\binom{-u}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+2u}.\end{aligned} \tag{4.44}$$

Now,

$$\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor} g_{2u+1}\sum_{k=1}^{\infty}\binom{-\frac{2u+1}{2}}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k+2u+1}$$
$$=\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor} g_{2u+1}\sum_{t=u+1}^{\infty}\binom{-\frac{2u+1}{2}}{t-u}s^{t-u}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}$$
$$=\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor} g_{2u+1}\sum_{t=u+1}^{\lfloor\frac{w-2}{2}\rfloor}\binom{-\frac{2u+1}{2}}{t-u}s^{t-u}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}+$$
$$\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor} g_{2u+1}\sum_{t=\lceil\frac{w-1}{2}\rceil}^{\infty}\binom{-\frac{2u+1}{2}}{t-u}s^{t-u}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}$$
$$=\sum_{t=1}^{\lfloor\frac{w-2}{2}\rfloor}\sum_{u=0}^{t-1}\overline{g}_{2u+1}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}+\underbrace{\sum_{u=1}^{\lceil\frac{w-1}{2}\rceil}\sum_{t=\lceil\frac{w-1}{2}\rceil-u+1}^{\infty}g_{2u-1}\binom{-\frac{2u-1}{2}}{t}s^t\Big(\frac{1}{\sqrt{n}}\Big)^{2t+2u-1}}_{:=\mathcal{S}_o(w,n,s)}. \tag{4.45}$$

Next, we proceed to estimate the absolute value of the error sum $\mathcal{S}_o(w,n,s)$ for $s\in\mathbb{Z}_{\geq 1}$.

$$|\mathcal{S}_o(w,n,s)|$$
$$\leq \sum_{u=1}^{\lceil\frac{w-1}{2}\rceil}|g_{2u-1}|\Big(\frac{1}{\sqrt{n}}\Big)^{2u-1}\left|\sum_{t=\lceil\frac{w-1}{2}\rceil-u+1}^{\infty}\binom{-\frac{2u-1}{2}}{t}\frac{s^t}{n^t}\right|$$
$$<\ 4\sum_{u=1}^{\lceil\frac{w-1}{2}\rceil}|g_{2u-1}|\Big(\frac{1}{\sqrt{n}}\Big)^{2u-1}\sqrt{\frac{u}{\lceil\frac{w-1}{2}\rceil}}\binom{\lceil\frac{w-1}{2}\rceil}{u-1}\Big(\frac{s}{n}\Big)^{\lceil\frac{w-1}{2}\rceil-u+1}$$
$$\left(\text{by substitution } (m,s,n)\mapsto\Big(\Big\lceil\frac{w-1}{2}\Big\rceil-u+1,u,\frac{n}{s}\Big)\ \text{ in }\ (4.25)\right)$$
$$\leq\ 4\left(\sum_{u=1}^{\lceil\frac{w-1}{2}\rceil}|g_{2u-1}|\binom{\lceil\frac{w-1}{2}\rceil}{u-1}\frac{1}{s^u}\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2\lceil\frac{w-1}{2}\rceil+1}s^{\lceil\frac{w-1}{2}\rceil+1}$$

$$
\begin{aligned}
&\le 4\left(\sum_{u=1}^{\lceil \frac{w-1}{2}\rceil} \frac{1}{(2u-1)(24\alpha)^{u-1}}\binom{\lceil\frac{w-1}{2}\rceil}{u-1}\frac{1}{s^u}\right)\left(\frac{1}{\sqrt{n}}\right)^w s^{\lceil\frac{w-1}{2}\rceil+1} \quad \big(\text{by Lemma } 4.3.1\big)\\
&= 4\left(\sum_{u=0}^{\lceil \frac{w-1}{2}\rceil-1} \frac{1}{2u+1}\binom{\lceil\frac{w-1}{2}\rceil}{u}\frac{1}{(24\alpha s)^u}\right)\left(\frac{1}{\sqrt{n}}\right)^w s^{\lceil\frac{w-1}{2}\rceil}\\
&\le \frac{16}{3}\left(\sum_{u=0}^{\lceil \frac{w-1}{2}\rceil-1} \frac{1}{2u+2}\binom{\lceil\frac{w-1}{2}\rceil}{u}\frac{1}{(24\alpha s)^u}\right)\left(\frac{1}{\sqrt{n}}\right)^w s^{\lceil\frac{w-1}{2}\rceil}\\
&\qquad\qquad\qquad\qquad\left(\text{because } \frac{4}{2u+1}\le\frac{8}{3(u+1)} \text{ for all } u\ge 1\right)\\
&= \frac{16}{3}\frac{1}{2\left(\lceil\frac{w-1}{2}\rceil+1\right)}\left(\left(\left(1+\frac{1}{24\alpha s}\right)^{\lceil\frac{w-1}{2}\rceil+1}-1\right)24\alpha s-\left(\frac{1}{24\alpha s}\right)^{\lceil\frac{w-1}{2}\rceil}\right)\left(\frac{1}{\sqrt{n}}\right)^w s^{\lceil\frac{w-1}{2}\rceil}\\
&< \frac{16}{3w}\left(\left(1+\frac{1}{24\alpha s}\right)^{\lceil\frac{w-1}{2}\rceil+1}-1\right)24\alpha s\left(\frac{1}{\sqrt{n}}\right)^w s^{\lceil\frac{w-1}{2}\rceil} < \frac{28}{w}\left(s+\frac{1}{24\alpha}\right)^{\lceil\frac{w-1}{2}\rceil+1}\left(\frac{1}{\sqrt{n}}\right)^w.
\end{aligned} \tag{4.46}
$$

Similar to (4.45), we get

$$
\begin{aligned}
&\sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor} g_{2u}\sum_{k=1}^{\infty}\binom{-u}{k}s^k\left(\frac{1}{\sqrt{n}}\right)^{2k+2u}\\
&= \sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{t=u+1}^{\infty} g_{2u}\binom{-u}{t-u}s^{t-u}\left(\frac{1}{\sqrt{n}}\right)^{2t}\\
&= \sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{t=u+1}^{\lfloor\frac{w-1}{2}\rfloor} g_{2u}\binom{-u}{t-u}s^{t-u}\left(\frac{1}{\sqrt{n}}\right)^{2t}+\sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{t=\lceil\frac{w}{2}\rceil}^{\infty} g_{2u}\binom{-u}{t-u}s^{t-u}\left(\frac{1}{\sqrt{n}}\right)^{2t}\\
&= \sum_{t=2}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{u=1}^{t-1}\overline{g}_{2u}(s;t)\left(\frac{1}{\sqrt{n}}\right)^{2t}+\underbrace{\sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{t=\lceil\frac{w}{2}\rceil-u}^{\infty} g_{2u}\binom{-u}{t}s^t\left(\frac{1}{\sqrt{n}}\right)^{2t+2u}}_{:=\mathcal{S}_e(w,n,s)}.
\end{aligned} \tag{4.47}
$$

Consequently for $s \in \mathbb{Z}_{\geq 1}$,

$$
\begin{aligned}
|\mathcal{S}_e(w,n,s)| &\leq \sum_{u=1}^{\lfloor \frac{w-1}{2} \rfloor} |g_{2u}| \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \Bigg| \sum_{t=\lceil \frac{w}{2} \rceil - u}^{\infty} \binom{-u}{t} \frac{s^t}{n^t} \Bigg| \\
&< \ 2 \sum_{u=1}^{\lfloor \frac{w-1}{2} \rfloor} |g_{2u}| \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \Big(\frac{s}{n}\Big)^{\lceil \frac{w}{2} \rceil - u} \\
&\qquad\qquad \Bigg(\text{by substitution } (m,s,n) \mapsto \Big(\Big\lceil \frac{w}{2} \Big\rceil - u, u, \frac{n}{s}\Big) \ \text{ in } \ (4.27)\Bigg) \\
&= 2\Bigg(\sum_{u=1}^{\lfloor \frac{w-1}{2} \rfloor} |g_{2u}| \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \frac{1}{s^u} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \\
&\leq 2\Bigg(\sum_{u=1}^{\lfloor \frac{w-1}{2} \rfloor} \frac{1}{2u} \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \frac{1}{(24\alpha s)^u} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \quad \Big(\text{by Lemma } 4.3.1\Big) \\
&= \ 2\Bigg(\sum_{u=1}^{\lceil \frac{w}{2} - 1 \rceil} \frac{1}{2u} \binom{\lceil \frac{w}{2} \rceil - 1}{u-1} \frac{1}{(24\alpha s)^u} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \\
&= \ \frac{1}{w} \Bigg(\Big(1 + \frac{1}{24\alpha s}\Big)^{\lceil \frac{w}{2} \rceil} - 1 - \Big(\frac{1}{24\alpha s}\Big)^{\lceil \frac{w}{2} \rceil} \Bigg) s^{\lceil \frac{w}{2} \rceil} \Big(\frac{1}{\sqrt{n}}\Big)^{w} \\
&< \frac{1}{w} \Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w-1}{2} \rceil + 1} \Big(\frac{1}{\sqrt{n}}\Big)^{w}.
\end{aligned} \tag{4.48}
$$

From (4.44), (4.45), and (4.47), we obtain

$$
\sum_{u=1}^{w-1} g_u \Big(\frac{1}{\sqrt{n+s}}\Big)^{u} - P^3_{n,s}(w) = \mathcal{S}_o(w,n,s) + \mathcal{S}_e(w,n,s), \tag{4.49}
$$

and taking absolute on both sides of (4.49) and applying (4.46) and (4.48), it follows that

$$
\begin{aligned}
\Bigg| \sum_{u=1}^{w-1} g_u \Big(\frac{1}{\sqrt{n+s}}\Big)^{u} - P^3_{n,s}(w) \Bigg| &= \Big| \mathcal{S}_o(w,n,s) + \mathcal{S}_e(w,n,s) \Big| \\
&\leq \Big| \mathcal{S}_o(w,n,s) \Big| + \Big| \mathcal{S}_e(w,n,s) \Big| \\
&< \frac{29}{w} \Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w-1}{2} \rceil + 1} \Big(\frac{1}{\sqrt{n}}\Big)^{w}.
\end{aligned} \tag{4.50}
$$

Note that in (4.43), the equality holds for $s = 0$ because first, $P^3_{n,0}(w) = 0$ and secondly, the error term $\mathcal{S}_o(w, n, 0)$ (resp. $\mathcal{S}_e(w, n, 0)$) in (4.45) (resp. in (4.47)) is identically zero and therefore, we conclude that $E^3_{n,0}(w) = 0$. □

Lemma 4.3.6. *Let γ_1, γ_2 as in Equation (4.24). For $(n, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, and $w \in \mathbb{Z}_{\geq 2}$, then*

$$-\frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil} w}\left(\frac{1}{\sqrt{n}}\right)^w \leq -\frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil} w}\left(\frac{1}{\sqrt{n+s}}\right)^w \tag{4.51}$$

and

$$\frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w}\left(\frac{1}{\sqrt{n+s}}\right)^w \leq \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w}\left(\frac{1}{\sqrt{n}}\right)^w. \tag{4.52}$$

Proof. The proof of both (4.51) and (4.52) is immediate from the fact that $\frac{1}{\sqrt{n+s}} \leq \frac{1}{\sqrt{n}}$ for all $(n, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$. □

Definition 4.3.7. *Let the coefficient sequence $(g_n)_{n\geq 1}$ be as in Lemma 4.2.2 and $(\overline{g}_n(s;t))_{n\geq 1}$ be as in Lemma 4.3.5. Then for $(n, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$ and $U \in \mathbb{Z}_{\geq 1}$, we define*

$$P_{n,s}(U) := -\log 4\sqrt{3} - \log n + \pi\sqrt{\frac{2n}{3}} + \sum_{u=1}^{U} \widetilde{g}_{u,s}\left(\frac{1}{\sqrt{n}}\right)^u, \tag{4.53}$$

where

$$\widetilde{g}_{2u,s} := \frac{(-s)^u}{u} + g_{2u} + \sum_{k=1}^{u-1} \overline{g}_{2k}(s;u) \quad \text{for all} \quad 1 \leq u \leq \lfloor U/2 \rfloor$$

and

$$\widetilde{g}_{2u+1,s} := \pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1} s^{u+1} + g_{2u+1} + \sum_{k=0}^{u-1} \overline{g}_{2k+1}(s;u) \quad \text{for all} \quad 0 \leq u \leq \lfloor (U-1)/2 \rfloor.$$

Definition 4.3.8. *Let γ_1, γ_2 be as in (4.24). For $(n, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, $w \in \mathbb{Z}_{\geq 2}$, and $n > 2s$, we define*

$$E^{\mathcal{U}}_{n,s}(w) := \left(45\left(s + \frac{1}{24\alpha}\right)^{\lceil \frac{w+1}{2} \rceil}\delta_s + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil}}\right)\frac{1}{w}\left(\frac{1}{\sqrt{n}}\right)^w$$

and

$$E^{\mathcal{L}}_{n,s}(w) := \left(45\Big(s+\frac{1}{24\alpha}\Big)^{\lceil\frac{w+1}{2}\rceil}\delta_s + \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^w.$$

Theorem 4.3.9. *Let $P_{n,s}(U)$ be as in Definition 4.3.7 and $E^{\mathcal{L}}_{n,s}(w), E^{\mathcal{U}}_{n,s}(w)$ be as in Definition 4.3.8. If $(n,s)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$, $w\in\mathbb{Z}_{\geq 2}$, and $n>\max\{g(w)-s, 2s\}$, then*

$$P_{n,s}(w-1) - E^{\mathcal{L}}_{n,s}(w) < \log p(n+s) < P_{n,s}(w-1) + E^{\mathcal{U}}_{n,s}(w). \tag{4.54}$$

Proof. From (4.23), it follows that for $\lceil\frac{w}{2}\rceil \geq \gamma_0$ and $n > g(w)-s$,

$$P_{n+s}(w-1) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n+s}}\Big)^w < \log p(n+s) < P_{n+s}(w-1) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n+s}}\Big)^w, \tag{4.55}$$

where (by Definition 4.2.6)

$$P_{n+s}(w-1) = -\log 4\sqrt{3} - \log(n+s) + \pi\sqrt{\frac{2(n+s)}{3}} + \sum_{u=1}^{w-1} g_u\Big(\frac{1}{\sqrt{n+s}}\Big)^u.$$

Applying Lemma 4.3.6 into (4.55), we obtain

$$P_{n+s}(w-1) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w < \log p(n+s) < P_{n+s}(w-1) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w. \tag{4.56}$$

Invoking Lemmas 4.3.3, 4.3.4, and 4.3.5 into (4.56), it follows that

$$\begin{aligned} -\log 4\sqrt{3} + \sum_{i=1}^{3} P^i_{n,s}(w) - \sum_{i=1}^{3} E^i_{n,s}(w) - \frac{\gamma_1}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w &< \log p(n+s) \\ < -\log 4\sqrt{3} + \sum_{i=1}^{3} P^i_{n,s}(w) + \sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w&. \end{aligned} \tag{4.57}$$

For $s \geq 1$,

$$\begin{aligned} &\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\Big(\frac{1}{\sqrt{n}}\Big)^w \\ &= \left(\frac{8s^{\lceil\frac{w+1}{2}\rceil}}{\lceil w/2\rceil} + \frac{29}{w}\Big(s+\frac{1}{24\alpha}\Big)^{\lceil\frac{w-1}{2}\rceil+1} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2\rceil}w}\right)\Big(\frac{1}{\sqrt{n}}\Big)^w \end{aligned}$$

$$= \left(\frac{8s^{\lceil \frac{w+1}{2} \rceil}}{\lceil w/2 \rceil} + \frac{29}{w}\Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2} \rceil} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w}\right)\Big(\frac{1}{\sqrt{n}}\Big)^w$$

$$\leq \left(\frac{16s^{\lceil \frac{w+1}{2} \rceil}}{w} + \frac{29}{w}\Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2} \rceil} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w}\right)\Big(\frac{1}{\sqrt{n}}\Big)^w$$

$$< \left(45\Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2} \rceil} + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^w, \tag{4.58}$$

and for $s = 0$,

$$\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w = \frac{\gamma_2}{(24\alpha)^{\lceil w/2 \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w. \tag{4.59}$$

Similarly, for $s \geq 1$,

$$\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w < \left(45\Big(s + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2} \rceil} + \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^w, \tag{4.60}$$

and for $s = 0$,

$$\sum_{i=1}^{3} E^i_{n,s}(w) + \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w = \frac{\gamma_1}{(24\alpha)^{\lceil w/2 \rceil} w}\Big(\frac{1}{\sqrt{n}}\Big)^w. \tag{4.61}$$

Putting (4.58)-(4.61) together into (4.57), we get

$$-\log 4\sqrt{3} + \sum_{i=1}^{3} P^i_{n,s}(w) - E^{\mathcal{L}}_{n,s}(w) < \log p(n+s) < -\log 4\sqrt{3} + \sum_{i=1}^{3} P^i_{n,s}(w) + E^{\mathcal{U}}_{n,s}(w). \tag{4.62}$$

From Lemmas 4.3.3-4.3.5, it follows that

$$\begin{aligned}
&-\log 4\sqrt{3}+\sum_{i=1}^{3}P_{n,s}^{i}(w)\\
&=-\log 4\sqrt{3}-\log n+\pi\sqrt{\frac{2n}{3}}+\sum_{u=1}^{w-1}g_u\Big(\frac{1}{\sqrt{n}}\Big)^{u}\\
&+\left(\sum_{k=1}^{\lfloor\frac{w-1}{2}\rfloor}\frac{(-1)^k s^k}{k}\Big(\frac{1}{\sqrt{n}}\Big)^{2k}+\sum_{t=2}^{\lfloor\frac{w-1}{2}\rfloor}\sum_{u=1}^{t-1}\overline{g}_{2u}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\right)\\
&+\left(\pi\sqrt{\frac{2}{3}}\sum_{k=1}^{\lfloor\frac{w}{2}\rfloor}\binom{1/2}{k}s^k\Big(\frac{1}{\sqrt{n}}\Big)^{2k-1}+\sum_{t=1}^{\lfloor\frac{w-2}{2}\rfloor}\sum_{u=0}^{t-1}\overline{g}_{2u+1}(s;t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\right)\\
&=-\log 4\sqrt{3}-\log n+\pi\sqrt{\frac{2n}{3}}+\sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}\left(\frac{(-s)^u}{u}+g_{2u}+\sum_{k=1}^{u-1}\overline{g}_{2k}(s;u)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2u}\\
&+\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}\left(\pi\sqrt{\frac{2}{3}}\binom{1/2}{k+1}s^{k+1}+g_{2u+1}+\sum_{k=0}^{u-1}\overline{g}_{2k+1}(s;u)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2u+1}\\
&=-\log 4\sqrt{3}-\log n+\pi\sqrt{\frac{2n}{3}}+\sum_{u=1}^{\lfloor\frac{w-1}{2}\rfloor}\widetilde{g}_{2u,s}\Big(\frac{1}{\sqrt{n}}\Big)^{2u}+\sum_{u=0}^{\lfloor\frac{w-2}{2}\rfloor}\widetilde{g}_{2u+1,s}\Big(\frac{1}{\sqrt{n}}\Big)^{2u+1}\\
&=P_{n,s}(w-1). \qquad (4.63)
\end{aligned}$$

From (4.62) and (4.63), we conclude the proof of (4.54). □

Next, we proceed to estimate $\sum_{i=1}^{T}\log p(n+s_i)$.

Definition 4.3.10. *For $n,T\in\mathbb{Z}_{\geq 1}$ and $\vec{\boldsymbol{s}}:=(s_1,s_2,\ldots,s_T)\in\mathbb{Z}_{\geq 0}^{T}$, we define*

$$\log p(n;\vec{\boldsymbol{s}}):=\sum_{i=1}^{T}\log p(n+s_i).$$

Definition 4.3.11. *Let the coefficient sequence $(g_n)_{n\geq 1}$ be as in Lemma 4.2.2, $(\overline{g}_n(s;t))_{n\geq 1}$ be as in Lemma 4.3.5, and $\vec{\boldsymbol{s}}$ be as in Definition 4.3.10. For $n,T\in\mathbb{Z}_{\geq 1}$ and $U\in\mathbb{Z}_{\geq 1}$, we define*

$$P_{n,\vec{s}}(U):=-T\cdot\log 4\sqrt{3}-T\cdot\log n+T\cdot\pi\sqrt{\frac{2n}{3}}+\sum_{u=1}^{U}\widetilde{g}_{u,\vec{s}}\Big(\frac{1}{\sqrt{n}}\Big)^{u}, \qquad (4.64)$$

where

$$\widetilde{g}_{2u,\vec{s}} := \frac{1}{u}\sum_{i=1}^{T}(-s_i)^u + T\cdot g_{2u} + \sum_{i=1}^{T}\sum_{k=1}^{u-1}\overline{g}_{2k}(s_i;u) \quad \text{for all} \quad 1 \le u \le \lfloor U/2 \rfloor$$

and for all $0 \le u \le \lfloor (U-1)/2 \rfloor$,

$$\widetilde{g}_{2u+1,\vec{s}} := \pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1}\sum_{i=1}^{T} s_i^{u+1} + T\cdot g_{2u+1} + \sum_{i=1}^{T}\sum_{k=0}^{u-1}\overline{g}_{2k+1}(s_i;u).$$

Definition 4.3.12. *Let* γ_1, γ_2 *be as in* (4.24) *and* $\vec{\boldsymbol{s}}$ *be as in Definition 4.3.10. For each* $\{s_i\}_{1\le i\le T}$, δ_{s_i} *be as in Definition 4.3.2. For* $n, T \in \mathbb{Z}_{\ge 1}$, $w \in \mathbb{Z}_{\ge 2}$, *and* $n > 2s_i$, *we define*

$$E^{\mathcal{U}}_{n,\vec{s}}(w) := \left(45\sum_{i=1}^{T}\Big(s_i + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2}\rceil}\delta_{s_i} + \frac{T\cdot\gamma_2}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}$$

and

$$E^{\mathcal{L}}_{n,\vec{s}}(w) := \left(45\sum_{i=1}^{T}\Big(s_i + \frac{1}{24\alpha}\Big)^{\lceil \frac{w+1}{2}\rceil}\delta_{s_i} + \frac{T\cdot\gamma_1}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}\Big(\frac{1}{\sqrt{n}}\Big)^{w}.$$

A generalized version of Theorem 4.3.9 is as follows:

Theorem 4.3.13. *Let* $\log p(n;\vec{\boldsymbol{s}})$ *be as in Definition 4.3.10,* $P_{n,\vec{s}}(U)$ *be as in Definition 4.3.11, and let* $g(k)$ *be as in Definition 4.2.5. Let* $E^{\mathcal{L}}_{n,\vec{s}}(w)$ *and* $E^{\mathcal{U}}_{n,\vec{s}}(w)$ *be as in Definition 4.3.12. If* $n, T \in \mathbb{Z}_{\ge 1}$, $w \in \mathbb{Z}_{\ge 2}$, *and*

$$n > \max_{1\le i\le T}\{g(w) - \min_{1\le i\le T}\{s_i\}, 2s_i\} := g(w;\vec{\boldsymbol{s}}),$$

then

$$P_{n,\vec{s}}(w-1) - E^{\mathcal{L}}_{n,\vec{s}}(w) < \log p(n;\vec{\boldsymbol{s}}) < P_{n,\vec{s}}(w-1) + E^{\mathcal{U}}_{n,\vec{s}}(w). \tag{4.65}$$

Proof. Applying (4.54) for each $\{s_i\}_{1\le i\le T}$ and summing up, we get (4.65). □

Remark 4.3.14. *A few applications of Theorem 4.3.13 are listed below.*

1. *Choosing* $w = 5$ *(resp.* $w = 7$*), we obtain* $(p(n))_{n\ge 26}$ *is* log-concave *(resp. (4.7)).*

2. *Define* $u_n := \frac{p(n)p(n+2)}{p(n+1)^2}$ *and let* N *be any positive integer. Then choosing* $w = N$*, we have a full asymptotic expansion of* $\log u_n$ *with a precise estimation of the error bound after truncation of the asymptotic expansion at a point* N*.*

3. *Applying* $\vec{\boldsymbol{s}} = \{m, m\}$ *and* $\vec{\boldsymbol{r}} = \{0, 2m\}$ *to (4.65), and estimation of*

$$P_{n,\vec{\boldsymbol{s}}}(4) + E^{\mathcal{L}}_{n,\vec{\boldsymbol{s}}}(5) - P_{n,\vec{\boldsymbol{r}}}(4) - E^{\mathcal{U}}_{n,\vec{\boldsymbol{s}}}(5),$$

   leads to the strong log*-concavity property of* $p(n)$*.*

4. *Without loss of generality, assume* $b = \lambda a$ *with* $\lambda \geq 1$ *in Theorem 4.1.1. By making the substitutions* $(n, \vec{\boldsymbol{s}}) = (a, 0)$*,* $(n, \vec{\boldsymbol{s}}) = (\lambda a, 0)$*, and* $(n, \vec{\boldsymbol{r}}) = (a(1+\lambda), 0)$ *to (4.65), we can retrieve (4.10).*

4.4 Asymptotics of $(-1)^{r-1}\Delta^r \log p(n)$

Lemma 4.4.1. *Let* $P_{n,s}(w-1)$ *be as in Theorem 4.3.9. Then for all* $r \geq 2$*,*

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} P_{n,i}(2r) = C_r \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - (r-1)! \Big(\frac{1}{\sqrt{n}}\Big)^{2r}, \tag{4.66}$$

where $C_r = \dfrac{\pi}{\sqrt{6}} \Big(\dfrac{1}{2}\Big)_{r-1}$ *and* $(a)_k$ *is the standard notation for the rising factorial.*

Proof. From Definition 4.3.7, it follows that

$$\begin{aligned}
&\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} P_{n,i}(2r) \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \Bigg(-\log 4\sqrt{3} - \log n + \sqrt{\frac{2n}{3}} + \sum_{u=1}^{2r} \widetilde{g}_{u,i} \Big(\frac{1}{\sqrt{n}}\Big)^{u} \Bigg) \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=1}^{2r} \widetilde{g}_{u,i} \Big(\frac{1}{\sqrt{n}}\Big)^{u} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=1}^{2r-2} \widetilde{g}_{u,i} \Big(\frac{1}{\sqrt{n}}\Big)^{u} + \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2r-1,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \\
&\qquad + \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2r,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2r}.
\end{aligned} \tag{4.67}$$

Following the notation from [68], here $\left\{ {n \atop m} \right\}$ denotes the Stirling number of second kind. For all integers $1 \leq u \leq 2r-2$ and $u \equiv 0 \pmod 2$, we have

$$
\begin{aligned}
&\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=1}^{r-1} \widetilde{g}_{2u,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \\
=& \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=1}^{r-1} \left[\frac{(-i)^u}{u} + g_{2u} + \sum_{k=1}^{u-1} \overline{g}_{2k}(i;u) \right] \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \\
=& \sum_{u=1}^{r-1} \frac{(-1)^u}{u} (-1)^{r+1} r! \left\{ {u \atop r} \right\} \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \\
+& \sum_{u=1}^{r-1} \sum_{k=1}^{u-1} g_{2k} \binom{-k}{u-k} \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} i^{u-k} \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \\
=& \sum_{u=1}^{r-1} \frac{(-1)^u}{u} (-1)^{r+1} r! \left\{ {u \atop r} \right\} \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \\
+& \sum_{u=1}^{r-1} \sum_{k=1}^{u-1} g_{2k} \binom{-k}{u-k} (-1)^{r+1} r! \left\{ {u-k \atop r} \right\} \Big(\frac{1}{\sqrt{n}}\Big)^{2u} \\
=& \ 0 \quad \left(\text{as } \left\{ {n \atop m} \right\} = 0 \text{ for all } n < m \right).
\end{aligned} \tag{4.68}
$$

Similarly for all integers $1 \leq u \leq 2r-2$ and $u \equiv 1 \pmod 2$, we obtain

$$
\begin{aligned}
&\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=0}^{r-2} \widetilde{g}_{2u+1,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
=& \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=0}^{r-2} \left[\pi \sqrt{\frac{2}{3}} \binom{1/2}{u+1} i^{u+1} + g_{2u+1} + \sum_{k=0}^{u-1} \overline{g}_{2k+1}(i;u) \right] \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
=& \sum_{u=0}^{r-2} \pi \sqrt{\frac{2}{3}} \binom{1/2}{u+1} (-1)^{r+1} r! \left\{ {u+1 \atop r} \right\} \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
&+ \sum_{u=0}^{r-2} \sum_{k=0}^{u-1} g_{2k+1} \binom{-k-1/2}{u-k} (-1)^{r+1} r! \left\{ {u-k \atop r} \right\} \Big(\frac{1}{\sqrt{n}}\Big)^{2u+1} \\
=& \ 0 \quad \left(\text{as } \left\{ {n \atop m} \right\} = 0 \text{ for all } n < m \right).
\end{aligned} \tag{4.69}
$$

From (4.68) and (4.69), it follows that for all $1 \le u \le 2r-2$,

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \sum_{u=1}^{2r-2} \widetilde{g}_{u,i} \Big(\frac{1}{\sqrt{n}}\Big)^{u} = 0.$$

Now

$$\begin{aligned}
&\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2r-1,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \Bigg[\pi\sqrt{\frac{2}{3}}\binom{1/2}{r} i^{r} + g_{2r-1} + \sum_{k=0}^{r-2} \overline{g}_{2k+1}(i; r-1)\Bigg] \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \\
&= \Bigg[\pi\sqrt{\frac{2}{3}}\binom{1/2}{r}(-1)^{r+1} r! \begin{Bmatrix} r \\ r \end{Bmatrix} + \sum_{k=0}^{r-2} g_{2k+1} \binom{-k-1/2}{r-1-k} (-1)^{r+1} r! \begin{Bmatrix} r-1-k \\ r \end{Bmatrix}\Bigg] \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \\
&= \frac{\pi}{\sqrt{6}} \Big(\frac{1}{2}\Big)_{r-1} \Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} \quad \Bigg(\text{since } \begin{Bmatrix} r-1-k \\ r \end{Bmatrix} = 0 \text{ for all } 0 \le k \le r-2\Bigg). \qquad (4.70)
\end{aligned}$$

We finish the proof by showing that

$$\begin{aligned}
\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2r,i} \Big(\frac{1}{\sqrt{n}}\Big)^{2r} &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \Bigg[\frac{(-i)^r}{r} + g_{2r} + \sum_{k=1}^{r-1} \overline{g}_{2k}(i; r)\Bigg] \Big(\frac{1}{\sqrt{n}}\Big)^{2r} \\
&= \Bigg[-(r-1)! + \sum_{k=1}^{r-1} g_{2k} \binom{-k}{r-k} (-1)^{r+1} r! \begin{Bmatrix} r-k \\ r \end{Bmatrix}\Bigg] \Big(\frac{1}{\sqrt{n}}\Big)^{2r} \\
&= -(r-1)! \Big(\frac{1}{\sqrt{n}}\Big)^{2r}. \qquad (4.71)
\end{aligned}$$

□

Definition 4.4.2. *Let γ_1 be as in (4.24) and C_r be as in Lemma 4.4.1. Then for all $r \ge 2$, define*

$$\begin{aligned}
L_1(r) &:= \Bigg(\frac{\gamma_1}{(12\alpha)^{r+1}} + 45 \sum_{i=1}^{r} \binom{r}{i} \Big(i + \frac{1}{24\alpha}\Big)^{r+1}\Bigg) \frac{1}{2r+1}, \\
L(r) &:= (r-1)! + L_1(r),
\end{aligned}$$

and

$$N_L(r) := \max\Bigg\{\Big(\frac{L(r)}{C_r}\Big)^2, g(2r+1)\Bigg\}.$$

Lemma 4.4.3. *Let $L(r), N_L(r)$ be as in Definition 4.4.2 and C_r be as in Lemma 4.4.1. Then for all $n > N_L(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) > \log\left(1 + C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - L(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r}\right). \tag{4.72}$$

Proof. We split $(-1)^{r-1}\Delta^r \log p(n)$ as follows:

$$\begin{aligned}(-1)^{r-1}\Delta^r \log p(n) &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}\log p(n+i)\\ &= \sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}\log p(n+2i+1) - \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}\log p(n+2i).\end{aligned} \tag{4.73}$$

Applying Theorem 4.3.9 with $w = 2r+1$ to (4.73), we have for all $n > \max\limits_{0\le i\le r}\{g(2r+1)-i, 2i\} = g(2r+1)$,

$$\begin{aligned}&(-1)^{r-1}\Delta^r \log p(n)\\ &> \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}P_{n,i}(2r) - \sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{L}}_{n,2i+1}(2r+1) - \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{U}}_{n,2i}(2r+1)\\ &= C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - (r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r} - \sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{L}}_{n,2i+1}(2r+1)\\ &\qquad - \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{U}}_{n,2i}(2r+1) \quad \Big(\text{by Lemma 4.4.1}\Big).\end{aligned} \tag{4.74}$$

From Definition 4.3.8, it is clear that $E^{\mathcal{U}}_{n,s}(w) < E^{\mathcal{L}}_{n,s}(w)$ because $\gamma_2 < \gamma_1$. Therefore,

$$\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{L}}_{n,2i+1}(2r+1) + \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{U}}_{n,2i}(2r+1) < \sum_{i=0}^{r}\binom{r}{i}E^{\mathcal{L}}_{n,i}(2r+1) \tag{4.75}$$

and

$$\sum_{i=0}^{r}\binom{r}{i}E^{\mathcal{L}}_{n,i}(2r+1) = L_1(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r+1}. \tag{4.76}$$

From (4.74) and (4.76), it follows that

$$\begin{aligned}(-1)^{r-1}\Delta^r \log p(n) &> C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - (r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r} - L_1(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r+1}\\ &> C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - L(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r}, \qquad (4.77)\end{aligned}$$

and consequently for all $n > N_L(r)$, we get

$$(-1)^{r-1}\Delta^r \log p(n) > \log\left(1 + C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - L(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r}\right).$$

□

Definition 4.4.4. *Let $L_1(r)$ be as in Definition 4.4.2 and C_r be as in Lemma 4.4.1. Then for all $r \geq 2$, define*

$$N_U(r) := \max\left\{\Big(\frac{L_1(r)+1}{(r-1)!}\Big)^2, \Big(\frac{C_r^2}{2}\Big)^{2/2r-3}, g(2r+1)\right\}.$$

Lemma 4.4.5. *Let $L_1(r)$ be as in Definition 4.4.2, C_r be as in Lemma 4.4.1, and $N_U(r)$ be as in Definition 4.4.4. Then for all $n > N_U(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1}\right). \qquad (4.78)$$

Proof. Applying Theorem 4.3.9 with $w = 2r+1$ to (4.73), we have for all $n > g(2r+1)$,

$$\begin{aligned}&(-1)^{r-1}\Delta^r \log p(n)\\ &< \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1}P_{n,i}(2r) + \sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\binom{r}{2i+1}E^{\mathcal{U}}_{n,2i+1}(2r+1) + \sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}E^{\mathcal{L}}_{n,2i}(2r+1)\\ &< C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - (r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r} + \sum_{i=0}^{r}\binom{r}{i}E^{\mathcal{L}}_{n,i}(2r+1) \quad \Big(\text{by Lemma 4.4.1}\Big)\\ &= C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - (r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r} + L_1(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r+1}. \qquad (4.79)\end{aligned}$$

For all $n > N_U(r)$, it follows that

$$-(r-1)!\Big(\frac{1}{\sqrt{n}}\Big)^{2r} + L_1(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r+1} < -\frac{C_r^2}{2\ n^{2r-1}}. \tag{4.80}$$

From (4.79) and (4.80), it follows that for all $n > N_U(r)$,

$$(-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1}\right).$$

□

Theorem 4.4.6. *Let $L(r), N_L(r)$ be as in Definition 4.4.2 and $N_U(r)$ be as in Definition 4.4.4. Let C_r be as in Lemma 4.4.1. Then for all $n > N(r) := \max\Big\{N_L(r), N_U(r)\Big\}$,*

$$\log\left(1+C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1} - L(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2r}\right) < (-1)^{r-1}\Delta^r \log p(n) < \log\left(1+C_r\Big(\frac{1}{\sqrt{n}}\Big)^{2r-1}\right). \tag{4.81}$$

Proof. Lemmas 4.4.3 and 4.4.5 together imply (4.81). □

Theorem 4.4.7. *For all $r \geq 2$,*

$$(-1)^{r-1}\Delta^r \log p(n) \underset{n\to\infty}{\sim} \sum_{u=2r-1}^{\infty} G_u\Big(\frac{1}{\sqrt{n}}\Big)^u, \tag{4.82}$$

with for all $u \geq 1$

$$G_{2u} = \left[\frac{(-1)^u}{u}\left\{{u \atop r}\right\} + \sum_{k=1}^{u-r} g_{2k}\binom{-k}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r! \quad \textit{for all } u \geq r \tag{4.83}$$

and for all $u \geq r-1$,

$$G_{2u+1} = \left[\pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1}\left\{{u+1 \atop r}\right\} + \sum_{k=0}^{u-r} g_{2k+1}\binom{-k-1/2}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r!. \tag{4.84}$$

Proof. Following (4.67) and letting $w \to \infty$, we obtain

$$(-1)^{r-1}\Delta^r \log p(n) \underset{n\to\infty}{\sim} \sum_{u=2r-1}^{\infty} \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{u,i} \Big(\frac{1}{\sqrt{n}}\Big)^u. \tag{4.85}$$

For all $u \geq 2r-1$ and $u \equiv 0 \pmod 2$, we get

$$\begin{aligned}\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2u,i} &= \left[\frac{(-1)^u}{u} \left\{ {u \atop r} \right\} + \sum_{k=1}^{u-1} g_{2k} \binom{-k}{u-k} \left\{ {u-k \atop r} \right\}\right] (-1)^{r+1} r! \\ &= \left[\frac{(-1)^u}{u} \left\{ {u \atop r} \right\} + \sum_{k=1}^{u-r} g_{2k} \binom{-k}{u-k} \left\{ {u-k \atop r} \right\}\right] (-1)^{r+1} r!.\end{aligned}$$

Similarly, for all $u \geq 2r-1$ and $u \equiv 1 \pmod 2$, it follows that

$$\begin{aligned}&\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} \widetilde{g}_{2u+1,i} \\ &= \left[\pi\sqrt{\frac{2}{3}} \binom{1/2}{u+1} \left\{ {u+1 \atop r} \right\} + \sum_{k=0}^{u-r} g_{2k+1} \binom{-k-1/2}{u-k} \left\{ {u-k \atop r} \right\}\right] (-1)^{r+1} r!.\end{aligned}$$

□

4.5 A framework to verify multiplicative inequalities for $p(n)$

Here we list down the steps in order to make a decision on whether a given multiplicative inequality holds or not.

- (Step 0): Given $\prod_{i=1}^{T} p(n+s_i)$ and $\prod_{i=1}^{T} p(n+r_i)$ with $T \geq 1$. Without loss of generality, assume that s_i, r_i are non-negative integers for all $1 \leq i \leq T$. Transform the products into additive ones by applying the natural logarithm; i.e., $\sum_{i=1}^{T} \log p(n+s_i)$ and $\sum_{i=1}^{T} \log p(n+r_i)$.
- (Step 1): Choose $w = m+1$, where $(s_1, \ldots, s_T) \overset{m}{=} (r_1, \ldots, r_T)$. From (4.65), we observe that for each $1 \leq i \leq T$, $\log p(n+s_i)$ and $\log p(n+r_i)$ has the main

term $P_{n,\vec{\mathbf{s}}}(w-1)$ and $P_{n,\vec{\mathbf{r}}}(w-1)$ respectively. Consequently, each of these main terms are dominated by $T \cdot c \sum_{i=1}^{T} \sqrt{n+s_i}$ and $T \cdot c \sum_{i=1}^{T} \sqrt{n+r_i}$ with $c = \pi\sqrt{2/3}$ respectively. Therefore, in order to choose w, it is enough to compute the Taylor expansion of $\sum_{i=1}^{T} \left(\sqrt{n+s_i} - \sqrt{n+s_i}\right)$ which is given by:

$$\sum_{i=1}^{T} \left(\sqrt{n+s_i} - \sqrt{n+s_i}\right) = \sum_{m=1}^{\infty} \frac{\binom{1/2}{m}}{\sqrt{n}^{2m-1}} \sum_{i=1}^{T} (s_i^m - r_i^m). \tag{4.86}$$

So our optimal choice is such minimal $m \geq 1$ so that $\sum_{i=1}^{T} (s_i^m - r_i^m) \neq 0$.

- (Step 2): Applying $w = m+1$ as in the previous step to Theorem 4.3.13, it remains to verify whether

$$P_{n,\vec{\mathbf{s}}}(m) - E^{\mathcal{L}}_{n,\vec{\mathbf{s}}}(m+1) > P_{n,\vec{\mathbf{r}}}(m) + E^{\mathcal{U}}_{n,\vec{\mathbf{r}}}(m+1) \tag{4.87}$$

or

$$P_{n,\vec{\mathbf{r}}}(m) - E^{\mathcal{L}}_{n,\vec{\mathbf{r}}}(m+1) > P_{n,\vec{\mathbf{s}}}(m) + E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(m+1), \tag{4.88}$$

in order to decide whether $\sum_{i=1}^{T} \log p(n+s_i) \geq \sum_{i=1}^{T} \log p(n+r_i)$ or $\sum_{i=1}^{T} \log p(n+r_i) \geq \sum_{i=1}^{T} \log p(n+s_i)$ respectively.

4.6 Inequalities for $p(n; \vec{\mathbf{s}})$

Definition 4.6.1. *Let $\widetilde{g}_{u,\vec{\boldsymbol{s}}}$ be as in Definition 4.3.11, and $\vec{\boldsymbol{s}}$ be as in Definition 4.3.10. For $n, T, U \in \mathbb{Z}_{\geq 1}$, define*

$$\mathcal{M}(n;T) := \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^T,$$

and

$$\widetilde{P}_{n,\vec{\boldsymbol{s}}}(U) := \exp\left(\sum_{u=1}^{U} \widetilde{g}_{u,\vec{\boldsymbol{s}}} \left(\frac{1}{\sqrt{n}}\right)^u\right).$$

Definition 4.6.2. *Let* γ_1, γ_2 *be as in* (4.24) *and* $\vec{\boldsymbol{s}}$ *be as in Definition 4.3.10. For each* $\{s_i\}_{1\leq i\leq T}$, δ_{s_i} *be as in Definition 4.3.2. For* $n, T \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, *and* $n > 2s_i$, *we define*

$$C_{\mathcal{U}}(w;\vec{\boldsymbol{s}}) := \left(45\sum_{i=1}^{T}\Big(s_i+\frac{1}{24\alpha}\Big)^{\lceil\frac{w+1}{2}\rceil}\delta_{s_i}+\frac{T\cdot\gamma_2}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}$$

and

$$C_{\mathcal{L}}(w;\vec{\boldsymbol{s}}) := \left(45\sum_{i=1}^{T}\Big(s_i+\frac{1}{24\alpha}\Big)^{\lceil\frac{w+1}{2}\rceil}\delta_{s_i}+\frac{T\cdot\gamma_1}{(24\alpha)^{\lceil w/2\rceil}}\right)\frac{1}{w}.$$

Lemma 4.6.3. *Let* $\log p(n;\vec{\boldsymbol{s}})$ *be as in Definition 4.3.10, and let* $g(k)$ *be as in Definition 4.2.5. Let* $\mathcal{M}(n;T)$ *and* $\widetilde{P}_{n,\vec{\mathbf{s}}}(U)$ *be as in Definition 4.6.1. Let* $g(w;\vec{\boldsymbol{s}})$ *be as in Theorem 4.3.13, and* $C_{\mathcal{L}}(w;\vec{\boldsymbol{s}}), C_{\mathcal{U}}(w;\vec{\boldsymbol{s}})$ *be as in Definition 4.6.2. If* $n, T \in \mathbb{Z}_{\geq 1}$, $w \in \mathbb{Z}_{\geq 2}$, *and*

$$n > \max\left\{g(w;\vec{\boldsymbol{s}}), \Big(C_{\mathcal{L}}(w;\vec{\boldsymbol{s}})\Big)^{2/w}, \Big(C_{\mathcal{U}}(w;\vec{\boldsymbol{s}})\Big)^{2/w}\right\} := N_1(w;\vec{\boldsymbol{s}}),$$

then

$$\begin{aligned}\mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)\left(1-C_{\mathcal{L}}(w;\vec{\boldsymbol{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right) < p(n;\vec{\boldsymbol{s}}) <\\ \mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)\left(1+2\ C_{\mathcal{U}}(w;\vec{\boldsymbol{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right).\end{aligned} \tag{4.89}$$

Proof. Applying the exponential function on both sides of the inequality (4.65), we get for all $n > g(w;\vec{\mathbf{s}})$,

$$\mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)e^{-E^{\mathcal{L}}_{n,\vec{\mathbf{s}}}(w)} < p(n;\vec{\mathbf{s}}) < \mathcal{M}(n;T)\widetilde{P}_{n,\vec{\mathbf{s}}}(w-1)e^{E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w)}. \tag{4.90}$$

Now for all $n > \max\left\{\Big(C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big)^{2/w}, C_{\mathcal{U}}(w;\vec{\mathbf{s}})\Big)^{2/w}\right\}$, it follows that

$$0 < E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w) < 1 \ \text{ and } \ 0 < E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w) < 1. \tag{4.91}$$

For all $0 < x < 1$, we know that $e^x < 1+2x$ and $e^{-x} > 1-x$. Therefore from (4.91) and following Definition 4.3.12, we finally have

$$e^{E^{\mathcal{U}}_{n,\vec{\mathbf{s}}}(w)} < 1+2\ C_{\mathcal{U}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w} \ \text{ and } \ e^{-E^{\mathcal{L}}_{n,\vec{\mathbf{s}}}(w)} > 1-C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}. \tag{4.92}$$

Equations (4.90) and (4.92) together imply (4.89). □

Definition 4.6.4. *For $k \in \mathbb{Z}_{\geq 0}$, $w \geq 2$, and $\vec{\ell} := (\ell_1, \ldots, \ell_{w-1})$, define*

$$X(k) := \Big\{ \vec{\ell} \in \mathbb{Z}_{\geq 0}^{w-1} : \sum_{u=1}^{w-1} \ell_u = k \Big\},$$

$$X_{\mathcal{M}}(k) := \Big\{ \vec{\ell} \in X(k) : 0 \leq \sum_{u=1}^{w-1} u\ell_u \leq w-1 \Big\},$$

and

$$X_{\mathcal{E}}(k) := \Big\{ \vec{\ell} \in X(k) : \sum_{u=1}^{w-1} u\ell_u \geq w \Big\}.$$

Definition 4.6.5. *Let $X(k)$ and $X_{\mathcal{M}}(k)$ be as in Definition 4.6.4 and $\widetilde{g}_{u,\vec{s}}$ be as in Definition 4.3.11. Then for all $w \geq 2$, define*

$$\widehat{P}_{n,\vec{s}}(w-1) := \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{M}}(k)} F(k; w; \vec{\boldsymbol{s}}) \Big(\frac{1}{\sqrt{n}} \Big)^{\sum_{u=1}^{w-1} u\ell_u},$$

and

$$\widehat{E}_{n,\vec{s}}(w-1) := \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{E}}(k)} F(k; w; \vec{\boldsymbol{s}}) \Big(\frac{1}{\sqrt{n}} \Big)^{\sum_{u=1}^{w-1} u\ell_u},$$

where

$$F(k; w; \vec{\boldsymbol{s}}) := \binom{k}{\ell_1, \ldots, \ell_{w-1}} \prod_{u=1}^{w-1} \Big(\widetilde{g}_{u,\vec{s}} \Big)^{\ell_u},$$

with $\binom{k}{\ell_1,\ldots,\ell_{w-1}} = \frac{k!}{\ell_1! \cdots \ell_{w-1}!}$ is a multinomial coefficient.

Definition 4.6.6. *Let $X_{\mathcal{E}}(k)$ be as in Definition 4.6.4 and $F(k; w; \vec{\boldsymbol{s}})$ be as in Definition 4.6.5 and $\widetilde{g}_{u,\vec{s}}$ be as in Definition 4.3.11. For $w \geq 2$, define*

$$E(w; \vec{\boldsymbol{s}}) := \sum_{k=0}^{w-1} \frac{1}{k!} \sum_{\vec{\ell} \in X_{\mathcal{E}}(k)} \Big| F(k; w; \vec{\boldsymbol{s}}) \Big| + 3 \Big(\big| \widetilde{g}_{1,\vec{s}} \big| + 1 \Big)^{w}.$$

Lemma 4.6.7. *Let $\widetilde{P}_{n,\vec{s}}(U)$ be as in Definition 4.6.1 and $X_{\mathcal{E}}(k)$ be as in Definition 4.6.4. Let $\widehat{P}_{n,\vec{s}}(w-1), \widehat{P}_{n,\vec{s}}(w-1)$, and $F(k;w;\vec{\boldsymbol{s}})$ be as in Definition 4.6.5. Let $E(w;\vec{\boldsymbol{s}})$ be as in Definition 4.6.6. Then for all $w \geq 2$ and*

$$n > \max_{1 \leq u \leq w-1} \left\{ \Big((w-1) \big| \widetilde{g}_{u,\vec{s}} \big| \Big)^{2/u} \right\} := N_2(w;\vec{\boldsymbol{s}}),$$

we have

$$\Big| \widetilde{P}_{n,\vec{s}}(w-1) - \widehat{P}_{n,\vec{s}}(w-1) \Big| < E(w;\vec{\boldsymbol{s}}) \Big(\frac{1}{\sqrt{n}} \Big)^{w}. \tag{4.93}$$

Proof. Expanding $\widetilde{P}_{n,\vec{s}}(w-1)$ and splitting it as follows:

$$\begin{aligned} \widetilde{P}_{n,\vec{s}}(w-1) &= \widehat{P}_{n,\vec{s}}(w-1) + \widehat{E}_{n,\vec{s}}(w-1) + \sum_{k=w}^{\infty} \frac{1}{k!} \sum_{\vec{\ell} \in X(k)} F(k;w;\vec{\mathbf{s}}) \Big(\frac{1}{\sqrt{n}} \Big)^{\sum_{u=1}^{w-1} u\ell_u} \\ &= \widehat{P}_{n,\vec{s}}(w-1) + \widehat{E}_{n,\vec{s}}(w-1) + \sum_{k=w}^{\infty} \frac{1}{k!} \left(\sum_{u=1}^{w-1} \frac{\widetilde{g}_{u,\vec{s}}}{\sqrt{n}^{u}} \right)^{k}. \end{aligned} \tag{4.94}$$

Therefore

$$\begin{aligned} &\Big| \widetilde{P}_{n,\vec{s}}(w-1) - \widehat{P}_{n,\vec{s}}(w-1) \Big| \\ &\leq \Big| \widehat{E}_{n,\vec{s}}(w-1) \Big| + \left(\sum_{u=1}^{w-1} \frac{\big| \widetilde{g}_{u,\vec{s}} \big|}{\sqrt{n}^{u}} \right)^{w} \sum_{k=0}^{\infty} \frac{1}{(k+w)!} \left(\sum_{u=1}^{w-1} \frac{\big| \widetilde{g}_{u,\vec{s}} \big|}{\sqrt{n}^{u}} \right)^{k} \\ &= \Big| \widehat{E}_{n,\vec{s}}(w-1) \Big| + \Big(\frac{1}{\sqrt{n}} \Big)^{w} \left(\big| \widetilde{g}_{1,\vec{s}} \big| + \sum_{u=1}^{w-2} \frac{\big| \widetilde{g}_{u+1,\vec{s}} \big|}{\sqrt{n}^{u}} \right)^{w} \sum_{k=0}^{\infty} \frac{1}{(k+w)!} \left(\sum_{u=1}^{w-1} \frac{\big| \widetilde{g}_{u,\vec{s}} \big|}{\sqrt{n}^{u}} \right)^{k} \\ &< \Big| \widehat{E}_{n,\vec{s}}(w-1) \Big| + \Big(\frac{1}{\sqrt{n}} \Big)^{w} \Big(\big| \widetilde{g}_{1,\vec{s}} \big| + 1 \Big)^{w} \sum_{k=0}^{\infty} \frac{1}{(k+w)!} \quad \Big(\text{since} \quad n > N_2(w;\vec{\mathbf{s}}) \Big) \\ &\leq \Big| \widehat{E}_{n,\vec{s}}(w-1) \Big| + \frac{\Big(\big| \widetilde{g}_{1,\vec{s}} \big| + 1 \Big)^{w}}{w!} \Big(\frac{1}{\sqrt{n}} \Big)^{w} \sum_{k=0}^{\infty} \frac{1}{k!} \\ &< \Big| \widehat{E}_{n,\vec{s}}(w-1) \Big| + 3 \frac{\Big(\big| \widetilde{g}_{1,\vec{s}} \big| + 1 \Big)^{w}}{w!} \Big(\frac{1}{\sqrt{n}} \Big)^{w}. \end{aligned} \tag{4.95}$$

Now

$$\begin{aligned}\left|\widehat{E}_{n,\vec{\mathbf{s}}}(w-1)\right| &\le \sum_{k=0}^{w-1}\frac{1}{k!}\sum_{\vec{\ell}\in X_{\mathcal{E}}(k)}\left|F(k;w;\vec{\mathbf{s}})\right|\left(\frac{1}{\sqrt{n}}\right)^{\sum_{u=1}^{w-1}u\ell_u}\\ &\le \sum_{k=0}^{w-1}\frac{1}{k!}\sum_{\vec{\ell}\in X_{\mathcal{E}}(k)}\left|F(k;w;\vec{\mathbf{s}})\right|\left(\frac{1}{\sqrt{n}}\right)^{w}\quad\Big(\text{since }\ \vec{\ell}\in X_{\mathcal{E}}(k)\Big).\end{aligned}\tag{4.96}$$

Combining (4.95) and (4.96), we get (4.93). □

Definition 4.6.8. *Let $C_{\mathcal{U}}(w;\vec{\boldsymbol{s}})$ and $C_{\mathcal{L}}(w;\vec{\boldsymbol{s}})$ be as in Definition 4.6.2. Let $E(w;\vec{\boldsymbol{s}})$ be as in Definition 4.6.6. Then for all $w \ge 2$, define*

$$E_L(w;\vec{\boldsymbol{s}}) := 3\ C_{\mathcal{L}}(w;\vec{\boldsymbol{s}}) + E(w;\vec{\boldsymbol{s}}),$$

and

$$E_U(w;\vec{\boldsymbol{s}}) := 6\ C_{\mathcal{U}}(w;\vec{\boldsymbol{s}}) + E(w;\vec{\boldsymbol{s}})\Big(2\ C_{\mathcal{U}}(w;\vec{\boldsymbol{s}})+1\Big).$$

Theorem 4.6.9. *Let $\mathcal{M}(n;T)$ be as in Definition 4.6.1 and $\widehat{P}_{n,\vec{s}}(w-1)$ be as in Definition 4.6.5. Let $E^L_{n,\vec{s}}(w)$ and $E^U_{n,\vec{s}}(w)$ be as in Definition 4.6.8. Let $N_1(w;\vec{\boldsymbol{s}})$ and $N_2(w;\vec{\boldsymbol{s}})$ be as in Lemmas 4.6.3 and 4.6.7. Then for all $w \ge 2$ and*

$$N > \max\Big\{N_1(w;\vec{\boldsymbol{s}}), N_2(w;\vec{\boldsymbol{s}})\Big\} := N(w;\vec{\boldsymbol{s}}),$$

we have

$$\begin{aligned}\mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{s}}(w-1) - E_L(w;\vec{\boldsymbol{s}})\left(\frac{1}{\sqrt{n}}\right)^{w}\right) < p(n;\vec{\boldsymbol{s}}) <\\ \mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{s}}(w-1) + E_U(w;\vec{\boldsymbol{s}})\left(\frac{1}{\sqrt{n}}\right)^{w}\right).\end{aligned}\tag{4.97}$$

Proof. From Lemmas 4.6.3 and 4.6.7, for $n > N(w;\vec{\mathbf{s}})$, it follows that

$$p(n;\vec{\mathbf{s}}) < \mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{\mathbf{s}}}(w-1)+E(w;\vec{\mathbf{s}})\left(\frac{1}{\sqrt{n}}\right)^{w}\right)\left(1+2\ C_{\mathcal{U}}(w;\vec{\mathbf{s}})\left(\frac{1}{\sqrt{n}}\right)^{w}\right),\tag{4.98}$$

and

$$p(n;\vec{\mathbf{s}}) > \mathcal{M}(n;T)\left(\widehat{P}_{n,\vec{\mathbf{s}}}(w-1) - E(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right)\left(1 - C_{\mathcal{L}}(w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{w}\right). \quad (4.99)$$

Now

$$\begin{aligned}
\left|\widehat{P}_{n,\vec{\mathbf{s}}}(w-1)\right| &= \left|\sum_{k=0}^{w-1}\frac{1}{k!}\sum_{\vec{\ell}\in X_{\mathcal{M}}(k)} F(k;w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{\sum_{u=1}^{w-1} u\ell_u}\right| \\
&\le \left|\sum_{k=0}^{w-1}\frac{1}{k!}\sum_{\vec{\ell}\in X(k)} F(k;w;\vec{\mathbf{s}})\Big(\frac{1}{\sqrt{n}}\Big)^{\sum_{u=1}^{w-1} u\ell_u}\right| \quad \Big(\text{as } \; X_{\mathcal{M}}(k)\subseteq X(k)\Big) \\
&= \left|\sum_{k=0}^{w-1}\frac{1}{k!}\left(\sum_{u=1}^{w-1}\frac{\widetilde{g}_{u,\vec{\mathbf{s}}}}{\sqrt{n}^{u}}\right)^{k}\right| \le \sum_{k=0}^{w-1}\frac{1}{k!}\left(\sum_{u=1}^{w-1}\frac{\left|\widetilde{g}_{u,\vec{\mathbf{s}}}\right|}{\sqrt{n}^{u}}\right)^{k} \\
&< \sum_{k=0}^{w-1}\frac{1}{k!} \quad \Big(\text{as } n > N_2(w;\vec{\mathbf{s}})\Big) < 3. \qquad (4.100)
\end{aligned}$$

Applying (4.100) to (4.98), we arrive at the upper bound of (4.97). We get the lower bound of (4.97) by applying (4.100) to (4.99) and from the fact that $C_{\mathcal{L}}(w;\vec{\mathbf{s}})\cdot E(w;\vec{\mathbf{s}}) > 0$ for all $w \ge 2$. □

4.7 Conclusion

We conclude this chapter by pointing out the following aspects in which Theorem 4.6.9 remains incomplete.

1. Suppose we are given the following two functions defined by shifts of $p(n)$:

$$SP(n;S) := \sum_{j=1}^{M}\prod_{i=1}^{T} p(n+s_{i,j}) \;\text{ and }\; SP(n;R) := \sum_{j=1}^{M}\prod_{i=1}^{T} p(n+r_{i,j}),$$

   where $S = (s_{i,j})_{1\le i\le T, 1\le j\le M}$ and $R = (r_{i,j})_{1\le i\le T, 1\le j\le M}$. Now in order to decide whether $SP(n;S) \ge SP(n;R)$ for all $n \ge N(S,R)$, we need to estimate

$\prod_{i=1}^{T} p(n+s_{i,j})$ and $\prod_{i=1}^{T} p(n+r_{i,j})$ individually for each $1 \leq j \leq M$. In view of Theorem 4.97, estimation of two factors come into the prominence: computation of the term $\sum_{j=1}^{M}\Big(\widehat{P}_{n,\vec{\mathbf{s}}_j}(w-1)-\widehat{P}_{n,\vec{\mathbf{r}}_j}(w-1)\Big)$ with $\vec{\mathbf{s}}_j := (s_{1,j},\ldots,s_{T,j})$, $\vec{\mathbf{r}}_j := (r_{1,j},\ldots,r_{T,j})$, and approximation of the error term.

2. Depending on the truncation point w, one can compute the main term

$$\sum_{j=1}^{M}\Big(\widehat{P}_{n,\vec{\mathbf{s}}_j}(w-1)-\widehat{P}_{n,\vec{\mathbf{r}}_j}(w-1)\Big).$$

But computational complexity will arise in estimation of the error term because in order to approximate $\widehat{E}(w;\vec{\mathbf{s}}_j)$ for each j, one needs to have a good control over $X_{\mathcal{E}}(k)$ for $0 \leq k \leq w-1$. This seems to be difficult as w tends to infinity, growth of $|X_{\mathcal{E}}(k)|$ is exponential.

3. For example, in order to prove the higher order Turán inequality for $p(n)$, the minimal choice for w is 10 and consequently, by Theorem 4.6.9 with appropriate choices for $\vec{\mathbf{s}}$, it follows that

$$4\big(1-u_{n-1}\big)\big(1-u_n\big)-\big(1-u_nu_{n-1}\big)^2 = \frac{\pi^3}{12\sqrt{6}}\frac{1}{n^{9/2}}+O\Big(\frac{1}{n^5}\Big).$$

This concludes that $p(n)$ satisfies the higher order Turán inequalities for sufficiently large n although due to Chen, Jia, and Wang [37], we know that the inequality holds for all $n \geq 95$. So, from the aspect of the error bound computation in order to confirm such inequalities from a certain explicit point onward, our method is inaccessible.

4. Last, but not least, the above discussions naively suggest that for making a decision whether a given inequality for the partition function (of the above types) holds or not, we need to have a full asymptotic expansion for the shifted value of the partition function and explicit computation of the error bound after truncation the expansion at any positive integer w.

Chapter 5

Error bounds for the asymptotic expansion of the partition function

Asymptotic study on the partition function $p(n)$ began with the work of Hardy and Ramanujan. Later Rademacher obtained a convergent series for $p(n)$ and an error bound was given by Lehmer. Despite having this, a full asymptotic expansion for $p(n)$ with an explicit error bound is not known. Recently O'Sullivan studied the asymptotic expansion of $p^k(n)$-partitions into kth powers, initiated by Wright, and consequently obtained an asymptotic expansion for $p(n)$ along with a concise description of the coefficients involved in the expansion but without any estimation of the error term. Here we consider a detailed and comprehensive analysis on an estimation of the error term obtained by truncating the asymptotic expansion for $p(n)$ at any positive integer N. This gives rise to an infinite family of inequalities for $p(n)$ which finally answers to a question proposed by Chen. Our error term estimation predominantly relies on applications of algorithmic methods from symbolic summation.

5.1 Asymptotic expansion of the partition function

A partition of a positive integer n is a non-increasing sequence of positive integers which sum to n, and the partition function $p(n)$ counts the number of partitions of n. In their epoch-making breakthrough work in the theory of partitions, Hardy and

Ramanujan [76] proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad \text{as } n \to \infty. \tag{5.1}$$

They also proved that $p(n)$ is the integer nearest to

$$\frac{1}{2\sqrt{2}} \sum_{q=1}^{\nu} \sqrt{q} A_q(n) \psi_q(n), \tag{5.2}$$

where $A_q(n)$ is a certain exponential sum, $\nu = \nu(n)$ is of the order of $\sqrt{n}$, and

$$\psi_q(n) = \frac{d}{dn}\left(\exp\left\{\frac{C}{q}\lambda_n\right\}\right), \; \lambda_n = \sqrt{n - \frac{1}{24}}, \; C = \pi\sqrt{\frac{2n}{3}}.$$

Extending ν to infinity, Lehmer [96] proved that (5.2) is a divergent series. Rademacher [122, 124, 123] considered a modification of (5.2) that presents a convergent series for $p(n)$ which reads:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \frac{d}{dn}\left(\frac{\sinh\Big(C\lambda_n/k\Big)}{\lambda_n}\right). \tag{5.3}$$

Lehmer [99, 98] obtained an error bound after subtraction of the Nth partial sum from the convergent series (5.3).

The study of a full asymptotic expansion for $p(n)$ can be traced in two directions by considering two different classes that arise from imposing restrictions on parts of partitions. The two restricted families are $p^s(n)$, the number of partitions of n into perfect sth powers, and $p(n,k)$, the number of partitions of n into at most k parts. As an application of the "circle method", Hardy and Ramanujan [76, Section 7, 7.3] obtained the main term in the asymptotic expansion of $p^s(n)$. This of course retrieves (5.1) when we take $s = 1$. Wright [152, 153] extended the work of Hardy and Ramanujan and obtained a full asymptotic expansion for $p^s(n)$. Recently O'Sullivan [112] proposed a simplified proof of Wright's results on the asymptotic expansion of $p^s(n)$, and consequently obtained an asymptotic formula for $p(n)$.

Theorem 5.1.1. *[112, Proposition 4.4] Let n and R be positive integers. As $n \to \infty$,*

$$p(n) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(1 + \sum_{t=1}^{R-1} \frac{\omega_t}{\sqrt{n}^t} + O\Big(n^{-R/2}\Big)\right), \tag{5.4}$$

with an implied constant depending only on R, where

$$\omega_t = \frac{1}{(-4\sqrt{6})^t} \sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \left(\frac{\pi}{6}\right)^{t-2k}. \tag{5.5}$$

The binomial coefficient is defined as $\binom{x}{k} := x(x-1)\dots(x-k+1)/k!$ if $k \in \mathbb{Z}_{\geq 0}$, $\binom{x}{0} := 1$, and $\binom{x}{k} := 0$ if $k \in \mathbb{Z}_{<0}$. Szekeres [142] proposed an asymptotic expansion for $p(n,k)$ for n and k sufficiently large and considering $k = n$, one obtains the expansion for $p(n)$ as $p(n,k) = p(n)$. Canfield [33] proved Szekeres' result by using a recursion satisfied by $p(n,k)$ without using theory of complex functions and as a corollary, obtained the main term of the Hardy-Ramanujan formulas for $p(n)$, see (5.1). For a probabilistic approach to the asymptotic expansion of $p(n)$, we refer to [30].

The primary objective of this chapter is to obtain an explicit and computable error bound for the asymptotic expansion of $p(n)$. A main motivation to consider such a problem is that from the literature, including the works [152, 142, 33, 30, 112], we could not retrieve any information on the error bound for asymptotic expansion of $p(n)$. An advantage of getting a control over the error bound is that one can prove the log-concavity property of $p(n)$ directly from the asymptotic expansion as speculated by Chen [35, p. 121]. In the language of Theorem 5.4, Chen's question can be formulated as follows:

Question 5.1.2. *Do there exists d and n_0 such that*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(1 + \sum_{t=1}^{3} \frac{\omega_t}{\sqrt{n}^t} - \frac{d}{n^2}\right) < p(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(1 + \sum_{t=1}^{3} \frac{\omega_t}{\sqrt{n}^t} + \frac{d}{n^2}\right) \tag{5.6}$$

holds for all $n > n_0$?

Chen remarked that (5.6) implies that $p(n)$ is log-concave for sufficiently large n. Now in order to demystify the phrase "sufficiently large", explicit information about n_0 is required; a question being intricately connected with the computation of the error bound d. A similar phenomena can be found in O'Sullivan's work:

Theorem 5.1.3. *[112, Theorem 1.3, (1.15)] For each positive integer k there exists $\mathcal{D}_k$ so that for all $n \geq \mathcal{D}_k$,*

$$p^k(n)^2 \geq p^k(n+1) \cdot p^k(n-1) \cdot (1 + n^{-2}). \tag{5.7}$$

For $k = 1$, Theorem 5.1.3 merely implies that $p(n)$ is log-concave for sufficiently large n although we know that $(p(n))_{n\geq 26}$ is log-concave due to [111, 53]. Moreover, O'Sullivan [112, (5.17)] proved that for large enough n,

$$\frac{p(n+1)p(n-1)}{p(n)^2}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > 1,$$

settled by Chen, Wang, and Xie [39]. The first three authors and Zeng [22, Theorem 7.6] proved a stronger version of (5.7) using an infinite familiy of inequalities for $\log p(n)$.

We conclude this section by discussing the novelty of this chapter in brevity. In order to elucidate the term $O\left(n^{-R/2}\right)$ in (5.4), determination of the asymptotic growth of the coefficients ω_t in (5.5) is required; a task which looks deceptively simple. Our representation of ω_t is of the following form:

$$\omega_t = \sum_{u=0}^{t} \gamma(u) \sum_{s=0}^{u} \psi(s).$$

In an effort to estimate the inner sum $\sum_{s=0}^{u} \psi(s)$, the use of the symbolic summation tool `Sigma` [128] was essential. Schneider considered [128, 127, 129] a broader algorithmic framework that subsumes the theory of difference field and ring extensions together with the method of creative telescoping. This algorithmic tool began to be aimed at a wider class of multi-sums, most frequently encountered in problems of enumerative combinatorics. For example, in Andrews, Paule, and Schneider [11] we can see how `Sigma` assists to solve the TSPP-problem in an LU-reformulation by Andrews. Beyond the world of combinatorics, applications of `Sigma` transcends to solve a very general class of Feynman integrals which are of relevance for manifold physical processes in quantum field theory, see [1]. This chapter adds a new facet to the regime of applications of `Sigma`; in particular, its foray into asymptotic estimation for partition-like functions seems to begin with this work.

5.2 A roadmap for the reader

In this section we will provide a roadmap on the structure of this chapter; i.e., a navigation from the starting point to the final goal of this chapter, to facilitate for the reader to follow.

Using the Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound, Chen, Jia, and Wang [37, Lemma 2.2] proved that for all $n \geq 1207$,

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{10}}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{10}}\right), \quad (5.8)$$

where for $n \geq 1$, $\mu(n) := \frac{\pi}{6}\sqrt{24n-1}$; a definition which is kept throughout this chapter. More generally, due to the first three authors and Zeng, we have the following result.

Theorem 5.2.1. *[22, Theorem 4.4] For $k \in \mathbb{Z}_{\geq 2}$, define*

$$\widehat{g}(k) := \frac{1}{24}\left(\frac{36}{\pi^2}\cdot \nu(k)^2+1\right),$$

where $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}$. Then for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k)$ such that $(n,k) \neq (6,2)$, we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{k}}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{k}}\right). \quad (5.9)$$

The goal of this chapter is to derive an inequality of the form

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{k-1}\frac{g(t)}{\sqrt{n}^j}+\frac{L(k)}{\sqrt{n}^k}\right) < p(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{k-1}\frac{g(t)}{\sqrt{n}^t}+\frac{U(k)}{\sqrt{n}^k}\right), \quad (5.10)$$

stated precisely in Theorem 5.7.5, starting from the inequality (5.9). As a consequence we obtain Corollary 9.1.7 which will give an explicit answer to the problem stated in Question 5.1.2 and which, as a further consequence reveals that $p(n)$ is log-concave for all $n \geq 26$, see Remark 5.7.7.

The first step is to find explicitly the coefficients $g(t)$ such that

$$\frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}\right) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\sum_{t=0}^{\infty}\frac{g(t)}{\sqrt{n}^t}.$$

This is done in Section 5.3 by computing separately $g(2t)$ and $g(2t+1)$. In spite of having a double sum representation for $g(t)$, we will see that the coefficients $g(t)$ are indeed equal to ω_t as in Theorem 5.1.1.

The next step is to estimate the number $g(t)$ in the following form:

$$f(t) - l(t) \leq g(t) \leq f(t) + u(t). \tag{5.11}$$

Here $f(t)$ has the property that $\lim_{t\to\infty} \frac{g(t)}{f(t)} = 1$, $\lim_{t\to\infty} \frac{l(t)}{f(t)} = 0$, and $\lim_{t\to\infty} \frac{u(t)}{f(t)} = 0$. Precise descriptions for $f(t)$, $u(t)$, and $l(t)$ are given in Section 5.5 along with the inequalities of the form (5.11). In order to prove such inequalities, we will use the preliminary lemmas from Section 5.4 and the summation package `Sigma`.

Finally in Section 5.6, applying the bounds for $g(t)$, given in Section 5.5, we find $\widehat{L}_1(k), \widehat{U}_1(k)$ such that

$$\frac{\widehat{L}_1(k)}{\sqrt{n}^k} < \sum_{t=k}^{\infty} \frac{g(t)}{\sqrt{n}^t} < \frac{\widehat{U}_1(k)}{\sqrt{n}^k}.$$

Also we compute explicitly $\widehat{L}_2(k)$ and $\widehat{U}_2(k)$ such that

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \frac{\widehat{L}_2(k)}{\sqrt{n}^k} < \frac{\sqrt{12}\ e^{\mu(n)}}{24n-1} \frac{1}{\mu(n)^k} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \frac{\widehat{U}_2(k)}{\sqrt{n}^k}.$$

Combining the error bounds as $L(k) = \widehat{L}_1(k) + \widehat{L}_2(k)$ and $U(k) = \widehat{U}_1(k) + \widehat{U}_2(k)$, we arrive at the desired inequality (5.10) for $p(n)$.

5.3 Estimation of the coefficients $g(t)$

From Theorem 5.2.1, we have for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k)$ such that $(n, k) \neq (6, 2)$,

$$\frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^k}\Big) < p(n) < \frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^k}\Big). \tag{5.12}$$

Rewrite the major term $\frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)}\Big)$ in the following way:

$$\frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)}\Big) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \underbrace{e^{\pi\sqrt{2n/3}\ \left(\sqrt{1-\frac{1}{24n}}-1\right)}}_{:=A_1(n)} \underbrace{\Big(1 - \frac{1}{24n}\Big)^{-1}\Big(1 - \frac{1}{\mu(n)}\Big)}_{:=A_2(n)}. \tag{5.13}$$

Next we compute the Taylor expansion of the residue parts of $A_1(n)$ and $A_2(n)$, defined in (5.13).

Definition 5.3.1. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$e_1(t) := \begin{cases} 1, & \text{if } t = 0 \\ \frac{(-1)^t}{(24)^t} \frac{(1/2-t)_{t+1}}{t} \sum_{u=1}^{t} \frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!} \Big(\frac{\pi^2}{36}\Big)^u, & \text{otherwise} \end{cases}, \tag{5.14}$$

and

$$E_1\Big(\frac{1}{\sqrt{n}}\Big) := \sum_{t=0}^{\infty} e_1(t) \Big(\frac{1}{\sqrt{n}}\Big)^{2t}, \ n \geq 1. \tag{5.15}$$

Definition 5.3.2. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$o_1(t) := -\frac{\pi}{12\sqrt{6}} \Bigg(\frac{(-1)^t(1/2-t)_{t+1}}{(24)^t} \sum_{u=0}^{t} \frac{(-1)^u(-t)_u}{(t+u+1)!(2u)!} \Big(\frac{\pi^2}{36}\Big)^u \Bigg) \tag{5.16}$$

and

$$O_1\Big(\frac{1}{\sqrt{n}}\Big) := \sum_{t=0}^{\infty} o_1(t) \Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}, \ n \geq 1. \tag{5.17}$$

Lemma 5.3.3. *For $j, k \in \mathbb{Z}_{\geq 0}$,*

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{i/2}{j} = \begin{cases} 1, & j = k = 0 \\ (-1)^j 2^{k-2j} \frac{k}{j} \binom{2j-k-1}{j-k}, & \text{otherwise} \end{cases}. \tag{5.18}$$

Proof. The case $j = k = 0$ is trivial. By the inversion relation

$$f(k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} g(i) \Leftrightarrow g(k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(i),$$

(5.18) for $j \neq 0$ is equivalent to

$$\sum_{i=0}^{k} (-1)^{i+j} 2^{i-2j} \frac{i}{j} \binom{k}{i} \binom{2j-i-1}{j-i} = \binom{k/2}{j};$$

which can be proved (and derived) by any standard summation method, resp. algorithm. □

Lemma 5.3.4. *Let $A_1(n)$ be defined as in (5.13). Let $E_1(n)$ be as in Definition 5.3.1 and $O_1(n)$ as in Definition 5.3.2. Then*

$$A_1(n) = E_1\Big(\frac{1}{\sqrt{n}}\Big) + O_1\Big(\frac{1}{\sqrt{n}}\Big). \tag{5.19}$$

Proof. From Equation (5.13), we get

$$\begin{aligned}
A_1(n) &= e^{\pi\sqrt{2n/3}\,\left(\sqrt{1-\frac{1}{24n}}-1\right)} \\
&= \sum_{k=0}^{\infty} \frac{(\pi\sqrt{2n/3})^k}{k!}\left(\sqrt{1-\frac{1}{24n}}-1\right)^k \\
&= \sum_{k=0}^{\infty} \frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(\sqrt{1-\frac{1}{24n}}\right)^i \\
&= \sum_{k=0}^{\infty} \frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=0}^{\infty}\binom{i/2}{j}\frac{(-1)^j}{(24n)^j} \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{\infty} \frac{(\pi\sqrt{2/3})^k}{k!}\frac{(-1)^{k-i+j}}{(24)^j}\binom{k}{i}\binom{i/2}{j}(\sqrt{n})^{k-2j}. \qquad (5.20)
\end{aligned}$$

Define $S := \{(k,i,j) \in \mathbb{Z}^3_{\geq 0} : 0 \leq i \leq k\}$. In order to express $A_1(n)$ in the form $\sum_{m=0}^{\infty} a_m(\frac{1}{\sqrt{n}})^m$, we split the set S into a disjoint union of subsets; i.e., $S := \bigcup_{t\in\mathbb{Z}_{\geq 0}} V(t)$, where for each $t \in \mathbb{Z}_{\geq 0}$, $V(t) := \{(k,i,j) \in \mathbb{Z}^3_{\geq 0} : k-2j = -t\}$.

Notice that for $k > j$, by Lemma 5.3.3, $\sum_{i=0}^{k}\binom{k}{i}\binom{i/2}{j} = 0$. Furthermore, for each element $r = (k,i,j) \in S$, we define

$$S(r) := \frac{(\pi\sqrt{2/3})^k}{k!}\frac{(-1)^{k-i+j}}{(24)^j}\binom{k}{i}\binom{i/2}{j} \quad \text{and} \quad f(r) := k-2j.$$

Rewrite (5.20) as

$$\begin{aligned}
A_1(n) &= \sum_{r\in S} S(r)(\sqrt{n})^{f(r)} = \sum_{t=0}^{\infty}\sum_{r\in V(t)} S(r)\left(\frac{1}{\sqrt{n}}\right)^t \\
&= \sum_{t=0}^{\infty}\sum_{r\in V(2t)} S(r)\left(\frac{1}{\sqrt{n}}\right)^{2t} + \sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\left(\frac{1}{\sqrt{n}}\right)^{2t+1}. \qquad (5.21)
\end{aligned}$$

Now

$$\begin{aligned}
V(2t) &= \{(k,i,j) \in S : k-2j = -2t\} \\
&= \{(k,i,j) \in S : k \equiv 0 \pmod 2 \text{ and } k-2j = -2t\} \\
&= \{(2u,i,j) \in S : j = u+t\} = \{(2u,i,u+t) \in \mathbb{Z}^3_{\geq 0} : 0 \leq i \leq 2u\}. \qquad (5.22)
\end{aligned}$$

From (5.22), it follows that

$$\begin{aligned}
&\sum_{t=0}^{\infty} \sum_{r\in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\
=\ & \sum_{t=0}^{\infty} \frac{(-1)^t}{(24)^t}\left(\sum_{u=0}^{\infty}\frac{(2\pi^2/3)^u}{(2u)!}\frac{(-1)^u}{(24)^u}\sum_{i=0}^{2u}(-1)^i\binom{2u}{i}\binom{i/2}{u+t}\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\
=\ & \sum_{t=0}^{\infty} \frac{(-1)^t}{(24)^t}\left(\sum_{u=0}^{\infty}\frac{(-1)^u}{(2u)!}\Big(\frac{\pi}{6}\Big)^{2u}\underbrace{\sum_{i=0}^{2u}(-1)^i\binom{2u}{i}\binom{i/2}{u+t}}_{:=\mathcal{E}_1(u,t)}\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}.
\end{aligned} \tag{5.23}$$

By Lemma 5.3.3,

$$\mathcal{E}_1(u,t) = \begin{cases} 1, & \text{if } u=t=0 \\ 0, & \text{if } u>t \\ \frac{2u(1/2-t)_{t+1}(-t)_u}{t(t+u)!}, & \text{otherwise} \end{cases}.$$

Consequently, for all $t \geq 1$,

$$\sum_{u=0}^{t}\frac{(-1)^u}{(2u)!}\Big(\frac{\pi}{6}\Big)^{2u}\mathcal{E}_1(u,t) = \frac{(1/2-t)_{t+1}}{t}\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\Big(\frac{\pi^2}{36}\Big)^u. \tag{5.24}$$

It follows that

$$\begin{aligned}
&\sum_{t=0}^{\infty} \sum_{r\in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\
&= 1+\sum_{t=1}^{\infty}\left(\frac{(-1)^t}{(24)^t}\frac{(1/2-t)_{t+1}}{t}\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\Big(\frac{\pi^2}{36}\Big)^u\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} = E_1\Big(\frac{1}{\sqrt{n}}\Big).
\end{aligned} \tag{5.25}$$

Similar to (5.22), we have

$$V(2t+1) = \big\{(2u+1, i, u+t+1) \in \mathbb{Z}_{\geq 0}^3 : 0 \leq i \leq 2u+1\big\}, \tag{5.26}$$

and consequently, it follows that

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}$$
$$= \sum_{t=0}^{\infty}\frac{(-1)^t}{(24)^t}\Bigg(\sum_{u=0}^{\infty}\frac{(\pi\sqrt{2/3})^{2u+1}}{(2u+1)!}\frac{(-1)^u}{(24)^{u+1}}\underbrace{\sum_{i=0}^{2u+1}(-1)^i\binom{2u+1}{i}\binom{i/2}{u+t+1}\Bigg)}_{:=\mathcal{O}_1(u,t)}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{5.27}$$

By Lemma 5.3.3,

$$\mathcal{O}_1(u,t)=\begin{cases} 0, & \text{if } u>t\\ -\frac{(2u+1)(1/2-t)_{t+1}(-t)_u}{(t+u+1)!}, & \text{otherwise}\end{cases}.$$

It follows that

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}$$
$$= -\frac{\pi}{12\sqrt{6}}\sum_{t=0}^{\infty}\Bigg(\frac{(-1)^t(1/2-t)_{t+1}}{(24)^t}\sum_{u=0}^{t}\frac{(-1)^u(-t)_u}{(t+u+1)!(2u)!}\Big(\frac{\pi^2}{36}\Big)^u\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}$$
$$= O_1\Big(\frac{1}{\sqrt{n}}\Big). \tag{5.28}$$

From (5.21), (5.25), and (5.28), we get (5.19). □

Definition 5.3.5. *For $t\in\mathbb{Z}_{\geq 0}$, define*

$$E_2\Big(\frac{1}{\sqrt{n}}\Big) := \sum_{t=0}^{\infty} e_2(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \quad \textit{with} \quad e_2(t) := \frac{1}{(24)^t}. \tag{5.29}$$

Definition 5.3.6. *For $t\in\mathbb{Z}_{\geq 0}$, define*

$$O_2\Big(\frac{1}{\sqrt{n}}\Big) := \sum_{t=0}^{\infty} o_2(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \quad \textit{with} \quad o_2(t) := -\frac{6}{\pi\sqrt{24}}\binom{-3/2}{t}\frac{(-1)^t}{(24)^t}. \tag{5.30}$$

Lemma 5.3.7. *Let $A_2(n)$ be defined as in (5.13). Let $E_2(n)$ be as in Definition 5.3.5 and $O_2(n)$ as in Definition 5.3.6. Then*

$$A_2(n) = E_2\Big(\frac{1}{\sqrt{n}}\Big) + O_2\Big(\frac{1}{\sqrt{n}}\Big). \tag{5.31}$$

Proof. Recall the definition of $A_2(n)$ from (5.13) and expand it in the following way:

$$\begin{aligned}
A_2(n) &= \Big(1-\frac{1}{24n}\Big)^{-1}\Big(1-\frac{1}{\mu(n)}\Big) = \Big(1-\frac{1}{24n}\Big)^{-1} - \frac{6}{\pi\sqrt{24}}\frac{1}{\sqrt{n}}\Big(1-\frac{1}{24n}\Big)^{-3/2}\\
&= \sum_{t=0}^{\infty}\frac{1}{(24)^t}\Big(\frac{1}{\sqrt{n}}\Big)^{2t} - \frac{6}{\pi\sqrt{24}}\sum_{t=0}^{\infty}\binom{-3/2}{t}\frac{(-1)^t}{(24)^t}\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\
&= E_2\Big(\frac{1}{\sqrt{n}}\Big) + O_2\Big(\frac{1}{\sqrt{n}}\Big). \qquad (5.32)
\end{aligned}$$

This completes the proof of (5.31). □

Definition 5.3.8. *In view of the Definitions 5.3.1-5.3.6, we define*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}}\Big) := E_1\Big(\frac{1}{\sqrt{n}}\Big)E_2\Big(\frac{1}{\sqrt{n}}\Big), \qquad (5.33)$$

$$S_{e,2}\Big(\frac{1}{\sqrt{n}}\Big) := O_1\Big(\frac{1}{\sqrt{n}}\Big)O_2\Big(\frac{1}{\sqrt{n}}\Big), \qquad (5.34)$$

$$S_{o,1}\Big(\frac{1}{\sqrt{n}}\Big) := E_1\Big(\frac{1}{\sqrt{n}}\Big)O_2\Big(\frac{1}{\sqrt{n}}\Big), \qquad (5.35)$$

and

$$S_{o,2}\Big(\frac{1}{\sqrt{n}}\Big) := E_2\Big(\frac{1}{\sqrt{n}}\Big)O_1\Big(\frac{1}{\sqrt{n}}\Big). \qquad (5.36)$$

Lemma 5.3.9. *For each $i \in \{1,2\}$, let $S_{e,i}\Big(\frac{1}{\sqrt{n}}\Big)$ and $S_{o,i}\Big(\frac{1}{\sqrt{n}}\Big)$ be as in Definition 5.3.8. Then*

$$\frac{\sqrt{12}\,e^{\mu(n)}}{24n-1}\Big(1-\frac{1}{\mu(n)}\Big) = \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\sum_{i=1}^{2}\Bigg(S_{e,i}\Big(\frac{1}{\sqrt{n}}\Big) + S_{o,i}\Big(\frac{1}{\sqrt{n}}\Big)\Bigg). \qquad (5.37)$$

Proof. The proof follows immediately by applying Lemmas 5.3.4 and 5.3.7 to (5.13). □

Definition 5.3.10. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$S_1(t) := \sum_{s=1}^{t}\frac{(-1)^s(1/2-s)_{s+1}}{s}\sum_{u=1}^{s}\frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\Big(\frac{\pi^2}{36}\Big)^u, \qquad (5.38)$$

and

$$g_{e,1}(t) := \frac{1}{(24)^t}\Big(1+S_1(t)\Big). \qquad (5.39)$$

Lemma 5.3.11. *Let* $S_{e,1}\Big(\frac{1}{\sqrt{n}}\Big)$ *be as in* (5.33)*. Let* $g_{e,1}(t)$ *be as in Definition* 5.3.10*. Then*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}}\Big) = \sum_{t=0}^{\infty} g_{e,1}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{5.40}$$

Proof. From (5.15), (5.29), and (5.33), we have

$$\begin{aligned}
S_{e,1}\Big(\frac{1}{\sqrt{n}}\Big) &= E_1\Big(\frac{1}{\sqrt{n}}\Big)E_2\Big(\frac{1}{\sqrt{n}}\Big)\\
&= \left(1+\sum_{t=1}^{\infty} e_1(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\right)\left(1+\sum_{t=1}^{\infty} e_2(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\right)\\
&= 1+\sum_{t=1}^{\infty}\Big(e_1(t)+e_2(t)\Big)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}+\sum_{t=2}^{\infty}\left(\sum_{s=1}^{t-1} e_1(s)e_2(t-s)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\\
&= 1+\sum_{t=1}^{\infty}\left(e_1(t)+e_2(t)+\sum_{s=1}^{t-1} e_1(s)e_2(t-s)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}.
\end{aligned} \tag{5.41}$$

Combining (5.14) and (5.29), we obtain

$$\begin{aligned}
&e_1(t)+e_2(t)+\sum_{s=1}^{t-1} e_1(s)e_2(t-s)\\
=\ & \frac{(-1)^t(1/2-t)_{t+1}}{(24)^t\ t}\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\Big(\frac{\pi^2}{36}\Big)^u+\frac{1}{(24)^t}\\
+\ & \frac{1}{24^t}\sum_{s=1}^{t-1}\left(\frac{(-1)^s(1/2-s)_{s+1}}{s}\sum_{u=1}^{s}\frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\Big(\frac{\pi^2}{36}\Big)^u\right)\\
=\ & \frac{1}{(24)^t}\Big(1+S_1(t)\Big) = g_{e,1}(t),
\end{aligned} \tag{5.42}$$

which concludes the proof of (5.40). □

Definition 5.3.12. *For* $t \in \mathbb{Z}_{\geq 1}$*, define*

$$S_2(t) := \sum_{s=0}^{t-1}(1/2-s)_{s+1}\binom{-3/2}{t-s-1}\sum_{u=0}^{s}\frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\Big(\frac{\pi^2}{36}\Big)^u, \tag{5.43}$$

and

$$g_{e,2}(t) := \frac{(-1)^{t-1}}{(24)^t} S_2(t). \tag{5.44}$$

Lemma 5.3.13. *Let $S_{e,2}\Big(\frac{1}{\sqrt{n}}\Big)$ as in (5.34) and $g_{e,2}(t)$ as in Definition 5.3.12. Then*

$$S_{e,2}\Big(\frac{1}{\sqrt{n}}\Big) = \sum_{t=1}^{\infty} g_{e,2}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{5.45}$$

Proof. From (5.17), (5.30) and (5.34), we have

$$\begin{aligned} S_{e,2}\Big(\frac{1}{\sqrt{n}}\Big) &= O_1\Big(\frac{1}{\sqrt{n}}\Big)O_2\Big(\frac{1}{\sqrt{n}}\Big) \\ &= \left(\sum_{t=0}^{\infty} o_1(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\right)\left(\sum_{t=0}^{\infty} o_2(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\right) \\ &= \sum_{t=1}^{\infty}\left(\sum_{s=0}^{t-1} o_1(s)o_2(t-s-1)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\ &= \sum_{t=1}^{\infty} g_{e,2}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \quad \text{(by (5.16) and (5.30))}. \end{aligned} \tag{5.46}$$

□

Definition 5.3.14. *For $t \in \mathbb{Z}_{\geq 2}$, define*

$$S_3(t) := \sum_{s=1}^{t} \frac{(1/2-s)_{s+1}\binom{-3/2}{t-s}}{s} \sum_{u=1}^{s} \frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\Big(\frac{\pi^2}{36}\Big)^u, \tag{5.47}$$

and

$$g_{o,1}(t) := \begin{cases} -\dfrac{6}{\pi\sqrt{24}}\dfrac{(-1)^t}{(24)^t}\left(\binom{-3/2}{t} + S_3(t)\right), & \text{if } t \geq 2 \\ -\dfrac{432+\pi^2}{2304\sqrt{6}\pi}, & \text{if } t = 1\ . \\ -\dfrac{6}{\pi\sqrt{24}}, & \text{if } t = 0 \end{cases} \tag{5.48}$$

Lemma 5.3.15. *Let $S_{o,1}\Big(\frac{1}{\sqrt{n}}\Big)$ as in (5.35) and $g_{o,1}(t)$ be as in Definition 5.3.14. Then*

$$S_{o,1}\Big(\frac{1}{\sqrt{n}}\Big) = \sum_{t=0}^{\infty} g_{o,1}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{5.49}$$

Proof. From (5.15), (5.30) and (5.35), it follows that

$$\begin{aligned}
S_{o,1}\Big(\frac{1}{\sqrt{n}}\Big) &= E_1\Big(\frac{1}{\sqrt{n}}\Big)O_2\Big(\frac{1}{\sqrt{n}}\Big) \\
&= \frac{1}{\sqrt{n}}\left(1+\sum_{t=1}^{\infty} e_1(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\right)\left(-\frac{6}{\pi\sqrt{24}}+\sum_{t=1}^{\infty} o_2(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\right) \\
&= -\frac{6}{\pi\sqrt{24}}\frac{1}{\sqrt{n}} - \frac{432+\pi^2}{2304\sqrt{6}\pi}\frac{1}{\sqrt{n}^3} + \sum_{t=2}^{\infty} g_{o,1}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \\
&\qquad \text{(by (5.14) and (5.30))}. \qquad (5.50)
\end{aligned}$$

□

Definition 5.3.16. *For $t \in \mathbb{Z}_{\geq 1}$, define*

$$S_4(t) := \sum_{s=0}^{t}(-1)^s(1/2-s)_{s+1}\sum_{u=0}^{s}\frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\Big(\frac{\pi^2}{36}\Big)^u, \qquad (5.51)$$

and

$$g_{o,2}(t) := -\frac{\pi}{12\sqrt{6}}\frac{1}{(24)^t}S_4(t). \qquad (5.52)$$

Lemma 5.3.17. *Let $S_{o,2}\Big(\frac{1}{\sqrt{n}}\Big)$ be as in (5.36) and $g_{o,2}(t)$ be as in Definition 5.3.16. Then*

$$S_{o,2}\Big(\frac{1}{\sqrt{n}}\Big) = \sum_{t=0}^{\infty} g_{o,2}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \qquad (5.53)$$

Proof. From (5.17), (5.29) and (5.36), it follows that

$$\begin{aligned}
S_{o,1}\Big(\frac{1}{\sqrt{n}}\Big) &= O_1\Big(\frac{1}{\sqrt{n}}\Big)E_2\Big(\frac{1}{\sqrt{n}}\Big) \\
&= \sum_{t=0}^{\infty}\left(\sum_{s=0}^{t} o_1(s)e_2(t-s)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \\
&= \sum_{t=0}^{\infty} g_{o,2}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \ \text{(by (5.16) and (5.29))}. \qquad (5.54)
\end{aligned}$$

□

Definition 5.3.18. *For each $i \in \{1,2\}$, let $g_{e,i}(t)$ and $g_{o,i}(t)$ be as in Definitions 5.3.10-5.3.16. We define a power series*

$$G(n) := \sum_{t=0}^{\infty} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t = \sum_{t=0}^{\infty} g(2t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{t=0}^{\infty} g(2t+1)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1},$$

where

$$g(2t) := g_{e,1}(t) + g_{e,2}(t) \quad \text{and} \quad g(2t+1) := g_{o,1}(t) + g_{o,2}(t). \tag{5.55}$$

Lemma 5.3.19. *Let $G(n)$ be as in Definition 5.3.18. Then*

$$\frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\Big(1 - \frac{1}{\mu(n)}\Big) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \cdot G(n). \tag{5.56}$$

Proof. Applying Lemmas 5.3.11-5.3.17 to Lemma 5.3.9, we immediately obtain (5.56). □

Remark 5.3.20. *Note that using* `Sigma` *and* `GeneratingFunctions` *due to* Mallinger *[104], we observe that for all $t \geq 0$,*

$$g(2t) = g_{e,1}(t) + g_{e,2}(t) = \omega_{2t} \quad \text{and} \quad g(2t+1) = g_{o,1}(t) + g_{o,2}(t) = \omega_{2t+1}, \tag{5.57}$$

where ω_t is as in (5.5). Equivalently,

$$g(t) = \omega_t = \frac{1}{(-4\sqrt{6})^t} \sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \Big(\frac{\pi}{6}\Big)^{t-2k}. \tag{5.58}$$

However this was already clear from the uniqueness of the asymptotic expansion for $p(n)$ and its proof can be considered as an additional verification of our computations. The reader might wonder at this point why we did not use the single sum expression found by O'Sullivan to bound the remainder of the asymptotic expansion for $p(n)$. We tried this indeed, but could not obtain from ω_t an effective upper and lower bound. The summation package `Sigma` *could not rewrite ω_t as a definite sum which is crucial for our estimations. However going to the double sum expression $g(t)$,* `Sigma` *was able to give a definite sum expression for the inner sum as we will see later, and this enabled us to obtain effective upper and lower bounds in the sense that we described earlier. Namely, $l(t) < g(t) < u(t)$ and $\lim_{t\to\infty} \frac{l(t)}{g(t)} = \lim_{t\to\infty} \frac{u(t)}{g(t)} = 1$.*

5.4 Preliminary lemmas

This section presents all the preliminary facts needed for the proofs of the lemmas stated in Section 5.5. The proofs of Lemmas 5.4.1 to 5.4.6, except 5.4.4, are presented in Subsection 5.8.1.

Lemma 5.4.1. *Let $x_1, x_2, \ldots, x_n \leq 1$ and $y_1, \ldots, y_1$ be non-negative real numbers. Then*

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_n)}{(1+y_1)(1+y_2)\cdots(1+y_n)} \geq 1 - \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} y_j.$$

Lemma 5.4.2. *For $t \geq 1$ and non-negative integer $u \leq t$, we have*

$$\frac{1}{2t} \geq \frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} \geq \frac{1}{2t}\left(1 - \frac{u^2+\frac{1}{2}}{t}\right).$$

Lemma 5.4.3. *For $t \geq 1$ and non-negative integer $u \leq t$, we have*

$$\frac{2u+1}{2t} \geq \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i} \geq \frac{2u+1}{2t} - \frac{4u^3+6u^2+8u+3}{12t^2}.$$

Throughout the rest of this chapter,

$$\alpha := \frac{\pi}{6}.$$

Lemma 5.4.4. *We have*

$$\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} = \cosh(\alpha),\ \sum_{u=0}^{\infty}\frac{u\alpha^{2u}}{(2u)!} = \frac{1}{2}\alpha\sinh(\alpha),\ \sum_{u=0}^{\infty}\frac{u^2\alpha^{2u}}{(2u)!} = \frac{\alpha^2}{4}\cosh(\alpha) + \frac{\alpha}{4}\sinh(\alpha),$$

and

$$\sum_{u=0}^{\infty}\frac{u^3\alpha^{2u}}{(2u)!} = \frac{3\alpha^2}{8}\cosh(\alpha) + \frac{\alpha(\alpha^2+1)}{8}\sinh(\alpha).$$

Lemma 5.4.5. *Let $u \in \mathbb{Z}_{\geq 0}$. Assume that $a_{n+1} - a_n \geq b_{n+1} - b_n$ for all $n \geq u$, and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. *Then*

$$b_n \geq a_n \text{ for all } n \geq u.$$

Lemma 5.4.6. *For $t \geq 1$ and $k \in \{0, 1, 2, 3\}$ we have*

$$\sum_{u=t+1}^{\infty} \frac{u^k \alpha^{2u}}{(2u)!} \leq \frac{C_k}{t^2} \quad \text{with} \quad C_k = \frac{\alpha^4 2^k}{18}.$$

Lemma 5.4.7. *[22, Equation 7.5, Lemma 7.3] For $n, k, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$b_{k,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+k-1}} \binom{s+k-1}{s-1} \frac{1}{n^k},$$

then

$$0 < \sum_{t=k}^{\infty} \binom{-\frac{2s-1}{2}}{t} \frac{(-1)^k}{n^k} < b_{k,n}(s). \tag{5.59}$$

Lemma 5.4.8. *[22, Equation 7.9, Lemma 7.5] For $m, n, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$c_{m,n}(s) := \frac{2}{m} \frac{s^m}{n^m},$$

then

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty} \binom{1/2}{k} \frac{(-1)^k s^k}{n^k} < 0. \tag{5.60}$$

Lemma 5.4.9. *[22, Equation 7.7, Lemma 7.4] For $n, s \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{N}$ and $n > 2s$ let*

$$\beta_{m,n}(s) := \frac{2}{n^m} \binom{s+m-1}{s-1},$$

then

$$0 < \sum_{k=m}^{\infty} \binom{-s}{k} \frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{5.61}$$

5.5 Estimation of $\big(S_i(t)\big)$

For the sake of a compact representation the organization of this section is as follows. We first present the statements of the lemmas needed; then, in a separate subsection we present the proofs.

5.5.1 The Lemmas 5.5.1 to 5.5.4

Lemma 5.5.1. *Let $S_1(t)$ be as in Definition 5.3.10. Then for all $t \geq 1$,*

$$-\frac{1}{8t^2} < \frac{S_1(t)}{(-1)^t\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\big(\cosh(\alpha)-1\big) + \frac{1}{2t}\alpha\sinh(\alpha) < \frac{13}{25t^2}. \tag{5.62}$$

Lemma 5.5.2. *Let $S_2(t)$ be as in Definition 5.3.12. Then for all $t \geq 1$,*

$$-\frac{11}{10t} < \frac{S_2(t)}{\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha) + \frac{\sinh(\alpha)}{\alpha} < \frac{1}{t}. \tag{5.63}$$

Lemma 5.5.3. *Let $S_3(t)$ be as in Definition 5.3.14. Then for all $t \geq 2$,*

$$-\frac{71}{100t} < \frac{S_3(t)}{\binom{-\frac{3}{2}}{t}} + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\alpha\sinh(\alpha) + 1 - \cosh(\alpha) < \frac{12}{25t}. \tag{5.64}$$

Lemma 5.5.4. *Let $S_4(t)$ be as in Definition 5.3.16. Then for $t \geq 1$,*

$$-\frac{1}{3t^2} < \frac{S_4(t)}{(-1)^t\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha)}{\alpha} + \frac{1}{2t}\cosh(\alpha) < \frac{13}{20t^2}. \tag{5.65}$$

5.5.2 The Proofs of Lemmas 5.5.1 to 5.5.4

Proof of Lemma 5.5.1: We rewrite $S_1(t)$ as follows:

$$\begin{aligned} S_1(t) &= \sum_{u=1}^{t}\frac{(-1)^u\alpha^{2u}}{(2u-1)!}\sum_{s=u}^{t}\frac{(-1)^s}{s}\left(\frac{1}{2}-s\right)_{s+1}\frac{(-s)_u}{(s+u)!} \\ &= \sum_{u=1}^{t}\frac{(-1)^u\alpha^{2u}}{(2u-1)!}\underbrace{\sum_{s=0}^{t-u}\frac{(-1)^{s+u}}{s+u}\left(\frac{1}{2}-s-u\right)_{s+u+1}\frac{(-s-u)_u}{(s+2u)!}}_{=:S_1(t,u)}. \end{aligned} \tag{5.66}$$

We use the summation package `Sigma` (and its mechanization by `EvaluateMultiSums`)[2], to derive and prove that

$$S_1(t,u) = (-1)^t\binom{-\frac{3}{2}}{t}\frac{(-1)^u}{2u}A_1(t,u), \tag{5.67}$$

[2]For further explanations of this rigorous computer derviation we refer to Appendix 5.8.2 and Remark 5.8.1.

where

$$A_1(t,u) = \frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} - \left(\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{(1+2t)} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}\right).$$

Now by Lemmas 5.4.2 and 5.4.3,

$$\frac{1}{2t}+\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{2u+1}{2t}-\frac{u^2+\frac{1}{2}}{2t^2} \le A_1(t,u) \le \frac{1}{2t}+\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{2u+1}{2t}+\frac{4u^3+6u^2+8u+3}{12t^2}.$$

It is convenient to reorder the terms in this inequality with respect to the powers of u:

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2} - \frac{u}{t} - \frac{u^2}{2t^2} \le A_1(t,u) \le \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} + \frac{1}{4t^2} + u\left(\frac{2}{3t^2} - \frac{1}{t}\right) + \frac{u^2}{2t^2} + \frac{u^3}{3t^2}. \quad (5.68)$$

Combining (5.66) and (5.67), if follows that

$$S_1(t) = (-1)^t\binom{-\frac{3}{2}}{t}\sum_{u=1}^{t}\frac{\alpha^{2u}A_1(t,u)}{(2u)!}. \quad (5.69)$$

To derive a lower bound, combine (5.68) with (5.69) to get

$$\begin{aligned}
&\frac{S_1(t)}{(-1)^t\binom{-\frac{3}{2}}{t}} \\
\ge\ & \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\sum_{u=1}^{t}\frac{\alpha^{2u}}{(2u)!} - \frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha^{2u}}{(2u)!} - \frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha^{2u}}{(2u)!} \\
\ge & \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} - 1 - \sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u)!}\right) - \frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha^{2u}}{(2u)!} - \frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha^{2u}}{(2u)!}. \\
> & \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} - 1 - \frac{\alpha^4}{18t^2}\right) - \frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha^{2u}}{(2u)!} - \frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha^{2u}}{(2u)!} \\
& \qquad \left(\text{by Lemma 5.4.6 and } \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} > \frac{1}{4t^2} \text{ for all } t \ge 1\right) \\
= & \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\left(\cosh(\alpha) - 1 - \frac{\alpha^4}{18t^2}\right) - \frac{1}{2t}\alpha\sinh(\alpha) -
\end{aligned}$$

$$\frac{1}{2t^2}\Big(\frac{\alpha^2}{4}\cosh(\alpha)+\frac{\alpha}{4}\sinh(\alpha)\Big)\quad\Big(\text{by Lemma 5.4.4}\Big)$$

$$> \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\Big(\cosh(\alpha)-1\Big)-\frac{\alpha^4}{18t^2}-\frac{1}{2t}\alpha\sinh(\alpha)-$$

$$\frac{1}{2t^2}\Big(\frac{\alpha^2}{4}\cosh(\alpha)+\frac{\alpha}{4}\sinh(\alpha)\Big)\quad\left(\text{as }\ \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}<1\ \text{ for all }\ t\geq 1\right)$$

$$= \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha)-1)-\frac{1}{2t}\alpha\sinh(\alpha)-$$

$$\frac{1}{2t^2}\left(\frac{\cosh(\alpha)-1}{2}+\frac{\alpha^4}{9}+\frac{\alpha^2}{4}\cosh(\alpha)+\frac{\alpha}{4}\sinh(\alpha)\right)$$

$$> \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha)-1)-\frac{1}{2t}\alpha\sinh(\alpha)-\frac{1}{8t^2}$$

$$\left(\text{as }\frac{\cosh(\alpha)-1}{2}+\frac{\alpha^4}{9}+\frac{\alpha^2}{4}\cosh(\alpha)+\frac{\alpha}{4}\sinh(\alpha)<\frac{1}{4}\right). \tag{5.70}$$

Similarly, for the upper bound, we have for all $t \geq 1$,

$$\frac{S_1(t)}{(-1)^t\binom{-\frac{3}{2}}{t}}$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{t}\frac{\alpha^{2u}}{(2u)!}-\frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha^{2u}}{(2u)!}+\frac{1}{4t^2}\sum_{u=1}^{t}\frac{\alpha^{2u}}{(2u)!}+\frac{2}{3t^2}\sum_{u=1}^{t}\frac{u\alpha^{2u}}{(2u)!}+\frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha^{2u}}{(2u)!}+$$

$$\frac{1}{3t^2}\sum_{u=1}^{t}\frac{u^3\alpha^{2u}}{(2u)!}$$

$$= \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{\infty}\frac{\alpha^{2u}}{(2u)!}-\frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha^{2u}}{(2u)!}+\frac{1}{t}\sum_{u=t+1}^{\infty}\frac{u\alpha^{2u}}{(2u)!}+\frac{1}{4t^2}\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!}$$

$$+\frac{2}{3t^2}\sum_{u=0}^{\infty}\frac{u\alpha^{2u}}{(2u)!}+\frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha^{2u}}{(2u)!}+\frac{1}{3t^2}\sum_{u=0}^{\infty}\frac{u^3\alpha^{2u}}{(2u)!}$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha)-1)-\frac{1}{2t}\alpha\sinh(\alpha)+\frac{\alpha^4}{9t^3}+\frac{1}{4t^2}\cosh(\alpha)+\frac{1}{3t^2}\alpha\sinh(\alpha)+$$

$$\frac{1}{2t^2}\left(\frac{\alpha^2}{4}\cosh(\alpha)+\frac{\alpha}{4}\sinh(\alpha)\right)+\frac{1}{3t^2}\left(\frac{3\alpha^2}{8}\cosh(\alpha)+\frac{\alpha(\alpha^2+1)}{8}\sinh(\alpha)\right)$$
$$\Big(\text{by Lemmas 5.4.4 and 5.4.6}\Big)$$
$$=\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha)-1)-\frac{1}{2t}\alpha\sinh(\alpha)+ \frac{1}{t^2}\left(\frac{\alpha^4}{9t}+\frac{\alpha^2+1}{4}\cosh(\alpha)+\frac{\alpha(\alpha^2+12)}{24}\sinh(\alpha)\right)$$
$$\leq\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha)-1)-\frac{1}{2t}\alpha\sinh(\alpha)+ \frac{1}{t^2}\left(\frac{\alpha^4}{9}+\frac{\alpha^2+1}{4}\cosh(\alpha)+\frac{\alpha(\alpha^2+12)}{24}\sinh(\alpha)\right)$$
$$<\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha)-1)-\frac{1}{2t}\alpha\sinh(\alpha)+\frac{13}{25t^2}$$
$$\left(\text{as }\frac{\alpha^4}{9}+\frac{\alpha^2+1}{4}\cosh(\alpha)+\frac{\alpha(\alpha^2+12)}{24}\sinh(\alpha)<\frac{13}{25}\right). \tag{5.71}$$

By (5.70) and (5.71), for all $t \geq 1$, it follows that

$$-\frac{1}{8t^2}<\frac{S_1(t)}{(-1)^t\binom{-\frac{3}{2}}{t}}-\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha)-1)+\frac{1}{2t}\alpha\sinh(\alpha)<\frac{13}{25t^2}, \tag{5.72}$$

which concludes the proof. □

Proof of Lemma 5.5.2: Rewrite $S_2(t)$ as follows:

$$\begin{aligned} S_2(t) &= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha^{2u}}{(2u)!}\sum_{s=u}^{t-1}\left(\frac{1}{2}-s\right)_{s+1}\binom{-\frac{3}{2}}{t-s-1}\frac{(-s)_u}{(s+u+1)!} \\ &= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha^{2u}}{(2u)!}\underbrace{\sum_{s=0}^{t-u-1}\left(\frac{1}{2}-s-u\right)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u-1}\frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_2(t,u)} \end{aligned} \tag{5.73}$$

Using the summation package Sigma (and its mechanization by EvaluateMultiSums)[3] we derive and prove that

$$S_2(t,u) = \binom{-\frac{3}{2}}{t}(-1)^{u+1}\Big(A_{2,1}(t,u) + A_{2,2}(t,u)\Big), \tag{5.74}$$

where

$$A_{2,1}(t,u) = \frac{2t(t-u)(-t)_u(-1)^u}{(1+2t)(1+2u)(t+u)(t)_u}$$

and

$$A_{2,2}(t,u) = \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-1)^i(-t)_i}{(t+i)(t)_i}.$$

From (5.73) and (5.74) it follows that

$$S_2(t) = -\binom{-\frac{3}{2}}{t}\Big(s_{2,1}(t) + s_{2,2}(t)\Big), \tag{5.75}$$

where

$$s_{2,1}(t) = \sum_{u=0}^{t-1}\frac{\alpha^{2u}}{(2u)!}A_{2,1}(t,u) \quad\text{and}\quad s_{2,2}(t) = \sum_{u=0}^{t-1}\frac{\alpha^{2u}}{(2u)!}A_{2,2}(t,u). \tag{5.76}$$

By Lemma 5.4.2, we have

$$\frac{1}{1+2u} - \frac{u^2+u+\frac{1}{2}}{t(1+2u)} \le \frac{t-u}{t(1+2u)}\left(1 - \frac{u^2+\frac{1}{2}}{t}\right) \le A_{2,1}(t,u) \le \frac{t-u}{t(1+2u)}. \tag{5.77}$$

Plugging (5.77) into (5.76) we obtain

$$\sum_{u=0}^{t-1}\frac{\alpha^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{t-1}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha^{2u} \le s_{2,1}(t) \le \sum_{u=0}^{t-1}\frac{\alpha^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{t-1}\frac{u\alpha^{2u}}{(2u+1)!},$$

and consequently,

$$\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u+1)!} - \sum_{u=t}^{\infty}\frac{\alpha^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{\infty}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha^{2u} \le s_{2,1}(t) \le$$
$$\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u+1)!} - \frac{1}{t}\left(\sum_{u=0}^{\infty}\frac{u\alpha^{2u}}{(2u+1)!} - \sum_{u=t}^{\infty}\frac{u\alpha^{2u}}{(2u+1)!}\right). \tag{5.78}$$

[3] We refer again to Appendix 5.8.2 and Remark 5.8.1 to see the underlying machinery in action.

By Lemma 5.4.6,

$$\sum_{u=t}^{\infty} \frac{\alpha^{2u}}{(2u+1)!} = \frac{1}{\alpha^2} \sum_{u=t+1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} = \frac{2}{\alpha^2} \sum_{u=t+1}^{\infty} \frac{u\alpha^{2u}}{(2u)!} \leq \frac{2C_1}{\alpha^2 t^2} = \frac{2\alpha^2}{9t^2}, \tag{5.79}$$

and

$$\sum_{u=t}^{\infty} \frac{u\alpha^{2u}}{(2u+1)!} = \frac{2}{\alpha^2} \sum_{u=t+1}^{\infty} \frac{u(u-1)\alpha^{2u}}{(2u)!} \leq \frac{2}{\alpha^2} \sum_{u=t+1}^{\infty} \frac{u^2\alpha^{2u}}{(2u)!} \leq \frac{2C_2}{\alpha^2 t^2} = \frac{4\alpha^2}{9t^2}. \tag{5.80}$$

Plugging (5.79) and (5.80) into (5.78) gives

$$\begin{split} \sum_{u=0}^{\infty} \frac{\alpha^{2u}}{(2u+1)!} - \frac{2\alpha^2}{9t^2} - \frac{1}{t} \sum_{u=0}^{\infty} \frac{u^2+u+\frac{1}{2}}{(2u+1)!} \alpha^{2u} \leq s_{2,1}(t) \leq \\ \sum_{u=0}^{\infty} \frac{\alpha^{2u}}{(2u+1)!} - \frac{1}{t} \sum_{u=0}^{\infty} \frac{u\alpha^{2u}}{(2u+1)!} + \frac{4\alpha^2}{9t^3}. \end{split} \tag{5.81}$$

Using Lemma 5.4.4, (5.81) further reduces to

$$\begin{split} \frac{\sinh(\alpha)}{\alpha} - \frac{1}{t}\left(\frac{\cosh(\alpha)}{4} + \frac{\sinh(\alpha)}{4\alpha} + \frac{\alpha\sinh(\alpha)}{4} + \frac{2\alpha^2}{9}\right) \leq s_{2,1}(t) \leq \\ \frac{\sinh(\alpha)}{\alpha} - \frac{1}{t}\left(\frac{\cosh(\alpha)}{2} - \frac{\sinh(\alpha)}{2\alpha} - \frac{4\alpha^2}{9}\right). \end{split} \tag{5.82}$$

A numerical check shows that

$$\frac{\cosh(\alpha)}{4} + \frac{\sinh(\alpha)}{4\alpha} + \frac{\alpha\sinh(\alpha)}{4} + \frac{2\alpha^2}{9} < \frac{7}{10} \quad \text{and} \quad \frac{\cosh(\alpha)}{2} - \frac{\sinh(\alpha)}{2\alpha} - \frac{4\alpha^2}{9} > -\frac{3}{40}.$$

This, along with (5.82), gives

$$\frac{\sinh(\alpha)}{\alpha} - \frac{7}{10t} < s_{2,1}(t) < \frac{\sinh(\alpha)}{\alpha} + \frac{3}{40t}. \tag{5.83}$$

Next we employ Lemma 5.4.3 and get

$$\frac{2u+1}{2t} - \frac{4u^3+6u^2+8u+3}{12t^2} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} \leq A_{2,2}(t,u) \leq \frac{2u+1}{2t} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}. \tag{5.84}$$

Plugging (5.84) into (5.76), we obtain

$$\sum_{u=0}^{t-1} \frac{\alpha^{2u}}{(2u)!}\left(\frac{2u+1}{2t} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} - \frac{4u^3+6u^2+8u+3}{12t^2}\right) \le s_{2,2}(t) \le \sum_{u=0}^{t-1} \frac{\alpha^{2u}}{(2u)!}\left(\frac{2u+1}{2t} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\right),$$

which, using $p_3(u) := 4u^3 + 6u^2 + 8u + 3$, can be rewritten as

$$\frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u)!} - \frac{1}{2t}\sum_{u=t}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u)!} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} - \frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha^{2u}}{(2u)!}$$
$$\le s_{2,2}(t) \le \frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u)!} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} - \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=t}^{\infty}\frac{\alpha^{2u}}{(2u)!}. \tag{5.85}$$

By Lemma 5.4.6 we obtain

$$\sum_{u=t}^{\infty}\frac{\alpha^{2u}}{(2u)!} = \frac{1}{\alpha^2}\sum_{u=t+1}^{\infty}\frac{(2u-1)2u\alpha^{2u}}{(2u)!} \le \frac{4}{\alpha^2}\sum_{u=t+1}^{\infty}\frac{u^2\alpha^{2u}}{(2u)!} \le \frac{4C_2}{\alpha^2 t^2} = \frac{8\alpha^2}{9t^2} \tag{5.86}$$

and

$$\sum_{u=t}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u)!} = \frac{1}{\alpha^2}\sum_{u=t+1}^{\infty}\frac{2u(2u-1)^2\alpha^{2u}}{(2u)!} \le \frac{8}{\alpha^2}\sum_{u=t+1}^{\infty}\frac{u^3\alpha^{2u}}{(2u)!} \le \frac{8C_3}{\alpha^2 t^2} = \frac{32\alpha^2}{9t^2}. \tag{5.87}$$

Combining (5.86) and (5.87) with (5.85) gives

$$\frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u)!} - \frac{1}{2t}\frac{32\alpha^2}{9t^2} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} - \frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha^{2u}}{(2u)!}$$
$$\le s_{2,2}(t) \le \frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u)!} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} + \frac{(-1)^{t}}{\binom{-\frac{3}{2}}{t}}\frac{8\alpha^2}{9t^2}. \tag{5.88}$$

Furthermore, for all $t \ge 1$ we have $\binom{2t}{t} \ge \frac{4^t}{2\sqrt{t}}$ which implies

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} = \frac{2^{2t+1}}{t+1}\frac{1}{\binom{2t+2}{t+1}} < 1, \quad t \ge 1. \tag{5.89}$$

Applying (5.89) and Lemma 5.4.4 to (5.88), we obtain

$$\begin{aligned}\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{2t}\Big(\underbrace{\cosh(\alpha)+\alpha\sinh(\alpha)}_{=:\operatorname{csh}(\alpha)}\Big)-\frac{C_{2,2}(\alpha)}{t^2}\le s_{2,2}(t)\le \\ \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{2t}\operatorname{csh}(\alpha)+\frac{8\alpha^2}{9t^2},\end{aligned}\tag{5.90}$$

where

$$C_{2,2}(\alpha)=\frac{16\alpha^2}{9}+\frac{\alpha^2\cosh(\alpha)}{4}+\frac{\alpha^3\sinh(\alpha)}{24}+\frac{\cosh(\alpha)}{4}+\frac{\alpha\sinh(\alpha)}{2}<1\quad\text{and}\quad\frac{8\alpha^2}{9}<\frac{1}{4}.$$

Therefore

$$\begin{aligned}\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{2t}\operatorname{csh}(\alpha)-\frac{1}{t^2}\le s_{2,2}(t)\le \\ \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{2t}\operatorname{csh}(\alpha)+\frac{1}{4t^2}.\end{aligned}\tag{5.91}$$

Applying (5.83) and (5.91) to (5.75) we obtain

$$\begin{aligned}\frac{\sinh(\alpha)}{\alpha}-\frac{7}{10t}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{2t}\operatorname{csh}(\alpha)-\frac{1}{t^2}\le -\frac{S_2(t)}{\binom{\frac{-3}{2}}{t}}\le \\ \frac{\sinh(\alpha)}{\alpha}+\frac{3}{40t}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{2t}\operatorname{csh}(\alpha)+\frac{1}{4t^2},\end{aligned}$$

which implies that for $t\ge 1$,

$$\begin{aligned}\frac{\sinh(\alpha)}{\alpha}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{t}\left(-\frac{7}{10}+\frac{\operatorname{csh}(\alpha)}{2}-1\right)\le -\frac{S_2(t)}{\binom{\frac{-3}{2}}{t}}\le \\ \frac{\sinh(\alpha)}{\alpha}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)+\frac{1}{t}\left(\frac{3}{40}+\frac{\operatorname{csh}(\alpha)}{2}+\frac{1}{4}\right).\end{aligned}\tag{5.92}$$

Since

$$-\frac{7}{10}+\frac{\operatorname{csh}(\alpha)}{2}-1>-1\quad\text{and}\quad\frac{3}{40}+\frac{\operatorname{csh}(\alpha)}{2}+\frac{1}{4}<\frac{11}{10},$$

from (5.92), it follows that for all $t\ge 1$,

$$-\frac{1}{t}<-\frac{\sinh(\alpha)}{\alpha}-\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha)-\frac{S_2(t)}{\binom{-\frac{3}{2}}{t}}<\frac{11}{10t}.\tag{5.93}$$

Multiplying by -1 on both sides of (5.93), we get (5.63). $\square$

Proof of Lemma 5.5.3: Rewrite $S_3(t)$ as follows:

$$\begin{aligned} S_3(t) &= \sum_{u=1}^{t} \frac{(-1)^u \alpha^{2u}}{(2u-1)!} \sum_{s=u}^{t} \frac{1}{s} \left(\frac{1}{2} - s\right)_{s+1} \binom{-\frac{3}{2}}{t-s} \frac{(-s)_u}{(s+u)!} \\ &= \sum_{u=1}^{t} \frac{(-1)^u \alpha^{2u}}{(2u-1)!} \underbrace{\sum_{s=0}^{t-u} \frac{1}{s+u} \left(\frac{1}{2} - s - u\right)_{s+u+1} \binom{-\frac{3}{2}}{t-s-u} \frac{(-s-u)_u}{(s+2u)!}}_{=:S_3(t,u)} \end{aligned} \tag{5.94}$$

Using the summation package `Sigma` (and its mechanization by `EvaluateMultiSums`), the sum $S_3(t,u)$ can be rewritten[4] as an *indefinite* sum

$$S_3(t,u) = \binom{-\frac{3}{2}}{t} (-1)^u \Big(A_{3,1}(t,u) + A_{3,2}(t,u) \Big), \tag{5.95}$$

where

$$A_{3,1}(t,u) = \frac{t(1+2t-2u)(-t)_u(-1)^u}{2(1+2t)u(t+u)(t)_u}$$

and

$$A_{3,2}(t,u) = \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{1+2t} + \frac{2t}{1+2t} \sum_{i=1}^{u} \frac{(-t)_i(-1)^i}{(t+i)(t)_i}.$$

From (5.94) and (5.95), it follows that

$$S_3(t) = \binom{-\frac{3}{2}}{t} \Big(s_{3,1}(t) + s_{3,2}(t) \Big), \tag{5.96}$$

where

$$s_{3,1}(t) = \sum_{u=1}^{t} \frac{\alpha^{2u}}{(2u-1)!} A_{3,1}(t,u) \quad \text{and} \quad s_{3,2}(t) = \sum_{u=1}^{t} \frac{\alpha^{2u}}{(2u-1)!} A_{3,2}(t,u). \tag{5.97}$$

By Lemma 5.4.2, we have

$$-\frac{1+2t-2u}{2u} \frac{1}{2t} \frac{u^2+\frac{1}{2}}{t} \le A_{3,1}(t,u) - \frac{1+2t-2u}{2u} \frac{1}{2t} = A_{3,1}(t,u) - \frac{1}{2u} + \frac{2u-1}{4ut} \le 0. \tag{5.98}$$

[4] We refer again to Appendix 5.8.2 and Remark 5.8.1 to see the underlying machinery in action.

Equation (5.98) implies that

$$-\frac{3u^2+2u+\frac{1}{2}}{4ut} = -\frac{u^2+\frac{1}{2}}{2ut} - \frac{\frac{u^2}{2}+\frac{1}{4}}{2ut} - \frac{2u-1}{4ut} \le A_{3,1}(t,u) - \frac{1}{2u} \le -\frac{2u-1}{4ut} \le 0. \tag{5.99}$$

Plugging (5.99) into (5.97), we obtain

$$-\frac{1}{2t}\sum_{u=1}^{\infty}\frac{(3u^2+2u+\frac{1}{2})\alpha^{2u}}{(2u)!} \le -\frac{1}{2t}\sum_{u=1}^{t}\frac{(3u^2+2u+\frac{1}{2})\alpha^{2u}}{(2u)!} \le s_{3,1}(t) - \sum_{u=1}^{t}\frac{\alpha^{2u}}{(2u)!} \le 0,$$

and consequently,

$$-\frac{1}{2t}\sum_{u=1}^{\infty}\frac{(3u^2+2u+\frac{1}{2})\alpha^{2u}}{(2u)!} - \sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u)!} \le s_{3,1}(t) - \sum_{u=1}^{\infty}\frac{\alpha^{2u}}{(2u)!} \le -\sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u)!} \le 0. \tag{5.100}$$

Applying Lemmas 5.4.6 and 5.4.4 to (5.100) gives

$$-\frac{1}{2t} < -\frac{1}{t}\left(\frac{3\alpha^2\cosh(\alpha)+7\alpha\sinh(\alpha)+2\cosh(\alpha)-2}{8}+\frac{\alpha^4}{9}\right) \le s_{3,1}(t)+1-\cosh(\alpha) \le 0. \tag{5.101}$$

Next, by Lemma 5.4.3, we obtain

$$-\frac{4u^3+6u^2+8u+3}{12t^2} \le A_{3,2}(t,u) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t} \le 0. \tag{5.102}$$

Applying (5.102) to (5.97), it follows that

$$s_{3,2}(t) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{t}\frac{\alpha^{2u}}{(2u-1)!} - \frac{1}{2t}\sum_{u=1}^{t}\frac{(2u+1)\alpha^{2u}}{(2u-1)!} \le 0 \tag{5.103}$$

and

$$\begin{aligned} s_{3,2}(t) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{t}\frac{\alpha^{2u}}{(2u-1)!} - \frac{1}{2t}\sum_{u=1}^{t}\frac{(2u+1)\alpha^{2u}}{(2u-1)!} &\ge -\frac{1}{12t^2}\sum_{u=1}^{t}\frac{p_3(u)\alpha^{2u}}{(2u-1)!} \\ &\ge -\frac{1}{12t^2}\sum_{u=1}^{\infty}\frac{p_3(u)\alpha^{2u}}{(2u-1)!}, \end{aligned} \tag{5.104}$$

where $p_3(u) = 4u^3+6u^2+8u+3$ is as in (5.85). Equations (5.103) and (5.104) imply that

$$s_{3,2}(t) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{\infty}\frac{\alpha^{2u}}{(2u-1)!} - \frac{1}{2t}\sum_{u=1}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u-1)!} \le \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u-1)!}, \tag{5.105}$$

and

$$s_{3,2}(t) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \sum_{u=1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=1}^{\infty} \frac{(2u+1)\alpha^{2u}}{(2u-1)!} \geq - \frac{1}{12t^2} \sum_{u=1}^{\infty} \frac{p_3(u)\alpha^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha^{2u}}{(2u-1)!}. \tag{5.106}$$

By Lemma 5.4.6 we obtain

$$\sum_{u=t+1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} = 2 \sum_{u=t+1}^{\infty} \frac{u\alpha^{2u}}{(2u)!} \leq \frac{4\alpha^4}{3 \cdot 3! t^2} = \frac{2\alpha^4}{9t^2} \tag{5.107}$$

and

$$\sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha^{2u}}{(2u-1)!} = 2u \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha^{2u}}{(2u)!} \leq \frac{20\alpha^4}{3 \cdot 3! t^2} = \frac{10\alpha^4}{9t^2}. \tag{5.108}$$

Substituting (5.107)-(5.108) into (5.105) and (5.106), it follows that

$$s_{3,2}(t) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \sum_{u=1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=1}^{\infty} \frac{(2u+1)\alpha^{2u}}{(2u-1)!} \leq \frac{3}{2} \cdot \frac{2\alpha^4}{9t^2} = \frac{\alpha^4}{3t^2} \tag{5.109}$$

and

$$-\frac{1}{12t^2} \sum_{u=1}^{\infty} \frac{p_3(u)\alpha^{2u}}{(2u-1)!} - \frac{5\alpha^4}{9t^2} \leq -\frac{1}{12t^2} \sum_{u=1}^{\infty} \frac{p_3(u)\alpha^{2u}}{(2u-1)!} - \frac{1}{2t} \frac{10\alpha^4}{9t^2} \leq s_{3,2}(t) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \sum_{u=1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=1}^{\infty} \frac{(2u+1)\alpha^{2u}}{(2u-1)!}. \tag{5.110}$$

Using Lemma 5.4.4 into (5.109) and (5.110), we obtain

$$-\frac{61}{100t^2} < -\frac{1}{t^2}\left(\frac{3\alpha^3 \sinh(\alpha)}{8} + \frac{(\alpha^4 + 24\alpha^2)\cosh(\alpha)}{24} + \frac{3\alpha \sinh(\alpha)}{4} + \frac{5\alpha^4}{9} \right) \leq s_{3,2}(t) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \alpha \sinh(\alpha) - \frac{1}{2t} \mathrm{sch}(\alpha) \leq \frac{\alpha^4}{3t^2} < \frac{3}{100t^2}, \tag{5.111}$$

where $\operatorname{sch}(\alpha) := \alpha^2 \cosh(\alpha) + 2\alpha \sinh(\alpha)$. Combining (5.101) and (5.111), and then plugging into (5.96) it follows that

$$-\frac{1}{2t} - \frac{61}{100t^2} < \frac{S_3(t)}{\binom{-\frac{3}{2}}{t}} + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\alpha \sinh(\alpha) - \frac{1}{2t}\operatorname{sch}(\alpha) + 1 - \cosh(\alpha) < \frac{3}{100t^2}.$$

Since for $t \geq 2$,

$$-\frac{1}{2t} - \frac{61}{100t^2} + \frac{1}{2t}\operatorname{sch}(\alpha) > -\frac{71}{100t},$$

and

$$\frac{3}{100t^2} + \frac{1}{2t}\operatorname{sch}(\alpha) < \frac{12}{25t},$$

we finally get

$$\frac{12}{25t} > \frac{S_3(t)}{\binom{-\frac{3}{2}}{t}} + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\alpha \sinh(\alpha) + 1 - \cosh(\alpha) > -\frac{71}{100t}. \tag{5.112}$$

□

Proof of Lemma 5.5.4: Rewrite $S_4(t)$ as follows:

$$\begin{aligned} S_4(t) &= \sum_{u=0}^{t} \frac{(-1)^u \alpha^{2u}}{(2u)!} \sum_{s=u}^{t} (-1)^s \left(\frac{1}{2} - s\right)_{s+1} \frac{(-s)_u}{(s+u+1)!} \\ &= \sum_{u=0}^{t} \frac{(-1)^u \alpha^{2u}}{(2u)!} \underbrace{\sum_{s=0}^{t-u} (-1)^{s+u} \left(\frac{1}{2} - s - u\right)_{s+u+1} \frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_4(t,u)} \end{aligned} \tag{5.113}$$

Using again the summation package `Sigma` (and its mechanization by `EvaluateMultiSums`)[5], we rewrite $S_4(t,u)$ as an indefinite sum

$$S_4(t,u) = \binom{-\frac{3}{2}}{t} (-1)^{u+t} \Big(A_{4,1}(t,u) + A_{4,2}(t,u) \Big), \tag{5.114}$$

where

$$A_{4,1}(t,u) = \frac{t(-t)_u(-1)^u}{2(1+2t)(t+u)(t+u+1)(t)_u}$$

[5]We refer again to Appendix 5.8.2 and Remark 5.8.1 to see the underlying machinery in action.

and

$$A_{4,2}(t,u) = \frac{1}{1+2u}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{1+2t} - \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-1)^i(-t)_i}{(t+i)(t)_i}\right).$$

From (5.113) and (5.114) it follows that

$$S_4(t) = (-1)^t\binom{-\frac{3}{2}}{t}\Big(s_{4,1}(t) + s_{4,2}(t)\Big), \tag{5.115}$$

where

$$s_{4,1}(t) = \sum_{u=0}^{t}\frac{\alpha^{2u}}{(2u)!}A_{4,1}(t,u) \quad \text{and} \quad s_{4,2}(t) := \sum_{u=0}^{t}\frac{\alpha^{2u}}{(2u)!}A_{4,2}(t). \tag{5.116}$$

From Lemmas 5.4.1 and 5.4.2 we have

$$\frac{1}{4t^2}\left(1 - \frac{u^2+u+\frac{3}{2}}{t}\right) \le \frac{1}{2(t+u+1)}\frac{1}{2t}\left(1 - \frac{u^2+\frac{1}{2}}{t}\right) \le A_{4,1}(t,u) \le \frac{1}{2(t+u+1)}\frac{1}{2t} \le \frac{1}{4t^2}. \tag{5.117}$$

Combining (5.117) with (5.116), we obtain

$$\frac{1}{4t^2}\sum_{u=0}^{t}\frac{\alpha^{2u}}{(2u)!} - \frac{1}{4t^3}\sum_{u=0}^{t}\frac{(u^2+u+\frac{3}{2})\alpha^{2u}}{(2u)!} \le s_{4,1}(t) \le \frac{1}{4t^2}\sum_{u=0}^{t}\frac{\alpha^{2u}}{(2u)!},$$

and consequently, we get

$$\frac{1}{4t^2}\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} - \frac{1}{4t^2}\sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u)!} - \frac{1}{4t^3}\sum_{u=0}^{\infty}\frac{(u^2+u+\frac{3}{2})\alpha^{2u}}{(2u)!} \le s_{4,1}(t) \le \frac{1}{4t^2}\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!}. \tag{5.118}$$

Equation (5.118) together with Lemmas 5.4.6-5.4.4 imply

$$\frac{1}{4t^2}\cosh(\alpha) - \frac{3}{5t^3} \le \frac{1}{4t^2}\cosh(\alpha) - \frac{1}{t^3}\left(\frac{\alpha^4}{72} + \frac{(\alpha^2+6)\cosh(\alpha)}{16} + \frac{3\alpha\sinh(\alpha)}{16}\right) \le s_{4,1}(t) \le \frac{1}{4t^2}\cosh(\alpha). \tag{5.119}$$

Next, by Lemma 5.4.3, we obtain

$$0 \le A_{4,2}(t,u) - \frac{1}{1+2u}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t}\right) \le \frac{1}{1+2u}\frac{p_3(u)}{12t^2}, \tag{5.120}$$

where $p_3(u)$ is as in (5.85). Plugging (5.120) into (5.116), it follows that

$$0 \le s_{4,2}(t) - \sum_{u=0}^{\infty} \frac{\alpha^{2u}}{(2u+1)!}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t}\right) + \sum_{u=t+1}^{\infty} \frac{\alpha^{2u}}{(2u+1)!}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t}\right)$$
$$\frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha^{2u}}{(2u+1)!},$$

which implies that

$$-\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u+1)!} \le s_{4,2}(t) - \sum_{u=0}^{\infty} \frac{\alpha^{2u}}{(2u+1)!}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t}\right) \le \frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha^{2u}}{(2u+1)!} + \frac{1}{2t}\sum_{u=t+1}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u+1)!}. \tag{5.121}$$

By Lemma 5.4.6,

$$\sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u+1)!} \le \sum_{u=t+1}^{\infty}\frac{\alpha^{2u}}{(2u)!} \le \frac{\alpha^4}{3\cdot 3!t^2} = \frac{\alpha^4}{18t^2}, \tag{5.122}$$

and

$$\sum_{u=t+1}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u+1)!} \le \sum_{u=t+1}^{\infty}\frac{(2u+1)\alpha^{2u}}{(2u)!} \le \frac{C_0+2C_1}{t^2} = \frac{\alpha^4(1+4)}{3\cdot 3!t^2} = \frac{5\alpha^4}{18t^2}. \tag{5.123}$$

Applying (5.122) and (5.123) to (5.121) and using Lemma 5.4.4, we finally obtain

$$-\frac{3}{1000t^2} < -\frac{2}{3}\cdot\frac{\alpha^4}{18t^2} \le -\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\alpha^4}{18t^2} \le s_{4,2}(t) - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha)}{\alpha} + \frac{1}{2t}\cosh(\alpha) \le \frac{1}{t^2}\left(\frac{(\alpha^2+6)\cosh(\alpha)}{24} + \frac{\alpha\sinh(\alpha)}{8} + \frac{5\alpha^4}{36}\right) < \frac{7}{20t^2}. \tag{5.124}$$

From (5.119), (5.124), and (5.115), it follows that

$$\frac{1}{t^2}\left(-\frac{3}{1000}+\frac{1}{4}\cosh(\alpha)-\frac{3}{5}\right)\leq -\frac{3}{1000t^2}-\frac{3}{5t^3}+\frac{1}{4t^2}\cosh(\alpha)\leq \frac{S_4(t)}{(-1)^t\binom{-\frac{3}{2}}{t}}-\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha)}{\alpha}+\frac{1}{2t}\cosh(\alpha)<\frac{1}{t^2}\Big(\frac{7}{20}+\frac{1}{4}\cosh(\alpha)\Big).$$

This implies for $t \geq 1$,

$$\frac{13}{20t^2}>\frac{S_4(t)}{(-1)^t\binom{-\frac{3}{2}}{t}}-\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha)}{\alpha}+\frac{1}{2t}\cosh(\alpha)>-\frac{1}{3t^2}. \tag{5.125}$$

□

5.6 Error bounds

Lemma 5.6.1. *For all $n, k \in \mathbb{Z}_{\geq 1}$,*

$$\frac{1}{(24n)^k}<\sum_{t=k}^{\infty}\frac{1}{(24n)^t}\leq\frac{24}{23}\frac{1}{(24n)^k}. \tag{5.126}$$

Proof. The statement follows from

$$\sum_{t=k}^{\infty}\frac{1}{(24n)^t}=\frac{1}{(24n)^k}\frac{24n}{24n-1}\quad\text{and}\quad 1<\frac{24n}{24n-1}\leq\frac{24}{23}\quad\text{for all } n\geq 1.$$

□

Lemma 5.6.2. *For all $n, k, s \in \mathbb{Z}_{\geq 1}$,*

$$\frac{1}{(k+1)^{s-\frac{1}{2}}}\frac{1}{(24n)^k}<\sum_{t=k}^{\infty}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^s}\frac{1}{(24n)^t}<\frac{12}{5(k+1)^{s-\frac{1}{2}}}\frac{1}{(24n)^k}. \tag{5.127}$$

Proof. Rewrite the infinite sum as

$$\sum_{t=k}^{\infty}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^s}\frac{1}{(24n)^t}=\sum_{t=k}^{\infty}\frac{\binom{2t+2}{t+1}}{4^t}\frac{t+1}{2t^s}\frac{1}{(24n)^t}. \tag{5.128}$$

For all $t \geq 1$,

$$\frac{4^t}{2\sqrt{t}} \leq \binom{2t}{t} \leq \frac{4^t}{\sqrt{\pi t}}.$$

From (5.128) we get

$$\sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{t^s} \frac{1}{(24n)^t} \leq \sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{1}{(24n)^t} \leq \frac{4}{\sqrt{\pi}} \sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{2t^s} \frac{1}{(24n)^t}. \tag{5.129}$$

For all $k \geq 1$,

$$\sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{1}{(24n)^t} \geq \sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{t^s} \frac{1}{(24n)^t} > \sum_{t=k}^{\infty} \frac{1}{(t+1)^{s-\frac{1}{2}}} \frac{1}{(24n)^t} > \frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{1}{(24n)^k} \tag{5.130}$$

and

$$\begin{aligned} \sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{1}{(24n)^t} &\leq \frac{4}{\sqrt{\pi}} \sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{2t^s} \frac{1}{(24n)^t} \\ &< \frac{4}{\sqrt{\pi}} \sum_{t=k}^{\infty} \frac{1}{(t+1)^{s-\frac{1}{2}}} \frac{1}{(24n)^t} \\ &\leq \frac{4}{\sqrt{\pi}(k+1)^{s-\frac{1}{2}}} \sum_{t=k}^{\infty} \frac{1}{(24n)^t} \\ &< \frac{4 \cdot 24}{23 \cdot \sqrt{\pi}} \frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{1}{(24n)^k} \quad \text{(by (5.126))}. \\ &< \frac{12}{5} \frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{1}{(24n)^k}. \end{aligned} \tag{5.131}$$

Equations (5.130) and (5.131) imply (5.127). □

Lemma 5.6.3. *For $n \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 0}$,*

$$0 < \sum_{t=k}^{\infty} \binom{-\frac{3}{2}}{t} \frac{(-1)^t}{(24n)^t} < 4\sqrt{2} \frac{\sqrt{k+1}}{(24n)^k}. \tag{5.132}$$

Proof. Setting $(n, s) \mapsto (24n, 2)$ in (5.59), it follows that for all $n \geq 1$,

$$0 < \sum_{t=k}^{\infty} \binom{-\frac{3}{2}}{t} \frac{(-1)^t}{(24n)^t} < 4\sqrt{2} \frac{\sqrt{k+1}}{(24n)^k}.$$

□

Definition 5.6.4. *For all $k \geq 1$ define*

$$L_1(k) := \left(\cosh(\alpha) - \frac{6\alpha\sinh(\alpha)}{5\sqrt{k+1}} - \frac{3}{10(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24}}\right)^{2k}$$

and

$$U_1(k) := \left(\frac{24\cosh(\alpha)}{23} - \frac{\alpha\sinh(\alpha)}{2\sqrt{k+1}} + \frac{5}{4(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24}}\right)^{2k}.$$

Lemma 5.6.5. *Let $L_1(k)$ and $U_1(k)$ be as in Definition 5.6.4. Let $g_{e,1}(t)$ be as in Definition 5.3.10. Then for all $n, k \in \mathbb{Z}_{\geq 1}$,*

$$L_1(k)\left(\frac{1}{\sqrt{n}}\right)^{2k} < \sum_{t=k}^{\infty} g_{e,1}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t} < U_1(k)\left(\frac{1}{\sqrt{n}}\right)^{2k}. \tag{5.133}$$

Proof. From (5.39) and (5.62), it follows that for $t \geq 1$,

$$\cosh(\alpha) - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{2t}\alpha\sinh(\alpha) - \frac{1}{8}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2} < (24)^t g_{e,1}(t) = 1 + S_1(t)$$
$$< \cosh(\alpha) - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{2t}\alpha\sinh(\alpha) + \frac{13}{25}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}. \tag{5.134}$$

Now, applying (5.126) and (5.127) with $s = 1$ and 2, respectively, to (5.134), it follows that for all $k \geq 1$,

$$\sum_{t=k}^{\infty} g_{e,1}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t} > \left(\cosh(\alpha) - \frac{6\alpha\sinh(\alpha)}{5\sqrt{k+1}} - \frac{3}{10(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24n}}\right)^{2k}$$

and

$$\sum_{t=k}^{\infty} g_{e,1}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t} < \left(\frac{24\cosh(\alpha)}{23} - \frac{\alpha\sinh(\alpha)}{2\sqrt{k+1}} + \frac{13\cdot 12}{25\cdot 5}\frac{1}{(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24n}}\right)^{2k}$$
$$< \left(\frac{24\cosh(\alpha)}{23} - \frac{\alpha\sinh(\alpha)}{2\sqrt{k+1}} + \frac{5}{4}\frac{1}{(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24n}}\right)^{2k}.$$

□

Definition 5.6.6. *For all $k \geq 1$, define*

$$L_2(k) := \left(-\frac{24\cosh(\alpha)}{23} - \frac{12}{5\sqrt{k+1}}\right)\left(\frac{1}{\sqrt{24}}\right)^{2k}$$

and

$$U_2(k) := \left(-\cosh(\alpha) + \frac{4\sqrt{2}\sinh(\alpha)}{\alpha}\sqrt{k+1} + \frac{66}{25\sqrt{k+1}}\right)\left(\frac{1}{\sqrt{24}}\right)^{2k}.$$

Lemma 5.6.7. *Let $L_2(k)$ and $U_2(k)$ be as in Definition 5.6.6. Let $g_{e,2}(t)$ be as in Definition 5.3.12. Then for all $n, k \in \mathbb{Z}_{\geq 1}$,*

$$L_2(k)\left(\frac{1}{\sqrt{n}}\right)^{2k} < \sum_{t=k}^{\infty} g_{e,2}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t} < U_2(k)\left(\frac{1}{\sqrt{n}}\right)^{2k}. \tag{5.135}$$

Proof. From (5.44) and (5.63), it follows that for $t \geq 1$,

$$\begin{aligned}-\cosh(\alpha) + (-1)^t\binom{-\frac{3}{2}}{t}\frac{\sinh(\alpha)}{\alpha} - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} &< (24)^t g_{e,2}(t) = (-1)^{t-1}S_2(t)\\ &< -\cosh(\alpha) + (-1)^t\binom{-\frac{3}{2}}{t}\frac{\sinh(\alpha)}{\alpha} + \frac{11}{10}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}.\end{aligned} \tag{5.136}$$

Now, applying (5.126), (5.127) with $s = 1$ and (5.132) to (5.136), it follows that for all $k \geq 1$,

$$\sum_{t=k}^{\infty} g_{e,2}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t} > \left(-\frac{24\cosh(\alpha)}{23} - \frac{12}{5\sqrt{k+1}}\right)\left(\frac{1}{\sqrt{24n}}\right)^{2k}$$

and

$$\sum_{t=k}^{\infty} g_{e,2}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t} < \left(-\cosh(\alpha) + \frac{4\sqrt{2}\sinh(\alpha)}{\alpha}\sqrt{k+1} + \frac{66}{25}\frac{1}{\sqrt{k+1}}\right)\left(\frac{1}{\sqrt{24n}}\right)^{2k}.$$

□

Definition 5.6.8. *For all $k \geq 1$, define*

$$L_3(k) := \left(\frac{19}{10}\alpha\sinh(\alpha) - \frac{109}{10}\cosh(\alpha)\sqrt{k+1} - \frac{23}{10}\frac{1}{\sqrt{k+1}}\right)\Big(\frac{1}{\sqrt{24}}\Big)^{2k+1}$$

and

$$U_3(k) := \left(2\alpha\sinh(\alpha) + \frac{33}{10}\frac{1}{\sqrt{k+1}}\right)\Big(\frac{1}{\sqrt{24}}\Big)^{2k+1}.$$

Lemma 5.6.9. *Let $L_3(k)$ and $U_3(k)$ be as in Definition 5.6.8. Let $g_{o,1}(t)$ be as in Definition 5.3.14. Then for all $n, k \in \mathbb{Z}_{\geq 1}$,*

$$L_3(k)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1} < \sum_{t=k}^{\infty} g_{o,1}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} < U_3(k)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1}. \tag{5.137}$$

Proof. Define $c_1(t) := -\frac{6}{\pi}(-1)^t\binom{-\frac{3}{2}}{t}$. From (5.48) and (5.64), it follows that for $t \geq 2$,

$$\begin{aligned}
&\frac{6}{\pi}\alpha\sinh(\alpha) - \frac{6}{\pi}\cosh(\alpha)(-1)^t\binom{-\frac{3}{2}}{t} - \frac{12\cdot 6}{25\cdot\pi}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}\\
&< (\sqrt{24})^{2t+1}g_{o,1}(t) = c_1(t)\left(1 + \frac{S_3(t)}{\binom{-\frac{3}{2}}{t}}\right)\\
&< \frac{6}{\pi}\alpha\sinh(\alpha) - \frac{6}{\pi}\cosh(\alpha)(-1)^t\binom{-\frac{3}{2}}{t} + \frac{71\cdot 6}{100\cdot\pi}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}.
\end{aligned} \tag{5.138}$$

A numerical check confirms that (5.138) also holds for $t = 1$; see (5.48). Now, applying (5.126), (5.127) with $s = 1$, and (5.132) to (5.138), it follows that for all $k \geq 1$,

$$\sum_{t=k}^{\infty} g_{o,1}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} > \left(\frac{19}{10}\alpha\sinh(\alpha) - \frac{109}{10}\cosh(\alpha)\sqrt{k+1} - \frac{23}{10}\frac{1}{\sqrt{k+1}}\right)\Big(\frac{1}{\sqrt{24n}}\Big)^{2k+1}$$

and

$$\begin{aligned}
\sum_{t=k}^{\infty} g_{o,1}(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} &< \left(\frac{6\cdot 24}{23\cdot\pi}\alpha\sinh(\alpha) + \frac{71\cdot 6\cdot 12}{100\cdot 5\cdot\pi}\frac{1}{\sqrt{k+1}}\right)\Big(\frac{1}{\sqrt{24n}}\Big)^{2k+1}\\
&< \left(2\alpha\sinh(\alpha) + \frac{33}{10}\frac{1}{\sqrt{k+1}}\right)\Big(\frac{1}{\sqrt{24n}}\Big)^{2k+1}.
\end{aligned}$$

□

Definition 5.6.10. *For all $k \geq 1$, define*

$$L_4(k) := \left(\frac{1}{4}\frac{\cosh(\alpha)}{\sqrt{k+1}} - \frac{11}{20}\alpha\sinh(\alpha) - \frac{41}{50}\frac{1}{(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24}}\right)^{2k+1}$$

and

$$U_4(k) := \left(\frac{63}{100}\frac{\cosh(\alpha)}{\sqrt{k+1}} - \frac{13}{25}\alpha\sinh(\alpha) + \frac{21}{50}\frac{1}{(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24}}\right)^{2k+1}.$$

Lemma 5.6.11. *Let $L_4(k)$ and $U_4(k)$ be as in Definition 5.6.10. Let $g_{o,2}$ be as in Definition 5.3.16. Then for all $n, k \in \mathbb{Z}_{\geq 1}$,*

$$L_4(k)\left(\frac{1}{\sqrt{n}}\right)^{2k+1} < \sum_{t=k}^{\infty} g_{o,2}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t+1} < U_4(k)\left(\frac{1}{\sqrt{n}}\right)^{2k+1}. \tag{5.139}$$

Proof. Define $c_2(t) := -\frac{\pi}{6}(-1)^t\binom{-\frac{3}{2}}{t}$. From (5.52) and (5.65), it follows that for $t \geq 1$,

$$\begin{aligned}
&\frac{\pi}{6\cdot 2}\cosh(\alpha)\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} - \frac{\pi}{6}\alpha\sinh(\alpha) - \frac{13\cdot\pi}{20\cdot 6}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}\\
&< (\sqrt{24})^{2t+1}g_{o,2}(t) = c_2(t)\frac{S_4(t)}{(-1)^t\binom{-\frac{3}{2}}{t}}\\
&< \frac{\pi}{6\cdot 2}\cosh(\alpha)\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} - \frac{\pi}{6}\alpha\sinh(\alpha) + \frac{\pi}{6\cdot 3}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}.
\end{aligned} \tag{5.140}$$

Now, applying (5.126) and (5.127) with $s = 1$ and 2, respectively, to (5.138), it follows that for all $k \geq 1$,

$$\sum_{t=k}^{\infty} g_{o,2}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t+1} > \left(\frac{1}{4}\frac{\cosh(\alpha)}{\sqrt{k+1}} - \frac{11}{20}\alpha\sinh(\alpha) - \frac{41}{50}\frac{1}{(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24n}}\right)^{2k+1}$$

and

$$\sum_{t=k}^{\infty} g_{o,2}(t)\left(\frac{1}{\sqrt{n}}\right)^{2t+1} < \left(\frac{63}{100}\frac{\cosh(\alpha)}{\sqrt{k+1}} - \frac{13}{25}\alpha\sinh(\alpha) + \frac{21}{50}\frac{1}{(k+1)^{3/2}}\right)\left(\frac{1}{\sqrt{24n}}\right)^{2k+1}.$$

□

Definition 5.6.12. *For $k \geq 1$, define*

$$\widehat{L}_2(k) := \frac{1}{\alpha^k}\frac{1}{\sqrt{24}^k}\left(1-\frac{1}{4\sqrt{n}}\right) \quad \text{and} \quad \widehat{U}_2(k) := \frac{1}{\alpha^k}\frac{1}{\sqrt{24}^k}\left(1+\frac{k}{3n}\right).$$

Definition 5.6.13. *For $k \geq 1$, define*

$$n_0(k) := \frac{k+2}{24}.$$

Lemma 5.6.14. *Let $\widehat{L}_2(k)$, and $\widehat{U}_2(k)$ be as in Definition 5.6.12. Let $n_0(k)$ be as in Definition 5.6.13. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n > n_0(k)$,*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{L}_2(k)}{\sqrt{n}^k} < \frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\frac{1}{\mu(n)^k} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(k)}{\sqrt{n}^k}. \tag{5.141}$$

Proof. Define

$$\mathcal{E}(n,k) := \frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\frac{1}{\mu(n)^k},\ \mathcal{U}(n,k) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{1}{\sqrt{n}^k}$$

and

$$\mathcal{Q}(n,k) := \frac{\mathcal{E}(n,k)}{\mathcal{U}(n,k)} = \mathcal{Q}(n,k) = \frac{e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1}{24n}}-1\right)}}{\alpha^k}\frac{1}{\sqrt{24}^k}\left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}}.$$

Using (5.60) with $(m,n,s) \mapsto (1,24n,1)$, we obtain for all $n \geq 1$,

$$-\frac{1}{12n} < \sqrt{1-\frac{1}{24n}}-1 = \sum_{m=1}^{\infty}\binom{1/2}{m}\frac{(-1)^m}{(24n)^m} < 0,$$

which implies that for $n \geq 1$,

$$\left(1-\frac{1}{4\sqrt{n}}\right) < e^{-\frac{\pi}{12}\sqrt{\frac{2}{3n}}} < e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1}{24n}}-1\right)} < 1. \tag{5.142}$$

Hence

$$\frac{1}{(\alpha\cdot\sqrt{24})^k}\left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}}\left(1-\frac{1}{4\sqrt{n}}\right) < \mathcal{Q}(n,k) < \frac{1}{(\alpha\cdot\sqrt{24})^k}\left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}}. \tag{5.143}$$

In order to estimate $\left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}}$, we need to split into two cases depending on k is even or odd.

For $k = 2\ell$ with $\ell \in \mathbb{Z}_{\geq 0}$:

$$\left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}} = \left(1-\frac{1}{24n}\right)^{-(\ell+1)} = 1 + \sum_{j=1}^{\infty} \binom{-(\ell+1)}{j} \frac{(-1)^j}{(24n)^j}.$$

From (5.61) with $(m, s, n) \mapsto (1, \ell+1, 24n)$, for all $n > \frac{\ell+1}{12}$, we get

$$0 < \sum_{j=1}^{\infty} \binom{-(\ell+1)}{j} \frac{(-1)^j}{(24n)^j} < \beta_{1,24n}(\ell+1) = \frac{\ell+1}{12n},$$

which is equivalent to

$$1 < \left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}} < 1 + \frac{k+2}{24n} \quad \text{for all} \quad n > \frac{k+2}{24}. \tag{5.144}$$

For $k = 2\ell + 1$ with $\ell \in \mathbb{Z}_{\geq 0}$:

$$\left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}} = \left(1-\frac{1}{24n}\right)^{-\frac{2\ell+3}{2}} = 1 + \sum_{j=1}^{\infty} \binom{-\frac{2\ell+3}{2}}{j} \frac{(-1)^j}{(24n)^j}.$$

Using (5.59) with $(m, s, n) \mapsto (1, \ell+2, 24n)$, for all $n > \frac{\ell+2}{24}$, we get

$$0 < \sum_{j=1}^{\infty} \binom{-\frac{2\ell+3}{2}}{j} \frac{(-1)^j}{(24n)^j} < b_{1,24n}(\ell+2) = \frac{\ell+2}{6n}$$

which is equivalent to

$$1 < \left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}} < 1 + \frac{k+3}{12n} \leq 1 + \frac{k}{3n} \quad \text{for all} \quad n > \frac{k+3}{48}. \tag{5.145}$$

From (5.144) and (5.145), for all $n > \frac{k+2}{24}$ it follows that

$$1 < \left(1-\frac{1}{24n}\right)^{-\frac{k+2}{2}} < 1 + \frac{k+3}{12n} \leq 1 + \frac{k}{3n}. \tag{5.146}$$

Combining (5.143) and (5.146) concludes the proof.

□

5.7 An infinite family of inequalities for $p(n)$

Definition 5.7.1. *For $w \in \mathbb{Z}_{\geq 1}$, define*

$$\big(\gamma_0(w), \gamma_1(w)\big) := \begin{cases} (23, 24), & \text{if } w \text{ is even} \\ (15, 17), & \text{if } w \text{ is odd} \end{cases}.$$

Definition 5.7.2. *Let $\gamma_0(w)$ and $\gamma_1(w)\big)$ be as in Definition 5.7.1. Then for all $w \in \mathbb{Z}_{\geq 1}$, define*

$$L(w) := -\gamma_0(w)\frac{\sqrt{\lceil w/2 \rceil + 1}}{\sqrt{24}^{w}} \quad \text{and} \quad U(w) := \gamma_1(w)\frac{\sqrt{\lceil w/2 \rceil + 1}}{\sqrt{24}^{w}}.$$

Lemma 5.7.3. *Let $\widehat{g}(k)$ be as in Theorem 5.2.1 and $n_0(k)$ as in Definition 5.6.13. Let $g(t)$ be as in (5.58). Let $L(w)$ and $U(w)$ be as in Definition 5.7.2. If $m \in \mathbb{Z}_{\geq 1}$ and $n > \max\big\{1, n_0(2m), \widehat{g}(2m)\big\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(2m)}{\sqrt{n}^{2m}}\right) < p(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(2m)}{\sqrt{n}^{2m}}\right).$$

Proof. Recalling Definition 5.3.18, from Lemma 5.3.19, we have

$$\begin{aligned} \sum_{t=0}^{\infty} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t &= \sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=2m}^{\infty} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t \\ &= \sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty} g(2t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{t=m}^{\infty} g(2t+1)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \\ &= \sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty}(g_{e,1}(t) + g_{e,2}(t))\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \\ &\qquad \sum_{t=m}^{\infty}(g_{o,1}(t) + g_{o,2}(t))\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \end{aligned} \tag{5.147}$$

Using Lemmas 5.6.5-5.6.11 by assigning $k \mapsto m$, it follows that

$$\begin{aligned} \frac{L_1(m) + L_2(m)}{\sqrt{n}^{2m}} + \frac{L_3(m) + L_4(m)}{\sqrt{n}^{2m+1}} &< \sum_{t=2m}^{\infty} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t \\ &< \frac{U_1(m) + U_2(m)}{\sqrt{n}^{2m}} + \frac{U_3(m) + U_4(m)}{\sqrt{n}^{2m+1}}. \end{aligned} \tag{5.148}$$

Moreover, by Lemma 5.6.14 with $k = 2m$, it follows that

$$\frac{\sqrt{12}\ e^{\mu(n)}}{24n-1}\frac{1}{\mu(n)^{2m}} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(2m)}{\sqrt{n}^{2m}}. \tag{5.149}$$

Finally, from (5.148) and (5.149) along with the fact that $U_3(m) + U_4(m) > 0$, we obtain

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{2m}}\right) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{\sum_{i=1}^4 U_i(m) + \widehat{U}_2(2m)}{\sqrt{n}^{2m}}\right). \tag{5.150}$$

Since for all $m \geq 1$, $L_3(m) + L_4(m) < 0$, it follows that

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{2m}}\right) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{\sum_{i=1}^4 L_i(m) - \widehat{U}_2(2m)}{\sqrt{n}^{2m}}\right). \tag{5.151}$$

From Lemmas 5.6.5-5.6.11 and 5.6.14, for all $n \geq \max\{1, n_0(2m)\}$,

$$\sum_{i=1}^{4} U_i(m) + \widehat{U}_2(2m) \quad < \quad \left(4 + \frac{4}{\sqrt{m+1}} + \frac{2}{(m+1)^{3/2}} + 6\sqrt{m+1} + \frac{2m}{3\alpha^2 n}\right)\frac{1}{\sqrt{24}^{2m}}.$$

For all $1 \leq m \leq 10$ observe that $n_0(2m) < 1$ and therefore, $\frac{2m}{3\alpha^2 n} < \frac{20}{3\alpha^2} < 25$; whereas for $m \geq 11$, $n_0(2m) > 1$. Consequently, $\frac{2m}{3\alpha^2 n} < \frac{8m}{\alpha^2(m+1)} < \frac{8}{\alpha^2} < 10$; i.e., $\frac{2m}{3\alpha^2 n} < 25$.

Continuing our estimation

$$\begin{aligned}\sum_{i=1}^{4} U_i(m) + \widehat{U}_2(2m) \quad &< \quad \left(29 + \frac{4}{\sqrt{m+1}} + \frac{2}{(m+1)^{3/2}} + 6\sqrt{m+1}\right)\frac{1}{\sqrt{24}^{2m}} \\ &\leq \quad \frac{24\sqrt{m+1}}{\sqrt{24}^{2m}} = U(2m). \end{aligned} \tag{5.152}$$

Similarly, for all $n \geq \max\{1, n_0(2m)\}$,

$$\begin{aligned}\sum_{i=1}^{4} L_i(m) - \widehat{U}_2(2m) \quad &> \quad \left(-29 - \frac{4}{\sqrt{m+1}} - \frac{1}{2(m+1)^{3/2}} - 3\sqrt{m+1}\right)\frac{1}{\sqrt{24}^{2m}} \\ &\geq \quad -\frac{23\sqrt{m+1}}{\sqrt{24}^{2m}} = L(2m). \end{aligned} \tag{5.153}$$

Plugging (5.152) and (5.153) into (5.150) and (5.151), respectively, and applying Theorem 5.2.1, we get

$$p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{2m}}\right) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(2m)}{\sqrt{n}^{2m}}\right) \tag{5.154}$$

and

$$p(n) > \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{2m}}\right) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(2m)}{\sqrt{n}^{2m}}\right). \tag{5.155}$$

□

Lemma 5.7.4. *Let $\widehat{g}(k)$ be as in Theorem 5.2.1 and $n_0(k)$ as in Definition 5.6.13. Let $g(t)$ be as in Equation (5.58). Let $L(w)$ and $U(w)$ be as in Definition 5.7.2. If $m \in \mathbb{Z}_{\geq 0}$ and $n > \max\{1, n_0(2m+1), \widehat{g}(2m+1)\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(2m+1)}{\sqrt{n}^{2m+1}}\right) < p(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(2m+1)}{\sqrt{n}^{2m+1}}\right).$$

Proof. Recalling Definition 5.3.18, by Lemma 5.3.19 we have

$$\begin{aligned}
\sum_{t=0}^{\infty} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t &= \sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=2m+1}^{\infty} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t \\
&= \sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty} g(2t+1)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} + \sum_{t=m+1}^{\infty} g(2t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\
&= \sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty}(g_{o,1}(t) + g_{o,2}(t))\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} + \\
&\qquad \sum_{t=m+1}^{\infty} (g_{e,1}(t) + g_{e,2}(t))\Big(\frac{1}{\sqrt{n}}\Big)^{2t}.
\end{aligned} \tag{5.156}$$

Using Lemmas 5.6.5-5.6.7 by substituting $k \mapsto m+1$ and Lemmas 5.6.9-5.6.11 by substituting $k \mapsto m$, it follows that

$$\frac{L_1(m+1)+L_2(m+1)}{\sqrt{n}^{2m+2}} + \frac{L_3(m)+L_4(m)}{\sqrt{n}^{2m+1}} < \sum_{t=2m+1}^{\infty} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t$$
$$< \frac{U_1(m+1)+U_2(m+1)}{\sqrt{n}^{2m+2}} + \frac{U_3(m)+U_4(m)}{\sqrt{n}^{2m+1}}. \tag{5.157}$$

By Lemma 5.6.14 with $k = 2m+1$,

$$\frac{\sqrt{12}\, e^{\mu(n)}}{24n-1}\frac{1}{\mu(n)^{2m+1}} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(2m+1)}{\sqrt{n}^{2m+1}}. \tag{5.158}$$

From (5.157) and (5.158) along with the fact that $U_1(m)+U_2(m) > 0$, we obtain

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{2m+1}}\right) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{\widehat{U}(2m+1)}{\sqrt{n}^{2m+1}}\right), \tag{5.159}$$

where

$$\widehat{U}(2m+1) = U_1(m+1)+U_2(m+1)+U_3(m)+U_4(m)+\widehat{U}_2(2m+1).$$

Since for all $m \geq 0$, $L_1(m)+L_2(m) < 0$, it follows that

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{2m+1}}\right) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{\widehat{L}(2m+1)}{\sqrt{n}^{2m+1}}\right) \tag{5.160}$$

with

$$\widehat{L}(2m+1) = L_1(m+1)+L_2(m+1)+L_3(m)+L_4(m)-\widehat{U}_2(2m+1).$$

Next, we estimate $\widehat{U}_2(2m+1)$. Recall from Lemma 5.6.14 that for all $n > n_0(2m+1)$,

$$\widehat{U}_2(2m+1) < \frac{1}{\alpha^{2m+1}}\Big(1+\frac{2m+1}{3n}\Big)\frac{1}{\sqrt{24}^{2m+1}} < \frac{1}{\alpha}\Big(1+\frac{2m+1}{3n}\Big)\frac{1}{\sqrt{24}^{2m+1}}.$$

We note that for $0 \leq m \leq 10$; $n \geq 1 > n_0(2m+1)$, and therefore, $\frac{1}{\alpha}\Big(1+\frac{2m+1}{3n}\Big) < \frac{8}{\alpha}$; whereas for $m \geq 11$, $n > \frac{2m+3}{24}$. This implies that $\frac{1}{\alpha}\Big(1+\frac{2m+1}{3n}\Big) < \frac{9}{\alpha}$. Hence, for all $n \geq \max\{1, n_0(2m+1)\}$,

$$\widehat{U}_2(2m+1) < \frac{9}{\alpha}\frac{1}{\sqrt{24}^{2m+1}}. \tag{5.161}$$

From Lemmas 5.6.5-5.6.11 and 5.6.14, for all $n \geq \max\{1, n_0(2m+1)\}$, we get

$$\begin{aligned} \widehat{U}(2m+1) &< \left(18+\frac{5}{\sqrt{m+1}}+\frac{1}{(m+1)^{3/2}}+2\sqrt{m+2}\right)\frac{1}{\sqrt{24}^{2m+1}} \\ &\leq \frac{17\sqrt{m+2}}{\sqrt{24}^{2m+1}} = U(2m+1). \end{aligned} \tag{5.162}$$

Similarly for all $n \geq \max\{1, n_0(2m+1)\}$, it follows that

$$\begin{aligned} \widehat{L}(2m+1) &> \left(-17-\frac{3}{\sqrt{m+1}}-\frac{1}{(m+1)^{3/2}}-13\sqrt{m+2}\right)\frac{1}{\sqrt{24}^{2m+1}} \\ &\geq -\frac{15\sqrt{m+2}}{\sqrt{24}^{2m+1}} = L(2m+1). \end{aligned} \tag{5.163}$$

Plugging (5.162) and (5.163) into (5.159) and (5.160), respectively, and applying Theorem 5.2.1, we get

$$p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{2m+1}}\right) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(2m+1)}{\sqrt{n}^{2m+1}}\right) \tag{5.164}$$

and

$$p(n) > \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{2m+1}}\right) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(2m+1)}{\sqrt{n}^{2m+1}}\right). \tag{5.165}$$

□

Theorem 5.7.5. *Let $\widehat{g}(k)$ be as in Theorem 5.2.1 and $g(t)$ as in (5.58). Let $L(w)$ and $U(w)$ be as in Definition 5.7.2. If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$ and $n > \widehat{g}(w)$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(w)}{\sqrt{n}^{w}}\right) < p(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1} g(t)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(w)}{\sqrt{n}^{w}}\right). \tag{5.166}$$

Proof. Combining Lemmas 5.7.3 and 5.7.4 together with the fact that $\widehat{g}(k) > \max\{n_0(k), 1\}$, we arrive at (5.166). $\square$

Corollary 5.7.6. *For all $n \geq 116$, we have*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{3}\frac{g(t)}{\sqrt{n}^t} - \frac{1}{14n^2}\right) < p(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{3}\frac{g(t)}{\sqrt{n}^t} + \frac{1}{13n^2}\right), \tag{5.167}$$

where

$$g(0) = 1,\ g(1) = -\frac{\pi^2+72}{24\sqrt{6}\pi},\ g(2) = \frac{\pi^2+432}{6912},\ g(3) = -\frac{\pi^4+1296\pi^2+93312}{497664\sqrt{6}\pi}.$$

Proof. Plugging $w = 4$ into (5.166), we obtain the inequality (5.167). $\square$

Remark 5.7.7. *Corollary 5.7.6 provides an answer to the Question 5.1.2, asked by Chen. As a consequence from (5.167), one can derive that $p(n)$ is* log-concave *for all $n \geq 26$.*

5.8 Appendix

5.8.1 Proofs of the lemmas presented in Section 5.4.

Proof of Lemma 5.4.1: For $n = 1$ we have to prove

$$\frac{1-x_1}{1+y_1} \geq 1 - x_1 - y_1,$$

which is equivalent to

$$1 - x_1 \geq (1+y_1)(1-x_1-y_1) \geq 1 - x_1 - y_1^2 - x_1y_1 \Leftrightarrow 0 \geq -y_1^2 - x_1y_1.$$

This is always true because x_1, y_1 are non-negative real numbers. Now assume by induction that the statement is true for $n = N$. Next we prove the statement for $n = N+1$. For $n = N$, we have $P \geq (1-S)$ with

$$P := \frac{(1-x_1)(1-x_2)\cdots(1-x_N)}{(1+y_1)(1+y_2)\cdots(1+y_N)} \quad \text{and} \quad S := \sum_{j=1}^{N} x_j + \sum_{j=1}^{N} y_j.$$

This implies that

$$P\frac{1-x_{N+1}}{1+y_{N+1}} \geq (1-S)\frac{1-x_{N+1}}{1+y_{N+1}}.$$

Therefore it suffices to prove that

$$(1-S)\frac{1-x_{N+1}}{1+y_{N+1}} \geq 1-S-y_{N+1}-x_{N+1},$$

which is equivalent to

$$(1-S)(1-x_{N+1}) \geq (1-S-y_{N+1}-x_{N+1})(1+y_{N+1}).$$

Equivalently,

$$1-x_{N+1}-S+S\cdot x_{N+1} \geq 1-S-x_{N+1}-y_{N+1}^2-x_{N+1}y_{N+1}-S\cdot y_{N+1},$$

which amounts to say that

$$S\cdot x_{N+1} \geq -y_{N+1}^2-x_{N+1}y_{N+1}-S\cdot y_{N+1},$$

and this inequality holds because $x_{N+1}, y_{N+1}, S \geq 0$. □

Proof of Lemma 5.4.2: Expanding the quotient $\frac{(-1)^i(-t)_i}{(t)_i}$ as

$$(-1)^i\frac{(-t)_i}{(t)_i} = (-1)^i\prod_{j=1}^{i}\frac{-t+j-1}{t+j-1} = \prod_{j=1}^{i}\frac{t-j+1}{t+j-1},$$

we obtain

$$\frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} = \frac{t}{2(t+\frac{1}{2})(t+u)}\prod_{j=1}^{u}\frac{t-(j-1)}{t+j-1} = \frac{1}{2t(1+\frac{1}{2t})(1+\frac{u}{t})}\prod_{j=1}^{u}\frac{1-\frac{j-1}{t}}{1+\frac{j-1}{t}}.$$

Since $t \geq 1$ and $u < t$, it is clear that

$$\frac{1}{2t(1+\frac{1}{2t})(1+\frac{u}{t})}\prod_{j=1}^{u}\frac{1-\frac{j-1}{t}}{1+\frac{j-1}{t}} \leq \frac{1}{2t}. \tag{5.168}$$

By Lemma 5.4.1, it follows that

$$\frac{1}{2t(1+\frac{1}{2t})(1+\frac{u}{t})}\prod_{j=1}^{u}\frac{1-\frac{j-1}{t}}{1+\frac{j-1}{t}} \geq \frac{1}{2t}\Big(1-\frac{\frac{1}{2}+u+2\sum_{j=1}^{u}(j-1)}{t}\Big) = \frac{1}{2t}\Big(1-\frac{u^2+\frac{1}{2}}{t}\Big). \tag{5.169}$$

Combining (5.168) and (5.169) concludes the proof. □

Proof of Lemma 5.4.3: By Lemma 5.4.1,

$$\frac{1}{2t} \geq \frac{1}{1+2t} = \frac{1}{2t}\frac{1}{(1+\frac{1}{2t})} \geq \frac{1}{2t}\Big(1-\frac{1}{2t}\Big) \geq \frac{1}{2t}-\frac{1}{4t^2}. \tag{5.170}$$

Now

$$\frac{2t\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}}{1+2t} = \frac{1}{1+\frac{1}{2t}}\sum_{i=1}^{u}\frac{1}{t+i}\prod_{j=1}^{i}\frac{t-j+1}{t+j-1} = \frac{1}{t}\frac{1}{1+\frac{1}{2t}}\sum_{i=1}^{u}\frac{1}{1+\frac{i}{t}}\prod_{j=1}^{i}\frac{1-\frac{j-1}{t}}{1+\frac{j-1}{t}}.$$

As $t \geq 1$ and $u < t$, it directly follows that

$$\frac{1}{t}\frac{1}{1+\frac{1}{2t}}\sum_{i=1}^{u}\frac{1}{1+\frac{i}{t}}\prod_{j=1}^{i}\frac{1-\frac{j-1}{t}}{1+\frac{j-1}{t}} \leq \frac{u}{t}. \tag{5.171}$$

Applying Lemma 5.4.1, we obtain

$$\frac{1}{t}\frac{1}{1+\frac{1}{2t}}\sum_{i=1}^{u}\frac{1}{1+\frac{i}{t}}\prod_{j=1}^{i}\frac{1-\frac{j-1}{t}}{1+\frac{j-1}{t}} \geq \frac{1}{t}\sum_{i=1}^{u}1-\frac{\frac{1}{2}+i+2\sum_{j=1}^{i}(j-1)}{t} = \frac{u}{t}-\frac{u(2u^2+3u+4)}{6t^2}. \tag{5.172}$$

Finally, (5.170), (5.171), and (5.172) imply the desired inequality. □

Proof of Lemma 5.4.5: Let $n \geq u$ be fixed. We have to show that $b_n \geq a_n$. First we note that

$$a_{k+1}-a_n = \sum_{j=n}^{k}(a_{j+1}-a_j) \geq \sum_{j=n}^{k}(b_{j+1}-b_j) = b_{k+1}-b_n.$$

Consequently, for all $k \geq n$ we have

$$a_{k+1}-a_n \geq b_{k+1}-b_n \Leftrightarrow b_n-b_{k+1} \geq a_n-a_{k+1}.$$

This implies that

$$b_n = \lim_{k\to\infty}(b_n-b_{k+1}) \geq \lim_{k\to\infty}(a_n-a_{k+1}) = a_n.$$

□

Proof of Lemma 5.4.6: We apply Lemma 5.4.5 with $a_n = \sum_{u=n+1}^{\infty}\frac{u^k\alpha^{2u}}{(2u)!}$ and $b_n = \frac{C_k}{n^2}$:

$$a_{n+1}-a_n = -\frac{(n+1)^k\alpha^{2n+2}}{(2n+2)!} \quad\text{and}\quad b_{n+1}-b_n = -\frac{C_k(2n+1)}{n^2(n+1)^2}.$$

Therefore $b_{n+1} - b_n \leq a_{n+1} - a_n$ is equivalent to

$$\frac{(n+1)^k \alpha^{2n+2}}{(2n+2)!} \leq \frac{C_k(2n+1)}{n^2(n+1)^2} \Leftrightarrow f(n) := \frac{n^2(n+1)^{k+2}\alpha^{2n+2}}{(2n+1)(2n+2)!} \leq C_k.$$

In order to prove $f(n) \leq C_k$, it suffices to prove $f(m) \leq C_k$, where m is such that $f(m)$ is maximal. Hence in order to find such a m, we find the first m such that $f(m+1) \leq f(m)$. This is equivalent to finding $\frac{f(m+1)}{f(m)} \leq 1$, also as we will see there is only one such maximum. Then $\max_{n\in\mathbb{N}} f(n) = f(m)$. Now

$$\frac{f(n+1)}{f(n)} = \frac{(n+1)^2(n+2)^{k+2}\alpha^{2n+4}}{(2n+3)(2n+4)!} \frac{(2n+1)(2n+2)!}{n^2(n+1)^{k+2}\alpha^{2n+2}} = \frac{\alpha^2(n+2)^{k+2}(2n+1)}{(2n+4)(2n+3)^2(n+1)^k n^2}.$$

Using Mathematica's implementation of Cylindrical Algebraic Decomposition [44], we obtain that

$$\frac{\alpha^2(n+2)^{k+2}(2n+1)}{(2n+4)(2n+3)^2(n+1)^k n^2} \leq 1, \text{ for all } \alpha^2 \leq \frac{800}{729}.$$

As $\alpha^2 = \frac{\pi^2}{36} < \frac{800}{729}$, $\max_{n\in\mathbb{N}} f(n) = f(1)$; i.e., $f(n) \leq f(1) = C_k$. □

5.8.2 The Sigma simplification of $S_3(t,u)$ in Lemma 5.5.3

Using the symbolic summation package `Sigma` [128] and its underlying machinery in the setting of difference rings [129] the inner sum $S_3(t,u)$ can be simplified as follows. Recall from (5.94) that

$$S_3(t,u) = \sum_{s=0}^{t-u} \frac{1}{s+u}\left(\frac{1}{2} - s - u\right)_{s+u+1} \binom{-\frac{3}{2}}{t-s-u} \frac{(-s-u)_u}{(s+2u)!}.$$

After loading `Sigma` into the computer algebra system Mathematica

In[7]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-JKU

we input the sum under consideration

In[8]:= $\mathbf{mySum3 = \sum_{s=0}^{t-u} \frac{1}{s+u}\left(\frac{1}{2} - s - u\right)_{s+u+1} \binom{-\frac{3}{2}}{t-s-u} \frac{(-s-u)_u}{(s+2u)!};}$

and compute a recurrence of it by executing

In[9]:= **rec3 = GenerateRecurrence[mySum3]**

```
Out[9]=  (t - u)uSUM[u] + 2(2 + t)(1 + u)SUM[u + 1] + (2 + u)(2 + t + u)SUM[u + 2] == 0
```

As a result we get a homogeneous linear recurrence of order 2 for $S_3(t, u) =$ `SUM[u](=mySum3)`. Internally, Zeilberger's creative telescoping paradigm [120] is applied which not only provides a recurrence but delivers simultaneously a proof certificate that guarantees the correctness of the result.

Verification of the recurrence. Denote the summand of $S_3(t, u)$ by $f(t, u, s)$; i.e. set

$$f(t,u,s) = \frac{1}{s+u}\Big(\frac{1}{2} - s - u\Big)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u}\frac{(-s-u)_u}{(s+2u)!}.$$

Then one can verify that the polynomials $a_0(t, u) = u(t-u)$, $a_1(t, u) = 2(2+t)(1+u)$ and $a_2(t, u) = (2+u)(2+t+u)$ (free of the summation variable s) and the expression

$$g(t,u,s) = -\frac{\gamma(t,u,s)s\binom{-\frac{3}{2}}{-s+t-u}(-s-u)_u\big(\frac{1}{2}-s-u\big)_{1+s+u}}{(s+2u)!(s+u)(1+s+2u)(2+s+2u)(3+s+2u)(-1+2s-2t+2u)}$$

with

$$\begin{aligned}\gamma(t,u,s) = &-6s - 6s^2 + 16s^3 + 8s^4 + 6t - 6st - 46s^2t - 20s^3t + 12t^2 + 30st^2 + 12s^2t^2\\ &- 12u - 22su + 66s^2u + 64s^3u + 8s^4u - 7tu - 138stu - 126s^2tu - 16s^3tu\\ &+ 52t^2u + 57st^2u + 8s^2t^2u - 27u^2 + 88su^2 + 172s^2u^2 + 44s^3u^2 - 108tu^2\\ &- 235stu^2 - 68s^2tu^2 + 57t^2u^2 + 24st^2u^2 + 32u^3 + 192su^3 + 92s^2u^3\\ &- 140tu^3 - 98stu^3 + 18t^2u^3 + 75u^4 + 86su^4 - 48tu^4 + 30u^5\end{aligned}$$

satisfy the summand recurrence

$$g(t,u,s+1) - g(t,u,s) = a_0(t,u)f(t,u,s) + a_1(t,u)f(t,u+1,s) + a_2(t,u)f(t,u+2,s) \tag{5.173}$$

for all $0 \leq s \leq t - u$ with $t \geq u$. The components of the summand recurrence can be obtained with the function call `CreativeTelescoping[mySum3]`. Summing the verified equation (5.173) over s from 0 to $t - u$ yields the output recurrence `Out[9]`, which at the same time yields a proof for the correctness of `Out[9]`. □

We remark that `Sigma`'s creative telescoping approach works not only for hypergeometric sums (here one could use, for instance, also the Paule-Schorn implementation [118] of Zeilberger's algorithm [120]), but can be applied in the general setting of difference rings which allows to treat summands built by indefinite nested sums and products. More involved examples in the context of plane partitions can be found, e.g., in [11].

We are now in the position to solve the output reccurrence Out[9] with the function call

In[10]:= **recSol = SolveRecurrence[rec3, SUM[u]]**

Out[10]= $\left\{\{0,(-1)^{u}\},\{0,\frac{(2+t-u)}{u(2+t+u)}\frac{(-t)_{u}}{(2+t)_{u}}+2(-1)^{u}\sum_{i=1}^{u}\frac{(-1)^{i}(-t)_{i}}{(2+i+t)(2+t)_{i}}\},\{1,0\}\right\}$

This means that we found two linearly independent solutions (the list entries whose first entry is a zero) that span the full solution space, i.e., the general solution to Out[9] is

$$G(t,u)=c_1(t)\,(-1)^u+c_2\Big(\frac{(2+t-u)}{u(2+t+u)}\frac{(-t)_u}{(2+t)_u}+2(-1)^u\sum_{i=1}^{u}\frac{(-1)^i(-t)_i}{(2+i+t)(2+t)_i}\Big) \tag{5.174}$$

where the c_1, c_2 are constants being free of u. For further details on the underlying machinery (inspired by [120]) we refer to [129].

Verification of the general solution. The correctness of the solutions can be verified by plugging them into the recurrence Out[9] and applying (iteratively) the shift relations

$$\begin{aligned}(-1)^{u+1}&=-(-1)^u,\\ (-t)_{1+u}&=(-t+u)(-t)_u,\\ (2+t)_{1+u}&=(2+t+u)(2+t)_u,\\ \sum_{i=1}^{1+u}\frac{(-1)^i(-t)_i}{(2+i+t)(2+t)_i}&=\sum_{i=1}^{u}\frac{(-1)^i(-t)_i}{(2+i+t)(2+t)_i}+\frac{(-1)^u(t-u)(-t)_u}{(2+t+u)(3+t+u)(2+t)_u}.\end{aligned}$$

Then simple rational function arithmetic shows that the obtained expression collapses to zero. □

Finally, we compute the first two initial values (by another round of symbolic summation) and find that

$$\begin{aligned}S_3(t,1)&=(-1)^t-\frac{(t+2)}{2(1+t)}\binom{-\frac{3}{2}}{t},\\ S_3(t,2)&=-(-1)^t+\frac{(8+7t+t^2)}{4(1+t)(2+t)}\binom{-\frac{3}{2}}{t}.\end{aligned} \tag{5.175}$$

With this information we can set $c_1=-(-1)^t+\frac{(3t+4)}{2(t+1)(t+2)}\binom{-\frac{3}{2}}{t}$ and $c_2=\frac{1}{2}\binom{-\frac{3}{2}}{t}$ so that the general solution (5.174) agrees with $S_3(t,u)$ for $u=1,2$. Since $S_3(t,u)$ and the specialized general solution are both solutions of the recurrence Out[9] and the first two initial values agree, they are identical for all $u \geq 0$ with $u \leq t$. This last step of combining the solutions accordingly can be accomplished by inserting the list of two initial values

In[11]:= $\mathbf{initialL} = \left\{(-1)^{t} - \frac{(t+2)}{2(1+t)}\binom{-\frac{3}{2}}{t}, -(-1)^{t} + \frac{(8+7t+t^{2})}{4(1+t)(2+t)}\binom{-\frac{3}{2}}{t}\right\};$

and then executing the command

In[12]:= **FindLinearCombination[recSol3, {1, initialL}, u, 2]**

Out[12]= $-(-1)^{t}(-1)^{u} + \frac{1}{2}\binom{-\frac{3}{2}}{t}\left(\frac{(2+t-u)}{u(2+t+u)}\frac{(-t)_{u}}{(2+t)_{u}} + (-1)^{u}\left(\frac{1}{1+t} + \frac{2}{2+t} + 2\sum_{i=1}^{u}\frac{(-1)^{i}(-t)_{i}}{(2+i+t)(2+t)_{i}}\right)\right)$

Carrying out all the steps above (including also the calculation of the initial values) can be rather cumbersome. In order to support the user with the simplification of such problems, the package

In[13]:= **<< EvaluateMultiSums.m**

EvaluateMultiSum by Carsten Schneider © RISC-JKU

has been developed. More precisely, by applying the command `EvaluateMultiSums` to the input sum `mySum3`$(= S_3(t,u))$, all the above steps are carried out automatically and one obtains in one stroke the desired result:

In[14]:= **sol3 = EvaluateMultiSum[mySum3, {}, {u, t}, {0, 1}, {t, ∞}]**

Out[14]= $-(-1)^{t}(-1)^{u} + \frac{1}{2}\binom{-\frac{3}{2}}{t}\left(\frac{(2+t-u)}{u(2+t+u)}\frac{(-t)_{u}}{(2+t)_{u}} + (-1)^{u}\left(\frac{1}{1+t} + \frac{2}{2+t} + 2\sum_{i=1}^{u}\frac{(-1)^{i}(-t)_{i}}{(2+i+t)(2+t)_{i}}\right)\right)$

Since we prefer to rewrite the found expression in terms of the Pochhammer symbol $(t)_u$, we execute the final simplification step with the function call

In[15]:= **SigmaReduce[sol3, u, Tower → {(t)$_u$}]**

Out[15]= $-(-1)^{t}(-1)^{u} + \binom{-\frac{3}{2}}{t}\left(\frac{t(1+2t-2u)}{2(1+2t)u(t+u)}\frac{(-t)_{u}}{(t)_{u}} + (-1)^{u}\left(\frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-1)^{i}(-t)_{i}}{(i+t)(t)_{i}}\right)\right)$

Remark 5.8.1. *We should mention that there is no particular reason for explaining the details of* `Sigma` *application only for* $S_3(t,u)$. *The simplification of the sums* $S_1(t,u)$, $S_2(t,u)$, *and* $S_4(t,u)$, *as in* (5.66), (5.73), *and* (5.113), *respectively, works completely analogously.*

5.9 Concluding remarks

We conclude this chapter with a list of possible future work based on the method devised in this chapter and its further applications.

1. A prudent application of our method might lead to obtaining full asymptotic expansion and respective error bounds for a broad class of functions; for example: $q(n)$-partitions into distinct parts, $p^s(n)$-partitions into perfect sth powers, k-colored partitions, k-regular partitions, Andrews' spt-function, $\alpha(n)$-nth coefficient of Ramanujan's third order mock theta function $f(q)$, the coefficient sequence of Klein's j-function, etc.

2. More generally consider the class of Dedekind η-quotients which fit perfectly into [41, Thm. 1.1] or [138, Thm. 1.1]. Therefore one can also obtain a full asymptotic expansion and infinite families of inequalities for the coefficient sequence arising from the Fourier expansion of the considered Dedekind η-function.

3. Theorem 5.7.5 can be utilized as a black box in order to prove inequalities pertaining to the partition function by constructing an unified framework. A major class of inequalities for $p(n)$ can be separated into the following two categories among many others:

   (a) Turán inequalities and its higher order analogues related to the real rootedness of Jensen polynomials associated to $p(n)$, studied in [53], [37], and [69].

   (b) Linear homogeneous inequalities for $p(n)$; i.e.,

   $$\sum_{i=1}^{r} p(n+x_i) \leq \sum_{i=1}^{s} p(n+y_i).$$

   For more details we refer to [83, 108].

4. More generally, it would be interesting to design a constructive method to decide whether for some positive integer N a relation of the form

   $$\sum_{j=1}^{M_1} \prod_{i=1}^{T_1} p(n+s_i^{(j)}) \leq \sum_{j=1}^{M_2} \prod_{i=1}^{T_2} p(n+r_i^{(j)})$$

   holds for all $n \geq N$ or not.

Chapter 6

Invariants of the quartic binary form and proofs of Chen's conjectures for partition function inequalities

The Turán inequality and the higher order Turán inequalities for $p(n)$ have been one of the more predominant themes in recent years. Griffin, Ono, Rolen, and Zagier proved that for every integer $d \geq 1$, there exists an integer $N(d)$ such that the Jensen polynomial of degree d and shift n associated with the partition function, denoted by $J_p^{d,n}(x)$, has only distinct real roots for all $n \geq N(d)$, conjectured by Chen, Jia, and Wang. Larson and Wagner have provided an estimate for $N(d)$. This implies that the discriminant of $J_p^{d,n}(x)$ is positive; i.e., $\mathrm{Disc}_x(J_p^{d,n}) > 0$. For $d = 2$, $\mathrm{Disc}_x(J_p^{d,n}) > 0$ when $n \geq N(d)$ is equivalent to the fact that $(p(n))_{n \geq 26}$ is log-concave. In 2017, Chen undertook a comprehensive investigation of inequalities for $p(n)$ through the lens of the invariant theory of binary forms of degree n. Positivity of the invariant of a quadratic binary form (resp. cubic binary form) associated with $p(n)$ reflects that the sequence $(p(n))_{n \geq 26}$ satisfies the Turán inequalities (resp. $(p(n))_{n \geq 95}$ satisfies the higher order Turán inequalities). Chen further studied the two invariants for a quartic binary form where its coefficients are shifted values of integer partitions and conjectured four inequalities for $p(n)$. In this chapter, we confirm the conjectures of Chen.

6.1 Inequalities for $p(n)$ and invariants of binary forms

Throughout this chapter, we consider only sequences of real numbers. A sequence $(a_n)_{n\geq 0}$ is said to satisfy the Turán inequalities or to be log-concave, if

$$a_n^2 - a_{n-1}a_{n+1} \geq 0 \quad \text{for all} \quad n \geq 1, \tag{6.1}$$

see [135]. We say that a sequence $(a_n)_{n\geq 0}$ is said to satisfy the higher order Turán inequalities if for all $n \geq 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 \geq 0. \tag{6.2}$$

The Turán inequalities and the higher order Turán inequalities are related to the Laguerre-Pólya class of real entire functions [56, 140]. A real entire function

$$\psi(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!} \tag{6.3}$$

is said to be in Laguerre-Pólya class, denoted by $\psi(x) \in \mathcal{LP}$, if it is of the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \Big(1 + \frac{x}{x_k}\Big) e^{-\frac{x}{x_k}},$$

where c, β, x_k are real numbers, $\alpha \geq 0$, $m \in \mathbb{Z}_{\geq 0}$, and $\sum_{k=1}^{\infty} x_k^{-2}$ converges. Any sequence of polynomials with only real zeroes, say $(P_n(x))_{n\geq 0}$, converges uniformly to a function $P(x) \in \mathcal{LP}$. For a more detailed study on the theory of the $\mathcal{LP}$ class, we refer to [125]. Jensen [80] proved that a real entire function $\psi(x)$ is in $\mathcal{LP}$ class if and only if for any $d \in \mathbb{Z}_{\geq 1}$, the Jensen polynomial of degree d associated with a sequence $(a_n)_{n\geq 0}$:

$$J_a^d(x) = \sum_{k=0}^{d} \binom{d}{k} a_k x^k$$

has only real zeroes. Pólya and Schur [130] proved that for a real entire function $\psi(x) \in \mathcal{LP}$ and for any $n \geq \mathbb{Z}_{\geq 0}$, the n-th derivative $\psi^{(n)}(x)$ of $\psi(x)$ also belongs to the $\mathcal{LP}$ class, that is, the Jensen polynomial associated with $\psi^{(n)}(x)$

$$J_a^{d,n}(x) = \sum_{k=0}^{d} \binom{d}{k} a_{n+k} x^k$$

has only real zeroes. Observe that for $d = 2$ and for all nonnegative integers n, the real-rootedness of $J_a^{d,n}(x)$ implies that the discriminant $4(a_{n+1}^2 - a_n a_{n+2})$ is nonnegative. Pólya's work [121] on $\mathcal{LP}$ class is closely connected with the Riemann hypothesis. He showed that the Riemann hypothesis is equivalent to the real rootedness of Jensen polynomial $J_a^{d,n}(x)$ for all nonnegative integers d and n, where the coefficient sequence $\{a_n\}_{n\geq 0}$ is defined by

$$(-1+4z^2)\ \Lambda\Big(\frac{1}{2}+z\Big) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^{2n},$$

with $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s)$, where ζ denotes the Riemann zeta function and Γ denotes the Gamma function. In 2019, Griffin, Ono, Rolen, and Zagier [69, Theorem 1] proved that for all $d \geq 1$, $J_a^{d,n}(x)$ has only real roots for all sufficiently large n.

Now we discuss in brief the inequalities of the partition function. A partition of a positive integer n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. Estimates on the partition function systematically began with the work of Hardy and Ramanujan [76] in 1918 and independently by Uspensky [144] in 1920:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty. \tag{6.4}$$

Hardy and Ramanujan's proof involved an important tool called the Circle Method which has manifold applications in analytic number theory. For a well-documented exposition of this collaboration, see [101]. During 1937-1943, Rademacher [122, 124, 123] improved the work of Hardy and Ramanujan and found a convergent series for $p(n)$ and Lehmer's [99, 98] considerations were on the estimation for the remainder term of the series for $p(n)$. The Hardy-Ramanujan-Rademacher formula reads

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] + R_2(n, N), \tag{6.5}$$

where

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \quad A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i s(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \left\lfloor \frac{\mu}{k} \right\rfloor - \frac{1}{2}\right)\left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2}\right),$$

and

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}}\left[\left(\frac{N}{\mu(n)}\right)^3 \sinh\frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2\right]. \tag{6.6}$$

Independently Nicolas [111] and DeSalvo and Pak [53, Theorem 1.1] proved that the partition function $(p(n))_{n\geq 26}$ is log-concave, conjectured by Chen [35]. DeSalvo and Pak [53, Theorem 4.1] also proved that for all $n \geq 2$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}, \tag{6.7}$$

conjectured by Chen [35]. Further, they improved the term $(1+\frac{1}{n})$ in (6.7) and proved that for all $n \geq 7$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \tag{6.8}$$

see [53, p. 4.2]. DeSalvo and Pak [53] finally came up with the conjecture that the coefficient of $1/n^{3/2}$ in (6.8) can be improved to $\pi/\sqrt{24}$; i.e., for all $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \tag{6.9}$$

which was proved by Chen, Wang and Xie [39, Sec. 2]. Paule, Radu, Zeng, and the author [22, Theorem 7.6] confirmed that the coefficient of $1/n^{3/2}$ is $\pi/\sqrt{24}$, which is optimal; i.e., they proved that for all $n \geq 120$,

$$p(n)^2 > \left(1+\frac{\pi}{\sqrt{24}n^{3/2}} - \frac{1}{n^2}\right)p(n-1)p(n+1). \tag{6.10}$$

Chen [36] conjectured that $p(n)$ satisfies the higher order Turán inequalities for all $n \geq 95$ which was proved by Chen, Jia, and Wang [37, Theorem 1.3] and analogous to the inequality (4.7), they conjectured that for all $n \geq 2$,

$$4(1-u_n)(1-u_{n+1}) < \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)(1-u_nu_{n+1})^2 \quad \text{with} \quad u_n := \frac{p(n+1)p(n-1)}{p(n)^2}, \tag{6.11}$$

settled by Larson and Wagner [95, Theorem 1.2]. In [37], Chen, Jia, and Wang conjectured[1] that for any integer $d \geq 1$ there exists an integer $N(d)$ such that the Jensen polynomial of degree d and shift n associated with $p(n)$ has only real roots for $n > N(d)$ which was settled by Griffin, Ono, Rolen, and Zagier [69, Theorem 5] and inspired by their work, Larson and Wagner [95, Theorem 1.3] proved that $N(d) \leq (3d)^{24d}(50d)^{3d^2}$. Proofs of the inequalities, stated before, primarily rely on the Hardy-Ramanujan-Rademacher formula (4.2) and Lehmer's error bound (4.3) but with different methodologies.

While studying on higher order Turán inequality for $p(n)$, Chen [36] undertook a comprehensive study on inequalities pertaining to invariants of a binary form. A binary form $P(x, y)$ of degree d is a homogeneous polynomial of degree d in two variables x and y is defined by

$$P_d(x, y) := \sum_{i=0}^{d} \binom{n}{i} a_i x^i y^{n-i},$$

where $(a_i)_{1 \leq i \leq n} \in \mathbb{C}^n$. But we restrict a_i to be real numbers. The binary form $P_d(x, y)$ is transformed into a new binary form, say $Q(\overline{x}, \overline{y})$ with

$$Q_d(\overline{x}, \overline{y}) = \sum_{i=0}^{d} \binom{n}{i} c_i \overline{x}^i \overline{y}^{n-i}$$

under the action of $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in GL_2(\mathbb{R})$ as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}.$$

The transformed coefficients $(c_i)_{0 \leq i \leq d}$ are polynomials in $(a_i)_{0 \leq i \leq d}$ and entries of the matrix M. For $k \in \mathbb{Z}_{\geq 0}$, a polynomial $I(a_0, a_1, \ldots, a_d)$ in the coefficients $(a_i)_{0 \leq i \leq d}$ is called an invariant of index of k of the binary form $P_d(x, y)$ if for any $M \in GL_2(\mathbb{R})$,

$$I(\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_d) = (\det M)^k I(a_0, a_1, \ldots, a_n).$$

For a more detailed study on the theory of invariants, see, for example, Hilbert [78], Kung and Rota [92], and Sturmfels [137]. We observe that $I(a_0, a_1, a_2) = a_1^2 - a_0 a_2$ is an invariant of the quadratic binary form

$$P_2(x, y) = a_2 x^2 + 2a_1 xy + a_0 y^2$$

[1]Independently conjectured by K. Ono

and the discriminant is $4I(a_0, a_1, a_2)$. For a sequence $(a_n)_{n\geq 0}$, define

$$I_{n-1}(a_0, a_1, a_2) := I(a_{n-1}, a_n, a_{n+1}) = a_n^2 - a_{n-1}a_{n+1}.$$

Therefore, if we choose $a_n = p(n)$, then $I_{n-1}(p(0), p(1), p(2)) > 0$ for all $n \geq 26$ is the same thing as saying $(p(n))_{n\geq 26}$ is log-concave. For degree 3,

$$I(a_0, a_1, a_2, a_3) = 4(a_1^2 - a_0a_2)(a_2^2 - a_1a_3) - (a_1a_2 - a_0a_3)^2$$

is an invariant of the cubic binary form $P_3(x, y) = a_3x^3 + 3a_2x^2y + 3a_1xy^2 + a_0y^3$ and the discriminant is $27I(a_0, a_1, a_2, a_3)$. Similarly, setting $a_n = p(n)$, the positivity of $I_{n-1}(a_0, a_1, a_2, a_3)$ for all $n \geq 95$ is equivalent to state that $(p(n))_{n\geq 95}$ satisfies the higher order Turán inequality. Two invariants of the quartic binary form

$$P_4(x, y) = a_4x^4 + 4a_3x^3y + 6a_2x^2y^2 + 4a_1xy^3 + a_0y^4$$

are of the following form

$$\begin{aligned} A(a_0, a_1, a_2, a_3, a_4) &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\ B(a_0, a_1, a_2, a_3, a_4) &= -a_0a_2a_4 + a_2^3 + a_0a_3^2 + a_1^2a_4 - 2a_1a_2a_3. \end{aligned}$$

Setting $a_n = p(n)$, Chen [36] conjectured that

$$A(a_{n-1}, a_n, a_{n+1}, a_{n+2}, a_{n+3}) > 0 \quad \text{and} \quad B(a_{n-1}, a_n, a_{n+1}, a_{n+2}, a_{n+3}) > 0,$$

along with the associated companion inequalities in the spirit of (6.9) and (6.11). Here we list all the four conjectures with $a_n = p(n)$.

Conjecture 6.1.1 (eq. (6.17), [36]).

$$a_{n-1}a_{n+3} + 3a_{n+1}^2 > 4a_na_{n+2} \quad \textit{for all} \quad n \geq 185. \tag{6.12}$$

Conjecture 6.1.2 (Conjecture 6.15, [36]). *We have*

$$4\Big(1 + \frac{\pi^2}{16n^3}\Big)a_na_{n+2} > a_{n-1}a_{n+3} + 3a_{n+1}^2 \quad \textit{for all} \quad n \geq 218. \tag{6.13}$$

Conjecture 6.1.3 (eq. (6.18), [36]).

$$a_{n+1}^3 + a_{n-1}a_{n+2}^2 + a_n^2a_{n+3} > 2a_na_{n+1}a_{n+2} + a_{n-1}a_{n+1}a_{n+3} \quad \textit{for all} \quad n \geq 221. \tag{6.14}$$

Conjecture 6.1.4 (Conjecture 6.16, [36]). *We have for all $n \geq 244$,*

$$\left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\right)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3}) > a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3}. \tag{6.15}$$

We prove all four conjectures along with the confirmation that the rate of decay $\pi^2/16n^3$ (resp. $\pi^3/72\sqrt{6}n^{9/2}$) in (6.1.2) (resp. in (6.1.4)) is the optimal one, as stated in Theorem 6.1.5 (resp. Theorem 6.1.7). We also ensure that the rate of decay is $\pi/\sqrt{24}n^{3/2}$ in the context of (6.11) can not be improved further by proving Theorem 6.1.9.

A major part of this chapter is devoted to obtaining an infinite family of inequalities for $p(n-\ell)$ for a non-negative integer ℓ, stated in Theorem 6.4.5, so that under a unified framework, we can prove inequalities for $p(n)$ stated below. Work done in Sections 6.3 and 6.4 incarnates the theme of work presented in [21].

Let $a_n := p(n)$.

Theorem 6.1.5. *For all $n \geq 218$,*

$$4\left(1+\frac{\pi^2}{16n^3}\right)a_na_{n+2} > a_{n-1}a_{n+3}+3a_{n+1}^2 > 4\left(1+\frac{\pi^2}{16n^3}-\frac{6}{n^{7/2}}\right)a_na_{n+2}. \tag{6.16}$$

Corollary 6.1.6. *Conjecture 6.1.1 and 6.1.2 is true.*

Theorem 6.1.7. *For all $n \geq 244$,*

$$\begin{aligned}\left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\right)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3}) &> a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3}\\ &> \left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}-\frac{8}{n^5}\right)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3}).\end{aligned} \tag{6.17}$$

Corollary 6.1.8. *Conjecture 6.1.3 and 6.1.4 is true.*

Theorem 6.1.9. *For all $n \geq 115$,*

$$\begin{aligned}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)(a_na_{n+1}-a_{n-1}a_{n+2})^2 &> 4(a_n^2-a_{n-1}a_{n+1})(a_{n+1}^2-a_na_{n+2})\\ &> \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{3}{n^2}\right)(a_na_{n+1}-a_{n-1}a_{n+2})^2.\end{aligned} \tag{6.18}$$

Remark 6.1.10. *We observe that Theorem 6.1.9 immediately implies the following three statements:*

1. $(p(n))_{n\geq 95}$ *satisfies the higher order Turán inequalities [37, Theorem 1.3].*

2. *For all* $n \geq 2$, (6.11) *holds [95, Theorem 1.2].*

3. $\frac{\pi}{\sqrt{24}n^{3/2}}$ *is the optimal rate of decay of the quotient*

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2})/(a_n a_{n+1} - a_{n-1}a_{n+2})^2.$$

The rest of this chapter is organized as follows. In Section 6.2, we shall present a couple of lemmas from [22, 21] that will be helpful in later sections. Following the work done by Paule, Radu, Schneider, and the author [21], Section 6.3 prepares the setup by determining the coefficients in the asymptotic expansion of $p(n-\ell)$ along with its estimates. An infinite family of inequalities for $p(n-\ell)$ is presented in Section 6.4. Section 6.5 presents proofs of the Theorems 6.1.5, 6.1.7, and 6.1.9. We conclude this chapter with a brief discussion on the further applications of this work, given in Section 6.7.

6.2 Preliminaries

This section presents all the preliminary lemmas required for the proofs of the lemmas presented in subsequent sections.

Lemma 6.2.1. *[21, Lemma 3.3] For* $j,\ k \in \mathbb{Z}_{\geq 0}$,

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{i/2}{j} = \begin{cases} 1, & j=k=0 \\ (-1)^j 2^{k-2j}\frac{k}{j}\binom{2j-k-1}{j-k}, & \text{otherwise} \end{cases}. \tag{6.19}$$

Lemma 6.2.2. *[21, Lemma 4.1] Let* $x_1, x_2, \dots, x_n \leq 1$ *and* $y_1, \dots, y_1$ *be non-negative real numbers. Then*

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_n)}{(1+y_1)(1+y_2)\cdots(1+y_n)} \geq 1 - \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} y_j.$$

Lemma 6.2.3. *[21, Lemma 4.2] For* $t \geq 1$ *and non-negative integer* $u \leq t$, *we have*

$$\frac{1}{2t} \geq \frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} \geq \frac{1}{2t}\left(1 - \frac{u^2+\frac{1}{2}}{t}\right).$$

Lemma 6.2.4. *[21, Lemma 4.3] For $t \geq 1$ and non-negative integer $u \leq t$, we have*

$$\frac{2u+1}{2t} \geq \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i} \geq \frac{2u+1}{2t} - \frac{4u^3+6u^2+8u+3}{12t^2}.$$

Throughout the rest of this chapter,

$$\alpha_\ell := \frac{\pi}{6}\sqrt{1+24\ell}.$$

Lemma 6.2.5. *We have*

$$\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} = \cosh(\alpha_\ell),\ \sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!} = \frac{1}{2}\alpha_\ell\sinh(\alpha_\ell),\ \sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!} = \frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell) + \frac{\alpha_\ell}{4}\sinh(\alpha_\ell),$$

and

$$\sum_{u=0}^{\infty}\frac{u^3\alpha_\ell^{2u}}{(2u)!} = \frac{3\alpha_\ell^2}{8}\cosh(\alpha_\ell) + \frac{\alpha_\ell(\alpha_\ell^2+1)}{8}\sinh(\alpha_\ell).$$

Lemma 6.2.6. *[21, Lemma 4.5] Let $u \in \mathbb{Z}_{\geq 0}$. Assume that $a_{n+1} - a_n \geq b_{n+1} - b_n$ for all $n \geq u$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Then*

$$b_n \geq a_n \text{ for all } n \geq u.$$

Lemma 6.2.7. *For $t \geq 1$ and $k \in \{0,1,2,3\}$ we have*

$$\sum_{u=t+1}^{\infty}\frac{u^k\alpha_\ell^{2u}}{(2u)!} \leq \frac{C_k(\ell)}{t^2},$$

where

$$C_k(\ell) = \begin{cases} C_k = \dfrac{\alpha_\ell^4\cdot 2^k}{18}, & \ell = 0 \\ \dfrac{\lceil\sqrt{\ell}\rceil^2\Big(1+\lceil\sqrt{\ell}\rceil\Big)^{k+2}\alpha_\ell^{2(1+\lceil\sqrt{\ell}\rceil)}}{(1+2\lceil\sqrt{\ell}\rceil)(2+2\lceil\sqrt{\ell}\rceil)!}, & \ell \geq 1 \end{cases}.$$

Proof. Applying Lemma 6.2.6 with $a_n = \sum_{u=n+1}^{\infty}\frac{u^k\alpha_\ell^{2u}}{(2u)!}$ and $b_n = \frac{C_k(\ell)}{n^2}$, $b_{n+1} - b_n \leq a_{n+1} - a_n$ is equivalent to show that $f(n) := \frac{n^2(n+1)^{k+2}\alpha_\ell^{2n+2}}{(2n+1)(2n+2)!} \leq C_k(\ell)$. To prove $f(n) \leq C_k(\ell)$, it is sufficient to show that $f(m) \leq C_k(\ell)$ for a minimal m such that $f(m)$ is maximal. In order to find such m, it is enough to that $\frac{f(n+1)}{f(n)} \leq 1$ for all

$n \geq \max\{\lceil\sqrt{\ell}\rceil, 1\}$, and therefore, $\max\limits_{n\in\mathbb{Z}_{\geq 0}} f(n) = f(\lceil\sqrt{\ell}\rceil) = C_k(\ell)$ for all $\ell \geq 1$ and for $\ell = 0$, $\max\limits_{n\in\mathbb{Z}_{\geq 0}} f(n) = f(1) = C_k(0)$. Now, $\frac{f(n+1)}{f(n)} = \frac{\alpha_\ell^2(n+2)^{k+2}(2n+1)}{(2n+4)(2n+3)^2(n+1)^k n^2} \leq 1$ holds for all all $n \geq \max\{\lceil\sqrt{\ell}\rceil, 1\}$. $\square$

Lemma 6.2.8. *[22, Equation 7.5, Lemma 7.3] For $n, k, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$b_{k,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+k-1}}\binom{s+k-1}{s-1}\frac{1}{n^k},$$

then

$$0 < \sum_{t=k}^{\infty}\binom{-\frac{2s-1}{2}}{t}\frac{(-1)^k}{n^k} < b_{k,n}(s). \tag{6.20}$$

Lemma 6.2.9. *[22, Equation 7.9, Lemma 7.5] For $m, n, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$c_{m,n}(s) := \frac{2}{m}\frac{s^m}{n^m},$$

then

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty}\binom{1/2}{k}\frac{(-1)^k s^k}{n^k} < 0. \tag{6.21}$$

Lemma 6.2.10. *[22, Equation 7.7, Lemma 7.4] For $n, s \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{N}$ and $n > 2s$ let*

$$\beta_{m,n}(s) := \frac{2}{n^m}\binom{s+m-1}{s-1},$$

then

$$0 < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{6.22}$$

6.3 Set up

Using the Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound, we have the following inequality for $p(n)$ due to Chen, Jia, and Wang.

Lemma 6.3.1. *[37, Lemma 2.2] For all $n \geq 1206$,*

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{10}}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{10}}\right), \tag{6.23}$$

where for $n \geq 1$, $\mu(n) := \frac{\pi}{6}\sqrt{24n-1}$.

The definition of $\mu(n)$ is kept throughout this chapter. Paule, Radu, Zeng, and the author extended Lemma 6.3.1 as follows.

Theorem 6.3.2. *[22, Theorem 4.4] For $k \in \mathbb{Z}_{\geq 2}$, define*

$$\widehat{g}(k) := \frac{1}{24}\left(\frac{36}{\pi^2} \cdot \nu(k)^2 + 1\right),$$

where $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}$. Then for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k)$ such that $(n,k) \neq (6,2)$, we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^k}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^k}\right). \tag{6.24}$$

By making the shift $n-\ell$ in $p(n)$ for any $\ell \geq 0$, we obtain the following result.

Theorem 6.3.3. *Let $\ell \in \mathbb{Z}_{\geq 0}$. For $k \in \mathbb{Z}_{\geq 2}$, let $\widehat{g}(k)$ be as in Theorem 6.3.2. Then for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k) + \ell$ such that $(n,k) \neq (6,2)$, we have*

$$\frac{\sqrt{12}e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1 - \frac{1}{\mu(n-\ell)} - \frac{1}{\mu(n-\ell)^k}\right) < p(n-\ell) < \frac{\sqrt{12}e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1 - \frac{1}{\mu(n-\ell)} + \frac{1}{\mu(n-\ell)^k}\right). \tag{6.25}$$

Rewrite the term $\dfrac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1 - \dfrac{1}{\mu(n-\ell)}\right)$ in the following way:

$$\begin{aligned}
&\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1 - \frac{1}{\mu(n-\ell)}\right)\\
&= \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \underbrace{e^{\pi\sqrt{2n/3}\ \left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)}}_{:=A_1(n,\ell)} \underbrace{\left(1 - \frac{1+24\ell}{24n}\right)^{-1}\left(1 - \frac{1}{\mu(n-\ell)}\right)}_{:=A_2(n,\ell)}.
\end{aligned} \tag{6.26}$$

Now we compute the Taylor expansion of the residue parts of $A_1(n,\ell)$ and $A_2(n,\ell)$, defined in (6.26).

Definition 6.3.4. *For $t, \ell \in \mathbb{Z}_{\geq 0}$, define*

$$e_1(t,\ell) := \begin{cases} 1, & \text{if } t=0 \\ \dfrac{(-1)^t(1+24\ell)^t}{(24)^t}\dfrac{(1/2-t)_{t+1}}{t}\displaystyle\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\alpha_\ell^{2u}, & \text{otherwise} \end{cases}, \tag{6.27}$$

and

$$E_1\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} e_1(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}, \;\; n \geq 1. \tag{6.28}$$

Definition 6.3.5. *For $t, \ell \in \mathbb{Z}_{\geq 0}$, define*

$$o_1(t,\ell) := -\frac{\pi}{12\sqrt{6}}(1+24\ell)\left(\frac{(-1)^t(1/2-t)_{t+1}(1+24\ell)^t}{(24)^t}\sum_{u=0}^{t}\frac{(-1)^u(-t)_u}{(t+u+1)!(2u)!}\alpha_\ell^{2u}\right) \tag{6.29}$$

and

$$O_1\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} o_1(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}, \;\; n \geq 1. \tag{6.30}$$

Lemma 6.3.6. *Let $A_1(n,\ell)$ be defined as in (6.26). Let $E_1(n,\ell)$ be as in Definition 6.3.4 and $O_1(n,\ell)$ as in Definition 6.3.5. Then*

$$A_1(n,\ell) = E_1\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{6.31}$$

Proof. From (6.26), we get

$$\begin{aligned} A_1(n,\ell) &= e^{\pi\sqrt{2n/3}\,\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)} \\ &= \sum_{k=0}^{\infty}\frac{(\pi\sqrt{2n/3})^k}{k!}\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)^k \\ &= \sum_{k=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(\sqrt{1-\frac{1+24\ell}{24n}}\right)^i \\ &= \sum_{k=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=0}^{\infty}\binom{i/2}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j} \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}\frac{(-1)^{k-i+j}(1+24\ell)^j}{(24)^j}\binom{k}{i}\binom{i/2}{j}(\sqrt{n})^{k-2j}. \end{aligned} \tag{6.32}$$

Split $S := \{(k,i,j) \in \mathbb{Z}^3_{\geq 0} : 0 \leq i \leq k\} := \bigcup_{t \in \mathbb{Z}_{\geq 0}} V(t)$, where for each $t \in \mathbb{Z}_{\geq 0}$,

$$V(2t) = \{(2u, i, u+t) \in \mathbb{Z}^3_{\geq 0} : 0 \leq i \leq 2u\}$$

and

$$V(2t+1) = \{(2u+1, i, u+t+1) \in \mathbb{Z}^3_{\geq 0} : 0 \leq i \leq 2u+1\}.$$

By Lemma 6.2.1, we have $\sum_{i=0}^{k} \binom{k}{i}\binom{i/2}{j} = 0$ for $k > j$. For $r = (k,i,j) \in S$, we define

$$S(r) := \frac{(\pi\sqrt{2/3})^k}{k!} \frac{(-1)^{k-i+j}(1+24\ell)^j}{(24)^j} \binom{k}{i}\binom{i/2}{j} \quad \text{and} \quad f(r) := k - 2j.$$

Rewrite (6.32) as

$$A_1(n,\ell) = \sum_{t=0}^{\infty} \sum_{r \in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{t=0}^{\infty} \sum_{r \in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{6.33}$$

Now

$$\sum_{t=0}^{\infty} \sum_{r \in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} = \sum_{t=0}^{\infty} \frac{(-1)^t(1+24\ell)^t}{(24)^t} \left(\sum_{u=0}^{\infty} \frac{(-1)^u}{(2u)!}\alpha_\ell^{2u}\mathcal{E}_1(u,t)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}, \tag{6.34}$$

where by Lemma 6.2.1,

$$\mathcal{E}_1(u,t) := \sum_{i=0}^{2u}(-1)^i\binom{2u}{i}\binom{i/2}{u+t} = \begin{cases} 1, & \text{if } u = t = 0 \\ 0, & \text{if } u > t \\ \frac{2u(1/2-t)_{t+1}(-t)_u}{t(t+u)!}, & \text{otherwise} \end{cases}.$$

Consequently, we have

$$\sum_{t=0}^{\infty} \sum_{r \in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} = E_1\Big(\frac{1}{\sqrt{n}}, \ell\Big). \tag{6.35}$$

Simplifying,

$$\sum_{t=0}^{\infty} \sum_{r \in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}$$
$$= -\frac{\pi(1+24\ell)}{12\sqrt{6}} \sum_{t=0}^{\infty} \frac{(-1)^t(1+24\ell)^t}{(24)^t} \left(\sum_{u=0}^{\infty} \frac{(-1)^u}{(2u+1)!}\alpha_\ell^{2u}\mathcal{O}_1(u,t)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}, \tag{6.36}$$

where by Lemma 6.2.1,

$$\mathcal{O}_1(u,t) := \sum_{i=0}^{2u+1}(-1)^i\binom{2u+1}{i}\binom{i/2}{u+t+1} = \begin{cases} 0, & \text{if } u > t \\ -\frac{(2u+1)(1/2-t)_{t+1}(-t)_u}{(t+u+1)!}, & \text{otherwise} \end{cases}.$$

Therefore, we have

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} = O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{6.37}$$

From (6.33), (6.35), and (6.37), we get (6.31). □

Definition 6.3.7. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} e_2(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \text{ with } e_2(t,\ell) := \frac{(1+24\ell)^t}{(24)^t}. \tag{6.38}$$

Definition 6.3.8. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$O_2\Big(\frac{1}{\sqrt{n}}\Big) := \sum_{t=0}^{\infty} o_2(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \text{ with } o_2(t) := -\frac{6}{\pi\sqrt{24}}\binom{-3/2}{t}\frac{(-1)^t(1+24\ell)^t}{(24)^t}. \tag{6.39}$$

Lemma 6.3.9. *Let $A_2(n,\ell)$ be defined as in (6.26). Let $E_2(n,\ell)$ be as in Definition 6.3.7 and $O_2(n,\ell)$ as in Definition 6.3.8. Then*

$$A_2(n,\ell) = E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_2\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{6.40}$$

Proof. Following the definition of $A_2(n,\ell)$ from (6.26) and expand it as follows:

$$\begin{aligned} A_2(n,\ell) &= \Big(1-\frac{1+24\ell}{24n}\Big)^{-1} - \frac{6}{\pi\sqrt{24}}\frac{1}{\sqrt{n}}\Big(1-\frac{1+24\ell}{24n}\Big)^{-3/2} \\ &= E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_2\Big(\frac{1}{\sqrt{n}},\ell\Big). \end{aligned} \tag{6.41}$$

This completes the proof of (6.40). □

Definition 6.3.10. *Following the Definitions 6.3.4-6.3.8, we define*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)E_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{6.42}$$

$$S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) := O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{6.43}$$

$$S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{6.44}$$

and

$$S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_2\Big(\frac{1}{\sqrt{n}},\ell\Big)O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{6.45}$$

Lemma 6.3.11. *For each $i \in \{1,2\}$, let $S_{e,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ and $S_{o,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ be as in Definition 6.3.10. Then*

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big) = \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\sum_{i=1}^{2}\Bigg(S_{e,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)+S_{o,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)\Bigg). \tag{6.46}$$

Proof. The proof follows immediately by applying Lemmas 6.3.6 and 6.3.9 to (6.26). □

6.3.1 Coefficients in the asymptotic expansion of $p(n-\ell)$

Definition 6.3.12. *For $t,\ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_1(t,\ell) := \sum_{s=1}^{t}\frac{(-1)^s(1/2-s)_{s+1}}{s}\sum_{u=1}^{s}\frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\alpha_\ell^{2u}, \tag{6.47}$$

and

$$g_{e,1}(t,\ell) := \frac{(1+24\ell)^t}{(24)^t}\Big(1+S_1(t,\ell)\Big). \tag{6.48}$$

Lemma 6.3.13. *Let $S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ be as in (6.42). Let $g_{e,1}(t,\ell)$ be as in Definition 6.3.12. Then*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) = \sum_{t=0}^{\infty}g_{e,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{6.49}$$

Proof. From (6.28), (6.38), and (6.42), we have

$$S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) \quad = \quad 1+\sum_{t=1}^{\infty}\left(e_1(t,\ell)+e_2(t,\ell)+\sum_{s=1}^{t-1}e_1(s,\ell)e_2(t-s,\ell)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{6.50}$$

Combining (6.27) and (6.38), we obtain

$$e_1(t)+e_2(t)+\sum_{s=1}^{t-1}e_1(s)e_2(t-s) = \frac{(1+24\ell)^t}{(24)^t}\Big(1+S_1(t,\ell)\Big) = g_{e,1}(t,\ell), \tag{6.51}$$

which concludes the proof of (6.49). ☐

Definition 6.3.14. *For $t \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_2(t,\ell) := \sum_{s=0}^{t-1}(1/2-s)_{s+1}\binom{-3/2}{t-s-1}\sum_{u=0}^{s}\frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\alpha_\ell^{2u}, \tag{6.52}$$

and

$$g_{e,2}(t,\ell) := \frac{(-1)^{t-1}(1+24\ell)^t}{(24)^t}S_2(t,\ell). \tag{6.53}$$

Lemma 6.3.15. *Let $S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ as in (6.43) and $g_{e,2}(t,\ell)$ as in Definition 6.3.14. Then*

$$S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) = \sum_{t=1}^{\infty}g_{e,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{6.54}$$

Proof. From (6.39), (6.40) and (6.43), we have

$$\begin{aligned} S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) &= O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big) \\ &= \sum_{t=1}^{\infty}\left(\sum_{s=0}^{t-1}o_1(s,\ell)o_2(t-s-1,\ell)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\ &= \sum_{t=1}^{\infty}g_{e,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \text{ (by (6.29) and (6.39))}. \end{aligned} \tag{6.55}$$

☐

Definition 6.3.16. *For $t \in \mathbb{Z}_{\geq 2}$ and $\ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_3(t,\ell) := \sum_{s=1}^{t} \frac{(1/2-s)_{s+1}\binom{-3/2}{t-s}}{s} \sum_{u=1}^{s} \frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\alpha_\ell^{2u}, \tag{6.56}$$

and

$$g_{o,1}(t,\ell) := \begin{cases} -\dfrac{6}{\pi\sqrt{24}}\dfrac{(-1)^t(1+24\ell)^t}{(24)^t}\left(\binom{-3/2}{t} + S_3(t,\ell)\right), & \text{if } t \geq 2 \\ -\dfrac{432+(1+24\ell)\pi^2}{2304\sqrt{6}\pi}, & \text{if } t = 1 \\ -\dfrac{6}{\pi\sqrt{24}}, & \text{if } t = 0 \end{cases}. \tag{6.57}$$

Lemma 6.3.17. *Let $S_{o,1}\left(\frac{1}{\sqrt{n}},\ell\right)$ as in (6.44) and $g_{o,1}(t,\ell)$ be as in Definition 6.3.16. Then*

$$S_{o,1}\left(\frac{1}{\sqrt{n}},\ell\right) = \sum_{t=0}^{\infty} g_{o,1}(t,\ell)\left(\frac{1}{\sqrt{n}}\right)^{2t+1}. \tag{6.58}$$

Proof. From (6.28), (6.39) and (6.44), it follows that

$$\begin{aligned} S_{o,1}\left(\frac{1}{\sqrt{n}},\ell\right) &= E_1\left(\frac{1}{\sqrt{n}},\ell\right)O_2\left(\frac{1}{\sqrt{n}},\ell\right) \\ &= g_{o,1}(0,\ell)\frac{1}{\sqrt{n}} + g_{o,1}(1,\ell)\frac{1}{\sqrt{n}^3} + \\ &\qquad \sum_{t=2}^{\infty}\left(o_2(t) + \sum_{s=1}^{t} e_1(s,\ell)o_2(t-s,\ell)\right)\left(\frac{1}{\sqrt{n}}\right)^{2t+1} \\ &= g_{o,1}(0,\ell)\frac{1}{\sqrt{n}} + g_{o,1}(1,\ell)\frac{1}{\sqrt{n}^3} + \sum_{t=2}^{\infty} g_{o,1}(t,\ell)\left(\frac{1}{\sqrt{n}}\right)^{2t+1} \\ &\qquad \text{(by (6.27) and (6.39))}. \end{aligned} \tag{6.59}$$

□

Definition 6.3.18. *For $t \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_4(t,\ell) := \sum_{s=0}^{t}(-1)^s(1/2-s)_{s+1}\sum_{u=0}^{s}\frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\alpha_\ell^{2u}, \tag{6.60}$$

and

$$g_{o,2}(t,\ell) := -\frac{\pi(1+24\ell)}{12\sqrt{6}}\frac{(1+24\ell)^t}{(24)^t}S_4(t,\ell). \tag{6.61}$$

Lemma 6.3.19. *Let $S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ be as in (6.45) and $g_{o,2}(t,\ell)$ be as in Definition 6.3.18. Then*

$$S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)=\sum_{t=0}^{\infty}g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{6.62}$$

Proof. From (6.29), (6.38) and (6.45), it follows that

$$\begin{aligned}S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) &= O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)E_2\Big(\frac{1}{\sqrt{n}},\ell\Big)\\ &= \sum_{t=0}^{\infty}\Bigg(\sum_{s=0}^{t}o_1(s,\ell)e_2(t-s,\ell)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\ &= \sum_{t=0}^{\infty}g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\ \text{(by (6.30) and (6.38))}. \end{aligned}\tag{6.63}$$

□

Definition 6.3.20. *For each $i\in\{1,2\}$, let $g_{e,i}(t,\ell)$ and $g_{o,i}(t,\ell)$ be as in Definitions 6.3.12-6.3.18. We define a power series*

$$G(n,\ell):=\sum_{t=0}^{\infty}g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{t}=\sum_{t=0}^{\infty}g(2t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}+\sum_{t=0}^{\infty}g(2t+1,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1},$$

where

$$g(2t,\ell):=g_{e,1}(t,\ell)+g_{e,2}(t,\ell)\quad\text{and}\quad g(2t+1,\ell):=g_{o,1}(t,\ell)+g_{o,2}(t,\ell). \tag{6.64}$$

Lemma 6.3.21. *Let $G(n,\ell)$ be as in Definition 6.3.20. Then*

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big)=\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\cdot G(n,\ell). \tag{6.65}$$

Proof. Applying Lemmas 6.3.13-6.3.19 to Lemma 6.3.9, we have (6.65). □

Remark 6.3.22. *Using* `Sigma` *due to Schneider [128] and* `GeneratingFunctions` *due to* Mallinger *[104], we observe that for all $t\geq 0$,*

$$g(2t,\ell)=g_{e,1}(t,\ell)+g_{e,2}(t,\ell)=\omega_{2t,\ell}\quad\text{and}\quad g(2t+1,\ell)=g_{o,1}(t,\ell)+g_{o,2}(t,\ell)=\omega_{2t+1,\ell}, \tag{6.66}$$

where

$$g(t,\ell)=\omega_{t,\ell}=\frac{(1+24\ell)^t}{(-4\sqrt{6})^t}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\Big(\frac{\pi}{6}\Big)^{t-2k}\frac{1}{(1+24\ell)^k}. \tag{6.67}$$

Note that for $\ell=0$, we retrieve ω_t as in O'Sullivan's [112, Proposition 4.4] work.

6.3.2 Estimation of $(S_i(t,\ell))$

We present the Lemmas 6.3.24-6.3.30 which will be needed in the Subsection 6.3.3. A brief sketch of proofs of these lemmas are presented in the Section 6.6.

Definition 6.3.23. *Let $C_k(\ell)$ be as in Lemma 6.2.7. Define*

$$C_1^{\mathcal{L}}(\ell) := \frac{\cosh(\alpha_\ell)-1}{4} + C_0(\ell) + \frac{\alpha_\ell^2\cosh(\alpha_\ell)+\alpha_\ell\sinh(\alpha_\ell)}{8},$$
$$C_1^{\mathcal{U}}(\ell) := C_1(\ell) + \frac{\alpha_\ell^2+1}{4}\cosh(\alpha_\ell) + \frac{\alpha_\ell(\alpha_\ell^2+12)}{24}\sinh(\alpha_\ell).$$

Lemma 6.3.24. *Let $S_1(t,\ell)$ be as in Definition 6.3.12 and $C_1^{\mathcal{L}}(\ell), C_1^{\mathcal{U}}(\ell)$ as in Definition 6.3.23. Then for all $t \geq 1$,*

$$-\frac{C_1^{\mathcal{L}}(\ell)}{t^2} < \frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\big(\cosh(\alpha_\ell)-1\big) + \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) < \frac{C_1^{\mathcal{U}}(\ell)}{t^2}. \tag{6.68}$$

Definition 6.3.25. *Let $C_k(\ell)$ be as in Lemma 6.2.7. Define*

$$\begin{aligned}
C_{2,1}^{\mathcal{L}}(\ell) &:= \frac{\cosh(\alpha_\ell)}{4} + \frac{\sinh(\alpha_\ell)}{4\alpha_\ell} + \frac{\alpha_\ell\sinh(\alpha_\ell)}{4} + \frac{2C_1(\ell)}{\alpha_\ell^2},\\
C_{2,1}^{\mathcal{U}}(\ell) &:= -\frac{\cosh(\alpha_\ell)}{2} + \frac{\sinh(\alpha_\ell)}{2\alpha_\ell} + \frac{2C_2(\ell)}{\alpha_\ell^2},\\
\mathrm{csh}(\ell) &:= \cosh(\alpha_\ell) + \alpha_\ell\sinh(\alpha_\ell),\\
C_{2,2}(\ell) &:= \frac{8C_3(\ell)}{\alpha_\ell^2} + \frac{(\alpha_\ell^2+1)\cosh(\alpha_\ell)}{4} + \frac{(\alpha_\ell^3+12\alpha_\ell)\sinh(\alpha_\ell)}{24},\\
C_2^{\mathcal{L}}(\ell) &:= C_{2,1}^{\mathcal{U}}(\ell) + \frac{\mathrm{csh}(\ell)}{2} + \frac{4C_2(\ell)}{\alpha_\ell^2},\\
C_2^{\mathcal{U}}(\ell) &:= C_{2,1}^{\mathcal{L}}(\ell) - \frac{\mathrm{csh}(\ell)}{2} + C_{2,2}(\ell).
\end{aligned}$$

Lemma 6.3.26. *Let $S_2(t,\ell)$ be as in Definition 6.3.14 and $C_2^{\mathcal{L}}(\ell), C_2^{\mathcal{U}}(\ell)$ as in Definition 6.3.25. Then for all $t \geq 1$,*

$$-\frac{C_2^{\mathcal{L}}(\ell)}{t} < \frac{S_2(t,\ell)}{\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell) + \frac{\sinh(\alpha_\ell)}{\alpha_\ell} < \frac{C_2^{\mathcal{U}}(\ell)}{t}. \tag{6.69}$$

Definition 6.3.27. *Let $C_k(\ell)$ be as in Lemma 6.2.7. Define*

$$C_{3,1}(\ell) := \frac{3\alpha_\ell^2\cosh(\alpha_\ell)+7\alpha_\ell\sinh(\alpha_\ell)+2\cosh(\alpha_\ell)-2}{8} + C_0(\ell),$$

$$\begin{aligned}
C_{3,2}(\ell) &:= \frac{9\alpha_\ell^3\sinh(\alpha_\ell)+(\alpha_\ell^4+24\alpha_\ell^2)\cosh(\alpha_\ell)+18\alpha_\ell\sinh(\alpha_\ell)}{24}+2C_2(\ell)+C_1(\ell),\\
\mathrm{sch}(\ell) &:= \alpha_\ell^2\cosh(\alpha_\ell)+2\alpha_\ell\sinh(\alpha_\ell),\\
C_3^{\mathcal{L}}(\ell) &:= C_{3,1}(\ell)+C_{3,2}(\ell)-\frac{\mathrm{sch}(\ell)}{2},\\
C_3^{\mathcal{U}}(\ell) &:= 3C_1(\ell)+\frac{\mathrm{sch}(\ell)}{2}.
\end{aligned}$$

Lemma 6.3.28. *Let $S_3(t,\ell)$ be as in Definition 6.3.16 and $C_3^{\mathcal{L}}(\ell), C_3^{\mathcal{U}}(\ell)$ as in Definition 6.3.27. Then for all $t \geq 2$,*

$$-\frac{C_3^{\mathcal{L}}(\ell)}{t} < \frac{S_3(t,\ell)}{\binom{-\frac{3}{2}}{t}} + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\alpha_\ell\sinh(\alpha_\ell)+1-\cosh(\alpha_\ell) < \frac{C_3^{\mathcal{U}}(\ell)}{t}. \tag{6.70}$$

Definition 6.3.29. *Let $C_k(\ell)$ be as in Lemma 6.2.7. Define*

$$\begin{aligned}
C_{4,1}(\ell) &:= \frac{\alpha_\ell^4}{72}+\frac{(\alpha_\ell^2+6)\cosh(\alpha_\ell)+3\alpha_\ell\sinh(\alpha_\ell)}{16},\\
C_4^{\mathcal{L}}(\ell) &:= C_{4,1}(\ell)-\frac{\cosh(\alpha_\ell)}{4}+\frac{2C_0(\ell)}{3},\\
C_4^{\mathcal{U}}(\ell) &:= \frac{(\alpha_\ell^2+12)\cosh(\alpha_\ell)+3\alpha_\ell\sinh(\alpha_\ell)+12C_0(\ell)}{24}.
\end{aligned}$$

Lemma 6.3.30. *Let $S_4(t,\ell)$ be as in Definition 6.3.18 and $C_4^{\mathcal{L}}(\ell), C_4^{\mathcal{U}}(\ell)$ as in Definition 6.3.29. Then for $t \geq 1$,*

$$-\frac{C_4^{\mathcal{L}}(\ell)}{t^2} < \frac{S_4(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha_\ell)}{\alpha_\ell}+\frac{1}{2t}\cosh(\alpha_\ell) < \frac{C_4^{\mathcal{U}}(\ell)}{t^2}. \tag{6.71}$$

6.3.3 Error bounds

Lemma 6.3.31. *For all $k \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathbb{Z}_{\geq 0}$, and $n \geq \ell+1$,*

$$\frac{(1+24\ell)^k}{(24n)^k} < \sum_{t=k}^{\infty}\frac{(1+24\ell)^t}{(24n)^t} \leq \frac{24(\ell+1)}{23}\frac{(1+24\ell)^k}{(24n)^k}. \tag{6.72}$$

Proof. Equation (6.72) follows from

$$\sum_{t=k}^{\infty}\frac{(1+24\ell)^t}{(24n)^t} = \frac{(1+24\ell)^k}{(24n)^k}\frac{24n}{24n-24\ell-1}$$

and

$$1 < \frac{24n}{24n-24\ell-1} \leq \frac{24(\ell+1)}{23} \quad \text{for all } n \geq \ell+1.$$

□

Lemma 6.3.32. *For all $n, k, s \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathbb{Z}_{\geq 0}$, and $n \geq \ell + 1$,*

$$\frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^k}{(24n)^k} < \sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} < \frac{12(\ell+1)}{5(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^t}{(24n)^k}. \quad (6.73)$$

Proof. We observe that

$$\sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{1}{(24n)^t} = \sum_{t=k}^{\infty} \frac{\binom{2t+2}{t+1}}{4^t} \frac{t+1}{2t^s} \frac{(1+24\ell)^t}{(24n)^t}. \quad (6.74)$$

For all $t \geq 1$,

$$\frac{4^t}{2\sqrt{t}} \leq \binom{2t}{t} \leq \frac{4^t}{\sqrt{\pi t}}.$$

From (6.74) we obtain

$$\sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} \leq \sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{1}{(24n)^t} \leq \frac{4}{\sqrt{\pi}} \sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{2t^s} \frac{(1+24\ell)^t}{(24n)^t}. \quad (6.75)$$

For all $k \geq 1$,

$$\sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} \geq \sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} > \frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^k}{(24n)^k} \quad (6.76)$$

and

$$\begin{aligned}
\sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} &< \frac{4}{\sqrt{\pi}} \sum_{t=k}^{\infty} \frac{1}{(t+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^t}{(24n)^t} \\
&\leq \frac{4}{\sqrt{\pi}(k+1)^{s-\frac{1}{2}}} \sum_{t=k}^{\infty} \frac{(1+24\ell)^t}{(24n)^t} \\
&< \frac{4 \cdot 24(\ell+1)}{23 \cdot \sqrt{\pi}} \frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^k}{(24n)^k} \quad \text{(by (6.72))}. \\
&< \frac{12}{5} \frac{(\ell+1)}{(k+1)^{s-\frac{1}{2}}} \frac{1}{(24n)^k}. \qquad (6.77)
\end{aligned}$$

Equations (6.76) and (6.77) imply (6.73). □

Lemma 6.3.33. *For $n \in \mathbb{Z}_{\geq 1}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, and $n \geq 4\ell + 1$,*

$$0 < \sum_{t=k}^{\infty} \binom{-\frac{3}{2}}{t} \frac{(-1)^t (1+24\ell)^t}{(24n)^t} < 4\sqrt{2} \frac{\sqrt{k+1}(1+24\ell)^k}{(24n)^k}. \tag{6.78}$$

Proof. Setting $(n, s) \mapsto (\frac{24n}{24\ell+1}, 2)$ in (6.20), it follows that for all $n \geq 4\ell + 1$,

$$0 < \sum_{t=k}^{\infty} \binom{-\frac{3}{2}}{t} \frac{(-1)^t}{(24n)^t} < 4\sqrt{2} \frac{\sqrt{k+1}(1+24\ell)^k}{(24n)^k}.$$

□

Definition 6.3.34. *Let $C_1^{\mathcal{L}}(\ell)$ and $C_1^{\mathcal{U}}(\ell)$ be as in Definition 6.3.23. Then for all $k \geq 1$ and $\ell \geq 0$, define*

$$L_1(k, \ell) := \left(\cosh(\alpha_\ell) - \frac{6\alpha_\ell \sinh(\alpha_\ell)(\ell+1)}{5\sqrt{k+1}} - \frac{12(\ell+1)}{5(k+1)^{3/2}} C_1^{\mathcal{L}}(\ell) \right) \left(\sqrt{\frac{1+24\ell}{24n}} \right)^{2k}$$

and

$$U_1(k, \ell) := \left(\frac{24(\ell+1)\cosh(\alpha_\ell)}{23} - \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{k+1}} + \frac{12(\ell+1)}{5(k+1)^{3/2}} C_1^{\mathcal{U}}(\ell) \right) \left(\sqrt{\frac{1+24\ell}{24n}} \right)^{2k}.$$

Lemma 6.3.35. *Let $L_1(k, \ell)$ and $U_1(k, \ell)$ be as in Definition 6.3.34. Let $g_{e,1}(t, \ell)$ be as in Definition 6.3.12. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell + 1$,*

$$L_1(k, \ell) \left(\frac{1}{\sqrt{n}} \right)^{2k} < \sum_{t=k}^{\infty} g_{e,1}(t, \ell) \left(\frac{1}{\sqrt{n}} \right)^{2t} < U_1(k, \ell) \left(\frac{1}{\sqrt{n}} \right)^{2k}. \tag{6.79}$$

Proof. From (6.48) and (6.68), it follows that for $t \geq 1$,

$$\begin{aligned} \cosh(\alpha_\ell) - \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{2t} \alpha_\ell \sinh(\alpha_\ell) - \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^2} C_1^{\mathcal{L}}(\ell) &< \left(\frac{24}{1+24\ell} \right)^t g_{e,1}(t) = 1 + S_1(t, \ell) \\ &< \cosh(\alpha_\ell) - \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{2t} \alpha_\ell \sinh(\alpha_\ell) + \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^2} C_1^{\mathcal{U}}(\ell). \end{aligned} \tag{6.80}$$

Applying (6.72) and (6.73) with $s = 1$ and 2, respectively, to (6.80), we obtain (6.79). □

Definition 6.3.36. *Let $C_2^{\mathcal{L}}(\ell)$ and $C_2^{\mathcal{U}}(\ell)$ be as in Definition 6.3.25. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_2(k,\ell) := \left(-\frac{24(\ell+1)\cosh(\alpha_\ell)}{23} - \frac{12(\ell+1)}{5\sqrt{k+1}}C_2^{\mathcal{U}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k}$$

and

$$U_2(k,\ell) := \left(-\cosh(\alpha_\ell) + \frac{4\sqrt{2}\sinh(\alpha_\ell)}{\alpha_\ell}\sqrt{k+1} + \frac{12(\ell+1)}{5\sqrt{k+1}}C_2^{\mathcal{L}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k}.$$

Lemma 6.3.37. *Let $L_2(k,\ell)$ and $U_2(k,\ell)$ be as in Definition 6.3.36. Let $g_{e,2}(t,\ell)$ be as in Definition 6.3.14. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_2(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k} < \sum_{t=k}^{\infty} g_{e,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} < U_2(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k}. \tag{6.81}$$

Proof. From (6.53) and (6.69), it follows that for $t \geq 1$,

$$\begin{aligned}
&-\cosh(\alpha_\ell) + (-1)^t\binom{-\frac{3}{2}}{t}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}C_2^{\mathcal{U}}(\ell)\\
&< \Big(\frac{1+24\ell}{24}\Big)^t g_{e,2}(t,\ell) = (-1)^{t-1}S_2(t,\ell) \qquad (6.82)\\
&< -\cosh(\alpha_\ell) + (-1)^t\binom{-\frac{3}{2}}{t}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}C_2^{\mathcal{L}}(\ell).
\end{aligned}$$

Applying (6.72), (6.73) with $s = 1$ and (6.78) to (6.82), we get (6.81). □

Definition 6.3.38. *Let $C_3^{\mathcal{L}}(\ell)$ and $C_3^{\mathcal{U}}(\ell)$ be as in Definition 6.3.27. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_3(k,\ell) := \left(\frac{6\alpha_\ell\sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} - \frac{24\sqrt{2}\cosh(\alpha_\ell)\sqrt{k+1}}{\pi\sqrt{1+24\ell}} - \frac{72(\ell+1)}{5\pi\sqrt{1+24\ell}}\frac{C_3^{\mathcal{U}}(\ell)}{\sqrt{k+1}}\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k+1}$$

and

$$U_3(k,\ell) := \left(\frac{6\cdot 24(\ell+1)}{23\pi\sqrt{1+24\ell}}\alpha_\ell\sinh(\alpha_\ell) + \frac{72(\ell+1)}{5\pi\sqrt{1+24\ell}}\frac{C_3^{\mathcal{L}}(\ell)}{\sqrt{k+1}}\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k+1}.$$

Lemma 6.3.39. *Let $L_3(k,\ell)$ and $U_3(k,\ell)$ be as in Definition 6.3.38. Let $g_{o,1}(t,\ell)$ be as in Definition 6.3.16. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_3(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1} < \sum_{t=k}^{\infty} g_{o,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} < U_3(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1}. \tag{6.83}$$

Proof. Define $c_1(t,\ell) := -\dfrac{6}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t}$. From (6.57) and (6.70), it follows that for $t \geq 2$,

$$\begin{aligned}
&\frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} - \frac{6\cosh(\alpha_\ell)}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t} - \frac{6C_3^{\mathcal{U}}(\ell)}{\pi\sqrt{1+24\ell}}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}\\
&< \left(\sqrt{\frac{24}{24\ell+1}}\right)^{2t+1} g_{o,1}(t,\ell) = c_1(t,\ell)\left(1+\frac{S_3(t,\ell)}{\binom{-\frac{3}{2}}{t}}\right)\\
&< \frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} - \frac{6\cosh(\alpha_\ell)}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t} + \frac{6C_3^{\mathcal{L}}(\ell)}{\pi\sqrt{1+24\ell}}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}.
\end{aligned} \tag{6.84}$$

We observe that (6.84) also holds for $t \in \{0,1\}$; see (6.57). Now, applying (6.72), (6.73) with $s=1$, and (6.78) to (6.84), we conclude the proof. ☐

Definition 6.3.40. *Let $C_4^{\mathcal{L}}(\ell)$ and $C_4^{\mathcal{U}}(\ell)$ be as in Definition 6.3.29. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_4(k,\ell) := \frac{\pi\sqrt{1+24\ell}}{6}\left(\frac{\cosh(\alpha_\ell)}{2\sqrt{k+1}} - \frac{24(\ell+1)\sinh(\alpha_\ell)}{23\alpha_\ell} - \frac{12(\ell+1)C_4^{\mathcal{U}}(\ell)}{5(k+1)^{3/2}}\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k+1}$$

and

$$U_4(k,\ell) := \frac{\pi\sqrt{1+24\ell}}{6}\left(\frac{6(\ell+1)\cosh(\alpha_\ell)}{5\sqrt{k+1}} - \frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{12(\ell+1)C_4^{\mathcal{L}}(\ell)}{5(k+1)^{3/2}}\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k+1}.$$

Lemma 6.3.41. *Let $L_4(k,\ell)$ and $U_4(k,\ell)$ be as in Definition 6.3.40. Let $g_{o,2}(t,\ell)$ be as in Definition 6.3.18. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_4(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1} < \sum_{t=k}^{\infty} g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} < U_4(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1}. \tag{6.85}$$

Proof. Define $c_2(t,\ell) := -\dfrac{\pi\sqrt{1+24\ell}}{6}(-1)^t\binom{-\frac{3}{2}}{t}$. From (6.61) and (6.71), it follows that for $t \geq 1$,

$$\frac{\pi\sqrt{1+24\ell}\cosh(\alpha_\ell)}{12}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} - \frac{\pi\sqrt{1+24\ell}\sinh(\alpha_\ell)}{6\alpha_\ell} - \frac{\pi\sqrt{1+24\ell}\ C_4^{\mathcal{U}}(\ell)}{6}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}$$
$$< \left(\sqrt{\frac{24}{24\ell+1}}\right)^{2t+1} g_{o,2}(t,\ell) = c_2(t,\ell)\frac{S_4(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}}$$
$$< \frac{\pi\sqrt{1+24\ell}\cosh(\alpha_\ell)}{12}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} - \frac{\pi\sqrt{1+24\ell}\sinh(\alpha_\ell)}{6\alpha_\ell} + \frac{\pi\sqrt{1+24\ell}\ C_4^{\mathcal{L}}(\ell)}{6}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}. \tag{6.86}$$

Now, applying (6.72) and (6.73) with $s = 1$ and 2, respectively, to (6.86), we have (6.85). ☐

Definition 6.3.42. *For $k \geq 1$ and $\ell \geq 0$, define*

$$n_0(k,\ell) := \max_{k\geq 1, \ell\geq 0}\left\{\frac{(24\ell+1)^2}{16}, \frac{(k+3)(24\ell+1)}{24}\right\}.$$

Definition 6.3.43. *Let $n_0(k,\ell)$ be as in Definition 6.3.42. For $k \geq 1$ and $\ell \geq 0$, define*

$$\widehat{L}_2(k,\ell) := \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1 - \frac{1+24\ell}{4\sqrt{n_0(k,l)}}\right) \text{ and } \widehat{U}_2(k,\ell) := \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1 + \frac{k(1+24\ell)}{3\cdot n_0(k,l)}\right).$$

Lemma 6.3.44. *Let $\widehat{L}_2(k,\ell)$, and $\widehat{U}_2(k,\ell)$ be as in Definition 6.3.43. Let $n_0(k,\ell)$ be as in Definition 6.3.42. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n > n_0(k,\ell)$,*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{L}_2(k,\ell)}{\sqrt{n}^k} < \frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^k} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(k,\ell)}{\sqrt{n}^k}. \tag{6.87}$$

Proof. For all $k \geq 1$ and $\ell \geq 0$, define

$$\mathcal{E}(n,k,\ell) := \frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^k},\ \mathcal{U}(n,k,\ell) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{1}{\sqrt{n}^k}$$

and

$$\mathcal{Q}(n,k,\ell) := \frac{\mathcal{E}(n,k,\ell)}{\mathcal{U}(n,k,\ell)} = \frac{e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)}}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}.$$

Using (6.21) with $(m,n,s)\mapsto(1,24n,24\ell+1)$, we obtain for all $n\geq 2\ell+1$,

$$-\frac{1+24\ell}{12n} < \sqrt{1-\frac{1}{24n}}-1 = \sum_{m=1}^{\infty}\binom{1/2}{m}\frac{(-1)^m}{(24n)^m} < 0,$$

and consequently for $n\geq n_0(k,\ell)$,

$$\left(1-\frac{1+24\ell}{4\sqrt{n_0(k,\ell)}}\right) < e^{-\frac{\pi(1+24\ell)}{6\sqrt{6n}}} < e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1}{24n}}-1\right)} < 1. \tag{6.88}$$

Therefore

$$\frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}\left(1-\frac{1}{4\sqrt{n_0(k,\ell)}}\right) < \mathcal{Q}(n,k,\ell) < \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}. \tag{6.89}$$

We estimate $\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}$ by splitting it into two cases depending on whether k is even or odd.

For $k=2r$ with $r\in\mathbb{Z}_{\geq 0}$:

$$\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}} = \left(1-\frac{1+24\ell}{24n}\right)^{-(r+1)} = 1+\sum_{j=1}^{\infty}\binom{-(r+1)}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}.$$

From (6.22) with $(m,s,n)\mapsto(1,r+1,\frac{24n}{24\ell+1})$, for all $n>\frac{(r+1)(1+24\ell)}{12}$, we get

$$0<\sum_{j=1}^{\infty}\binom{-(r+1)}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j} < \frac{(r+1)(24\ell+1)}{12n},$$

which is equivalent to

$$1<\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}} < 1+\frac{(k+2)(24\ell+1)}{24n} \quad \text{for all} \quad n>n_0(k,\ell). \tag{6.90}$$

For $k = 2r+1$ with $r \in \mathbb{Z}_{\geq 0}$:

$$\Big(1 - \frac{1+24\ell}{24n}\Big)^{-\frac{k+2}{2}} = \Big(1 - \frac{1+24\ell}{24n}\Big)^{-\frac{2r+3}{2}} = 1 + \sum_{j=1}^{\infty} \binom{-\frac{2r+3}{2}}{j} \frac{(-1)^j (1+24\ell)^j}{(24n)^j}.$$

Using (6.20) with $(m,s,n) \mapsto (1, r+2, \dfrac{24n}{24\ell+1})$, for all $n > \frac{(r+2)(1+24\ell)}{12}$, we get

$$0 < \sum_{j=1}^{\infty} \binom{-\frac{2\ell+3}{2}}{j} \frac{(-1)^j}{(24n)^j} < \frac{(r+2)(1+24\ell)}{6n}$$

which is equivalent to

$$1 < \Big(1 - \frac{1+24\ell}{24n}\Big)^{-\frac{k+2}{2}} < 1 + \frac{k(1+24\ell)}{3n} \quad \text{for all} \quad n > n_0(k,\ell). \tag{6.91}$$

From (6.90) and (6.91), for all $n > n_0(k,\ell)$ it follows that

$$1 < \Big(1 - \frac{1+24\ell}{24n}\Big)^{-\frac{k+2}{2}} < 1 + \frac{k(1+24\ell)}{3 \cdot n_0(k,\ell)}. \tag{6.92}$$

From (6.89) and (6.92), we conclude the proof.

□

6.4 Inequalities for $p(n-\ell)$

Definition 6.4.1. *Let $(L_i(k,\ell))_{1\leq i\leq 4}$ and $(U_i(k,\ell))_{1\leq i\leq 4}$ be as in Definitions 6.3.34-6.3.40. Let $\widehat{U}_2(k,\ell)$ be as in Definition 6.3.43. Then for all $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$, define*

$$L(w,\ell) := L_1\Big(\Big\lceil \frac{w}{2} \Big\rceil, \ell\Big) + L_2\Big(\Big\lceil \frac{w}{2} \Big\rceil, \ell\Big) + L_3\Big(\Big\lfloor \frac{w}{2} \Big\rfloor, \ell\Big) + L_4\Big(\Big\lfloor \frac{w}{2} \Big\rfloor, \ell\Big) - \widehat{U}_2(w,\ell)$$

and

$$U(w,\ell) := U_1\Big(\Big\lceil \frac{w}{2} \Big\rceil, \ell\Big) + U_2\Big(\Big\lceil \frac{w}{2} \Big\rceil, \ell\Big) + U_3\Big(\Big\lfloor \frac{w}{2} \Big\rfloor, \ell\Big) + U_4\Big(\Big\lfloor \frac{w}{2} \Big\rfloor, \ell\Big) + \widehat{U}_2(w,\ell).$$

Lemma 6.4.2. *Let $\widehat{g}(k)$ be as in Theorem 6.3.2 and $n_0(k,\ell)$ as in Definition 6.3.42. Let $g(t,\ell)$ be as in (6.67). Let $L(w,\ell)$ and $U(w,\ell)$ be as in Definition 6.4.1. If $m \in \mathbb{Z}_{\geq 1}$ and $n > \max\{1, n_0(2m,\ell), \widehat{g}(2m)+\ell\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \Bigg(\sum_{t=0}^{2m-1} \frac{g(t,\ell)}{\sqrt{n}^t} + \frac{L(2m,\ell)}{\sqrt{n}^{2m}} \Bigg) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \Bigg(\sum_{t=0}^{2m-1} \frac{g(t,\ell)}{\sqrt{n}^t} + \frac{U(2m,\ell)}{\sqrt{n}^{2m}} \Bigg).$$

Proof. Following Definition 6.3.20 and from Lemma 6.3.21, we have

$$\begin{aligned}\sum_{t=0}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=2m}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t \\ &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty}(g_{e,1}(t,\ell)+g_{e,2}(t,\ell))\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\ &\qquad + \sum_{t=m}^{\infty}(g_{o,1}(t,\ell)+g_{o,2}(t,\ell))\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}.\end{aligned} \tag{6.93}$$

Using Lemmas 6.3.35-6.3.41 by making the substitution $k \mapsto m$, it follows that

$$\begin{aligned}&\frac{L_1(m,\ell)+L_2(m,\ell)}{\sqrt{n}^{2m}} + \frac{L_3(m,\ell)+L_4(m,\ell)}{\sqrt{n}^{2m+1}} \\ &< \sum_{t=2m}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t < \frac{U_1(m,\ell)+U_2(m,\ell)}{\sqrt{n}^{2m}} + \frac{U_3(m,\ell)+U_4(m,\ell)}{\sqrt{n}^{2m+1}}.\end{aligned} \tag{6.94}$$

Moreover, by Lemma 6.3.44 with $k = 2m$, it follows that

$$\frac{\sqrt{12}\; e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^{2m}} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(2m,\ell)}{\sqrt{n}^{2m}}. \tag{6.95}$$

Combining (6.94) and (6.95), and applying to Theorem 6.3.2, we conclude the proof. □

Lemma 6.4.3. *Let $\widehat{g}(k)$ be as in Theorem 6.3.2 and $n_0(k,\ell)$ as in Definition 6.3.42. Let $g(t,\ell)$ be as in Equation (6.67). Let $L(w,\ell)$ and $U(w,\ell)$ be as in Definition 6.4.1. If $m \in \mathbb{Z}_{\geq 0}$ and $n > \max\{1, n_0(2m+1,\ell), \widehat{g}(2m+1)+\ell\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m}\frac{g(t,\ell)}{\sqrt{n}^t}+\frac{L(2m+1,\ell)}{\sqrt{n}^{2m+1}}\right) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m}\frac{g(t,\ell)}{\sqrt{n}^t}+\frac{U(2m+1,\ell)}{\sqrt{n}^{2m+1}}\right).$$

Proof. The proof is analogous to the proof of Lemma 6.4.2. □

Definition 6.4.4. *Let $g(t,\ell)$ be as in (6.67), $L(w,\ell), U(w,\ell)$ as in Definition 6.4.1. If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$, define*

$$\mathcal{L}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(w,\ell)}{\sqrt{n}^w} \quad \text{and} \quad \mathcal{U}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(w,\ell)}{\sqrt{n}^w}.$$

Theorem 6.4.5. *Let $\widehat{g}(k)$ be as in Theorem 6.3.2 and $n_0(k,\ell)$ as in Definition 6.3.42. Let $\mathcal{L}_n(w,\ell)$ and $\mathcal{U}_n(w,\ell)$ be as in Definition 6.4.4. If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$ and $n > \max\{\widehat{g}(w)+\ell, n_0(w,\ell)\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell). \tag{6.96}$$

Proof. Putting Lemmas 6.4.2 and 6.4.3 together, we obtain (6.96). □

6.5 Proofs of Bill Chen's conjectures

Proof of Theorem 6.1.5: To prove the lower bound of (6.16), it is equivalent to show that

$$p(n-4)p(n)+3p(n-2)^2 > 4\left(1+\frac{\pi^2}{16(n-3)^3}-\frac{6}{(n-3)^{7/2}}\right)p(n-3)p(n-1). \tag{6.97}$$

Since $1+\dfrac{\pi^2}{16n^3}-\dfrac{5}{n^{7/2}} > 1+\dfrac{\pi^2}{16(n-3)^3}-\dfrac{6}{(n-3)^{7/2}}$ for all $n \geq 5$, it is enough to show that

$$p(n-4)p(n)+3p(n-2)^2 > 4\left(1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}}\right)p(n-3)p(n-1). \tag{6.98}$$

Choosing $w = 12$ and applying Theorem 6.4.5, for all $n \geq 2329$, we have

$$p(n-4)p(n)+3p(n-2)^2 > \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{L}_n(12,4)\cdot\mathcal{L}_n(12,0)+3\,\mathcal{L}_n^2(12,2)\right), \tag{6.99}$$

and

$$p(n-3)p(n-1) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{U}_n(12,3)\cdot\mathcal{U}_n(12,1)\right). \tag{6.100}$$

Therefore, it suffices to show that

$$\mathcal{L}_n(12,4)\cdot\mathcal{L}_n(12,0)+3\,\mathcal{L}_n^2(12,2) > 4\left(1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}}\right)\mathcal{U}_n(12,3)\cdot\mathcal{U}_n(12,1). \tag{6.101}$$

Using the `Reduce`[1] command within Mathematica, it can be easily checked that for all $n \geq 625$, (6.101) holds.

Similarly, to prove the upper bound of (6.16), it is equivalent to prove that

$$p(n-4)p(n)+3p(n-2)^2 < 4\left(1+\frac{\pi^2}{16(n-3)^3}\right)p(n-3)p(n-1). \tag{6.102}$$

Since $1+\frac{\pi^2}{16n^3} < 1+\frac{\pi^2}{16(n-3)^3}$ for all $n \geq 4$, it is enough to show that

$$p(n-4)p(n)+3p(n-2)^2 < 4\left(1+\frac{\pi^2}{16n^3}\right)p(n-3)p(n-1). \tag{6.103}$$

Choosing $w = 12$ and applying Theorem 6.4.5, for all $n \geq 2329$, we have

$$p(n-4)p(n)+3p(n-2)^2 < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{U}_n(12,4)\cdot\mathcal{U}_n(12,0)+3\,\mathcal{U}_n^2(12,2)\right), \tag{6.104}$$

and

$$p(n-3)p(n-1) > \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{L}_n(12,3)\cdot\mathcal{L}_n(12,1)\right). \tag{6.105}$$

Therefore, it suffices to show that

$$\mathcal{U}_n(12,4)\cdot\mathcal{U}_n(12,0)+3\,\mathcal{U}_n^2(12,2) < 4\left(1+\frac{\pi^2}{16n^3}\right)\mathcal{L}_n(12,3)\cdot\mathcal{L}_n(12,1). \tag{6.106}$$

In a similar way as stated before, it can be easily checked that for all $n \geq 784$, (6.101) holds. We conclude the proof of Theorem 6.1.5 by verifying the inequality (6.16) for all $218 \leq n \leq 2328$ with Mathematica. □

Proof of Theorem 6.1.7: To prove the lower bound of (6.17), it is equivalent to show that

$$p(n-2)^3+p(n-4)p(n-1)^2+p(n-3)^2p(n) >$$
$$\left(1+\frac{\pi^3}{72\sqrt{6}(n-3)^{9/2}}-\frac{8}{(n-3)^5}\right)\big(2p(n-3)p(n-2)p(n-1)+p(n-4)p(n-2)p(n)\big). \tag{6.107}$$

[1]`Reduce` uses cylindrical algebraic decomposition for polynomials over real domains which is based on Collin's algorithm [44]. Cylindrical Algebraic Decomposition (CAD) is an algorithm which proves that a given polynomial in several variables is positive (non-negative).

As $1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}-\frac{7}{n^5}>1+\frac{\pi^3}{72\sqrt{6}(n-3)^{9/2}}-\frac{8}{(n-3)^5}$ for all $n\geq 4$, it suffices to show that

$$p(n-2)^3+p(n-4)p(n-1)^2+p(n-3)^2p(n)> \left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}-\frac{7}{n^5}\right)\big(2p(n-3)p(n-2)p(n-1)+p(n-4)p(n-2)p(n)\big). \tag{6.108}$$

Choosing $w=15$ and applying Theorem 6.4.5, for all $n\geq 4047$, we have

$$p(n-2)^3+p(n-4)p(n-1)^2+p(n-3)^2p(n)> \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^3\left(\mathcal{L}_n^3(15,2)+\mathcal{L}_n(15,4)\cdot\mathcal{L}_n^2(15,1)+\mathcal{L}_n^2(15,3)\cdot\mathcal{L}_n(15,0)\right) \tag{6.109}$$

and

$$2p(n-3)p(n-2)p(n-1)+p(n-4)p(n-2)p(n)< \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^3\left(2\cdot\mathcal{U}_n(15,3)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,1)+\mathcal{U}_n(15,4)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,0)\right). \tag{6.110}$$

Similar to the proof of (6.101), it can be easily checked that for all $n\geq 1444$,

$$\mathcal{L}_n^3(15,2)+\mathcal{L}_n(15,4)\cdot\mathcal{L}_n^2(15,1)+\mathcal{L}_n^2(15,3)\cdot\mathcal{L}_n(15,0)>\left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}-\frac{7}{n^5}\right)\times \left(2\cdot\mathcal{U}_n(15,3)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,1)+\mathcal{U}_n(15,4)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,0)\right). \tag{6.111}$$

Analogously, one can prove that for all $n\geq 2916$,

$$\mathcal{U}_n^3(15,2)+\mathcal{U}_n(15,4)\cdot\mathcal{U}_n^2(15,1)+\mathcal{U}_n^2(15,3)\cdot\mathcal{U}_n(15,0)<\left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\right)\times \left(2\cdot\mathcal{L}_n(15,3)\cdot\mathcal{L}_n(15,2)\cdot\mathcal{L}_n(15,1)+\mathcal{L}_n(15,4)\cdot\mathcal{L}_n(15,2)\cdot\mathcal{L}_n(15,0)\right) \tag{6.112}$$

which is sufficient to prove the upper bound of (6.17). We conclude the proof of Theorem 6.1.7 by verifying the inequality (6.17) for all $244 \leq n \leq 4047$ with Mathematica. $\square$

Proof of Theorem 6.1.9: Corresponding to (6.18), we show

$$\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)\Big(p(n-2)p(n-1)-p(n-3)p(n)\Big)^2 > \\ 4\Big(p(n-2)^2-p(n-3)p(n-1)\Big)\Big(p(n-1)^2-p(n-2)p(n)\Big) \tag{6.113}$$

and

$$4\Big(p(n-2)^2-p(n-3)p(n-1)\Big)\Big(p(n-1)^2-p(n-2)p(n)\Big) > \\ \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{2}{n^2}\right)\Big(p(n-2)p(n-1)-p(n-3)p(n)\Big)^2. \tag{6.114}$$

Applying Theorem 6.4.5 with $w = 13$, and following the similar method worked out in the proof of Theorem 6.1.5, we obtain (6.18) for all $n \geq 2842$. For $115 \leq n \leq 2841$, we verified (6.18) numerically with Mathematica. $\square$

6.6 Appendix

In the proofs of Lemmas 6.3.24-6.3.30, we follow the same notations and the proof strategy as in [21, Subsection 5.2].

Proof of Lemma 6.3.24: Following Definition 6.3.12, write $S_1(t,\ell)$ as follows:

$$\begin{aligned} S_1(t,\ell) &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\sum_{s=u}^{t}\frac{(-1)^s}{s}\left(\frac{1}{2}-s\right)_{s+1}\frac{(-s)_u}{(s+u)!} \\ &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\underbrace{\sum_{s=0}^{t-u}\frac{(-1)^{s+u}}{s+u}\left(\frac{1}{2}-s-u\right)_{s+u+1}\frac{(-s-u)_u}{(s+2u)!}}_{=:S_1(t,u)}. \end{aligned}$$

From [21, eq. (5.6)], we have

$$S_1(t,u) = (-1)^t\binom{-\frac{3}{2}}{t}\frac{(-1)^u}{2u}A_1(t,u), \tag{6.115}$$

where

$$A_1(t,u) = \frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} - \left(\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{(1+2t)} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}\right).$$

Now by Lemmas 6.2.3 and 6.2.4,

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2} - \frac{u}{t} - \frac{u^2}{2t^2} \le A_1(t,u) \le \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} + \frac{1}{4t^2} + u\Big(\frac{2}{3t^2} - \frac{1}{t}\Big) + \frac{u^2}{2t^2} + \frac{u^3}{3t^2}. \quad (6.116)$$

From (6.115), it follows that

$$S_1(t,\ell) = (-1)^t\binom{-\frac{3}{2}}{t}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}A_1(t,u)}{(2u)!}. \quad (6.117)$$

Applying (6.116) to (6.117), we get the following lower bound of $S_1(t,\ell)$,

$$\begin{aligned}
&\frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} \\
&\ge \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!} - \frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha_\ell^{2u}}{(2u)!} \\
&\ge \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} - 1 - \sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}\right) - \frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!} - \frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!}. \\
&> \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} - 1 - \frac{C_0(\ell)}{t^2}\right) - \frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!} - \frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!} \\
&\qquad\qquad \left(\text{by Lemma 6.2.7 and } \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} > \frac{1}{4t^2} \text{ for all } t \ge 1\right) \\
&> \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\Big(\cosh(\alpha_\ell) - 1\Big) - \frac{C_0(\ell)}{t^2} - \frac{\alpha_\ell\sinh(\alpha_\ell)}{2t} \\
&\qquad\qquad - \frac{1}{2t^2}\Big(\frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell) + \frac{\alpha_\ell}{4}\sinh(\alpha_\ell)\Big) \\
&\qquad\qquad \left(\text{by Lemma 6.2.5 and } \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2} < 1 \text{ for all } t \ge 1\right) \\
&= \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell) - 1) - \frac{\alpha_\ell\sinh(\alpha_\ell)}{2t} - \frac{C_1^{\mathcal{L}}(\ell)}{2t^2} \quad \Big(\text{by Definition 6.3.23}\Big).
\end{aligned} \quad (6.118)$$

For the upper bound estimation, we have for all $t \geq 1$,

$$\begin{aligned}
&\frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} \\
&\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!} + \frac{1}{4t^2}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!} + \frac{2}{3t^2}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!} + \\
&\qquad\qquad \frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha_\ell^{2u}}{(2u)!} + \frac{1}{3t^2}\sum_{u=1}^{t}\frac{u^3\alpha_\ell^{2u}}{(2u)!} \\
&\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) + \frac{C_1(\ell)}{t^3} + \frac{1}{4t^2}\cosh(\alpha_\ell) + \frac{1}{3t^2}\alpha_\ell\sinh(\alpha_\ell) + \\
&\quad \frac{1}{2t^2}\left(\frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell) + \frac{\alpha_\ell}{4}\sinh(\alpha_\ell)\right) + \frac{1}{3t^2}\left(\frac{3\alpha_\ell^2}{8}\cosh(\alpha_\ell) + \frac{\alpha_\ell(\alpha_\ell^2+1)}{8}\sinh(\alpha_\ell)\right) \\
&\qquad\qquad \Big(\text{by Lemmas 6.2.5 and 6.2.7}\Big) \\
&\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) + \frac{C_1^{\mathcal{U}}(\ell)}{t^2} \quad \Big(\text{by Definition 6.3.23}\Big). \qquad (6.119)
\end{aligned}$$

Combining (6.118) and (6.119), we arrive at (6.68) which concludes the proof. ☐

Proof of Lemma 6.3.26: Following Definition 6.3.14, write $S_2(t,\ell)$ as follows:

$$\begin{aligned}
S_2(t,\ell) &= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha_\ell^{2u}}{(2u)!}\sum_{s=u}^{t-1}\left(\frac{1}{2}-s\right)_{s+1}\binom{-\frac{3}{2}}{t-s-1}\frac{(-s)_u}{(s+u+1)!} \\
&= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha_\ell^{2u}}{(2u)!}\underbrace{\sum_{s=0}^{t-u-1}\left(\frac{1}{2}-s-u\right)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u-1}\frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_2(t,u)}.
\end{aligned} \qquad (6.120)$$

From [21, eq. (5.13)], we have

$$S_2(t,u) = \binom{-\frac{3}{2}}{t}(-1)^{u+1}\Big(A_{2,1}(t,u) + A_{2,2}(t,u)\Big), \qquad (6.121)$$

where

$$A_{2,1}(t,u) = \frac{2t(t-u)(-t)_u(-1)^u}{(1+2t)(1+2u)(t+u)(t)_u}$$

and

$$A_{2,2}(t,u) = \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-1)^i(-t)_i}{(t+i)(t)_i}.$$

Combining (6.120) and (6.121), we get

$$S_2(t,\ell) = -\binom{-\frac{3}{2}}{t}\Big(s_{2,1}(t,\ell) + s_{2,2}(t,\ell)\Big), \tag{6.122}$$

where

$$s_{2,1}(t,\ell) = \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u)!}A_{2,1}(t,u) \quad \text{and} \quad s_{2,2}(t,\ell) = \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u)!}A_{2,2}(t,u). \tag{6.123}$$

By Lemma 6.2.3, we have

$$\frac{1}{1+2u} - \frac{u^2+u+\frac{1}{2}}{t(1+2u)} \le A_{2,1}(t,u) \le \frac{t-u}{t(1+2u)}. \tag{6.124}$$

Applying (6.124) into (6.123) we obtain

$$\sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{t-1}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha_\ell^{2u} \le s_{2,1}(t) \le \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{t-1}\frac{u\alpha_\ell^{2u}}{(2u+1)!},$$

and consequently,

$$\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{\infty}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha_\ell^{2u} \le s_{2,1}(t,\ell) \le$$
$$\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\left(\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!} - \sum_{u=t}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!}\right). \tag{6.125}$$

By Lemma 6.2.7, it follows that

$$\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} \le \frac{2C_1(\ell)}{\alpha_\ell^2\, t^2} \quad \text{and} \quad \sum_{u=t}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!} \le \frac{2C_2(\ell)}{\alpha_\ell^2 t^2}. \tag{6.126}$$

Applying (6.126) into (6.125) and by Lemma 6.2.5, we obtain

$$\frac{\sinh(\alpha_\ell)}{\alpha_\ell} - \frac{C_{2,1}^{\mathcal{L}}(\ell)}{t} \le s_{2,1}(t,\ell) \le \frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{C_{2,1}^{\mathcal{U}}(\ell)}{t}. \tag{6.127}$$

Next we apply Lemma 6.2.4 and get

$$\frac{2u+1}{2t} - \frac{4u^3+6u^2+8u+3}{12t^2} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} \le A_{2,2}(t,u) \le \frac{2u+1}{2t} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}. \quad (6.128)$$

Plugging (6.128) into (6.123), we obtain

$$\frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} - \frac{1}{2t}\sum_{u=t}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha_\ell^{2u}}{(2u)!}$$
$$\le s_{2,2}(t,\ell) \le \frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}, \quad (6.129)$$

where $p_3(u) = 4u^3+6u^2+8u+3$. By Lemma 6.2.7 we obtain

$$\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} \le \frac{4C_2(\ell)}{\alpha_\ell^2\, t^2} \quad \text{and} \quad \sum_{u=t}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} \le \frac{8C_3(\ell)}{\alpha_\ell^2\, t^2}. \quad (6.130)$$

Note that for all $t \ge 1$,

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} = \frac{2^{2t+1}}{t+1}\frac{1}{\binom{2t+2}{t+1}} < 1. \quad (6.131)$$

Combining (6.130) with (6.131) and applying Lemma 6.2.7 to (6.129), we obtain

$$\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell) + \frac{\operatorname{csh}(\alpha_\ell)}{2t} - \frac{C_{2,2}(\alpha_\ell)}{t^2} \le s_{2,2}(t,\ell) \le \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell) + \frac{\operatorname{csh}(\alpha_\ell)}{2t} + \frac{4C_2(\ell)}{\alpha_\ell^2\, t^2}. \quad (6.132)$$

Applying (6.127) and (6.132) to (6.122), we obtain (6.69). □

Proof of Lemma 6.3.28: Recalling Definition 6.3.16, rewrite $S_3(t,\ell)$ as follows:

$$\begin{aligned} S_3(t,\ell) &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\sum_{s=u}^{t}\frac{1}{s}\left(\frac{1}{2}-s\right)_{s+1}\binom{-\frac{3}{2}}{t-s}\frac{(-s)_u}{(s+u)!} \\ &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\underbrace{\sum_{s=0}^{t-u}\frac{1}{s+u}\left(\frac{1}{2}-s-u\right)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u}\frac{(-s-u)_u}{(s+2u)!}}_{=:S_3(t,u)}. \end{aligned} \quad (6.133)$$

From [21, eq. (5.34)], we have

$$S_3(t,u) = \binom{-\frac{3}{2}}{t}(-1)^u\Big(A_{3,1}(t,u) + A_{3,2}(t,u)\Big), \tag{6.134}$$

where

$$A_{3,1}(t,u) = \frac{t(1+2t-2u)(-t)_u(-1)^u}{2(1+2t)u(t+u)(t)_u}$$

and

$$A_{3,2}(t,u) = \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}.$$

From (6.133) and (6.134), it follows that

$$S_3(t,\ell) = \binom{-\frac{3}{2}}{t}\Big(s_{3,1}(t) + s_{3,2}(t)\Big), \tag{6.135}$$

with

$$s_{3,1}(t,\ell) = \sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u-1)!}A_{3,1}(t,u) \quad \text{and} \quad s_{3,2}(t,\ell) = \sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u-1)!}A_{3,2}(t,u). \tag{6.136}$$

By Lemma 6.2.3, we have

$$-\frac{3u^2+2u+\frac{1}{2}}{4ut} \le A_{3,1}(t,u) - \frac{1}{2u} \le 0. \tag{6.137}$$

Applying (6.137) into (6.136) and by Lemmas 6.2.7 and 6.2.5, we obtain

$$-\frac{C_{3,1}(\ell)}{t} \le s_{3,1}(t,\ell) + 1 - \cosh(\alpha_\ell) \le 0. \tag{6.138}$$

Now, by Lemma 6.2.4, we obtain

$$-\frac{4u^3+6u^2+8u+3}{12t^2} \le A_{3,2}(t,u) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t} \le 0. \tag{6.139}$$

Applying (6.139) to (6.136), it follows that

$$s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t}\sum_{u=1}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} \le \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u-1)!}, \tag{6.140}$$

and

$$s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \sum_{u=1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} \geq - \frac{1}{12t^2} \sum_{u=1}^{\infty} \frac{p_3(u)\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!}, \tag{6.141}$$

where $p_3(u) = 4u^3 + 6u^2 + 8u + 3$ is as in (6.129). By Lemma 6.2.7 we obtain

$$\sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u-1)!} \leq \frac{2C_1(\ell)}{t^2} \quad \text{and} \quad \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} \leq \frac{4C_2(\ell) + 2C_1(\ell)}{t^2}. \tag{6.142}$$

Applying (6.142) and Lemma 6.2.5 into (6.140) and (6.141), we have

$$-\frac{C_{3,2}(\ell)}{t^2} \leq s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \alpha_\ell \sinh(\alpha_\ell) - \frac{1}{2t} \mathrm{sch}(\alpha_\ell) \leq \frac{3C_1(\ell)}{t^2}. \tag{6.143}$$

Applying (6.138) and (6.143) into (6.135) we arrive at (6.70). $\square$

Proof of Lemma 6.3.30: Following Definition 6.3.18, write $S_4(t,\ell)$ as follows:

$$\begin{aligned} S_4(t,\ell) &= \sum_{u=0}^{t} \frac{(-1)^u \alpha_\ell^{2u}}{(2u)!} \sum_{s=u}^{t} (-1)^s \left(\frac{1}{2} - s\right)_{s+1} \frac{(-s)_u}{(s+u+1)!} \\ &= \sum_{u=0}^{t} \frac{(-1)^u \alpha_\ell^{2u}}{(2u)!} \underbrace{\sum_{s=0}^{t-u} (-1)^{s+u} \left(\frac{1}{2} - s - u\right)_{s+u+1} \frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_4(t,u)}. \end{aligned} \tag{6.144}$$

From [21, eq. (5.53)], we have

$$S_4(t,u) = \binom{-\frac{3}{2}}{t} (-1)^{u+t} \Big(A_{4,1}(t,u) + A_{4,2}(t,u) \Big), \tag{6.145}$$

where

$$A_{4,1}(t,u) = \frac{t(-t)_u(-1)^u}{2(1+2t)(t+u)(t+u+1)(t)_u}$$

and

$$A_{4,2}(t,u) = \frac{1}{1+2u} \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{1+2t} - \frac{2t}{1+2t} \sum_{i=1}^{u} \frac{(-1)^i(-t)_i}{(t+i)(t)_i} \right).$$

From (6.144) and (6.145) it follows that

$$S_4(t,\ell) = (-1)^t \binom{-\frac{3}{2}}{t} \Big(s_{4,1}(t,\ell) + s_{4,2}(t,\ell) \Big), \tag{6.146}$$

where

$$s_{4,1}(t,\ell) = \sum_{u=0}^{t} \frac{\alpha_\ell^{2u}}{(2u)!} A_{4,1}(t,u) \quad \text{and} \quad s_{4,2}(t) := \sum_{u=0}^{t} \frac{\alpha_\ell^{2u}}{(2u)!} A_{4,2}(t). \tag{6.147}$$

Lemmas 6.2.2 and 6.2.3 imply that

$$\frac{1}{4t^2}\left(1 - \frac{u^2+u+\frac{3}{2}}{t}\right) \le A_{4,1}(t,u) \le \frac{1}{4t^2}. \tag{6.148}$$

From (6.148) and (6.147), we obtain

$$\frac{1}{4t^2}\sum_{u=0}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{4t^2}\sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{4t^3}\sum_{u=0}^{\infty} \frac{(u^2+u+\frac{3}{2})\alpha_\ell^{2u}}{(2u)!} \le s_{4,1}(t,\ell) \le \frac{1}{4t^2}\sum_{u=0}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!}. \tag{6.149}$$

Applying Lemmas 6.2.7 and 6.2.5 to (6.149), it follows that

$$\frac{1}{4t^2}\cosh(\alpha_\ell) - \frac{C_{4,1}(\ell)}{t^3} \le s_{4,1}(t,\ell) \le \frac{1}{4t^2}\cosh(\alpha_\ell). \tag{6.150}$$

Now, by Lemma 6.2.4, we obtain

$$0 \le A_{4,2}(t,u) - \frac{1}{1+2u}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t} \right) \le \frac{1}{1+2u}\frac{p_3(u)}{12t^2}, \tag{6.151}$$

where $p_3(u)$ is as in (6.129). Plugging (6.151) into (6.147), it follows that

$$\begin{aligned} -\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u+1)!} \le & s_{4,2}(t,\ell) - \sum_{u=0}^{\infty} \frac{\alpha_\ell^{2u}}{(2u+1)!}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t} \right) \le \\ & \frac{1}{12t^2}\sum_{u=0}^{\infty} \frac{p_3(u)\alpha_\ell^{2u}}{(2u+1)!} + \frac{1}{2t}\sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u+1)!}. \end{aligned} \tag{6.152}$$

Using Lemma 6.2.7, we get

$$\sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u+1)!} \le \frac{C_0(\ell)}{t^2} \quad \text{and} \quad \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u+1)!} = \sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!} \le \frac{C_0(\ell)}{t^2}. \tag{6.153}$$

Plugging (6.153) to (6.152) and using Lemma 6.2.5, we finally obtain

$$-\frac{2C_0(\ell)}{3t^2} \leq s_{4,2}(t,\ell) - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{\cosh(\alpha_\ell)}{2t} \leq \frac{(\alpha^2+6)\cosh(\alpha_\ell) + 3\alpha_\ell \sinh(\alpha_\ell) + 12C_0(\ell)}{24t^2}. \tag{6.154}$$

We conclude the proof by combining (6.150), (6.154), and (6.146). □

6.7 Further applications

6.7.1 Higher order Laguerre inequalities for $p(n)$

The partition function $p(n)$ satisfies Laguerre inequality of order m if

$$L_m(p(n)) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{k+m}\binom{2m}{k}p(n+k)p(2m-k+n).$$

In [145], Wagner proved the mth order Laguerre inequalities for the partition function as $n \to \infty$ for all $m \geq 2$. He proposed a conjecture for the cut offs $(N_L(m))_{1\leq m\leq 10}$ such that for all $n \geq N_L(m)$, $p(n)$ satisfies the mth order Laguerre inequalities. Wang and Yang [149] settled the case $m = 2$. Dou and Wang [58] gave an explicit bounds for $(N_L(m))_{3\leq m\leq 10}$ and consequently, Wagner's conjecture for the cases $m = 3$ and 4 have been settled.
For $2 \leq m \leq 11$, let $N_L(m)$ denotes the cut-off conjectured by Wagner, $w(m)$ denotes the truncation point as given in Theorem 6.4.5, $N_B(m)$ denotes the cut-off from which point on we are able to show that $(p(n))_{n\geq N_B(m)}$ satisfies Laguerre inequalities of order m, and $T(m)$ denotes the time (in seconds) taken in computation with 'Reduce' command in Mathematica.

Enumeration of cut-off				
m	$N_L(m)$	$w(m)$	$N_B(m)$	$T(m)$
2	184	11	1873	0.76
3	531	15	4049	1.53
4	1102	20	8164	4.61
5	1923	23	11436	7.51
6	3014	30	21577	11.46

Enumeration of cut-off				
m	$N_L(m)$	$w(m)$	$N_B(m)$	$T(m)$
7	4391	34	29034	25.34
8	6070	39	40138	40.88
9	8063	45	56180	126.91
10	10382	50	71893	177.34
11	13037	55	89803	366.15

Applying Theorem 6.4.5 with $(w(m))_{2\leq m\leq 11}$, we obtain the following theorem.

Theorem 6.7.1. *For $2 \leq m \leq 11$,*

$$L_m(p(n-2m)) > 0 \quad \text{for all} \quad n > N_L(m). \tag{6.155}$$

Remark 6.7.2. *In spite of having Wagner's proof on positivity of $L_m(p(n))$ as $n \to \infty$, a natural question arises: what is the growth of $L_m(p(n))$ as $n \to \infty$?*

6.7.2 Asymptotic growth of $\Delta_j^r p(n)$

Let Δ be the difference operator defined on a sequence $(a(n))_{n\geq 0}$ by $\Delta(a(n)) := a(n+1)-a(n)$. A r-fold applications of Δ is denoted by Δ^r. Recently, Gomez, Males, and Rolen [66] generalized the Δ operator by introducing a shift parameter j, defined as $\Delta_j^2(a(n)) := a(n) - 2a(n-j) + a(n-2j)$, and studied the positivity of $\Delta_j^2(p(n))$. Consequently, they also proved that $N_k(m,n) - N_k(m+1,n) > 0$, where the k-rank function $N_k(m,n)$ which counts the number of partitions of n into at least $(k-1)$ successive Durfee squares with k-rank equal to m (see [64]). Following Theorem 6.4.5, we obtain the asymptotic expansion of $\Delta_j^r(p(n)) := \sum_{m=0}^{r}(-1)^m\binom{m}{r}p(n-m\cdot j)$ for any positive integer r, which finally completes the work of Odlyzko [114] on $\Delta^r p(n)$ (set $j=1$) by proving its asymptotic growth. Works related to the positivity of $\Delta^r p(n)$ can be found in [67, 71, 5, 87].

Following the notation from [68], here $\left\{ n \atop m \right\}$ denotes the Stirling number of the second kind.

Lemma 6.7.3. *Let $g(t,\ell)$ be as in Equation 6.67. Then for all $r \geq 1$,*

$$\sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=0}^{r+1}\frac{g(t,m\cdot j)}{\sqrt{n}^t} = \Big(\frac{\pi\cdot j}{\sqrt{6}}\Big)^r\frac{1}{\sqrt{n}^r} - \Big(\frac{\pi\cdot j}{\sqrt{6}}\Big)^{r-1}\frac{j}{4}\left[\frac{\pi^2}{36}(1+12jr)+(r^2+3r+2)\right]\frac{1}{\sqrt{n}^{r+1}}. \tag{6.156}$$

Proof. Following (6.67), we have

$$\begin{aligned}
&\sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=0}^{r+1}\frac{g(t,m\cdot j)}{\sqrt{n}^t}\\
=\ &\sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=0}^{r+1}\Big(\frac{1+24m\cdot j}{-4\sqrt{6n}}\Big)^t\times
\end{aligned}$$

$$
\sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \Big(\frac{\pi}{6}\Big)^{t-2k} \frac{1}{(1+24m\cdot j)^k}
$$

$$
\begin{aligned}
&= \sum_{t=0}^{r+1} \sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \Big(\frac{\pi}{6}\Big)^{t-2k} \Big(\frac{1}{-4\sqrt{6n}}\Big)^{t} \times \\
&\qquad\qquad \sum_{m=0}^{r} (-1)^m \binom{r}{m} (1+24m\cdot j)^{t-k} \\
&= \sum_{t=0}^{r+1} \sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \Big(\frac{\pi}{6}\Big)^{t-2k} \Big(\frac{1}{-4\sqrt{6n}}\Big)^{t} \times \\
&\qquad\qquad \sum_{\ell=0}^{t-k} \binom{t-k}{\ell} (24j)^{\ell} \sum_{m=0}^{r} (-1)^m \binom{r}{m} m^{\ell} \\
&= \sum_{t=0}^{r+1} \sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \Big(\frac{\pi}{6}\Big)^{t-2k} \Big(\frac{1}{-4\sqrt{6n}}\Big)^{t} \sum_{\ell=0}^{t-k} \binom{t-k}{\ell} (24j)^{\ell} (-1)^r r! \begin{Bmatrix} \ell \\ r \end{Bmatrix}.
\end{aligned}
\tag{6.157}
$$

We observe that for $\left\{ {\ell \atop r} \right\} = 0$ for all $\ell < r$. Therefore, the minimal choice for $(t,k,\ell) = (r,0,r)$ so that the sum on the right hand side of (6.157) to be non-zero. For $t = r+1$, we have two choices; i.e., $(k,\ell) = (1,r)$ and for $k = 0$, $\ell \in \{r, r+1\}$. Therefore, we have

$$
\begin{aligned}
&\sum_{m=0}^{r} (-1)^m \binom{r}{m} \sum_{t=0}^{r+1} \frac{g(t, m\cdot j)}{\sqrt{n}^{t}} \\
&= \sum_{t=r}^{r+1} \sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \Big(\frac{\pi}{6}\Big)^{t-2k} \Big(\frac{1}{-4\sqrt{6n}}\Big)^{t} \times \\
&\qquad\qquad \sum_{\ell=0}^{t-k} \binom{t-k}{\ell} (24j)^{\ell} (-1)^r r! \begin{Bmatrix} \ell \\ r \end{Bmatrix} \\
&= \Big(\frac{\pi\cdot j}{\sqrt{6}}\Big)^{r} \frac{1}{\sqrt{n}^{r}} + \\
&\qquad \left[\sum_{k=0}^{1} \binom{r+2}{k} \frac{r+2-k}{(r+2-2k)!} \Big(\frac{\pi}{6}\Big)^{r+1-2k} \sum_{\ell=r}^{r+1-k} (24j)^{\ell} (-1)^r r! \begin{Bmatrix} \ell \\ r \end{Bmatrix} \right] \frac{1}{(-4\sqrt{6n})^{r+1}}
\end{aligned}
$$

$$= \left(\frac{\pi \cdot j}{\sqrt{6}}\right)^r \frac{1}{\sqrt{n}^r} - \left(\frac{\pi \cdot j}{\sqrt{6}}\right)^{r-1} \frac{j}{4}\left[\frac{\pi^2}{36}(1+12jr) + (r^2+3r+2)\right]\frac{1}{\sqrt{n}^{r+1}}. \qquad (6.158)$$

$\square$

Definition 6.7.4. *For all $r \geq 1$, define*

$$C_r(j) := \left(\frac{\pi \cdot j}{\sqrt{6}}\right)^r,$$

$$C_{r+1}(j) := \left(\frac{\pi \cdot j}{\sqrt{6}}\right)^{r-1} \frac{j}{4}\left[\frac{\pi^2}{36}(1+12jr) + (r^2+3r+2)\right],$$

$$\widetilde{U}_r(j) := \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r}{2m} U(r+2, 2m \cdot j) - \sum_{m=0}^{\lfloor (r-1)/2 \rfloor} \binom{r}{2m+1} L(r+2, (2m+1)j)$$

and

$$\widetilde{L}_r(j) := \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r}{2m} L(r+2, 2m \cdot j) - \sum_{m=0}^{\lfloor (r-1)/2 \rfloor} \binom{r}{2m+1} U(r+2, (2m+1)j).$$

Lemma 6.7.5. *For all $n > \max\left\{\widehat{g}(r+2) + r \cdot j, n_0(r+2, r \cdot j)\right\}$, we have*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\widetilde{M}_r(n,j) + \frac{\widetilde{L}_r(j)}{\sqrt{n}^{r+2}}\right) < \Delta_j^r(p(n)) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\widetilde{M}_r(n,j) + \frac{\widetilde{U}_r(j)}{\sqrt{n}^{r+2}}\right), \qquad (6.159)$$

where

$$M_r(n,j) = \frac{C_r(j)}{\sqrt{n}^r} - \frac{C_{r+1}(j)}{\sqrt{n}^{r+1}}.$$

Proof. We split $\Delta_j^r(p(n))$ as follows:

$$\begin{aligned} \Delta_j^r(p(n)) &= \sum_{m=0}^{r} (-1)^m \binom{m}{r} p(n - m \cdot j) \\ &= \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r}{2m} p(n - 2m \cdot j) - \sum_{m=0}^{\lfloor (r-1)/2 \rfloor} \binom{r}{2m+1} p(n - (2m+1) \cdot j). \end{aligned} \qquad (6.160)$$

Applying Theorem 6.4.5 for each of the above two factors, we obtain (6.159). $\square$

From Lemma 6.7.5, we obtain the following theorem.

Theorem 6.7.6. *For all* $r, j \in \mathbb{Z}_{\geq 1}$,

$$\Delta_j^r(p(n)) \sim \frac{\left(\pi \cdot j/\sqrt{6}\right)^r}{4\sqrt{3}} \frac{e^{\pi\sqrt{2n/3}}}{\sqrt{n}^{r+2}} \quad \text{as} \quad n \to \infty. \tag{6.161}$$

Corollary 6.7.7. *For* $j = 1$ *and* $r \in \mathbb{Z}_{\geq 1}$, *we have*

$$\Delta^r(p(n)) \sim \frac{\left(\pi/\sqrt{6}\right)^r}{4\sqrt{3}} \frac{e^{\pi\sqrt{2n/3}}}{\sqrt{n}^{r+2}} \quad \text{as} \quad n \to \infty.$$

Theorem 6.7.8. *For all* $r, j \in \mathbb{Z}_{\geq 1}$,

$$\Delta_j^r(p(n)) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(\frac{\pi \cdot j}{\sqrt{6n}}\right)^r \sum_{t=0}^{\infty} \frac{\widetilde{g}_{r,j}(t)}{\sqrt{n}^t} \quad \text{as} \quad n \to \infty, \tag{6.162}$$

where

$$\widetilde{g}_{r,j}(t) = \frac{(t+r+1)r!}{(-4\sqrt{6})^t} \sum_{k=0}^{t} \sum_{\ell=0}^{t-k} \binom{t+r}{\ell+r} \binom{t-\ell}{k} \frac{1}{(t+r+1-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^\ell \left\{ \begin{matrix} \ell+r \\ r \end{matrix} \right\}.$$

Proof. Letting $w \to \infty$, from (6.96) it follows that

$$\begin{aligned} \Delta_j^r(p(n)) &= \sum_{m=0}^{r} (-1)^m \binom{m}{r} p(n - m \cdot j) \\ &\underset{n\to\infty}{\sim} \sum_{m=0}^{r} (-1)^m \binom{r}{m} \sum_{t=0}^{\infty} \frac{g(t, m \cdot j)}{\sqrt{n}^t}. \end{aligned} \tag{6.163}$$

From Lemma 6.7.3, for $0 \leq t \leq r-1$ we have,

$$\sum_{m=0}^{r} (-1)^m \binom{r}{m} \sum_{t=0}^{r-1} \frac{g(t, m \cdot j)}{\sqrt{n}^t} = 0. \tag{6.164}$$

From (6.163) and (6.164), as $n \to \infty$ we get,

$\Delta_j^r(p(n))$

$$
\begin{aligned}
&\sim \sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=r}^{\infty}\frac{g(t,m\cdot j)}{\sqrt{n}^t}\\
&= \sum_{t=r}^{\infty}\sum_{k=0}^{(t+1)/2}\frac{1}{(-4\sqrt{6n})^t}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}\times\\
&\qquad\qquad\qquad\qquad\sum_{m=0}^{r}(-1)^m\binom{r}{m}(1+24m\cdot j)^{t-k}\\
&= \sum_{t=r}^{\infty}\sum_{k=0}^{(t+1)/2}\sum_{\ell=0}^{t-k}\frac{1}{(-4\sqrt{6n})^t}\binom{t+1}{k}\binom{t-k}{\ell}\frac{t+1-k}{(t+1-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}(24j)^{\ell}\times\\
&\qquad\qquad\qquad\qquad\sum_{m=0}^{r}(-1)^m\binom{r}{m}m^{\ell}\\
&= (-1)^r r!\sum_{t=r}^{\infty}\sum_{k=0}^{(t+1)/2}\sum_{\ell=0}^{t-k}\frac{1}{(-4\sqrt{6n})^t}\binom{t+1}{k}\binom{t-k}{\ell}\frac{t+1-k}{(t+1-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}(24j)^{\ell}\left\{\begin{matrix}\ell\\ r\end{matrix}\right\}\\
&:= (-1)^r r!\sum_{t=r}^{\infty}A(t,r) = (-1)^r r!\sum_{t=0}^{\infty}A(t+r,r). \qquad (6.165)
\end{aligned}
$$

Now

$$
\begin{aligned}
&A(t+r,r)\\
&=\frac{(4\pi\cdot j)^r}{(-4\sqrt{6n})^{r+t}}\sum_{k=0}^{t}\sum_{\ell=0}^{t-k}\binom{t+r}{\ell+r}\binom{t-\ell}{k}\frac{t+r+1}{(t+r+1-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}(24j)^{\ell}\left\{\begin{matrix}\ell+r\\ r\end{matrix}\right\}.
\end{aligned}
\qquad (6.166)
$$

Applying (6.166) to (6.165), we finally obtain (6.162). □

Corollary 6.7.9. *For* $j \in \mathbb{Z}_{\geq 1}$,

$$
\Delta_j^1(p(n)) \sim \frac{e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}}\pi j\sum_{t=0}^{\infty}\frac{\widetilde{g}_{1,j}(t)}{\sqrt{n}^t} \quad \text{as} \quad n\to\infty, \qquad (6.167)
$$

where

$$
\widetilde{g}_{1,j}(t) = \frac{(t+2)}{(-4\sqrt{6})^t}\sum_{k=0}^{t}\sum_{\ell=0}^{t-k}\binom{t+1}{\ell+1}\binom{t-\ell}{k}\frac{1}{(t+2-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}(24j)^{\ell}(\ell+1).
$$

Remark 6.7.10. *Replacing $n \mapsto n-k-m+1 := n_k$ and plugging $j = 1$ in Corollary 6.7.9, for all $m > n/2$, we have the full asymptotic expansion of $N_k(m,n)$ with respect to the base $\frac{1}{\sqrt{n_k}^t}$. But in order to get the asymptotic expansion with respect to the base $\frac{1}{\sqrt{n}^t}$, we directly employ Theorem 6.4.5 and obtain for $m > n/2$,*

$$N_k(m,n) \underset{n\to\infty}{\sim} \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \sum_{t=0}^{\infty} \frac{\overline{g}_k(t)}{\sqrt{n}^t}, \tag{6.168}$$

where

$$\overline{g}_k(t) := g(t, k+m-1) - g(t, k+m).$$

For $k = 1, 2$ we get the asymptotic expansion of $M(m,n)$ and $N(m,n)$ respectively.

Corollary 6.7.11. *For $j \in \mathbb{Z}_{\geq 1}$,*

$$\Delta_j^2(p(n)) \sim \frac{e^{\pi\sqrt{2n/3}}}{24\sqrt{3}n^2}\pi^2 j^2 \sum_{t=0}^{\infty} \frac{\widetilde{g}_{2,j}(t)}{\sqrt{n}^t} \quad \text{as} \quad n \to \infty, \tag{6.169}$$

where

$$\widetilde{g}_{2,j}(t) = \frac{(2t+6)}{(-4\sqrt{6})^t} \sum_{k=0}^{t} \sum_{\ell=0}^{t-k} \binom{t+2}{\ell+2} \binom{t-\ell}{k} \frac{1}{(t+3-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^{\ell} \big(2^{\ell+1} - 1\big).$$

Remark 6.7.12. *By making the substitution $n \mapsto n-k-m+1 := n_k$ and plugging $j = 1$ in Corollary 6.7.11, for all $m > n/2$, we have the full asymptotic expansion of $N_k(m,n) - N_k(m+1,n)$ with respect to the base $\frac{1}{\sqrt{n_k}^t}$. But in order to get the asymptotic expansion with respect to the base $\frac{1}{\sqrt{n}^t}$, we directly employ Theorem 6.4.5 and obtain for $m > n/2$,*

$$N_k(m,n) - N_k(m+1,n) \underset{n\to\infty}{\sim} \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \sum_{t=0}^{\infty} \frac{\widetilde{g}_k(t)}{\sqrt{n}^t}, \tag{6.170}$$

where

$$\widetilde{g}_k(t) := g(t, k+m-1) - 2g(t, k+m) + g(t, k+m+1).$$

For $k = 1, 2$ we get the asymptotic expansion of $M(m,n) - M(m+1)$ and $N(m,n) - N(m+1,n)$ respectively.

6.7.3 Higher order log-concavity for $p(n)$

Recall the definition of r-fold log-concavity of a sequence $(a_n)_{n\geq 0}$ from Section 2.5.

Theorem 6.7.13. *For* $r \in \{1,2,3\}$ *and* $n > \max\Big\{\widehat{g}(3{\cdot}2^r-2)+2r, n_0(3{\cdot}2^r-r, 2r)\Big\} := N(r)$,

$$\mathcal{L}^r(p(n-r)) = \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^{2^r}\left(\frac{\pi^{2^r-1}}{\sqrt{2}^{r^2+r+1}\sqrt{3}^{r^2-r+1}\sqrt{n}^{3(2^r-1)}}+O\left(\frac{1}{\sqrt{n}^{3\cdot 2^r-2}}\right)\right) \tag{6.171}$$

Proof. For $r=1$, $\mathcal{L}(p(n-1)) = p(n-1)^2 - p(n)p(n-2)$. Applying Theorem 6.4.5 with $w=4$, for all $n > N(4) = 151$ we have

$$\begin{aligned}\mathcal{L}(p(n-1)) &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\sum_{t=0}^{3}\frac{g(t,1)^2-g(t,0)g(t,2)}{\sqrt{n}^t}+O\Big(\frac{1}{n^2}\Big)\right)\\ &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\frac{\pi}{2\sqrt{6}n^{3/2}}+O\Big(\frac{1}{n^2}\Big)\right).\end{aligned} \tag{6.172}$$

Define

$$g_2(t,\ell) := g(t,\ell)^2 - g(t,\ell-1)g(t,\ell+1).$$

Now for $r=2$, applying Theorem 6.4.5 with $w=10$, for all $n > N(10) = 1473$ it follows that

$$\begin{aligned}\mathcal{L}^2(p(n-2)) &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\sum_{t=0}^{9}\frac{g_2(t,2)^2-g_2(t,3)g_2(t,1)}{\sqrt{n}^t}+O\Big(\frac{1}{n^5}\Big)\right)\\ &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\frac{\pi^3}{24\sqrt{6}n^{9/2}}+O\Big(\frac{1}{n^5}\Big)\right).\end{aligned} \tag{6.173}$$

Define

$$g_3(t,\ell) := g_2(t,\ell)^2 - g_2(t,\ell-1)g_2(t,\ell+1).$$

Finally for $r=3$, from Theorem 6.4.5 with $w=22$, for all $n > N(22) = 10273$ we get

$$\begin{aligned}\mathcal{L}^3(p(n-3)) &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^8\left(\sum_{t=0}^{21}\frac{g_3(t,3)^2-g_3(t,4)g_3(t,2)}{\sqrt{n}^t}+O\Big(\frac{1}{n^{11}}\Big)\right)\\ &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^8\left(\frac{\pi^7}{1728\sqrt{6}n^{21/2}}+O\Big(\frac{1}{n^{11}}\Big)\right).\end{aligned} \tag{6.174}$$

□

Remark 6.7.14. *For $r \in \{1,2\}$, we obtain the following two inequalities for $\mathcal{L}^r(p(n-r))$ using (6.96). For all $n > 676$,*

$$\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\frac{\pi}{2\sqrt{6}n^{3/2}}-\frac{4}{n^2}\right) < \mathcal{L}^1(p(n-1)) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\frac{\pi}{2\sqrt{6}n^{3/2}}+\frac{4}{n^2}\right); \tag{6.175}$$

and for all $n > 5499$,

$$\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\frac{\pi^3}{24\sqrt{6}n^{9/2}}-\frac{10}{n^5}\right) < \mathcal{L}^2(p(n-2)) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\frac{\pi^3}{24\sqrt{6}n^{9/2}}+\frac{10}{n^5}\right). \tag{6.176}$$

Equations (6.175) and (6.176) retrieve that $(p(n))_{n\geq 26}$ is log-*concave and $(p(n))_{n\geq 222}$ is* 2-log-*concave respectively along with the asymptotic growths.*
Following the proof of Theorem 6.7.13, it suggests that for all $n > N(r)$,

$$\mathcal{L}^r(p(n-r)) = \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^{2^r}\left(\sum_{t=0}^{3(2^r-1)}\frac{g_r(t,r)^2-g_r(t,r+1)g_r(t,r-1)}{\sqrt{n}^t}+O\left(\frac{1}{\sqrt{n}^{3\cdot 2^r-2}}\right)\right),$$

where $g_r(t,r) = g_{r-1}(t,r)^2 - g_{r-1}(t,r-1)g_{r-1}(t,r+1)$ for all $r \geq 2$ and $g_1(t,r) = g(t,r)$.
Moreover, following (6.172)-(6.174), it further suggests that

$$\sum_{t=0}^{3(2^r-1)}\frac{g_r(t,r)^2-g_r(t,r+1)g_r(t,r-1)}{\sqrt{n}^t} = \frac{G_r}{\sqrt{n}^{3(2^r-1)}},$$

where $G_r = g_r(3(2^r-1),r)^2 - g_r(3(2^r-1),r+1)g_r(3(2^r-1),r-1)$. This finally leads us to make the following conjecture.

Conjecture 6.7.15. *For $r \in \mathbb{Z}_{\geq 1}$ and $n > \max\left\{\widehat{g}(3\cdot 2^r-2)+2r, n_0(3\cdot 2^r-r, 2r)\right\}$,*

$$\mathcal{L}^r p(n-r) \sim \frac{\pi^{2^r-1}}{\sqrt{2}^{r^2+r+1}\sqrt{3}^{r^2-r+1}\sqrt{n}^{3(2^r-1)}}\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^{2^r}, \quad \text{as} \quad n\to\infty. \tag{6.177}$$

Remark 6.7.16. *The* 2-log-*concavity for the partition function has been studied independently in [82, Theorem 1.6] and [79, Page 128]. Similar to the proofs of Theorems 6.1.5-6.1.9, $(p(n))_{n\geq 1873}$ is* 2-log-*concave also follows directly from Theorem 6.4.5 by choosing $w = 11$ and with Mathematica, we confirm that $(p(n))_{n\geq 221}$ is* 2-log-*concave.*

We conclude this section with a list of further possible ideas that emerged from our work.

1. Partition inequalities arising from truncated theta series that has been documented in [9, 10, 59] among many research works done by Andrews, Guo, Merca, Yee, Zeng, to name a few. Despite having combinatorial proofs of such inequalities for $p(n)$, it seems that no such inequalities have been traced via the analytic approach. Theorem 6.4.5 might play a key role in proving these inequalities. More generally, given non-trivial linear homogeneous partition inequalities considered by Merca and Katriel [83, 108], it would be nice to develop an algorithm by making an appropriate choice for w and applying Theorem 6.4.5 to decide whether such a given inequality holds or not.

2. Starting from the estimates of Dawsey and Masri [50] on Andrews' spt function, one can follow the similar method as worked out in this chapter to settle all the conjectures on inequalities for spt function pertaining to the invariants of a quartic binary form given by Chen [36].

Chapter 7

Inequalities for the partition function arising from truncated theta series

Positivity questions related to the partition function arising from classical theta identities have been studied in the combinatorial and q-series framework. Two such identities that emerge from truncation of Euler's pentagonal number theorem and an identity due to Gauss are the predominant ones among others. In this chapter, we prove the asymptotic growth of coefficients of truncation of theta series directly from inequalities for the shifted partition function rather than taking a detour to Wright's circle method. Recently, Andrews and Merca conjectured that for n odd or k even,

$$M_k(n) \geq (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p(n-j(2j+1))-p(n-(j+1)(2j+1))\Big),$$

where $M_k(n)=(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1)\Big)$. We confirm the conjecture for all $n \geq N(k)$ with explicit information about $N(k)$ by determining the asymptotic growth of the difference between the alternating sums presented in the above inequality. This in turn shows that the conjecture of Andrews and Merca is even true for the excluded case; i.e., n even and k odd with $n > N(k)$. Moreover we modify the error bound in the asymptotic expansion of $M_k(n)$, obtained by Chern. We also present an unified structure to obtain asymptotic growths up to any order as we please for such alternating sums involving the partition function.

7.1 Positivity of alternating sums involving the partition function

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\pi_1, \pi_2, \ldots, \pi_r$ such that $\sum_{i=1}^{r} \pi_i = n$. The partition $(\pi_1, \pi_2, \ldots, \pi_r)$ will be denoted by π, and we shall write $\pi \vdash n$ to denote that π is a partition of n. The partition function $p(n)$ is the number of partitions of n. Due to Euler, the generating function of $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout the rest of this section, we follow the standard notation for the q-shifted factorial

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

One of the more well known results in the theory of partitions is Euler's pentagonal number theorem [8, Equation (1.3.1)] which states that

$$(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k+1)/2}. \tag{7.1}$$

Applying the principle of mathematical induction and q-binomial theorem, Andrews and Merca [9] showed that the truncation of (7.1) has nonnegative coefficients.

Theorem 7.1.1. *[9, Theorem 1.1] For $n > 0$, $k \geq 1$,*

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \Big(p(n - j(3j+1)/2) - p(n - j(3j+5)/2 - 1) \Big) = M_k(n), \tag{7.2}$$

where $M_k(n)$ is the number of partitions of n in which k is the least integer that is not a part and there are more parts $> k$ than there are $< k$.

As a corollary of Theorem 7.1.1, they proved that $M_k(n) \geq 0$ with strict inequality for $n \geq k(3k+1)/2$, see [9, Corollary 1.3]. Yee [154] gave a combinatorial proof of Theorem 7.1.1. Burnette and Kolitsch [89, 90] gave combinatorial interpretation for $M_k(n)$ using partition pairs. In [147], Wang explained $M_k(n)$ as the difference between size of two sets of partitions based on its rank enumeration. An asymptotic estimation for $M_k(n)$ was given by Chern [40] using Wright's circle method.

Theorem 7.1.2. *[40, Theorem 1.1] Let $\epsilon > 0$ be arbitrarily small. Then as $n \to \infty$, we have, for $k \ll n^{1/8-\epsilon}$,*

$$M_k(n) = \frac{\pi}{12\sqrt{2}} k n^{-3/2} e^{2\pi\sqrt{n}/\sqrt{6}} + O\Big(k^3 n^{-7/4} e^{2\pi\sqrt{n}/\sqrt{6}}\Big). \tag{7.3}$$

Applying an extended version of Bailey's transform, Bachraoui [59, Corollary 1 and 2] obtained the following two inequalities for the partition function in the spirit of Andrews and Merca.

Apart from Euler's pentagonal number theorem, the following is another classical theta identity [8, Equation (2.2.13)] due to Gauss (or sometimes Jacobi):

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \sum_{j=0}^{\infty} (-q)^{j(j+1)/2}. \tag{7.4}$$

Starting from Rogers-Fine identity, Andrews and Merca [10] retrieved Theorem 7.1.1 and studying the truncated version of (7.4), obtained the following result.

Theorem 7.1.3. *[10, Theorem 1.9] For $n, k \geq 1$,*

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{j=0}^{2k-1} (-q)^{j(j+1)/2} = 1 - (-1)^k \frac{(-q;q^2)_k}{(q^2;q^2)_k} \sum_{j=0}^{\infty} \frac{q^{k(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2k+2j+2};q^2)_\infty}. \tag{7.5}$$

Consequently, they proved the following infinite family of inequalities for the partition function.

Corollary 7.1.4. *[10, Corollary 11] If at least one of n and k is odd,*

$$\widetilde{M}_k(n) := (-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \Big(p(n - j(2j+1)) - p(n - (j+1)(2j+1))\Big) \geq 0. \tag{7.6}$$

Ballantine, Merca, Passary, and Yee [12, Theorem 3] gave a combinatorial interpretation for $\widetilde{M}_k(n)$ in term of overpartitions. Andrews and Merca proposed the following conjecture with regards to $M_k(n)$ and $\widetilde{M}_k(n)$.

Conjecture 7.1.5. (*Andrews-Merca*)*[10] For n odd or k even,*

$$M_k(n) \geq \widetilde{M}_k(n). \tag{7.7}$$

In [108, Theorem 1.2], Merca and Katriel studied a family of non-trivial homogeneous partition inequalities from the framework of Prouhet-Tarry-Escott problem [55, Chapter XXIV] that arises in Diophantine equations. Using this set up, they proved that Conjecture 7.1.5 is true for k odd and for sufficiently large n.

The main motivation of this chapter is to derive asymptotic growth of the aforementioned alternating sums involving the partition function. We construct an unified framework by employing the infinite family of inequalities obtained by the first author [16, Theorem 4.5] so as to get the desired asymptotic growth. Of course, the inequalities presented before are much stronger in the sense that it predicts the exact threshold,say $N(k)$ for n from which the inequality holds. For example, in context of Theorem 7.1.1, we already know that $M_k(n) > 0$ for all $n \geq k(3k+1)/2$ but here our goal is to get to the asymptotic growth. Nonetheless, we also derive an explicit threshold for n which is higher than the optimal one. Studies on truncated theta series identities already unfolded the combinatorial facets through the jargon of partitions, whereas in this chapter, we unearth the other facet of such problems by studying asymptotic analysis for the partition function.

Asymptotic analysis for the partition function had begun with the work of Hardy and Ramanujan [76] in 1918 that reads:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty. \tag{7.8}$$

Rademacher [123] improved the work of Hardy and Ramanujan by providing a convergent series for $p(n)$ and Lehmer [98] estimated the remainder term of the convergent series for $p(n)$. The Hardy-Ramanujan-Rademacher formula states that

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] + R_2(n, N), \tag{7.9}$$

where

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \quad A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i s(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \left\lfloor \frac{\mu}{k} \right\rfloor - \frac{1}{2} \right) \left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2} \right),$$

and

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)} \right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)} \right)^2 \right]. \tag{7.10}$$

After Rademacher's work on the partition function, numerous research papers have been written on inequalities for the partition function. Recently Paule, Radu, Schneider and the first author [21] obtained a full asymptotic expansion of $p(n)$ along with estimations of error bounds. Based on their work, an infinite family of inequalities for shifted partition function $p(n-\ell)$ for $\ell \geq 0$ is given in [16, Theorem 4.5] which is the key machinery in proving all of the theorems stated below.

Theorem 7.1.6. *Define for all $k \geq 1$,*

$$\mathcal{M}_k^1(n) := \frac{\pi k}{\sqrt{6n}} + \frac{k^3}{144n}\left(-36\pi^2 + \frac{23\pi^2 - 216}{k^2}\right) + \frac{k^5}{6912\sqrt{6}\pi n^{3/2}} \times \left(1296\pi^4 + \frac{31104\pi^2 - 2760\pi^4}{k^2} + \frac{31104 - 19872\pi^2 + 1681\pi^4}{k^4}\right).$$

Then for all $n > 121k^4$,

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^1(n) + \frac{\mathcal{E}_L^1(k)}{n^2}\Big) < M_k(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^1(n) + \frac{\mathcal{E}_U^1(k)}{n^2}\Big). \tag{7.11}$$

Explicit expressions for $\mathcal{E}_L^1(k)$ and $\mathcal{E}_U^1(k)$ are given in (7.31) and (7.34) respectively for k odd and even.

Corollary 7.1.7. *For $k \geq 1$ and $n > 121k^4$, as $n \to \infty$,*

$$M_k(n) \sim \frac{\pi e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}}k + \frac{e^{\pi\sqrt{2n/3}}}{576\sqrt{3}n^2}k^3\Big(\frac{23\pi^2 - 216}{k^2} - 36\pi^2\Big). \tag{7.12}$$

Remark 7.1.8. *Rewriting the asymptotic expansion (7.12) of $M_k(n)$ in the following way:*

$$M_k(n) \sim \frac{\pi e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}}k + O\Big(\frac{e^{\pi\sqrt{2n/3}}}{n^2}k^3\Big) \quad as \quad n \to \infty,$$

we observe that the growth of error bound is in indeed the optimal one in comparison with Theorem 7.1.2.

Remark 7.1.9. *From the lower bound in (7.11), one can retrieve positivity of $M_k(n)$ for $n > f_1(k)$ with minimal $f_1(k)$ such that $\mathcal{M}_k^1(n) + \frac{\mathcal{E}_L^1(k)}{n^2} > 0$ holds for all $n > f_1(k)$.*

Theorem 7.1.10. *Define for all $k \geq 1$,*

$$\mathcal{M}_k^2(n) := \frac{\pi k}{\sqrt{6n}} + \frac{k^3}{144n}\left(-48\pi^2 + \frac{35\pi^2 - 216}{k^2}\right) + \frac{k^5}{6912\sqrt{6}\pi n^{3/2}} \times \left(2304\pi^4 + \frac{41472\pi^2 - 5472\pi^4}{k^2} + \frac{31104 - 30240\pi^2 + 3385\pi^4}{k^4}\right).$$

Then for all $n > 169k^4$,

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^2(n) + \frac{\mathcal{E}_L^2(k)}{n^2}\Big) < \widetilde{M}_k(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^2(n) + \frac{\mathcal{E}_U^2(k)}{n^2}\Big). \tag{7.13}$$

Explicit expressions for $\mathcal{E}_L^2(k)$ and $\mathcal{E}_U^2(k)$ are given in (7.46) and (7.49) respectively for k odd and even.

Corollary 7.1.11. *For $k \geq 1$ and $n > 169k^4$, as $n \to \infty$,*

$$\widetilde{M}_k(n) \sim \frac{\pi e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}}k + \frac{e^{\pi\sqrt{2n/3}}}{576\sqrt{3}n^2}k^3\Big(\frac{35\pi^2 - 216}{k^2} - 48\pi^2\Big). \tag{7.14}$$

Remark 7.1.12. *Similar to Remark 7.1.16, from the lower bound in (7.13), one can prove positivity of $\widetilde{M}_k(n)$ for $n > f_2(k)$ such that $\mathcal{M}_k^2(n) + \dfrac{\mathcal{E}_L^2(k)}{n^2} > 0$ holds for all $n > f_2(k)$.*

Theorem 7.1.13. *Define for all $k \geq 1$,*

$$\begin{aligned}\mathcal{M}_k^3(n) &:= \mathcal{M}_k^1(n) - \mathcal{M}_k^2(n) \\ &= \frac{k^3 - k}{12n} - \frac{k^5}{6912\sqrt{6}\pi n^{3/2}}\left(1008\pi^4 + \frac{10368\pi^2 - 2712\pi^4}{k^2} + \frac{-10368\pi^2 + 1704\pi^4}{k^4}\right).\end{aligned}$$

Then for all $n > 169k^4$,

$$\begin{aligned}\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^3(n) + \frac{\mathcal{E}_L^1(k) - \mathcal{E}_U^2(k)}{n^2}\Big) &< M_k(n) - \widetilde{M}_k(n) < \\ &\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\Big(\mathcal{M}_k^3(n) + \frac{\mathcal{E}_U^1(k) - \mathcal{E}_L^2(k)}{n^2}\Big).\end{aligned} \tag{7.15}$$

Proof. Theorems 7.1.6 and 7.1.10 immediately imply (7.15). □

Corollary 7.1.14. *For $k \geq 1$ and $n > 169k^4$, as $n \to \infty$,*

$$M_k(n) - \widetilde{M}_k(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{48\sqrt{3}n^2}(k^3 - k). \tag{7.16}$$

Remark 7.1.15. *Proving $M_k(n) > \widetilde{M}_k(n)$ for $n \geq N(k)$, it is enough to show that $\mathcal{M}_k^3(n) + \dfrac{\mathcal{E}_L^1(k) - \mathcal{E}_U^2(k)}{n^2} > 0$ holds for all $n \geq N(k)$.*

Remark 7.1.16. *Note that for $k = 1$, $M_k(n) - \widetilde{M}_k(n) = 0$ because $M_k(n) = \widetilde{M}_k(n) = p(n) - p(n-1)$, whereas for all $k \geq 2$, (7.15) suggests that $M_k(n) - \widetilde{M}_k(n)$ is positive for $n \geq N(k)$. This observation helps us to relax the condition given in Conjecture 7.1.5; i.e., instead of restricting to either n odd or k even, we can assume for all n and k with $n \geq N(k)$ that subsumes the excluded case k odd and n even. Still it is worthwhile to point out that whenever we consider n odd or k even, (7.7) is true for all $n \geq 1$ and $k \geq 1$. But when we assume the case k odd and n even, (7.7) doesn't hold for all $n, k \geq 1$, in other words, it remains to determine the optimal $N(k)$.*

By numerical verification with Mathematica, we listed down the values of $(N(k))_{1 \leq k \leq 20}$ such that $M_{2k+1}(2n) > \widetilde{M}_{2k+1}(2n)$ for all $n \geq N(k)$.

k	1	2	3	4	5	6	7	8	9	10
$N(k)$	11	28	54	88	129	179	237	303	376	458

k	11	12	13	14	15	16	17	18	19	20
$N(k)$	548	646	752	866	988	1118	1256	1402	1558	1719

Based on the above data, a rough estimation predicts that as k become larger,

$$N(k) \approx \left\lfloor 4k^2 + 7k - \sqrt{k}\log k \right\rfloor - \left\lfloor \frac{k}{3} \right\rfloor := N_c(k).$$

Table of $N_c(k)$ is as follows:

k	1	2	3	4	5	6	7	8	9	10
$N_c(k)$	11	29	54	88	130	179	237	304	377	459

k	11	12	13	14	15	16	17	18	19	20
$N_c(k)$	550	647	753	868	989	1119	1258	1403	1558	1720

Extending the assumption of Conjecture 7.1.5, we propose the following question:

Problem 7.1.17. *For all $k \geq 1$ and $n \geq N_c(k)$, does the following inequality*

$$M_{2k+1}(2n) > \widetilde{M}_{2k+1}(2n) \tag{7.17}$$

hold?

The rest of the chapter is organized as follows. In Section 7.2, we give all the necessary definitions and inequalities for $p(n-\ell)$ for all $\ell \geq 0$ (see Theorem 7.2.5 below) so as to ease to follow the later section. Section 7.3 presents the proofs of Theorems 7.1.6 and 7.1.10.

7.2 Preliminaries

First, we shall recall a few definitions from [16] which will be useful in the estimations worked out in Section 7.3.

Definition 7.2.1. *Following [16, Theorem 3.2], for $k \in \mathbb{Z}_{\geq 2}$, we define*

$$\widehat{g}(k) := \frac{1}{24}\left(\frac{36}{\pi^2} \cdot \nu(k)^2 + 1\right), \tag{7.18}$$

where $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}.$

Definition 7.2.2. *[16, Definition 3.42] For all $k \geq 1$ and $\ell \geq 0$, define*

$$n_0(k,\ell) = \max_{k\geq 1, \ell \geq 0}\left\{\frac{(24\ell+1)^2}{16}, \frac{(k+3)(24\ell+1)}{24}\right\}.$$

Definition 7.2.3. *[16, Equation (3.45)] For all $\ell \geq 0$ and $t \geq 0$,*

$$g(t,\ell) = \frac{(1+24\ell)^t}{(-4\sqrt{6})^t}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}\frac{1}{(1+24\ell)^k}. \tag{7.19}$$

Definition 7.2.4. *[16, Definition 4.4] Let $g(t,\ell)$ be as in (7.2.1). If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$, define*

$$\mathcal{L}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(w,\ell)}{\sqrt{n}^w} \text{ and } \mathcal{U}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(w,\ell)}{\sqrt{n}^w}.$$

The explicit expressions for $L(w,\ell)$ and $U(w,\ell)$ are given in [16, Definition 4.1].

Theorem 7.2.5. *[16, Theorem 4.5] For $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$ and $n > \max\{\widehat{g}(w)+\ell, n_0(w,\ell)\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{U}_n(w,\ell). \tag{7.20}$$

7.3 Proof of Theorems 7.1.6-7.1.10

Proof of Theorem 7.1.6: Let $k \geq 1$ be an odd integer. Following (7.2), we write

$$M_{2k+1}(n) = M^e_{2k+1}(n) - M^o_{2k+1}(n), \tag{7.21}$$

where

$$M^e_{2k+1}(n) = \sum_{j=0}^{k}\Big(p(n-j(6j+1)) - p(n-j(6j+5)-1)\Big)$$

and

$$M^o_{2k+1}(n) = \sum_{j=0}^{k-1}\Big(p(n-(2j+1)(3j+2)) - p(n-(2j+1)(3j+4)-1)\Big).$$

Applying Theorem 7.2.5 with $w=4$, we obtain

$$M^e_{2k+1}(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k}\sum_{t=0}^{3}\Big(g(t,j(6j+1)) - g(t,j(6j+5)+1)\Big)\frac{1}{\sqrt{n}^t} + \frac{U^e_1(2k+1)}{n^2}\right) \tag{7.22}$$

and

$$M^e_{2k+1}(n) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{j=0}^{k}\sum_{t=0}^{3}\Big(g(t,j(6j+1)) - g(t,j(6j+5)+1)\Big)\frac{1}{\sqrt{n}^t} + \frac{L^e_1(2k+1)}{n^2}\right), \tag{7.23}$$

with

$$L_1^e(2k+1) = \sum_{j=0}^{k} L\big(4, j(6j+1)\big) - U\big(4, j(6j+5)+1\big) \tag{7.24}$$

and

$$U_1^e(2k+1) = \sum_{j=0}^{k} U\big(4, j(6j+1)\big) - L\big(4, j(6j+5)+1\big) \tag{7.25}$$

Analogously, for $M_{2k+1}^o(n)$, we get

$$M_{2k+1}^o(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \times \left(\sum_{j=0}^{k-1}\sum_{t=0}^{3}\Big(g(t,(2j+1)(3j+2)) - g(t,(2j+1)(3j+4)+1)\Big)\frac{1}{\sqrt{n}^t} + \frac{U_1^o(2k+1)}{n^2}\right) \tag{7.26}$$

and

$$M_{2k+1}^o(n) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \times \left(\sum_{j=0}^{k-1}\sum_{t=0}^{3}\Big(g(t,(2j+1)(3j+2)) - g(t,(2j+1)(3j+4)+1)\Big)\frac{1}{\sqrt{n}^t} + \frac{L_1^o(2k+1)}{n^2}\right), \tag{7.27}$$

with

$$L_1^o(2k+1) = \sum_{j=0}^{k-1} L\big(4, (2j+1)(3j+2)\big) - U\big(4, (2j+1)(3j+4)+1\big) \tag{7.28}$$

and

$$U_1^o(2k+1) = \sum_{j=0}^{k-1} U\big(4, (2j+1)(3j+2)\big) - L\big(4, (2j+1)(3j+4)+1\big). \tag{7.29}$$

Combining (7.22)-(7.29) and applying to (7.21), if follows that

$$\frac{\mathcal{E}_L^1(2k+1)}{n^2} < \frac{M_{2k+1}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} - \sum_{j=0}^{2k}(-1)^j \sum_{t=0}^{3} \frac{g(t,j(3j+1)/2) - g(t,j(3j+5)/2+1)}{\sqrt{n}^t} < \frac{\mathcal{E}_U^1(2k+1)}{n^2}, \tag{7.30}$$

with

$$\mathcal{E}_L^1(2k+1) = L_1^e(2k+1) - U_1^o(2k+1) \text{ and } \mathcal{E}_U^1(2k+1) = U_1^e(2k+1) - L_1^o(2k+1). \tag{7.31}$$

Next assume $k \geq 1$ is even. We split $M_{2k}(n)$ as follows:

$$M_{2k}(n) = -M_{2k}^e(n) + M_{2k}^o(n), \tag{7.32}$$

with

$$M_{2k}^e(n) = \sum_{j=0}^{k-1}\Big(p(n-j(6j+1)) - p(n-j(6j+5)-1)\Big)$$

and

$$M_{2k}^o(n) = \sum_{j=0}^{k-1}\Big(p(n-(2j+1)(3j+2)) - p(n-(2j+1)(3j+4)-1)\Big).$$

Applying (7.20) separately to $M_{2k}^e(n)$ and $M_{2k}^o(n)$, we get

$$\frac{\mathcal{E}_L^1(2k)}{n^2} < \frac{M_{2k}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} + \sum_{j=0}^{2k-1}(-1)^j \sum_{t=0}^{3} \frac{g(t,j(3j+1)/2) - g(t,j(3j+5)/2+1)}{\sqrt{n}^t} < \frac{\mathcal{E}_U^1(2k)}{n^2}, \tag{7.33}$$

where

$$\mathcal{E}_L^1(2k) = L_1^o(2k) - U_1^e(2k) \text{ and } \mathcal{E}_U^1(2k) = U_1^o(2k) - L_1^e(2k), \tag{7.34}$$

with

$$\begin{aligned}
L_1^e(2k) &= \sum_{j=0}^{k-1} L(4, j(6j+1)) - U(4, j(6j+5)+1) \\
U_1^e(2k) &= \sum_{j=0}^{k-1} U(4, j(6j+1)) - L(4, j(6j+5)+1) \\
L_1^o(2k) &= \sum_{j=0}^{k-1} L(4, (2j+1)(3j+2)) - U(4, (2j+1)(3j+4)+1) \\
U_1^o(2k) &= \sum_{j=0}^{k-1} U(4, (2j+1)(3j+2)) - L(4, (2j+1)(3j+4)+1).
\end{aligned}$$

Define $n_1(k) := \max\Big\{\widehat{g}(4) + (k-1)(3k+5)/2 + 1, n_0\Big(4, (k-1)(3k+5)/2+1\Big)\Big\}$. Putting (7.30) and (7.33) together, for all $n > n_1(k)$, it follows that

$$\frac{\mathcal{E}_L^1(k)}{n^2} < \frac{M_k(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} - (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j \sum_{t=0}^{3} \frac{g(t, j(3j+1)/2) - g(t, j(3j+5)/2+1)}{\sqrt{n}^t} < \frac{\mathcal{E}_U^1(k)}{n^2}. \tag{7.35}$$

Following (7.19), we get

$$\begin{aligned}
&(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j \sum_{t=0}^{3} \frac{g(t, j(3j+1)/2) - g(t, j(3j+5)/2+1)}{\sqrt{n}^t} \\
&= \frac{\pi k}{\sqrt{6n}} + \frac{k^3}{144n}\left(-36\pi^2 + \frac{23\pi^2 - 216}{k^2}\right) + \frac{k^5}{6912\sqrt{6}\pi n^{3/2}} \times \\
&\qquad \left(1296\pi^4 + \frac{31104\pi^2 - 2760\pi^4}{k^2} + \frac{31104 - 19872\pi^2 + 1681\pi^4}{k^4}\right) \\
&= \mathcal{M}_k^1(n).
\end{aligned} \tag{7.36}$$

Finally, it is easy to verify that for all $k \geq 1$,

$$n_1(k) \leq 121k^4.$$

This finishes the proof of Theorem 7.1.6. □

Proof of Theorem 7.1.10: Assume $k \geq 1$ is odd. Following (7.6), rewrite

$$\widetilde{M}_{2k+1}(n) = \widetilde{M}^e_{2k+1}(n) - \widetilde{M}^o_{2k+1}(n), \tag{7.37}$$

where

$$\widetilde{M}^e_{2k+1}(n) = \sum_{j=0}^{k} \Big(p(n - 2j(4j+1)) - p(n - (2j+1)(4j+1)) \Big)$$

and

$$\widetilde{M}^o_{2k+1}(n) = \sum_{j=0}^{k-1} \Big(p(n - (2j+1)(4j+3)) - p(n - (2j+2)(4j+3)) \Big).$$

Applying Theorem 7.2.5 with $w = 4$, it follows that

$$\widetilde{M}^e_{2k+1}(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(\sum_{j=0}^{k} \sum_{t=0}^{3} \Big(g(t, 2j(4j+1)) - g(t, (2j+1)(4j+1)) \Big) \frac{1}{\sqrt{n}^t} + \frac{U^e_2(2k+1)}{n^2} \right) \tag{7.38}$$

and

$$\widetilde{M}^e_{2k+1}(n) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(\sum_{j=0}^{k} \sum_{t=0}^{3} \Big(g(t, 2j(4j+1)) - g(t, (2j+1)(4j+1)) \Big) \frac{1}{\sqrt{n}^t} + \frac{L^e_2(2k+1)}{n^2} \right), \tag{7.39}$$

with

$$L^e_2(2k+1) = \sum_{j=0}^{k} L\big(4, 2j(4j+1)\big) - U\big(4, (2j+1)(4j+1)\big) \tag{7.40}$$

and

$$U^e_2(2k+1) = \sum_{j=0}^{k} U\big(4, 2j(4j+1)\big) - L\big(4, (2j+1)(4j+1)\big) \tag{7.41}$$

Similarly for $\widetilde{M}^o_{2k+1}(n)$, we obtain

$$\widetilde{M}^o_{2k+1}(n) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \times \left(\sum_{j=0}^{k-1} \sum_{t=0}^{3} \Big(g(t, (2j+1)(4j+3)) - g(t, (2j+2)(4j+3)) \Big) \frac{1}{\sqrt{n}^t} + \frac{U^o_2(2k+1)}{n^2} \right) \tag{7.42}$$

and

$$\widetilde{M}^o_{2k+1}(n) > \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \times \left(\sum_{j=0}^{k-1} \sum_{t=0}^{3} \Big(g(t,(2j+1)(4j+3)) - g(t,(2j+2)(4j+3)) \Big) \frac{1}{\sqrt{n}^t} + \frac{L^o_2(2k+1)}{n^2} \right),$$

with

$$L^o_2(2k+1) = \sum_{j=0}^{k-1} L\big(4,(2j+1)(4j+3)\big) - U\big(4,(2j+2)(4j+3)\big) \tag{7.43}$$

and

$$U^o_2(2k+1) = \sum_{j=0}^{k-1} U\big(4,(2j+1)(4j+3)\big) - L\big(4,(2j+2)(4j+3)\big). \tag{7.44}$$

Applying (7.38)-(7.44) to (7.37), if follows that

$$\frac{\mathcal{E}^2_L(2k+1)}{n^2} < \frac{\widetilde{M}_{2k+1}(n)}{\Big(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\Big)} - \sum_{j=0}^{2k} (-1)^j \sum_{t=0}^{3} \frac{g(t,j(2j+1)) - g(t,(j+1)(2j+1))}{\sqrt{n}^t} < \frac{\mathcal{E}^2_U(2k+1)}{n^2}, \tag{7.45}$$

with

$$\mathcal{E}^2_L(2k+1) = L^e_2(2k+1) - U^o_2(2k+1) \text{ and } \mathcal{E}^2_U(2k+1) = U^e_2(2k+1) - L^o_2(2k+1). \tag{7.46}$$

Now assume $k \geq 1$ is even. Split $\widetilde{M}_{2k}(n)$ as follows:

$$\widetilde{M}_{2k}(n) = -\widetilde{M}^e_{2k}(n) + \widetilde{M}^o_{2k}(n), \tag{7.47}$$

with

$$\widetilde{M}^e_{2k}(n) = \sum_{j=0}^{k-1} \Big(p(n - 2j(4j+1)) - p(n - (2j+1)(4j+1)) \Big)$$

and

$$\widetilde{M}^{o}_{2k}(n) = \sum_{j=0}^{k-1}\Big(p(n-(2j+1)(4j+3)) - p(n-(2j+2)(4j+3))\Big).$$

Applying (7.20) to $\widetilde{M}^{e}_{2k}(n)$ and $\widetilde{M}^{o}_{2k}(n)$, it follows that

$$\begin{aligned}\frac{\mathcal{E}^2_L(2k)}{n^2} <& \frac{\widetilde{M}_{2k}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} + \\ & \sum_{j=0}^{2k-1}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(2j+1)) - g(t,(j+1)(2j+1))}{\sqrt{n}^t} < \frac{\mathcal{E}^2_U(2k)}{n^2},\end{aligned} \tag{7.48}$$

where

$$\mathcal{E}^2_L(2k) = L^o_2(2k) - U^e_2(2k) \ \text{ and } \ \mathcal{E}^2_U(2k) = U^o_2(2k) - L^e_2(2k), \tag{7.49}$$

with

$$\begin{aligned} L^e_2(2k) &= \sum_{j=0}^{k-1} L(4,2j(4j+1)) - U(4,(2j+1)(4j+1)) \\ U^e_2(2k) &= \sum_{j=0}^{k-1} U(4,2j(4j+1)) - L(4,(2j+1)(4j+1)) \\ L^o_2(2k) &= \sum_{j=0}^{k-1} L(4,(2j+1)(4j+3)) - U(4,(2j+2)(4j+3)) \\ U^o_2(2k) &= \sum_{j=0}^{k-1} U(4,(2j+1)(4j+3)) - L(4,(2j+2)(4j+3)). \end{aligned}$$

Define $n_2(k) := \max\Big\{\widehat{g}(4) + k(2k-1), n_0\Big(4, k(2k-1)\Big)\Big\}$. Combining (7.45) and (7.48), for all $n > n_2(k)$, it follows that

$$\begin{aligned}\frac{\mathcal{E}^2_L(k)}{n^2} <& \frac{\widetilde{M}_{k}(n)}{\left(e^{\pi\sqrt{2n/3}}/4n\sqrt{3}\right)} - \\ & (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(2j+1)) - g(t,(j+1)(2j+1))}{\sqrt{n}^t} < \frac{\mathcal{E}^2_U(k)}{n^2}.\end{aligned} \tag{7.50}$$

Following (7.19), we have

$$
\begin{aligned}
&(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\sum_{t=0}^{3}\frac{g(t,j(2j+1))-g(t,(j+1)(2j+1))}{\sqrt{n}^t}\\
&=\frac{\pi k}{\sqrt{6n}}+\frac{k^3}{144n}\left(-48\pi^2+\frac{35\pi^2-216}{k^2}\right)+\frac{k^5}{6912\sqrt{6}\pi n^{3/2}}\times \\
&\qquad\left(2304\pi^4+\frac{41472\pi^2-5472\pi^4}{k^2}+\frac{31104-30240\pi^2+3385\pi^4}{k^4}\right)\\
&=\mathcal{M}_k^2(n).
\end{aligned}
\tag{7.51}
$$

We conclude the proof of Theorem 7.1.10 by verifying that for all $k \geq 1$, $n_2(k) \leq 169k^4$. □

7.4 Conclusion

We conclude this chapter by noting down a few possible follow ups.

1. Extending the inequality (7.35) (resp. (7.50)) by letting $w \to \infty$, we obtain the full asymptotic expansion of $M_k(n)$ (resp. of $\widetilde{M}_k(n)$).

2. We observe that all of the aforementioned inequalities with regard to the alternating sums for the partition function can be considered under the following framework:
$$\sum_{i=1}^{T} p(n+s_i) \geq \sum_{i=1}^{T} p(n+r_i),$$
where s_i, r_i are non-positive integers for all $1 \leq i \leq T$. In order to prove such inequalities, it is enough to choose the appropriate w in Theorem 7.2.5 and carry out similar work as done in Section 7.3. For the choice of w, it suffices to take the minimal $w_0 \geq 1$ such that $\displaystyle\sum_{i=1}^{T} g(w_0, s_i) - g(w_0, r_i) \neq 0$, where $g(t,\ell)$ as in (7.19).

3. Wang and Yee [146, Theorem 1.2] considered the sum representation of $(q;q)_\infty^2$ due to Hecke and showed positivity of the following alternating sum in the

2-colored partition function (denoted by $pp(n)$):

$$(-1)^m \sum_{n=0}^{m} \sum_{j=-n}^{n} (-1)^j \Big(pp(N_j - n(2n+1)) - pp(N_j - (n+1)(2n+1)) \Big), \quad (7.52)$$

where $N_j = N + j(3j+1)/2$. Recently Bringmann et. al. [32] studied the asymptotic expansion of k-colored partition function. Setting $k = 2$, one has the asymptotic expansion for $pp(n)$ and working out to derive the infinite family of inequalities for $pp(n-\ell)$ as in Theorem 7.2.5 which in turn finally show the asymptotic growth of (7.52). Whereas for $k = 3$, similar synthesis for the 3-colored partitions can be done to derive the asymptotic growth of

$$J_k(n) = (-1)^k \sum_{j=0}^{k} (-1)^j (2j+1) t\Big(n - j(j+1)/2\Big),$$

given in [10].

Chapter 8

Error bounds for the modified Bessel function of first kind of non-negative order

Let $I_\nu(x)$ be the modified Bessel function of order ν with real argument x. We present explicit error bounds for the asymptotic expansion of $I_\nu(x)$ with $x \geq 1$. Two cases, ν an integer and ν a half-integer are considered separately. In addition, we present a short discussion on the error analysis for $I_\nu(x)$ where ν is any non-negative real number.

8.1 Asymptotic expansion of $I_\nu(x)$ and scope of its applications

Consider Bessel's differential equation over the complex domain,

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0, \tag{8.1}$$

where ν is an arbitrary complex parameter. The solutions of this equation are termed as Bessel functions. In 1824, F. W. Bessel [24, 25] initiated a systematic rigorous analysis of such functions which was the starting point of a flourishing development along with a multitude of applications in connection with problems in number theory, integral transforms, differential equations, etc. The main object of this chapter is $I_\nu(z)$, a solution of

$$z^2y'' + zy' - (z^2 + \nu^2)y = 0, \tag{8.2}$$

the so-called modified version of (8.1), with series representation

$$I_\nu(z)=\sum_{m=0}^{\infty}\frac{(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{8.3}$$

In 1854 Kirchhoff [86] established an asymptotic expansion of $I_\nu(z)$: for fixed $\nu\in\mathbb{C}$,

$$I_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\left(1-\frac{4\nu^2-1}{8z}+\frac{(4\nu^2-1)(4\nu^2-9)}{2!(8z)^2}-\dots\right),\ |\arg z|<\frac{\pi}{2}.$$

Estimates for the error terms of asymptotic expansions of Bessel functions have been considered by Schläfli [126], Weber [151], Watson [150, p. 209-210], Meijer [107], Olver [116], Nemes [110], to name a few. For a more extensive study on the literature of the Bessel functions, we refer to [150].

This chapter focuses on deriving a family of inequalities for $I_\nu(x)$ with ν a non-negative integer or a half-integer, and x a real number ≥ 1. Why it is necessary to get such inequalities for $I_\nu(x)$? We have already mentioned that the theory of Bessel functions often sprout out in problems related to number theory. For example, $I_\nu(x)$ with non-negative integral or half-integral order ν appears in Hardy-Ramanujan-Rademacher type series expansions for coefficients of certain classes of Dedekind eta quotients; see for example [41, Thm. 1.1] or [138, Thm. 1.1]. These coefficients are quite often entangled with combinatorial features that emerge from the question whether a real polynomial associated with such sequences has roots all real. For example, consider the Jensen polynomial of degree d and shift n for a sequence $\{\alpha(n)\}_{n\geq 0}$ of real numbers, defined as

$$J_\alpha^{d,n}(x)=\sum_{j=0}^{d}\binom{d}{j}\alpha(n+j)x^j.$$

Now, to prove log-concavity (resp. higher order Turán inequalities) of $\alpha(n)$, it is equivalent to prove that $J_\alpha^{2,n}(x)$ (resp. $J_\alpha^{d,n}(x)$ for $d\geq 3$) has roots all real for all $n>N(d)$ where $N(d)$ is a positive real number depending on the degree d.

To answer these problems for a sequence, say $a_f(n)$, arising from the Fourier expansion of a periodic meromorphic function, say a Dedekind eta quotient $f(q)$, we would like to estimate $a_f(n)$ by computing a precise estimation of the associated Hardy-Ramanujan-Rademacher type series, say S_f. Now, in order to provide such a precise estimate for the main term obtained after truncating the series S_f to a finite number of terms, inequalities for $I_{\nu(f)}(x)$ are needed, where the index $\nu(f)$ is depending on f. For instance, Griffin, Ono, Rolen and Zagier [69] proved the following theorem.

Theorem 8.1.1. *[69, eq. 9] Let $\{a_f(n)\}_{n\geq 0}$ be a sequence of positive real numbers arising from the Fourier expansion of a periodic meromorphic function f. Suppose*

$$a_f(n) = A_f n^{\frac{k-1}{2}} I_{k-1}(4\pi\sqrt{mn}) + O\Big(n^C e^{2\pi\sqrt{mn}}\Big)$$

as $n \to \infty$ for some non-zero real constants $A_f, m,\ k,$ and C, where $I_\nu(x)$ is the modified Bessel function of the first kind of order ν. Then for $d \geq 1$, the Jensen polynomial $J^{d,n}_{a_f}(x)$ associated to $a_f(n)$ has only real roots for all sufficiently large n.

A concrete example with regard to log-concavity is this. In order to prove log-concavity of the colored partition function $p_k(n)$, conjectured by Chern, Fu and Tang [43, Conjecture 5.3], Bringmann et. al. estimated the error term by truncation of the asymptotic expansion of $I_\nu(x)$ at $N = 3$, which plays a key role in their proof of the conjecture [32, Conjecture 1]:

Theorem 8.1.2. *[32, Lemma 2.2 (4)] For $\nu \geq 2$ and $x \geq \frac{1}{120}(\nu + \frac{7}{2})^6$,*

$$\left|\frac{I_\nu(x)\sqrt{2\pi x}}{e^x} - 1 + \frac{4\nu^2-1}{8x} - \frac{(4\nu^2-1)(4\nu^2-9)}{128x^2} + \frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3072x^3}\right| < \frac{31\nu^8}{6x^4}. \tag{8.4}$$

Theorems 8.1.1 and 8.1.2 motivated us to study the inequalities for $I_\nu(x)$ by extending the truncation point to any positive integer N and estimating an error bound.

This chapter is organized in the following way. First we will give some basic notations and definitions which we use throughout the chapter. Section 8.2 presents lemmas, useful for the proofs given in later sections, followed by a brief illustration of the key features they possess. In Sections 8.3 and 8.4 we will discuss the method devised. Section 8.3 (resp. Section 8.4) presents the estimation of the error term of the asymptotic expansion of $I_\nu(x)$ with $\nu \in \mathbb{Z}_{\geq 0}$ (resp. $\nu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$), and derives Theorem 8.3.9 with the Corollary 8.3.10 (resp. Theorem 8.4.6). Section 8.5 is devoted to the study of the error analysis for any non-negative real index ν. The Appendix, Section 8.6, is divided into two subsections: Subsection 8.6.1 presents the proofs of three lemmas (from Section 8.2) and in Subsection 8.6.2 a Mathematica computation is presented which is needed for the completion of the proof of Corollary 8.3.10.

Let $a^{\underline{k}}$ denote the falling factorial,

$$a^{\underline{k}} = \begin{cases} a(a-1)\dots(a-k+1), & \text{if } k \in \mathbb{Z}_{>0} \\ 1, & \text{if } k = 0 \end{cases},$$

and the binomial coefficient is defined by $\binom{a}{m} = \frac{a^{\underline{m}}}{m!}$. In this framework, we restrict ourselves to $a \in \mathbb{R}$. Similarly, the rising factorial is defined by

$$a^{\overline{k}} = \begin{cases} a(a+1)\dots(a+k-1), & \text{if } k \in \mathbb{Z}_{>0} \\ 1, & \text{if } k = 0 \end{cases};$$

nevertheless, we mostly prefer to use the classical notation $(a)_k = a^{\overline{k}}$. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, the gamma function is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \tag{8.5}$$

For $\mathrm{Re}(z) \leq 0$, $\Gamma(z)$ is defined by analytic continuation. It is a meromorphic function with no zeros, and with simple poles of residue $(-1)^n/n!$ at $z = -n$ when $n \in \mathbb{Z}_{\geq 0}$. Note that $(a)_n = \Gamma(a+n)/\Gamma(a)$ for $a \notin \mathbb{Z}_{\leq 0}$. For a brief survey on the gamma function, readers may consult [2, Ch. 6.1], [115, Ch. 2.1] and [119]. The incomplete gamma functions $\gamma(a,z)$ and $\Gamma(a,z)$ are defined by

$$\gamma(a,z) \;=\; \int_0^z e^{-t} t^{a-1}\, dt, \quad \mathrm{Re}(a) > 0, \tag{8.6}$$

and

$$\Gamma(a,z) \;=\; \int_z^\infty e^{-t} t^{a-1}\, dt; \tag{8.7}$$

moreover,

$$\gamma(a,z) + \Gamma(a,z) = \Gamma(a), \quad a \notin \mathbb{Z}_{\leq 0}. \tag{8.8}$$

For our purpose, we need to consider $I_\nu(x)$ only for $\nu \in \mathbb{R}_{\geq 2}$ and $x \in \mathbb{R}_{\geq 1}$. To this end, we shall use the following representation of $I_\nu(x)$ [150, Ch. VII, 7.25],

$$I_\nu(x) = \frac{(\frac{x}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi e^{x\cos\theta} \sin^{2\nu}\theta\, d\theta. \tag{8.9}$$

8.2 Preliminary lemmas

This section presents all the preliminary facts needed for the proofs of the lemmas stated in Sections 8.3, 8.4 and 8.5. Lemma 8.2.1 helps us to estimate the integrand in $\gamma(a,x)$ and $\Gamma(a,x)$ for positive real numbers a and x. Using Lemmas 8.2.2 and 8.2.3 identifies the binomial coefficient $\binom{\nu-\frac{1}{2}}{m}$ with the standard binomial coefficients and as a consequence, we obtain an upper bound of the absolute value of $(-1)^m\binom{\nu-\frac{1}{2}}{m}$ in Lemma 8.2.4. The proofs of Lemmas 8.2.1 to 8.2.4 are presented in Subsection 8.6.1. Lemmas 8.2.6 and 8.2.7 illustrate the alternation of sign of the sums (8.14) and (8.15) depending on the parity of N for $\nu \in \mathbb{Z}_{\geq 0}$. Similar results are outlined in Lemmas 8.2.8 and 8.2.9 for $\nu \in \frac{1}{2}+\mathbb{Z}_{\geq 0}$.

Lemma 8.2.1. *For all* $(x,y) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,

$$e^{-x}\ x^y < \frac{1}{\sqrt{2\pi}}\ \Gamma(y)\sqrt{y}. \tag{8.10}$$

Lemma 8.2.2. *For* $(\nu,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,

$$\binom{\nu-\frac{1}{2}}{m} = \begin{cases} \frac{(-1)^{m-\nu}}{4^m}\frac{\binom{2\nu}{\nu}\binom{2m-2\nu}{m-\nu}}{\binom{m}{\nu}}, & \text{if } m > \nu \\ \frac{1}{4^m}\frac{\binom{2\nu}{\nu}\binom{\nu}{m}}{\binom{2\nu-2m}{\nu-m}}, & \text{if } m \leq \nu \end{cases}.$$

Lemma 8.2.3. *For* $\nu \in \mathbb{R}$ *and* $(k,N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,

$$\sum_{m=k}^{N}(-1)^m\binom{\nu-\frac{1}{2}}{m}\binom{m}{k} = 2\ (-1)^{N+1}(N+1)\binom{\nu-\frac{1}{2}}{N+1}\frac{1}{2k-2\nu+1}\binom{N}{k}. \tag{8.11}$$

Lemma 8.2.4. *For* $(\nu,m) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$,

$$\left|(-1)^m\binom{\nu-\frac{1}{2}}{m}\right| \leq \begin{cases} \frac{1}{\pi\sqrt{\nu(m-\nu)}}\frac{1}{\binom{m}{\nu}}, & \text{if } m > \nu \\ \frac{2}{\sqrt{\pi}}\binom{\nu}{m}, & \text{if } m \leq \nu \end{cases}.$$

Lemma 8.2.5. *For* $\alpha \in \mathbb{R}_{>1}$ *and* $(\nu,N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,

$$\sum_{m=0}^{N}\binom{\nu-\frac{1}{2}}{m}(-\alpha)^m = 2\ (-1)^{N+1}(N+1)\binom{\nu-\frac{1}{2}}{N+1}\sum_{m=0}^{N}\binom{N}{m}\frac{(\alpha-1)^m}{2m-2\nu+1}. \tag{8.12}$$

Proof. For $\beta := \alpha - 1$,

$$\begin{aligned}\sum_{m=0}^{N}\binom{\nu-\frac{1}{2}}{m}(-\alpha)^m &= \sum_{m=0}^{N}\sum_{k=0}^{m}(-1)^m\binom{\nu-\frac{1}{2}}{m}\binom{m}{k}\ \beta^k \\ &= \sum_{k=0}^{N}\sum_{m=k}^{N}(-1)^m\binom{\nu-\frac{1}{2}}{m}\binom{m}{k}\ \beta^k. \end{aligned} \tag{8.13}$$

From (8.13), using (8.11), the statement follows. □

Lemma 8.2.6. *For all $\alpha \in \mathbb{R}_{>1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 0}$ with $\nu \geq N+1$,*

$$(-1)^N \sum_{m=0}^{N} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m > 0. \tag{8.14}$$

Proof. Multiplying $(-1)^N$ on both sides of (8.12) and the fact that $\nu \geq N+1$ immediately implies (8.14). □

Lemma 8.2.7. *For all $\alpha \in (0, 1]$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 0}$ with $\nu \geq N+1$,*

$$(-1)^N \sum_{m=N+1}^{\nu} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m < 0. \tag{8.15}$$

Proof. Let

$$S(\nu) := (-1)^N \sum_{m=N+1}^{\nu} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m < 0.$$

We prove (8.15) by induction on $\nu \geq N+1$. For $\nu = N+1$,

$$S(N+1) = -\alpha^{N+1} \binom{N + \frac{1}{2}}{N+1} < 0$$
$$\left(\text{by Lemma 8.2.2 with } m = \nu = N+1 \text{ and } \binom{N+\frac{1}{2}}{N+1} > 0\right).$$

Assuming $S(T) < 0$ for $T \geq N+1$, we proceed to the case $\nu = T+1$. Using the Paule-Schorn [118] package fastZeil[1], after applying Lemma 8.2.2, we obtain

$$\begin{aligned} S(T+1) &= (1-\alpha)S(T) - \alpha^{N+1}\binom{T-\frac{1}{2}}{N} - (-1)^{T+N+1}\alpha^{T+1}\frac{4^{-T}}{2T+2}\binom{2T}{T} \\ &\leq -\alpha^{N+1}\binom{T-\frac{1}{2}}{N} - (-1)^{T+N+1}\alpha^{T+1}\frac{4^{-T}}{2T+2}\binom{2T}{T}. \end{aligned} \tag{8.16}$$

To see that the right hand side of (8.16) is strictly less than 0, we consider two cases depending on parity of T and N. If T and N have opposite parity; i.e., $T + N \equiv 1 \pmod 2$, then $S(T+1) < 0$, since

$$S(T+1) \leq -\alpha^{N+1}\binom{T-\frac{1}{2}}{N} - \alpha^{T+1}\frac{4^{-T}}{2T+2}\binom{2T}{T} < 0.$$

[1] The package is available at `https://combinatorics.risc.jku.at/software`.

Continuing with (8.16) in the case that $T \equiv N (\text{mod } 2)$, it follows that

$$\begin{aligned}
S(T+1) &\leq -\alpha^{N+1}\left(\binom{T-\frac{1}{2}}{N} - \frac{1}{2T+2}\frac{1}{4^T}\binom{2T}{T}\right) \quad (\text{as, } a \in (0,1] \text{ and } T > N) \\
&= -\alpha^{N+1}\left(\frac{1}{4^N}\frac{\binom{2T}{T}\binom{T}{N}}{\binom{2T-2N}{T-N}} - \frac{1}{2T+2}\frac{1}{4^T}\binom{2T}{T}\right) \\
&\qquad\qquad (\text{by Lemma 8.2.2 with } (m,\nu) \mapsto (N,T)) \\
&= -\alpha^{N+1}\frac{1}{4^T}\binom{2T}{T}\left(\frac{4^{T-N}}{\binom{2T-2N}{T-N}}\binom{T}{N} - \frac{1}{2T+2}\right) \\
&\leq -\alpha^{N+1}\frac{1}{4^T}\binom{2T}{T}\left(\binom{T}{N}\sqrt{\pi(T-N)} - \frac{1}{2T+2}\right) \\
&\qquad\qquad (\text{by (8.91) with } n \mapsto T-N) \\
&< 0 \quad (\text{since } T > N).
\end{aligned}$$

This finishes the proof of (8.15). ☐

Lemma 8.2.8. *For all $\alpha \in \mathbb{R}_{>1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 0}$ with $\nu \geq N$,*

$$(-1)^N \sum_{m=0}^{N} \binom{\nu}{m}(-\alpha)^m > 0. \tag{8.17}$$

Proof. For $\nu = N$,

$$(-1)^N \sum_{m=0}^{N} \binom{\nu}{m}(-\alpha)^m = (\alpha-1)^N > 0;$$

whereas for $\nu > N$, we apply Lemma (8.2.6) with the substitution $\nu \mapsto \nu + \frac{1}{2}$. ☐

Lemma 8.2.9. *For all $\alpha \in (0,1]$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 0}$ with $\nu \geq N$,*

$$(-1)^N \sum_{m=N+1}^{\nu} \binom{\nu}{m}(-\alpha)^m < 0. \tag{8.18}$$

Proof. Analogous to the proof of Lemma 8.2.7. ☐

The asymptotic expansion for $I_\nu(x)$ is well documented in the literature; see, for example, [2] or [27, p. 10.40.1]. Still, we recall it in brevity. Namely, in order to estimate the error term $E(\nu, N, x)$ in Lemma 8.3.1, the knowledge of both (8.19) and the variant (8.20) is required.

Lemma 8.2.10. *([150, Chapter VII, 7.25]) For $x \in \mathbb{R}_{\geq 1}$ and $\nu \in \mathbb{R}_{>-\frac{1}{2}}$,*

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu+\frac{1}{2})} \int_0^{2x} e^{-t} t^{\nu+m-\frac{1}{2}}\, dt. \tag{8.19}$$

Now from (8.19) we can rephrase to the asymptotic expansion of $I_\nu(x)$ in the following way,

$$\begin{aligned} \frac{\sqrt{2\pi x}}{e^x} I_\nu(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu+\frac{1}{2})} \Big(\int_0^{\infty} e^{-t} t^{\nu+m-\frac{1}{2}}\, dt - \int_{2x}^{\infty} e^{-t} t^{\nu+m-\frac{1}{2}}\, dt \Big) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m} \Big(\nu+\frac{1}{2}\Big)_m}{(2x)^m} - \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu+\frac{1}{2})} \int_{2x}^{\infty} e^{-t} t^{\nu+m-\frac{1}{2}}\, dt \\ &\underset{x\to\infty}{\sim} \sum_{m=0}^{\infty} (-1)^m \frac{\binom{\nu-\frac{1}{2}}{m} \Big(\nu+\frac{1}{2}\Big)_m}{(2x)^m}. \end{aligned}$$

Summarizing,

$$I_\nu(x) \underset{x\to\infty}{\sim} \frac{e^x}{\sqrt{2\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m a_m(\nu)}{x^m} \quad \text{with} \quad a_m(\nu) = \frac{\binom{\nu-\frac{1}{2}}{m} \Big(\nu+\frac{1}{2}\Big)_m}{2^m}. \tag{8.20}$$

8.3 Inequalities for modified Bessel function of integral order

In this section, we shall describe how one can obtain an infinite family of inequalities for $I_\nu(x)$, $\nu \in \mathbb{Z}_{\geq 0}$, as stated in Theorem (8.3.9). We first split the infinite series on the right hand side of (8.19). This results in the identity (8.22) where the left hand side presents the remainder term obtained from truncation of the asymptotic series expansion (8.20) of $\frac{\sqrt{2\pi x}}{e^x} I_\nu(x)$ after extracting its partial sum. In Lemma 8.3.1, we further dissect the remainder term $E(\nu, N, x)$ depending on $\nu \geq N+1$ or

$\nu \le N$. Lemmas 8.3.2-8.3.5 (resp. Lemmas 8.3.6-8.3.8) deal with the error analysis for $\nu \ge N+1$ (resp. for $\nu \le N$).

For $\nu \ge N+1$, using Lemmas 8.2.6 and 8.2.7, we obtain upper bounds for the absolute value of $E_{\nu,N,1}(x)$ and $E^{\nu}_{N,2}(x)$. In order to compute an upper bound of $|E^{\nu}_{N,3}(x)|$, we first estimate a bound for $|\psi^{\nu}_m(t;x)|$ using Lemma 8.2.1 and then estimate the sum by Lemma 8.2.4. Combining the upper bounds from Lemmas 8.3.2-8.3.4, we obtain the final bound for $|E(\nu, N, x)|$, as given in Lemma 8.3.5.

On the other hand, for $\nu \le N$, the remainder term $E(\nu, N, x)$ is divided into two components, denoted by $E_{\nu,N,1}(x)$ and $E^{N}_{\nu,2}(x)$. Here we carry out a different method to obtain an upper bound for $|E_{\nu,N,1}(x)|$, since $\phi^{\nu}_m(t;x)$ for $\nu \le N$ is different from the case $\nu \ge N+1$, see Lemma 8.2.2. Using Lemma 8.2.5, we shall finally get an upper bound for $|E_{\nu,N,1}(x)|$, as given in Lemma 8.3.6. Analogous to the computation for upper bound of $|E^{\nu}_{N,3}(x)|$, a similar estimation has been done for $|E^{N}_{\nu,2}(x)|$ to obtain (8.54). Lemmas 8.3.6 and 8.3.7 imply Lemma 8.3.8.

Finally, we state the main result of this chapter, Theorem 8.3.9, as an immediate consequence of Lemmas 8.3.5 and 8.3.8. From Theorem 8.3.9, we get an analogous result to [32, Lemma 2.2 (4)] for $N = 3$ and $\nu \in \mathbb{Z}_{\ge 0}$, as documented in Corollary 8.3.10.

In this subsection,

$$a_m(\nu) = \frac{\binom{\nu-\frac{1}{2}}{m}\left(\nu+\frac{1}{2}\right)_m}{2^m}, \tag{A}$$

as in (8.20).

Define

$$\phi^{\nu}_m(t;x) = \frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_{2x}^{\infty} e^{-t}t^{\nu+m-\frac{1}{2}}\,dt, \tag{PHI}$$

$$\psi^{\nu}_m(t;x) = \frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_{0}^{2x} e^{-t}t^{\nu+m-\frac{1}{2}}\,dt, \tag{PSI}$$

$$E_{\nu,N,1}(x) = -\sum_{m=0}^{N}\phi^{\nu}_m(t;x), \tag{E1}$$

$$E^{\nu}_{N,2}(x) = \sum_{m=N+1}^{\nu}\psi^{\nu}_m(t;x), \tag{E2}$$

$$E^{\nu}_{N,3}(x) = \sum_{m=\nu+1}^{\infty}\psi^{\nu}_m(t;x), \tag{E3}$$

and

$$E^N_{\nu,2}(x) = \sum_{m=N+1}^{\infty} \psi^\nu_m(t;x). \tag{E0}$$

Lemma 8.3.1. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,*

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = E(\nu, N, x),$$

with

$$E(\nu, N, x) = \begin{cases} E_{\nu,N,1}(x) + E^\nu_{N,2}(x) + E^\nu_{N,3}(x), & \text{if } \nu \geq N+1 \\ E_{\nu,N,1}(x) + E^N_{\nu,2}(x), & \text{if } \nu \leq N \end{cases}. \tag{8.21}$$

Proof. From (8.19) it follows that

$$\begin{aligned} \frac{\sqrt{2\pi x}}{e^x} I_\nu(x) &= \sum_{m=0}^{\infty} \psi^\nu_m(t;x) \\ &= \sum_{m=0}^{N} \psi^\nu_m(t;x) + \sum_{m=N+1}^{\infty} \psi^\nu_m(t;x) \\ &= \sum_{m=0}^{N} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m} (\nu+\frac{1}{2})_m}{(2x)^m} - \sum_{m=0}^{N} \phi^\nu_m(t;x) + \sum_{m=N+1}^{\infty} \psi^\nu_m(t;x). \end{aligned}$$

Therefore

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = - \sum_{m=0}^{N} \phi^\nu_m(t;x) + \sum_{m=N+1}^{\infty} \psi^\nu_m(t;x). \tag{8.22}$$

We split the right hand side of (8.22) according to $\nu \geq N+1$ and $\nu \leq N$ as follows. For $\nu \geq N+1$,

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = - \sum_{m=0}^{N} \phi^\nu_m(t;x) + \sum_{m=N+1}^{\nu} \psi^\nu_m(t;x) + \sum_{m=\nu+1}^{\infty} \psi^\nu_m(t;x); \tag{8.23}$$

whereas, for $\nu \leq N$,

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = - \sum_{m=0}^{N} \phi^\nu_m(t;x) + \sum_{m=N+1}^{\infty} \psi^\nu_m(t;x). \tag{8.24}$$

The relations (8.23) and (8.24) prove (8.21). □

Lemma 8.3.2. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$,*

$$|E_{\nu,N,1}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}} \; \Gamma(\nu+N+\frac{3}{2}, 2x), \tag{8.25}$$

where Γ is the incomplete gamma function from (8.7).

Proof. From Lemma 8.3.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$ we have

$$\begin{aligned} E_{\nu,N,1}(x) &= -\sum_{m=0}^{N} \phi_m^{\nu}(t;x) = -\int_{2x}^{\infty} \frac{e^{-t} t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} \Big(-\frac{t}{2x}\Big)^m \, dt \\ &= -\int_{2x}^{\infty} \frac{e^{-t} t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m \, dt \;\; \text{where } \theta := \frac{t}{2x}. \end{aligned} \tag{8.26}$$

Let N be even. Applying (8.14), we get

$$-\binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} < -\sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m < 0 \;\; \text{(by (8.14))} \tag{8.27}$$

Now from (8.26) and (8.27), by taking the integral, it follows that

$$-\phi_{N+1}^{\nu}(t;x) < -\sum_{m=0}^{N} \phi_m^{\nu}(t;x) < 0. \tag{8.28}$$

Similarly, for N odd,

$$\binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} > -\sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m > 0 \;\; \text{(by (8.14))}. \tag{8.29}$$

Using (8.26) and (8.29), we have

$$0 < -\sum_{m=0}^{N} \phi_m^{\nu}(t;x) < \phi_{N+1}^{\nu}(t;x). \tag{8.30}$$

(8.28) and (8.30) together imply (8.25). ∎

Lemma 8.3.3. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$,*

$$|E^{\nu}_{N,2}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\,\gamma(\nu+N+\frac{3}{2}, 2x), \tag{8.31}$$

where γ is the incomplete gamma function from (8.6).

Proof. For $\nu \geq N+1$, from Lemma 8.3.1, it follows that

$$\begin{aligned} E^{\nu}_{N,2}(x) = \sum_{m=N+1}^{\nu} \psi^{\nu}_{m}(t;x) &= \int_0^{2x} \frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=N+1}^{\nu} \binom{\nu-\frac{1}{2}}{m}\Big(-\frac{t}{2x}\Big)^m dt \\ &= \int_0^{2x} \frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=N+1}^{\nu} \binom{\nu-\frac{1}{2}}{m}(-\theta)^m dt \quad \text{where } \theta := \frac{t}{2x}. \end{aligned} \tag{8.32}$$

For N even,

$$-\binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} < \sum_{m=N+1}^{\nu} \binom{\nu-\frac{1}{2}}{m}(-\theta)^m < 0 \quad \text{(by (8.15))}. \tag{8.33}$$

From (8.32) and (8.33), we have

$$-\psi^{\nu}_{N+1}(t;x) < \sum_{m=N+1}^{\nu} \psi^{\nu}_{m}(t;x) < 0. \tag{8.34}$$

Likewise, for N odd,

$$0 < \sum_{m=N+1}^{\nu} \binom{\nu-\frac{1}{2}}{m}(-\theta)^m < \binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} \quad \text{(by (8.15))}, \tag{8.35}$$

and (8.32) and (8.35) together imply

$$0 < \sum_{m=N+1}^{\nu} \psi^{\nu}_{m}(t;x) < \psi^{\nu}_{N+1}(t;x). \tag{8.36}$$

Applying (8.34) and (8.36) concludes the proof. □

Define

$$E^{\nu}_{N+1} = \frac{\sqrt{2}}{\pi}\sqrt{(2N+5/2)(N+2)} \tag{8.37}$$

and

$$E^{\nu}_{N+2} = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{\nu+1}{\nu-N-1}}\sqrt{\nu+N+\frac{3}{2}}\left(\sqrt{\frac{1}{\nu-N-1}}-\sqrt{\frac{1}{\nu+1}}\right). \tag{8.38}$$

Lemma 8.3.4. *Let $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$. Then with (8.37) and (8.38) one has,*

$$|E^{\nu}_{N,3}(x)| < E^{\nu}_{N,3}\frac{a_{N+1}(\nu)}{x^{N+1}},$$

with

$$E^{\nu}_{N,3} = \begin{cases} E^{\nu}_{N+2}, & \text{if } \nu \geq N+2 \\ E^{\nu}_{N+1}, & \text{if } \nu = N+1 \end{cases}. \tag{8.39}$$

Proof.

$$\begin{aligned}
|E^{\nu}_{N,3}(x)| &= \Big|\sum_{m=\nu+1}^{\infty} \psi^{\nu}_{m}(t;x)\Big| \quad \text{(by Lemma 8.3.1)} \\
&\leq \sum_{m=\nu+1}^{\infty} \frac{\left|(-1)^m\binom{\nu-\frac{1}{2}}{m}\right|}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_0^{2x} e^{-t}t^{\nu+m-\frac{1}{2}}\, dt \\
&< \sum_{m=\nu+1}^{\infty} \frac{\left|(-1)^m\binom{\nu-\frac{1}{2}}{m}\right|}{(2x)^m\Gamma(\nu+\frac{1}{2})}\frac{\Gamma(\nu+N+\frac{3}{2})\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}}\int_0^{2x} t^{m-N-2}\, dt \\
&\qquad \left(\text{by } (x,y)\mapsto\left(t,\nu+N+\frac{3}{2}\right) \text{ in (8.10)}\right) \\
&= \frac{1}{\sqrt{2\pi}}\frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{(2x)^{N+1}}\binom{\nu-\frac{1}{2}}{N+1}\sum_{m=\nu+1}^{\infty}\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right|\frac{1}{m-N-1}.
\end{aligned} \tag{8.40}$$

Using Lemmas 8.2.2, 8.2.3, and (8.91) along with the fact that $\frac{1}{\binom{N}{k}} \leq \frac{k}{N}$ for all $N > k$ and $\sqrt{\frac{1}{1-\frac{\nu}{m}}} \leq \sqrt{\nu+1}$ for all $m \geq \nu+1$, it follows that

$$\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right| \leq \begin{cases} \frac{1}{\pi}(N+1)\sqrt{\frac{\nu+1}{\nu-N-1}}\frac{1}{m^{3/2}}, & \text{if } \nu \geq N+2 \\ \frac{1}{\sqrt{\pi}}(N+1)\sqrt{\nu+1}\frac{1}{m^{3/2}}, & \text{if } \nu = N+1 \end{cases}. \tag{8.41}$$

For $\nu \geq N+2$,

$$|E^{\nu}_{N,3}(x)|$$

$$
\begin{aligned}
&< \left(\frac{1}{\pi\sqrt{2\pi}}(N+1)\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\sum_{m=\nu+1}^{\infty}\frac{1}{m^{3/2}(m-N-1)}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (8.40) and (8.41))} \\
&< \left(\frac{1}{\pi\sqrt{2\pi}}(N+1)\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\int_{\nu}^{\infty}\frac{1}{t^{3/2}(t-N-1)}\,dt\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= \left(\frac{1}{\pi\sqrt{2\pi}}(N+1)\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\int_{\nu-N-1}^{\infty}\frac{1}{(t+N+1)^{3/2}\,t}\,dt\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\left(\frac{\operatorname{arcsinh}\left(\sqrt{\frac{N+1}{\nu-N-1}}\right)}{\sqrt{N+1}}-\sqrt{\frac{1}{\nu+1}}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&< \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\left(\sqrt{\frac{1}{\nu-N-1}}-\sqrt{\frac{1}{\nu+1}}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(\text{since } \operatorname{arcsinh}\ x < x \text{ for all } x > 0\right). \qquad (8.42)
\end{aligned}
$$

On the other hand, for $\nu = N+1$, it follows that

$$
\begin{aligned}
&|E_{N,3}^{\nu}(x)| \\
&< \left(\frac{1}{\pi\sqrt{2}}(N+1)\sqrt{(\nu+1)(\nu+N+\frac{3}{2})}\sum_{m=\nu+1}^{\infty}\frac{1}{m^{3/2}(m-N-1)}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (8.40) and (8.41))} \\
&= \left(\frac{1}{\pi\sqrt{2}}(N+1)\sqrt{(N+2)(2N+5/2)}\sum_{m=N+2}^{\infty}\frac{1}{m^{3/2}(m-N-1)}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&< \left(\frac{1}{\pi\sqrt{2}}(N+1)\sqrt{(N+2)(2N+5/2)}\left(\frac{1}{(N+2)^{3/2}}+\int_{N+2}^{\infty}\frac{1}{t^{3/2}(t-N-1)}\,dt\right)\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= \frac{\sqrt{2}}{\pi}\sqrt{(N+2)(2N+5/2)}\left(\frac{1}{(N+2)^{3/2}}+\frac{\operatorname{arcsinh}(\sqrt{N+1})}{\sqrt{N+1}}-\sqrt{\frac{1}{N+2}}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&< \frac{\sqrt{2}}{\pi}\sqrt{(2N+5/2)(N+2)}\frac{a_{N+1}(\nu)}{x^{N+1}}. \qquad (8.43)
\end{aligned}
$$

Finally, (8.42) and (8.43) imply (8.39). □

Lemma 8.3.5. *Let $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$. Then with (8.37) and (8.38) one has,*

$$|E(\nu, N, x)| < E_N^\nu \frac{a_{N+1}(\nu)}{x^{N+1}},$$

with

$$E_N^\nu = \begin{cases} 1 + E_{N+2}^\nu, & \text{if } \nu \geq N+2 \\ 1 + E_{N+1}^\nu, & \text{if } \nu = N+1 \end{cases}. \tag{8.44}$$

Proof. For $\nu \geq N+1$,

$$\begin{aligned} |E(\nu, N, x)| &\leq \left| E_{\nu,N,1}(x) + E_{N,2}^\nu(x) + E_{N,3}^\nu(x) \right| \quad \text{(by (8.21))} \\ &< \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}} \left(\Gamma(\nu+N+\frac{3}{2}, 2x) + \gamma(\nu+N+\frac{3}{2}, 2x) \right) + \\ &\qquad E_{N,3}^\nu \frac{a_{N+1}(\nu)}{x^{N+1}} \quad \text{(by (8.25), (8.31) and (8.39))} \\ &= \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}} \Gamma(\nu+N+\frac{3}{2}) + E_{N,3}^\nu \frac{a_{N+1}(\nu)}{x^{N+1}} \\ &= (1 + E_{N,3}^\nu) \frac{a_{N+1}(\nu)}{x^{N+1}}. \end{aligned} \tag{8.45}$$

From (8.39), we get (8.44). □

Lemma 8.3.6. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\nu \leq N$,*

$$|E_{\nu,N,1}(x)| < \frac{1}{\sqrt{2\pi}} E_{\nu,1}^N \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}} \ \ln(N+1),$$

with

$$E_{\nu,1}^N = \left(1 + \frac{(2\nu+1)(\nu+2)}{\ln(N+1)} + \frac{(2\nu+1)(\nu+2)}{N+2} \right). \tag{8.46}$$

Proof. From Lemma 8.3.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\nu \leq N$, we have

$$|E_{\nu,N,1}(x)| = \left| \sum_{m=0}^{N} \phi_m^\nu(t; x) \right| = \left| \int_{2x}^{\infty} \frac{e^{-t} t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m \, dt \right| \left(\theta := \frac{t}{2x} \right)$$

$$
\begin{aligned}
&\leq \int_{2x}^{\infty} \frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \left| \sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m \right| dt \\
&= 2(N+1) \left| \binom{\nu-\frac{1}{2}}{N+1} \right| \int_{2x}^{\infty} \frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \left| \sum_{m=0}^{N} \frac{1}{2m-2\nu+1} \binom{N}{m} \vartheta^m \right| dt \\
&\qquad \text{(by (8.12))}, \quad (8.47)
\end{aligned}
$$

where $\vartheta := \theta - 1$.
Define

$$
\mathcal{M}_N(\vartheta) = \sum_{m=0}^{N} \frac{1}{2m-2\nu+1} \binom{N}{m} \vartheta^m
$$

and

$$
S_N(\vartheta) = \sum_{m=0}^{N} \frac{1}{2m+2} \binom{N}{m} \vartheta^m = \frac{(\vartheta+1)^{N+1}-1}{2\vartheta(N+1)}. \qquad (8.48)
$$

Consequently,

$$
\begin{aligned}
&\frac{\left| \mathcal{M}_N(\vartheta) - S_N(\vartheta) \right|}{S_N(\vartheta)} \\
&= \frac{1}{S_N(\vartheta)} (2\nu+1) \left| \sum_{m=0}^{N} \frac{1}{(2m-2\nu+1)(2m+2)} \binom{N}{m} \vartheta^m \right| \\
&\leq \frac{1}{S_N(\vartheta)} \frac{(2\nu+1)(\nu+2)}{2} \sum_{m=0}^{N} \frac{1}{(m+1)(m+2)} \binom{N}{m} \vartheta^m \\
&\quad \left(\left| \frac{1}{2m-2\nu+1} \right| \leq \frac{\nu+2}{m+2} \text{ for all integers } \nu, m \text{ with } 0 \leq \nu, m \leq N \right) \\
&= \frac{(2\nu+1)(\nu+2)}{(\vartheta+1)^{N+1}-1} \left(\frac{(\vartheta+1)^{N+2} - \vartheta(N+2) - 1}{\vartheta(N+2)} \right) \\
&\quad \left(\text{because } \sum_{m=0}^{N} \frac{1}{(m+1)(m+2)} \binom{N}{m} \vartheta^m = \frac{(\vartheta+1)^{N+2} - \vartheta(N+2) - 1}{\vartheta^2(N+1)(N+2)} \right) \\
&= \frac{(2\nu+1)(\nu+2)}{N+2} \left(1 + \frac{\sum_{i=0}^{N} \theta^i - (N+1)}{\theta^{N+1}-1} \right) \text{ (by replacing } \vartheta + 1 = \theta). \quad (8.49)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \Big|\mathcal{M}_N(\vartheta)\Big| \\
= \; & S_N(\vartheta)\frac{\Big|\mathcal{M}_N(\vartheta) - S_N(\vartheta) + S_N(\vartheta)\Big|}{S_N(\vartheta)} \\
\leq \; & S_N(\vartheta)\left(1 + \frac{\Big|\mathcal{M}_N(\vartheta) - S_N(\vartheta)\Big|}{S_N(\vartheta)}\right) \\
= \; & \frac{1}{2(N+1)}\frac{\theta^{N+1}-1}{\theta-1}\left(1 + \frac{\Big|\mathcal{M}_N(\vartheta) - S_N(\vartheta)\Big|}{S_N(\vartheta)}\right) \\
\leq \; & \frac{1}{2(N+1)}\frac{\theta^{N+1}-1}{\theta-1}\left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\left(1 + \frac{\sum_{i=0}^{N}\theta^i - (N+1)}{\theta^{N+1}-1}\right)\right) \quad \text{(by (8.49))} \\
= \; & \frac{1}{2(N+1)}\left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right)\sum_{i=0}^{N}\theta^i + \frac{1}{2(N+1)}\frac{(2\nu+1)(\nu+2)}{N+2}\sum_{i=0}^{N-1}(N-i)\theta^i.
\end{aligned}
\tag{8.50}
$$

Now,

$$
\begin{aligned}
& |E_{\nu,N,1}(x)| \\
\leq \; & \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right)\int_{2x}^{\infty}\frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\sum_{i=0}^{N}\theta^i \; dt \\
+ \; & \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\frac{(2\nu+1)(\nu+2)}{N+2}\int_{2x}^{\infty}\frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\sum_{i=0}^{N-1}(N-i)\theta^i \; dt \quad \text{(by (8.47) and (8.50))} \\
= \; & \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right)\sum_{i=0}^{N}\frac{1}{\Gamma(\nu+\frac{1}{2})(2x)^i}\int_{2x}^{\infty}e^{-t}t^{\nu+i-\frac{1}{2}} \; dt \\
+ \; & \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\frac{(2\nu+1)(\nu+2)}{N+2}\sum_{i=0}^{N-1}\frac{N-i}{\Gamma(\nu+\frac{1}{2})(2x)^i}\int_{2x}^{\infty}e^{-t}t^{\nu+i-\frac{1}{2}} \; dt,
\end{aligned}
\tag{8.51}
$$

where $\theta = \frac{t}{2x}$.

In order to estimate the two sums with integrals on the right hand side of (8.51), define

$$
I_1(\nu, N, x) \;=\; \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right)\sum_{i=0}^{N}\frac{1}{\Gamma(\nu+\frac{1}{2})(2x)^i}\int_{2x}^{\infty}e^{-t}t^{\nu+i-\frac{1}{2}} \; dt
$$

and

$$I_2(\nu, N, x) \;=\; \frac{(2\nu+1)(\nu+2)}{N+2} \sum_{i=0}^{N-1} \frac{N-i}{\Gamma(\nu+\frac{1}{2})(2x)^i} \int_{2x}^{\infty} e^{-t} t^{\nu+i-\frac{1}{2}}\, dt.$$

Applying the substitution $(x, y) \mapsto (t, \nu + N + \frac{3}{2})$ in (8.10), it follows that

$$\begin{aligned} & I_1(\nu, N, x) \\ < \;& \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}} \sum_{i=0}^{N} \frac{1}{(2x)^i} \int_{2x}^{\infty} \frac{1}{t^{N-i+2}}\, dt \\ = \;& \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\;(2x)^{N+1}} \sum_{i=0}^{N} \frac{1}{N-i+1} \\ < \;& \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\;(2x)^{N+1}} \;\ln(N+1), \end{aligned} \tag{8.52}$$

and

$$\begin{aligned} I_2(\nu, N, x) \;&<\; \frac{(2\nu+1)(\nu+2)}{N+2} \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}} \sum_{i=0}^{N-1} \frac{N-i}{(2x)^i} \int_{2x}^{\infty} \frac{1}{t^{N-i+2}}\, dt \\ &=\; \frac{(2\nu+1)(\nu+2)}{N+2} \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\;(2x)^{N+1}} \sum_{i=0}^{N-1} \frac{N-i}{N-i+1} \\ &<\; (2\nu+1)(\nu+2) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\;(2x)^{N+1}}. \end{aligned} \tag{8.53}$$

By (8.51), (8.52) and (8.53), we obtain

$$|E_{\nu,N,1}(x)| < \frac{1}{\sqrt{2\pi}} E_{\nu,1}^{N} \,\frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}}\;\ln(N+1).$$

□

Lemma 8.3.7. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\nu \leq N$,*

$$|E_{\nu,2}^{N}(x)| < \left(\sqrt{2} + \frac{1}{\sqrt{\nu+N+\frac{3}{2}}}\right) \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}}. \tag{8.54}$$

Proof. From Lemma 8.3.1, we get

$$
\begin{aligned}
&\left|E_{\nu,2}^{N}(x)\right| \\
= &\Big|\sum_{m=N+1}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu+\frac{1}{2})} \int_0^{2x} e^{-t} t^{\nu+m-\frac{1}{2}}\, dt\Big| \\
\le &\frac{\left|\binom{\nu-\frac{1}{2}}{N+1}(-1)^{N+1}\right|}{(2x)^{N+1}\Gamma(\nu+\frac{1}{2})} \int_0^{2x} e^{-t} t^{\nu+N+\frac{1}{2}}\, dt + \sum_{m=N+2}^{\infty} \frac{\left|(-1)^m \binom{\nu-\frac{1}{2}}{m}\right|}{(2x)^m \Gamma(\nu+\frac{1}{2})} \int_0^{2x} e^{-t} t^{\nu+m-\frac{1}{2}}\, dt \\
< &\frac{|a_{N+1}(\nu)|}{x^{N+1}} + \frac{1}{\sqrt{2\pi}} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}} \sum_{m=N+2}^{\infty} \left| \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(-1)^{N+1}\binom{\nu-\frac{1}{2}}{N+1}} \right| \frac{1}{m-N-1} \\
&\qquad \Big(\text{by the substitution } (x,y) \mapsto (t, \nu+N+\frac{3}{2}) \text{ in (8.10)}\Big),
\end{aligned}
\tag{8.55}
$$

and using Lemma 8.2.3, it follows that

$$
\left| \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(-1)^{N+1}\binom{\nu-\frac{1}{2}}{N+1}} \right| \le \frac{2}{\sqrt{\pi}} \sqrt{\frac{N+1-\nu}{m-\nu}} \frac{\binom{N+1}{\nu}}{\binom{m}{\nu}} \quad (\text{by (8.91)}) < \frac{2}{\sqrt{\pi}} \sqrt{\frac{N+1}{m}} \tag{8.56}
$$

$$
\left(\text{since } \binom{N+1}{\nu} < \binom{m}{\nu} \text{ and } \sqrt{\frac{1}{1-\frac{\nu}{m}}} \le \sqrt{\frac{N+1}{N+1-\nu}} \text{ for all } m \ge N+2\right).
$$

Using (8.55) and (8.56), we see that

$$
\begin{aligned}
&\left|E_{\nu,2}^{N}(x)\right| \\
< &\frac{|a_{N+1}(\nu)|}{x^{N+1}} + \frac{\sqrt{2}}{\pi} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}} \sqrt{N+1} \sum_{m=N+2}^{\infty} \frac{1}{\sqrt{m}\,(m-N-1)} \\
< &\frac{|a_{N+1}(\nu)|}{x^{N+1}} + \frac{\sqrt{2}}{\pi} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}} \int_1^{\infty} \frac{\sqrt{N+1}}{t\,\sqrt{t+N+1}}\, dt \\
= &\frac{|a_{N+1}(\nu)|}{x^{N+1}} + \frac{2\sqrt{2}}{\pi} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}} \left(\frac{\pi}{2} - \arctan\Big(\frac{1}{\sqrt{N+1}}\Big) \right) \\
< &\frac{|a_{N+1}(\nu)|}{x^{N+1}} + \sqrt{2} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}} \\
= &\left(\sqrt{2} + \frac{1}{\sqrt{\nu+N+\frac{3}{2}}} \right) \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}}.
\end{aligned}
\tag{8.57}
$$

□

From (8.46) and (8.54), we have the final estimation for the error term with $\nu \leq N$, presented in the following lemma.

Lemma 8.3.8. *Let $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\nu \leq N$. Then with $E_{\nu,1}^N$ as in (8.46),*

$$|E(\nu, N, x)| < E_\nu^N \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}}\ \ln(N+1),$$

with

$$E_\nu^N = \frac{1}{\sqrt{2\pi}} E_{\nu,1}^N + \frac{1}{\ln(N+1)} \left(\sqrt{2} + \frac{1}{\sqrt{\nu + N + \frac{3}{2}}} \right). \tag{8.58}$$

Finally from Lemmas 8.3.5 and 8.3.8, we can bound the error term $E(\nu, N, x)$, given in (8.21), as follows.

Theorem 8.3.9. *Let $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Then using definitions (8.37)-(8.38) and (8.58),*

$$\left| \frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} \right| < E(\nu, N) \frac{|a_{N+1}(\nu)|}{x^{N+1}},$$

with

$$E(\nu, N) = \begin{cases} 1 + E_{N+2}^\nu, & \text{if } \nu \geq N+2 \\ 1 + E_{N+1}^\nu, & \text{if } \nu = N+1 \\ E_\nu^N \sqrt{\nu + N + \frac{3}{2}} \ln(N+1), & \text{if } \nu \leq N \end{cases}. \tag{8.59}$$

Corollary 8.3.10. *For $\nu \in \mathbb{Z}_{\geq 0}$, $N = 3$ and $x \in \mathbb{R}_{\geq 1}$,*

$$\left| \frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{3} \frac{(-1)^m a_m(\nu)}{x^m} \right| < E(\nu, 3, x),$$

with

$$E(\nu, 3, x) = \begin{cases} \frac{\nu^8}{382x^4}, & \text{if } \nu \geq 4 \\ \frac{\nu^8}{86x^4}, & \text{if } \nu = 3 \\ \frac{\nu^8}{25x^4}, & \text{if } \nu = 2 \\ \frac{12\nu^8}{5x^4}, & \text{if } \nu = 1 \\ \frac{1}{x^4}, & \text{if } \nu = 0 \end{cases}. \tag{8.60}$$

Proof. It suffices to estimate $E(\nu, N)|a_{N+1}(\nu)|$ for $N = 3$, defined in (8.59). For $\nu \in \{0, 1, 2, 3, 4\}$ and $N = 3$, by numerical checking in Mathematica, we confirm that

$$E(\nu, 3)|a_4(\nu)| < \begin{cases} \frac{\nu^8}{382}, & \text{if } \nu = 4 \\ \frac{\nu^8}{86}, & \text{if } \nu = 3 \\ \frac{\nu^8}{25}, & \text{if } \nu = 2 \\ \frac{12\nu^8}{5}, & \text{if } \nu = 1 \\ 1, & \text{if } \nu = 0 \end{cases}.$$

For the remaining case $\nu \geq 5$ we checked by Mathematica that $E(\nu, 3)|a_4(\nu)| < \frac{\nu^8}{382}$; see Subsection 8.6.2. $\square$

8.4 Inequalities for modified Bessel function of half-integral order

The section establishes inequalities for $I_{\nu+1/2}(x)$ with $\nu \in \mathbb{Z}_{\geq 2}$ and $x \in \mathbb{R}_{\geq 1}$. Again we use short hand notations from Section 8.3 as (A), (PHI), (PSI), etc. From (8.19) we obtain the asymptotic expansion of $\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x)$ in the form $\sum_{m=0}^{\infty} (-1)^m a_m(\nu + \frac{1}{2})/x^m$. Following a similar treatment as worked out in the proof of Lemma 8.3.1, we truncate the infinite series $\sum_{m=0}^{\infty} \psi_m^{\nu+1/2}(t; x)$ at some point $N > 0$ and consequently obtain two remainder terms, namely,

$$-\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t; x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t; x); \tag{8.61}$$

see also (8.62). Our next step is to obtain an upper bound of the absolute value of the remainder term by estimating the two finite sums (8.61). Using Lemma 8.2.8 (resp. 8.2.9), we obtain (8.65) (resp. (8.71)). Lemmas 8.4.4 and 8.4.5 together imply Theorem 8.4.6 which introduces an infinite family of inequalities for $\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x)$.

Recalling (8.20), note that

$$a_m(\nu + 1/2) = \frac{\binom{\nu}{m}(\nu + 1)_m}{2^m}.$$

Lemma 8.4.1. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu + 1/2)}{x^m} = -\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t; x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t; x). \tag{8.62}$$

Proof. From (8.19) it follows that

$$\begin{aligned}
\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) &= \sum_{m=0}^{\infty} \psi_m^{\nu+1/2}(t;x) \quad \Big(\text{by substitution } \nu \mapsto \nu + \frac{1}{2}\Big) \\
&= \sum_{m=0}^{\nu} \psi_m^{\nu+1/2}(t;x) \quad \Big(\text{as } \nu \in \mathbb{Z}_{\geq 2} \text{ and } \binom{\nu}{m} = 0 \text{ for } m > \nu\Big) \\
&= \sum_{m=0}^{N} \psi_m^{\nu+1/2}(t;x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) \\
&= \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu+1/2)}{(2x)^m} - \sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x).
\end{aligned}$$

□

Remark 8.4.2. *From (8.62), it is clear that throughout the rest of the section we have to consider the case $\nu \geq N$. This is because $\binom{\nu}{N} = 0$ for $\nu < N$ as pointed out in the proof of Lemma 8.4.1.*

We present identity (8.63) which serves for the error analysis for $N \in \mathbb{Z}_{\geq 1}$. To this end, following the Remark 8.4.2, we consider $\nu \in \mathbb{Z}_{\geq 2}$.

Lemma 8.4.3. *For $x \in \mathbb{R}_{\geq 1}$ and $\nu \in \{0, 1\}$,*

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x \quad \text{and} \quad I_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \Big(\cosh x - \frac{1}{x} \sinh x\Big). \tag{8.63}$$

Proof. We observe that

$$\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) = \sum_{m=0}^{\nu} \psi_m^{\nu+1/2}(t;x). \tag{8.64}$$

For $\nu = 0$ in (8.64), it follows that

$$I_{1/2}(x) = \frac{e^x}{\sqrt{2\pi x}} \int_0^{2x} e^{-t} \, dt = \sqrt{\frac{2}{\pi x}} \sinh x,$$

and for $\nu = 1$,

$$\begin{aligned}
I_{3/2}(x) &= \frac{e^x}{\sqrt{2\pi x}} \Big(\int_0^{2x} e^{-t} t \, dt - \frac{1}{2x} \int_0^{2x} e^{-t} t^2 \, dt\Big) \\
&= \frac{e^x}{\sqrt{2\pi x}} \Big(\Big(1 - \frac{1}{x}\Big) + e^{-2x}\Big(1 + \frac{1}{x}\Big)\Big) = \sqrt{\frac{2}{\pi x}} \Big(\cosh x - \frac{1}{x} \sinh x\Big).
\end{aligned}$$

□

Lemma 8.4.4. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\Big|-\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x)\Big| < \frac{\binom{\nu}{N+1}}{\nu!(2x)^{N+1}}\ \Gamma(\nu+N+2, 2x), \tag{8.65}$$

where Γ is the incomplete gamma function from (8.7).

Proof. From Lemma 8.4.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ we have

$$\begin{aligned} -\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x) &= -\int_{2x}^{\infty} \frac{e^{-t}t^{\nu}}{\nu!} \sum_{m=0}^{N} \binom{\nu}{m}\Big(-\frac{t}{2x}\Big)^m\, dt \\ &= -\int_{2x}^{\infty} \frac{e^{-t}t^{\nu}}{\nu!} \sum_{m=0}^{N} \binom{\nu}{m}(-\theta)^m\, dt \ \text{ where } \theta := \frac{t}{2x}. \end{aligned} \tag{8.66}$$

We first consider the case where N is an even positive integer. Then

$$-\binom{\nu}{N+1}\theta^{N+1} < -\sum_{m=0}^{N}\binom{\nu}{m}(-\theta)^m < 0 \quad \text{(by (8.17))}. \tag{8.67}$$

Now from (8.66) and (8.67), it follows that

$$-\phi_{N+1}^{\nu+1/2}(t;x) < -\sum_{m=0}^{N}\phi_m^{\nu+1/2}(t;x) < 0. \tag{8.68}$$

If N is an odd positive integer, it is immediate that

$$0 < -\sum_{m=0}^{N}\binom{\nu}{m}(-\theta)^m < \binom{\nu}{N+1}\theta^{N+1} \quad \text{(by (8.17))}. \tag{8.69}$$

By (8.66) and (8.69), we obtain

$$0 < -\sum_{m=0}^{N}\phi_m^{\nu+1/2}(t;x) < \phi_{N+1}^{\nu+1/2}(t;x). \tag{8.70}$$

(8.68) and (8.70) together imply (8.65). □

Lemma 8.4.5. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\Big|\sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x)\Big| < \frac{\binom{\nu}{N+1}}{\nu!(2x)^{N+1}}\ \gamma(\nu+N+2, 2x), \tag{8.71}$$

where γ is the incomplete gamma function from (8.6).

Proof. By Lemma 8.4.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{>1} \times \mathbb{Z}_{>0}$, it follows that

$$\begin{aligned} \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) &= \int_0^{2x} \frac{e^{-t}t^{\nu}}{\nu!} \sum_{m=N+1}^{\nu} \binom{\nu}{m} \Big(-\frac{t}{2x}\Big)^m \, dt \\ &= \int_0^{2x} \frac{e^{-t}t^{\nu}}{\nu!} \sum_{m=N+1}^{\nu} \binom{\nu}{m} (-\theta)^m \, dt \quad \text{where } \theta := \frac{t}{2x}. \end{aligned} \tag{8.72}$$

If N is an even positive integer, then it follows that

$$-\binom{\nu}{N+1}\theta^{N+1} < \sum_{m=N+1}^{\nu} \binom{\nu}{N} (-\theta)^m < 0 \quad \text{(by (8.18))}. \tag{8.73}$$

Consequently, from (8.72) and (8.73) it is immediate that

$$-\,\psi_{N+1}^{\nu+1/2}(t;x) < \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) < 0. \tag{8.74}$$

Similarly, if N is an odd positive integer we get

$$0 < \sum_{m=N+1}^{\nu} \binom{\nu}{N} (-\theta)^m < \binom{\nu}{N+1}\theta^{N+1} \quad \text{(by (8.18))}. \tag{8.75}$$

Equations (8.72) and (8.75) lead to the following inequality

$$0 < \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) < \psi_{N+1}^{\nu+1/2}(t;x). \tag{8.76}$$

Putting (8.74) and (8.76) together gives (8.71). □

Theorem 8.4.6. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\Big| \frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu+1/2)}{(2x)^m} \Big| < \frac{a_{N+1}(\nu+\frac{1}{2})}{x^{N+1}}. \tag{8.77}$$

Proof. For $\nu = N$, we observe that the (8.62) of Lemma 8.4.1 reduces to

$$\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu+1/2)}{(2x)^m} = -\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x).$$

From (8.65), it follows that

$$\left|\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m \binom{\nu}{m} (\nu+1)_m}{(2x)^m}\right| < \frac{\binom{\nu}{N+1}}{\nu!(2x)^{N+1}}\, \Gamma(\nu+N+2, 2x)$$
$$< \frac{\binom{\nu}{N+1}}{\nu!(2x)^{N+1}} \Gamma(\nu+N+2)$$
$$= \frac{a_{N+1}(\nu+1/2)}{x^{N+1}}.$$

Whereas for $\nu > N$, combining (8.65) and (8.71), we arrive at (8.77). □

8.5 Conclusion

We have studied the error analysis for $I_\nu(x)$, where ν either is a non-negative integer or a non-negative half integer. Our major results are the inequalities presented in Theorems 8.3.9 and 8.4.6. The main objective of this section is to carry out similar considerations as done in Section 8.3 but for $\nu \in \mathbb{R}_{\geq 0}$.

For $\nu \in \mathbb{R}_{\geq 0}$, define

$$\begin{aligned}
\widetilde{E}_{\nu,N,1}(x) &= -\sum_{m=0}^{N} \phi_m^\nu(t;x), \\
\widetilde{E}_{N,2}^{\nu}(x) &= \sum_{m=N+1}^{\lfloor \nu \rfloor} \psi_m^\nu(t;x), \\
\widetilde{E}_{N,3}^{\nu}(x) &= \sum_{m=\lfloor \nu \rfloor+1}^{\infty} \psi_m^\nu(t;x),
\end{aligned}$$

and

$$\widetilde{E}_{\nu,2}^{N}(x) = \sum_{m=N+1}^{\infty} \psi_m^\nu(t;x).$$

Throughout this section, for a given $x \in \mathbb{R}$, we follow the standard notation $\lfloor x \rfloor$ (resp. $\{x\}$) to denote integer part (resp. fractional part) of x. We split $E(\nu, N, x)$ depending on whether $\lfloor \nu \rfloor \geq N+1$ or $\lfloor \nu \rfloor \leq N$, as stated in Lemma 8.5.1. In (8.79), (8.80) and (8.81), we obtain upper bounds for $\widetilde{E}_{\nu,n,1}(x)$, $\widetilde{E}_{N,2}^{\nu}(x)$, and $\widetilde{E}_{N,3}^{\nu}(x)$

when $\lfloor \nu \rfloor \geq N+1$. From Lemmas 8.5.2, 8.5.3 and 8.5.4, we get an upper bound for $|\widetilde{E}(\nu, N, x)|$ in Lemma 8.5.5.

For $\lfloor \nu \rfloor \leq N$, we obtain an upper bound for $|E(\nu, N, x)|$ in Lemma 8.5.9 as a straightforward implication of Lemmas 8.5.6 and 8.5.8.

Lemmas 8.5.5 and 8.5.9 give rise to Theorem 8.5.10 for all $\nu \in \mathbb{R}_{\geq 0}$. Restricting $\nu \in \mathbb{Z}_{\geq 0}$ (resp. $\nu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$) in (8.89), we retrieve Theorem 8.3.9 (resp. Theorem 8.4.6).

Lemma 8.5.1. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,*

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = \widetilde{E}(\nu, N, x),$$

with

$$\widetilde{E}(\nu, N, x) = \begin{cases} \widetilde{E}_{\nu,N,1}(x) + \widetilde{E}^{\nu}_{N,2}(x) + \widetilde{E}^{\nu}_{N,3}(x), & \text{if } \lfloor \nu \rfloor \geq N+1 \\ \widetilde{E}_{\nu,N,1}(x) + \widetilde{E}^{N}_{\nu,2}(x), & \text{if } \lfloor \nu \rfloor \leq N \end{cases}, \tag{8.78}$$

and $a_m(\nu)$ be as in (8.20).

Lemma 8.5.2. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \geq N+1$,*

$$|\widetilde{E}_{\nu,N,1}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\, \Gamma(\nu + N + \frac{3}{2}, 2x), \tag{8.79}$$

where Γ is the incomplete gamma function from (8.7).

Proof. Analogous to the proof of Lemma 8.3.2, by Lemma 8.2.6, we have

$$(-1)^N \sum_{m=0}^{N} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m > 0,$$

which is also valid for $\nu \in \mathbb{R}_{\geq 2}$. This is due to the fact that Lemma 8.2.6 is an immediate implication of Lemma 8.2.5 which holds for all $\nu \in \mathbb{R}_{\geq 0}$. □

Lemma 8.5.3. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \geq N+1$,*

$$|\widetilde{E}^{\nu}_{N,2}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\, \gamma(\nu + N + \frac{3}{2}, 2x), \tag{8.80}$$

where γ is the incomplete gamma function from (8.6).

Proof. For $\lfloor \nu \rfloor \geq N+1$,

$$\widetilde{E}^{\nu}_{N,2}(x) = \int_0^{2x} \frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=N+1}^{\lfloor \nu \rfloor} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m dt \ \text{ where } \theta := \frac{t}{2x}.$$

Following up the proof of Lemma 8.3.3, we observe that it remains to prove for all $\nu \in \mathbb{R}_{\geq 2}$ and $\theta \in (0,1]$,

$$S(\lfloor \nu \rfloor) := (-1)^N \sum_{m=N+1}^{\lfloor \nu \rfloor} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m < 0.$$

Using the Paule-Schorn [118] package fastZeil, we obtain

$$S(\lfloor \nu \rfloor+1) = (1-\theta)S(\lfloor \nu \rfloor) - \theta^{N+1}\binom{\nu-\frac{1}{2}}{N} - (-1)^{\lfloor \nu \rfloor+N}\theta^{N+1}\binom{\nu-\frac{1}{2}}{\lfloor \nu \rfloor+1}.$$

The rest of the proof is analogous to the proof of Lemma 8.2.7. □

For the statements of Lemmas 8.5.4-8.5.9 and of Theorem 8.5.10 we use the following definitions,

$$\widetilde{E}^{\nu}_{N+1} = \frac{\sqrt{2}}{\pi}\sqrt{2N+\frac{5}{2}+\{\nu\}}\left(\sqrt{N+2}-1\right)\prod_{i=0}^{\lfloor \nu \rfloor-1}\left(1+\frac{\{\nu\}}{\lfloor \nu \rfloor-i-\frac{1}{2}}\right)$$

and

$$\widetilde{E}^{\nu}_{N+2} = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{\lfloor \nu \rfloor+1}{\lfloor \nu \rfloor-N-1}}\sqrt{\nu+N+\frac{3}{2}}\left(\sqrt{\frac{1}{\lfloor \nu \rfloor-N}}-\sqrt{\frac{1}{\lfloor \nu \rfloor+1}}\right)\times \prod_{i=0}^{\lfloor \nu \rfloor-1}\left(1+\frac{\{\nu\}}{\lfloor \nu \rfloor-i-\frac{1}{2}}\right).$$

Lemma 8.5.4. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \geq N+1$,*

$$|\widetilde{E}^{\nu}_{N,3}(x)| < \widetilde{E}^{\nu}_{N,3}\frac{a_{N+1}(\nu)}{x^{N+1}},$$

with

$$\widetilde{E}^{\nu}_{N,3} = \begin{cases} \widetilde{E}^{\nu}_{N+2}, & \text{if } \lfloor \nu \rfloor \geq N+2 \\ \widetilde{E}^{\nu}_{N+1}, & \text{if } \lfloor \nu \rfloor = N+1 \end{cases}. \tag{8.81}$$

Proof. Similar to (8.40) we get

$$|\widetilde{E}^{\nu}_{N,3}(x)| < \frac{1}{\sqrt{2\pi}}\frac{a_{N+1}(\nu)}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}\sum_{m=\nu+1}^{\infty}\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right|\frac{1}{m-N-1}. \qquad (8.82)$$

Now,

$$\begin{aligned}
&\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right| \\
=\ &\left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\frac{\prod_{i=0}^{m-1}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right)}{\prod_{i=0}^{N}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right)}\right| \\
=\ &\left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\right|\frac{\prod_{i=0}^{\lfloor\nu\rfloor-1}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right)\Big|(1-2\{\nu\})\Big|\prod_{i=1}^{m-\lfloor\nu\rfloor-1}\left(1-\frac{\{\nu\}}{i+\frac{1}{2}}\right)}{\prod_{i=0}^{N}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right)} \\
<\ &\left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\right|\prod_{i=0}^{\lfloor\nu\rfloor-1}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right). \qquad (8.83)
\end{aligned}$$

Applying (8.41) with the substitution $\nu\mapsto\lfloor\nu\rfloor$, it follows that

$$\left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\right| \le \begin{cases}\frac{1}{\pi}(N+1)\sqrt{\frac{\lfloor\nu\rfloor+1}{\lfloor\nu\rfloor-N-1}}\frac{1}{m^{3/2}}, & \text{if } \lfloor\nu\rfloor\ge N+2\\ \frac{1}{\sqrt{\pi}}(N+1)\sqrt{\lfloor\nu\rfloor+1}\frac{1}{m^{3/2}}, & \text{if } \lfloor\nu\rfloor = N+1\end{cases}. \qquad (8.84)$$

Substituting (8.83) and (8.84) into (8.82) and proceeding analogously as for the estimation worked out in (8.42) and (8.43), we get (8.81). □

Lemma 8.5.5. *For $x\in\mathbb{R}_{\ge1}$ and $(\nu,N)\in\mathbb{R}_{\ge2}\times\mathbb{Z}_{\ge1}$ with $\lfloor\nu\rfloor\ge N+1$,*

$$|\widetilde{E}(\nu,N,x)| < \widetilde{E}^{\nu}_{N}\frac{a_{N+1}(\nu)}{x^{N+1}},$$

with

$$\widetilde{E}^{\nu}_{N} = \begin{cases}1+\widetilde{E}^{\nu}_{N+2}, & \textit{if } \lfloor\nu\rfloor\ge N+2\\ 1+\widetilde{E}^{\nu}_{N+1}, & \textit{if } \lfloor\nu\rfloor = N+1\end{cases}. \qquad (8.85)$$

Lemma 8.5.6. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \leq N$,*

$$|\widetilde{E}_{\nu,N,1}(x)| < \frac{1}{\sqrt{2\pi}} \widetilde{E}_{\nu,1}^{N} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}}\ \ln(N+1),$$

with

$$\widetilde{E}_{\nu,1}^{N} = \left(1 + \left(\frac{1}{\ln(N+1)} + \frac{1}{N+2}\right) \frac{(2\nu+1)(\lceil \nu \rceil + 2)}{|(1-2\{\nu\})|}\right). \tag{8.86}$$

Proof. For all $\nu \in \mathbb{R}_{\geq 0}$ with $\lfloor \nu \rfloor \leq N$ and non-negative integers $0 \leq m \leq N$, it follows that

$$\left|\frac{1}{2m - 2\nu + 1}\right| \leq \frac{\lceil \nu \rceil + 2}{m+2} \left|\frac{1}{1 - 2\{\nu\}}\right|.$$

For the rest, one can follow the same line of arguments as presented in the proof of Lemma 8.3.6. □

Remark 8.5.7. *Observe that on the right hand side of (8.86), the term $|(1-2\{\nu\})|$ is in the denominator. The factor $(1-2\{\nu\})$ makes trouble if and only if $\{\nu\} = 1/2$. But as we have already pointed out in the Remark 8.4.2 that for half-integral order one has to consider only the case $\nu \geq N$. In short, for the $\lfloor \nu \rfloor \leq N$ case, it is being understood that $\{\nu\} \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.*

Lemma 8.5.8. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \leq N$,*

$$|\widetilde{E}_{N,2}^{\nu}(x)| < \left(\sqrt{2}\left|\frac{\prod_{i=0}^{\lfloor \nu \rfloor}\left(1 + \frac{\{\nu\}}{i - \frac{1}{2}}\right)}{\prod_{i=0}^{N - \lfloor \nu \rfloor}\left(1 - \frac{\{\nu\}}{i + \frac{1}{2}}\right)}\right| + \frac{1}{\sqrt{\nu + N + \frac{3}{2}}}\right) \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}}. \tag{8.87}$$

Proof. Analogous to (8.55), it follows that

$$\begin{aligned}&\left|\widetilde{E}_{\nu,2}^{N}(x)\right| \\ &< \frac{|a_{N+1}(\nu)|}{x^{N+1}} + \frac{1}{\sqrt{2\pi}} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}} \sum_{m=N+2}^{\infty} \left|\frac{(-1)^m \binom{\nu - \frac{1}{2}}{m}}{(-1)^{N+1}\binom{\nu - \frac{1}{2}}{N+1}}\right| \frac{1}{m - N - 1}.\end{aligned}$$

Therefore, it is sufficient to estimate $\left|\frac{(-1)^m \binom{\nu - \frac{1}{2}}{m}}{(-1)^{N+1}\binom{\nu - \frac{1}{2}}{N+1}}\right|$ to get (8.87). Similar to (8.83), we see that

$$\left|\frac{(-1)^m \binom{\nu - \frac{1}{2}}{m}}{(-1)^{N+1}\binom{\nu - \frac{1}{2}}{N+1}}\right| < \left|\frac{(-1)^m \binom{\lfloor \nu \rfloor - \frac{1}{2}}{m}}{(-1)^{N+1}\binom{\lfloor \nu \rfloor - \frac{1}{2}}{N+1}}\right| \left|\frac{\prod_{i=0}^{\lfloor \nu \rfloor}\left(1 + \frac{\{\nu\}}{i - \frac{1}{2}}\right)}{\prod_{i=0}^{N - \lfloor \nu \rfloor}\left(1 - \frac{\{\nu\}}{i + \frac{1}{2}}\right)}\right|.$$

□

Lemma 8.5.9. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \leq N$,*

$$|\widetilde{E}(\nu, N, x)| < \widetilde{E}_\nu^N \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}}\ \ln(N+1),$$

with

$$\widetilde{E}_\nu^N = \frac{1}{\sqrt{2\pi}} \widetilde{E}_{\nu,1}^N + \frac{1}{\ln(N+1)} \left(\sqrt{2} \left| \frac{\prod_{i=0}^{\lfloor \nu \rfloor} \left(1 + \frac{\{\nu\}}{i - \frac{1}{2}}\right)}{\prod_{i=0}^{N - \lfloor \nu \rfloor} \left(1 - \frac{\{\nu\}}{i + \frac{1}{2}}\right)} \right| + \frac{1}{\sqrt{\nu + N + \frac{3}{2}}} \right). \tag{8.88}$$

Combining Lemmas 8.5.5 and 8.5.9, we arrive at the following theorem.

Theorem 8.5.10. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$,*

$$\Big| \frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} \Big| < \widetilde{E}(\nu, N) \frac{|a_{N+1}(\nu)|}{x^{N+1}},$$

with

$$\widetilde{E}(\nu, N) = \begin{cases} 1 + \widetilde{E}_{N+2}^\nu, & \text{if } \lfloor \nu \rfloor \geq N+2 \\ 1 + \widetilde{E}_{N+1}^\nu, & \text{if } \lfloor \nu \rfloor = N+1 \\ \widetilde{E}_\nu^N \sqrt{\nu + N + \frac{3}{2}} \ln(N+1), & \text{if } \lfloor \nu \rfloor \leq N \end{cases}. \tag{8.89}$$

8.6 Appendix

8.6.1 Proofs of some lemmas presented in Section 8.2.

Proof of Lemma 8.2.1: We define $f(x) = \frac{x^y}{e^x}$. Now $f'(x) = \frac{yx^{y-1} - x^y}{e^x}$ and $f'(x) = 0$ at $x = y$. Note that $f''(x) = \frac{y(y-1)x^{y-2} - 2yx^{y-1} + x^y}{e^x}$ and consequently, $f''(y) = -\frac{y^{y-1}}{e^y} < 0$ for all $y \in \mathbb{R}_{>0}$. So, $f(x)$ attains its maximum at $x = y$; i.e., $f(x) \leq f(y) = (\frac{y}{e})^y$. From [27, eq. 5.6.1], we have

$$1 < (2\pi)^{-1/2}\ x^{1/2-x}\ e^x\ \Gamma(x) \text{ for } x \in \mathbb{R}_{>0}. \tag{8.90}$$

By the substitution $x \mapsto y$ in (8.90) and using the maximum value of $f(x)$, it follows that

$$\frac{\Gamma(y)}{\sqrt{\frac{2\pi}{y}}} > (\frac{y}{e})^y \geq f(x),$$

which implies (8.10).

□

Proof of Lemma 8.2.2: For $m > \nu$,

$$\begin{aligned}
\binom{\nu-\frac{1}{2}}{m} &= \frac{(2\nu-1)(2\nu-3)\dots(2\nu-2m+1)}{2^m\ m!} \\
&= \frac{1}{2^m\ m!}\frac{\nu!}{2^\nu}\binom{2\nu}{\nu}(-1)^{m-\nu}\frac{(2m-2\nu)!}{2^{m-\nu}\ (m-\nu)!} = \frac{(-1)^{m-\nu}}{4^m}\frac{\binom{2\nu}{\nu}\binom{2m-2\nu}{m-\nu}}{\binom{m}{\nu}},
\end{aligned}$$

and for $m \le \nu$,

$$\begin{aligned}
\binom{\nu-\frac{1}{2}}{m} &= \frac{(2\nu-1)(2\nu-3)\dots(2\nu-2m+1)}{2^m\ m!} \\
&= \frac{1}{2^m\ m!}\frac{(2\nu)!}{(2\nu-2m)!}\frac{(\nu-m)!}{\nu!} = \frac{1}{4^m}\frac{\binom{2\nu}{\nu}\binom{\nu}{m}}{\binom{2\nu-2m}{\nu-m}}.
\end{aligned}$$

□

Proof of Lemma 8.2.3:

$$\begin{aligned}
\sum_{m=k}^{N}(-1)^m\binom{\nu-\frac{1}{2}}{m}\binom{m}{k} &= \sum_{m=k}^{N}(-1)^m\binom{\nu-\frac{1}{2}}{k}\binom{\nu-\frac{1}{2}-k}{m-k} \quad \text{(by [68, (5.21)])} \\
&= \binom{\nu-\frac{1}{2}}{k}(-1)^k\sum_{m=0}^{N-k}\binom{\nu-\frac{1}{2}-k}{m}(-1)^m \\
&= (-1)^N\binom{\nu-\frac{1}{2}}{k}\binom{\nu-\frac{3}{2}-k}{N-k} \quad \text{(by [68, (5.16)])},
\end{aligned}$$

and

$$2(-1)^{N+1}(N+1)\binom{\nu-\frac{1}{2}}{N+1}\frac{1}{2k-2\nu+1}\binom{N}{k} = (-1)^N\binom{\nu-\frac{1}{2}}{k}\binom{\nu-\frac{3}{2}-k}{N-k}$$

(by [68, (5.21)]).

□

Proof of Lemma 8.2.4: First, observe that for all $n \in \mathbb{Z}_{\ge 1}$,

$$\frac{4^n}{2\sqrt{n}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}. \tag{8.91}$$

Now for $m > \nu$,

$$\begin{aligned}
\left|(-1)^m \binom{\nu - \frac{1}{2}}{m}\right| &= \frac{1}{4^m} \frac{\binom{2\nu}{\nu}\binom{2m-2\nu}{m-\nu}}{\binom{m}{\nu}} \quad \text{(by Lemma 8.2.2)} \\
&\leq \frac{1}{4^m} \frac{4^\nu}{\sqrt{\pi\nu}} \frac{4^{m-\nu}}{\sqrt{\pi(m-\nu)}} \frac{1}{\binom{m}{\nu}} \quad \text{(by (8.91))} \\
&= \frac{1}{\pi\sqrt{\nu(m-\nu)}} \frac{1}{\binom{m}{\nu}},
\end{aligned}$$

and for $m \leq \nu$,

$$\begin{aligned}
\left|(-1)^m \binom{\nu - \frac{1}{2}}{m}\right| &= \frac{1}{4^m} \frac{\binom{2\nu}{\nu}\binom{\nu}{m}}{\binom{2\nu-2m}{\nu-m}} \quad \text{(by Lemma 8.2.2)} \\
&\leq \frac{1}{4^m} \frac{4^\nu}{\sqrt{\pi\nu}} \frac{2\sqrt{\nu-m}}{4^{\nu-m}} \binom{\nu}{m} \quad \text{(by (8.91))} \\
&\leq \frac{2}{\sqrt{\pi}} \binom{\nu}{m} \quad \left(\text{since } \sqrt{\frac{\nu-m}{\nu}} \leq 1\right).
\end{aligned}$$

□

8.6.2 Mathematica computation for the proof of Corollary 8.3.10.

We complete the proof of Corollary 8.3.10 by checking $E(\nu, 3)a_4(\nu) < \frac{\nu^8}{382}$ for all $\nu \geq 5$ with Mathematica using Cylindrical Algebraic Decomposition [44]. In order to do this computation, our input for $a_m(\nu)$ (resp. for $E(\nu, N)$ with $\nu > N+1$) is $a[v, m]$ (resp. $E1[v, N]$) in Mathematica.

In[16]:= $\mathbf{a[v,m] := \dfrac{\text{Binomial}[v-\frac{1}{2},m]\ \text{Pochhammer}[v+\frac{1}{2},m]}{2^m}}$

In[17]:= $\mathbf{E1[v,N] := \left(1+\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{v+1}{v-N-1}}\sqrt{v+N+\frac{3}{2}}\left(\sqrt{\frac{1}{v-N}}-\sqrt{\frac{1}{v+1}}\right)\right)a[v,N+1]}$

In[18]:= $\mathbf{\text{CylindricalDecomposition}[\{\frac{v^8}{382} > E1[v,3], v \geq 5\}, v]}$

Out[18]= $v \geq 5$

Part III

Combinatorial Inequalities

Chapter 9

Positivity of the second shifted difference of partitions and overpartitions

This chapter is devoted to the study of inequalities related to the second shifted difference of the number of integer partitions $p(n)$ and of overpartitions $\overline{p}(n)$ by an elementary combinatorial approach. Recently Gomez, Males, and Rolen proved the positivity of $\Delta_j^2(p(n)) = p(n) - 2p(n-j) + p(n-2j)$ by employing the Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound. Our goal is to prove $\Delta_j^2(p(n)) \geq 0$ (resp. $\Delta_j^2(\overline{p}(n)) > 0$) by an explicit description of a non-empty subset, say $X_p^2(n,j)$ of the set of integer partitions $P(n)$ (resp. $X_{\overline{p}}^2(n,j)$ and the set of overpartitions $\overline{P}(n)$) with $|X_p^2(n,j)| = \Delta_j^2(p(n))$ (resp. $|X_{\overline{p}}^2(n,j)| = \Delta_j^2(\overline{p}(n))$) .

9.1 Introduction

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that $\sum_{i=1}^{\ell} \lambda_i = n$, denoted by $\lambda \vdash n$. The set of partitions of n is denoted by $P(n)$ and $|P(n)| = p(n)$. For $\lambda \vdash n$, we define $\ell(\lambda)$ to be the total number of parts of λ and $\text{mult}_\lambda(\lambda_i)$ to be the multiplicity of the part λ_i in λ. For $\lambda \vdash n$ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu \vdash m$ with $\mu = (\mu_1, \ldots, \mu_{\ell'})$, define the union $\lambda \cup \mu \vdash m+n$ to be the partition with parts $\{\lambda_i, \mu_j\}$ arranged in nonincreasing order. Inequalities for the partition function have been studied in many directions and proofs of such inequalities were by employing analytic tools as the Hardy-Ramanujan-Rademacher formula for $p(n)$, see [76, 122, 124, 123], and Lehmer's error bound [99, 98]. Let Δ be the backward difference operator defined on a sequence $a(n)$ by

$\Delta(a(n)) := a(n) - a(n-1)$ and, for $r \geq 1$, $\Delta^r(a(n)) := \Delta\Big(\Delta^{r-1}(a(n))\Big)$. In 1977, Good [67] conjectured that $\Delta^r(p(n))$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive. Using the Hardy-Ramanujan-Rademacher series for $p(n)$, Gupta [71] proved that for any given $r \in \mathbb{Z}_{\geq 1}$, $\Delta^r(p(n)) > 0$ for sufficiently large n. In 1988, Odlyzko [114] proved the conjecture of Good and obtained the following asymptotic formula for $n(r)$:

$$n(r) \sim \frac{6}{\pi^2} r^2 \log^2 r \quad \text{as} \quad r \to \infty.$$

For a more detailed study on $\Delta(p(n))$, we refer to [3]. Recently, Gomez, Males and Rolen studied the second order j-shifted difference of $p(n)$, defined by

$$\Delta_j^2(p(n)) = p(n) - 2p(n-j) + p(n-2j)$$

and proved the following theorem.

Theorem 9.1.1 (Theorem 1.2, [66])**.** *Let $n \geq 2$ and $j \leq \frac{1}{4}\sqrt{n - \frac{1}{24}}$. Then we have that*

$$\Delta_j^2(p(n)) \geq 0.$$

In other words, $p(n)$ satisfies the extended convexity result

$$p(n) + p(n-2j) \geq 2p(n-j).$$

An overpartition of n is a nonincreasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. We denote the number of overpartitions of n by $\overline{p}(n)$ and the set of overpartitions of n by $\overline{P}(n)$. For example, the 4 overpartitions of 2 are $2, \overline{2}, 1+1, \overline{1}+1$. The Study on overpartitions dates back to MacMahon [103] but under different nomenclature an extensive study on the overpartitions began with the work of Corteel and Lovejoy [46]. A Hardy-Ramanujan-Rademacher type series expansion for $\overline{p}(n)$ was due to Zuckerman [156]. Recently, Wang, Xie, and Zhang [148] proved that $\Delta^r(\overline{p}(n)) > 0$ for $n \geq n(r)$, where $n(r)$ is a positive integer depending on r.

The main motivation of this chapter is to prove Theorem 9.1.1 using a combinatorial approach rather than the analytic one; i.e., by studying an asymptotic estimate of $\frac{p(n-j)}{p(n)}$ as in [66, Theorem 1.1]. Moreover, we will show $\Delta_j^2(p(n)) \geq 0$ for all $n \geq 2j$, a weaker assumption in comparison to $n \geq \max\{2, 16j^2 + \frac{1}{24}\}$ assumed in Theorem 9.1.1. Moreover, we show $\Delta_j^2(\overline{p}(n)) > 0$ with a similar combinatorial approach as that for $p(n)$. Gomez, Males, and Rolen [66] proved the positivity of $\Delta_j^2(p(n))$ using asymptotic estimate of the quotient $p(n-j)/p(n)$ whereas our main objective is to show that $(\Delta_j^2(p(n)))_{n\geq 2j}$ (resp. $(\Delta_j^2(\overline{p}(n)))_{n\geq 2j}$) can be enumerated by a non-empty proper subset of $P(n)$ (resp. of $\overline{P}(n)$) so as to prove positivity of the respective sequences.

We organize the chapter in the following way. Below we list all the theorems, Theorems 9.1.3-9.1.9, with two corollaries Corollaries 9.1.7 and 9.1.10. The proofs of Theorems 9.1.3-9.1.9 are given in Section 9.2.

Definition 9.1.2. *For all positive integers n and j, define*

$$X_a^1(n,j) = A(n) \setminus A(n-j) \quad \text{and} \quad |X_a^1(n,j)| = \Delta_j^1(a(n)),$$
$$X_a^2(n,j) = X_a^1(n,j) \setminus X_a^1(n-j,j) \quad \text{and} \quad |X_a^2(n,j)| = \Delta_j^2(a(n)),$$

where $|A(n)| := a(n)$.

In our context, $A(n)$ is $P(n)$, resp. $\overline{P}(n)$; consequently, we will consider $X_a^i(n,j) = X_p^i(n,j)$, resp. $X_a^i(n,j) = X_{\overline{p}}^i(n,j)$.

Theorem 9.1.3. *For all positive integers n and j with $n \geq j$,*

$$X_p^1(n,j) = \Big\{\lambda \in P(n) : 0 \leq \lambda_1 - \lambda_2 \leq j-1\Big\}. \tag{9.1}$$

Remark 9.1.4. *Plugging in $j=1$ into Theorem 9.1.3, $X_p^1(n,j)$ is described as the set of non-unitary partitions of n as well as the set of partitions of $n-1$ in which the least part occurs exactly once [134, A002865].*

Theorem 9.1.5. *For all positive integers n and j with $n \geq 2j$,*

$$X_p^2(n,j) = \Big\{\lambda \in X_p^1(n,j) : 0 \leq \mathit{mult}_\lambda(1) \leq j-1\Big\}. \tag{9.2}$$

Remark 9.1.6. *Plugging in $j=1$ into Theorem 9.1.5, $X_p^2(n,j)$ is described as the set of partitions of $n-2$ with all parts >1 and with the largest part occurring more than once [134, A053445].*

Corollary 9.1.7. *For all positive integers n and j with $n \geq 2j$,*

$$\Delta_j^2(p(n)) \geq 0. \tag{9.3}$$

Proof. For $j=1$ and $n \in \{3,5,7\}$, $X_p^2(n,j) = \emptyset$ and so $\Delta_j^2(p(n)) = 0$ and for $n=2$, $\Delta_j^2(p(n)) = 1$. Next, if $n = 2k$ with $k \geq 2$, then $\lambda = (k,k) \in X_p^2(2k,1)$, and if $n = 2k+1$ with $k \geq 4$,

$$\lambda = \left(\left\lceil \frac{2k+1}{3} \right\rceil, \left\lceil \frac{2k+1}{3} \right\rceil, (2k+1) - 2\left\lceil \frac{2k+1}{3} \right\rceil\right) \in X_p^2(2k+1,1),$$

as $(2k+1) - 2\left\lceil \frac{2k+1}{3} \right\rceil > 1$ for all $k \geq 4$. So, $\Delta_1^2(p(n)) \geq 0$ for all $n \geq 2j$ with $j = 1$.

Finally, for $j \geq 2$ and $n = 2m \geq 2j$, observe that $\lambda = (m, m) \in X_p^2(n, j)$ and for $n = 2m+1 > 2j$, $\lambda = (m+1, m) \in X_p^2(n, j)$. Therefore, $\Delta_1^2(p(n)) > 0$ for all $n \geq 2j$ with $j \geq 2$. □

Theorem 9.1.8. *For all positive integers n and j with $n \geq j$,*

$$\begin{aligned} X_{\overline{p}}^1(n, j) = &\Big\{\lambda \in \overline{P}(n) : 0 \leq \lambda_1 - \lambda_2 \leq j - 1 \textit{ and } \lambda_1, \lambda_2 \textit{ may be overlined}\Big\} \\ &\cup \Big\{\lambda \in \overline{P}(n) : \lambda_1 - \lambda_2 = j \textit{ and } \lambda_2 \textit{ is overlined}\Big\}. \end{aligned} \tag{9.4}$$

Theorem 9.1.9. *For all positive integers n and j with $n \geq 2j$,*

$$X_{\overline{p}}^2(n, j) = \Big\{\lambda \in X_{\overline{p}}^1(n, j) : 0 \leq mult_\lambda(1) \leq j - 1 \textit{ and } 0 \leq mult_\lambda(\overline{1}) \leq 1\Big\}. \tag{9.5}$$

Corollary 9.1.10. *For all positive integers n and j with $n \geq 2j$,*

$$\Delta_j^2(\overline{p}(n)) > 0. \tag{9.6}$$

Proof. For $j = 1$ and $n = 2$, $\Delta_j^2(\overline{p}(n)) = 1$. For $j \geq 1$, $n = 2k \geq 2j$ with $k \in \mathbb{Z}_{\geq 2}$, $\lambda = (\overline{k}, k) \in X_{\overline{p}}^2(n, j)$ and when $n = 2k+1 > 2j$ with $k \in \mathbb{Z}_{\geq 1}$, $\lambda = (\overline{k+1}, \overline{k}) \in X_{\overline{p}}^2(n, j)$. This concludes the proof. □

9.2 Proofs of Theorems 9.1.3-9.1.9

Proof of Theorem 9.1.3: For all positive integers n, j with $n \geq j$, we define an injective map $i_1 : P(n-j) \longrightarrow P(n)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto i_1(\lambda) = (\lambda_1 + j, \lambda_2, \ldots, \lambda_r). \tag{9.7}$$

It is immediate that $i_1(\lambda) \in P(n)$, and the image set can be described as

$$\Im(i_1) = \Big\{\pi \in P(n) : \pi_1 - \pi_2 \geq j\Big\}.$$

Note that i_1 is an injective map: for any two partitions, say, for $\lambda, \mu \in P(n-j)$, there are two possible cases, either $\ell(\lambda) = \ell(\mu)$ or $\ell(\lambda) \neq \ell(\mu)$. When $\ell(\lambda) \neq \ell(\mu)$,

$\ell(i_1(\lambda)) \neq \ell(i_1(\mu))$ and therefore i_1 is injective. If $\ell(\lambda) = \ell(\mu)$, then $i_1(\lambda) = i_1(\mu)$ immediately implies that $\lambda_m = \mu_m$ for all $1 \leq m \leq \ell(\lambda)$. Hence,

$$P(n) \setminus i_1(P(n-j)) = \Big\{\pi \in P(n) : 0 \leq \pi_1 - \pi_2 \leq j\Big\} = X_p^1(n,j).$$

□

Proof of Theorem 9.1.5: For all positive integers n, j with $n \geq 2j$, we first define an injective map $i_2 : X_p^1(n-j,j) \longrightarrow X_p^1(n,j)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto i_2(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_r) \cup (\underbrace{1, 1, \ldots, 1}_{j \text{ times}}). \tag{9.8}$$

Now $i_2(\lambda) \in X_p^1(n,j)$ and consequently,

$$\Im(i_2) = \Big\{\pi \in X_p^1(n,j) : \mathrm{mult}_\pi(1) \geq j\Big\}.$$

Clearly, i_2 is an injective map, since we adjoin the partition of j with all parts being 1 to any partition $\lambda \in X_p^1(n-j,j)$. Therefore,

$$X_p^1(n,j) \setminus i_2(X_p^1(n-j,j)) = \Big\{\pi \in X_p^1(n,j) : 0 \leq \mathrm{mult}_\pi(1) \leq j-1\Big\} = X_p^2(n,j).$$

□

Proof of Theorem 9.1.8: For all positive integers n, j with $n \geq j$, we define an injective map $\bar{i}_1 : \overline{P}(n-j) \longrightarrow \overline{P}(n)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto \bar{i}_1(\lambda) = (\lambda_1 + j, \lambda_2, \ldots, \lambda_r) \in \overline{P}(n). \tag{9.9}$$

Here we consider two separate cases depending on whether $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$.

For $\lambda_1 = \lambda_2$, we observe that only the first occurence of λ_1 can be overlined and the image of $\bar{i}_1$ is given by

$$\Im(\bar{i}_1) = \Big\{\pi \in \overline{P}(n) : \pi_1 - \pi_2 = j \;\text{ and }\; \pi_2 \;\text{ is not overlined}\Big\}.$$

For the other case $\lambda_1 \neq \lambda_2$,

$$\Im(\bar{i}_1) = \Big\{\pi \in \overline{P}(n) : \pi_1 - \pi_2 \geq j \;\text{ and }\; \pi_1, \pi_2 \;\text{ may be overlined}\Big\}.$$

Clearly, $\bar{i}_1$ is an injective map in each of the cases. Therefore

$$\begin{aligned}\overline{P}(n) \setminus \bar{i}_1(\overline{P}(n-j)) =& \Big\{\pi \in \overline{P}(n) : 0 \leq \pi_1 - \pi_2 \leq j-1 \text{ and } \;\pi_1, \pi_2 \text{ may be overlined}\Big\} \\ & \cup \Big\{\pi \in \overline{P}(n) : \pi_1 - \pi_2 = j \text{ and } \pi_2 \text{ is overlined}\Big\} \\ =& \overline{X}_{\overline{p}}^1(n,j).\end{aligned}$$

□

Proof of Theorem 9.1.9: For all positive integers n, j with $n \geq 2j$, we define an injective map $\bar{i}_2 : \overline{X}_p^1(n-j, j) \longrightarrow \overline{X}_p^1(n, j)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto \bar{i}_2(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_r) \cup (\underbrace{1, 1, \ldots, 1}_{j \text{ times}}) \in \overline{X}_p^1(n, j). \tag{9.10}$$

Consequently,

$$\Im(\bar{i}_2) = \Big\{\pi \in X_p^1(n, j) : \text{mult}_\pi(1) \geq j\Big\}.$$

Note that i_2 is an injective map as we adjoin the overpartition of j with all parts being 1 to any overpartition $\lambda \in \overline{X}_p^1(n-j, j)$. Therefore,

$$\begin{aligned}\overline{X}_p^1(n, j) \setminus \bar{i}_2(\overline{X}_p^1(n-j, j)) &= \Big\{\pi \in X_p^1(n, j) : 0 \leq \text{mult}_\pi(1) \leq j-1 \ \text{ and } \ 0 \leq \text{mult}_\pi(\overline{1}) \leq 1\Big\} \\ &= \overline{X}_p^2(n, j),\end{aligned}$$

since if $\overline{1}$ is a part of an overpartition, say $\pi \in \overline{P}(n)$, then according to the definition $0 \leq \text{mult}_\pi(\overline{1}) \leq 1$.

□

Chapter 10

Parity bias of parts in partitions and restricted partitions

Let $p_o(n)$ (resp. $p_e(n)$) denote the number of partitions of n with more odd parts (resp. even parts) than even parts (resp. odd parts). Recently, Kim, Kim and Lovejoy proved that $p_o(n) > p_e(n)$ for all $n > 2$ and conjectured that $d_o(n) > d_e(n)$ for all $n > 19$ where $d_o(n)$ (resp. $d_e(n)$) denote the number of partitions into distinct parts having more odd parts (resp. even parts) than even parts (resp. odd parts). In this chapter we provide combinatorial proofs for both the result and the conjecture of Kim, Kim and Lovejoy. In addition, we show that if we restrict the smallest part of the partition to be 2, then the parity bias is reversed. That is, if $q_o(n)$ (resp. $q_e(n)$) denote the number of partitions of n with more odd parts (resp. even parts) than even parts (resp. odd parts) where the smallest part is at least 2, then we have $q_o(n) < q_e(n)$ for all $n > 7$. We also look at some more parity biases in partitions with restricted parts.

10.1 Parity on parts of integer partitions

In the theory of partitions, inequalities arising between two classes of partitions have a long tradition of study, for instance Alder's conjecture [4] and the Ehrenpreis problem [6] are the most famous examples in this direction. In recent years there have been a number of results about partition inequalities. For instance, work in this direction has been done by McLaughlin [105], Chern, Fu, and Tang [43], Berkovich and Uncu [23] among others. Very recently, Kim, Kim, and Lovejoy [85] have given interesting inequalities which show bias in parity of the partition functions. Further results on parity bias have been found by Kim and Kim [84], Chern [42], and [20]. Proofs of such results employ a wide range of techniques ranging from q-series methods, to combinatorial constructions and maps to classical asymptotic analysis. By parity bias we mean the tendency of partitions to have more parts of a particular parity than the other.

Let $p_o(n)$ (resp. $p_e(n)$) denote the number of partitions of n with more odd parts (resp. even parts) than even parts (resp. odd parts). Kim, Kim, and Lovejoy [85] proved that $p_o(n) > p_e(n)$ and conjectured that $d_o(n) > d_e(n)$ for all $n > 19$ where $d_o(n)$ (resp. $d_e(n)$) denote the number of partitions into distinct parts having more odd parts (resp. even parts) than even parts (resp. odd parts). The primary goal of the present chapter is to prove these two inequalities combinatorially.

In fact, our method can be amended to prove other results where biases in parity are found for restricted partitions. If $q_o(n)$ (resp. $q_e(n)$) denote the number of partitions of n with more odd parts (resp. even parts) than even parts (resp. odd parts) where the smallest part is at least 2, then we have $q_o(n) < q_e(n)$ for $n > 7$ (see Theorem 10.1.5 below). These parity biases seems to also occur for more restricted partition functions and we also explore some of these themes towards the end with a few conjectures.

We define a partition λ of a non-negative integer n to be an integer sequence $(\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. We say that λ is a partition of n, denoted by $\lambda \vdash n$ and $\sum_{i=1}^{\ell} \lambda_i = n$. The set of partition of n is denoted by $P(n)$ and $|P(n)| = p(n)$. For $\lambda \vdash n$, we define $a(\lambda)$ to be the largest part of λ, $\ell(\lambda)$ to be the total number of parts of λ and $\text{mult}_\lambda(\lambda_i) := m_i$ to be the multiplicity of the part λ_i in λ. We also use $\lambda = (\lambda_1^{m_1} \ldots \lambda_\ell^{m_\ell})$ as an alternative notation for partition. For $\lambda \vdash n$ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu \vdash m$ with $\mu = (\mu_1, \ldots, \mu_{\ell'})$, define the union $\lambda \cup \mu \vdash m + n$ to be the partition with parts $\{\lambda_i, \mu_j\}$ arranged in non-increasing order. For a partition $\lambda \vdash n$, we split λ into λ_e and λ_o respectively into even and odd parts; i.e., $\lambda = \lambda_e \cup \lambda_o$. We denote by $\ell_e(\lambda)$ (resp. $\ell_o(\lambda)$) to be the number of even parts (resp. odd parts) of λ and $\ell(\lambda) = \ell_e(\lambda) + \ell_o(\lambda)$.

The following sets of partitions are of interest in this chapter.

Definition 10.1.1.

$$\begin{aligned}
D(n) &:= \{\lambda \in P(n) : \mathit{mult}_\lambda(\lambda_i) = 1 \text{ for all } i\}, \\
P_e(n) &:= \{\lambda \in P(n) : \ell_e(\lambda) > \ell_o(\lambda)\}, \\
P_o(n) &:= \{\lambda \in P(n) : \ell_o(\lambda) > \ell_e(\lambda)\}, \\
D_e(n) &:= P_e(n) \cap D(n), \\
D_o(n) &:= P_o(n) \cap D(n), \\
Q(n) &:= \{\lambda \in P(n) : \lambda_i \neq 1 \text{ for all } i\},
\end{aligned}$$

$$\begin{aligned} Q_e(n) &:= \{\lambda \in Q(n) : \ell_e(\lambda) > \ell_o(\lambda)\}, \\ Q_o(n) &:= \{\lambda \in Q(n) : \ell_o(\lambda) > \ell_e(\lambda)\}, \\ DQ_e(n) &:= Q_e(n) \cap D(n), \\ \textit{and} \quad DQ_o(n) &:= Q_o(n) \cap D(n). \end{aligned}$$

Definition 10.1.2. *For all the sets defined above, their cardinalities will be denoted by the lower case letters. For instance, $|P_e(n)| = p_e(n)$, $|DQ_e(n)| = dq_e(n)$ and so on.*

Now, we state formally the main results proved in this chapter.

Theorem 10.1.3 (Theorem 1, [85])**.** *For all positive integers $n \neq 2$, we have*

$$p_o(n) > p_e(n).$$

Theorem 10.1.4 (Conjectured, [85])**.** *For all positive integers $n > 19$, we have*

$$d_o(n) > d_e(n).$$

Theorem 10.1.5. *For all positive integers $n > 7$, we have*

$$q_o(n) < q_e(n).$$

For a nonempty set $S \subsetneq \mathbb{Z}_{\geq 0}$, define

$$\begin{aligned} P_e^S(n) &:= \{\lambda \in P_e(n) : \lambda_i \notin S\} \\ \text{and} \quad P_o^S(n) &:= \{\lambda \in P_o(n) : \lambda_i \notin S\}. \end{aligned}$$

Consequently, denote the number of partitions in $P_e^S(n)$ (resp. $P_o^S(n)$) by $p_e^S(n)$ (resp. $p_o^S(n)$). The above definition leads us to the following results that describes not only the parity of parts but also its arithmetic by putting a constrain on its support.

Theorem 10.1.6. *For all $n \geq 1$ we have*

$$p_o^{\{2\}}(n) > p_e^{\{2\}}(n).$$

Theorem 10.1.7. *If $S = \{1, 2\}$, then for all integers $n > 8$, we have*

$$p_o^S(n) > p_e^S(n).$$

Before we move on further, let us describe the fundamental principle behind proofs of Theorems 10.1.3-10.1.7. Let X and Y be two given sets and our goal is to prove that $|Y| > |X|$. We choose a subset X_0 ($\subsetneq X$) and define an injective map $f : X_0 \to Y$. Then to prove $|Y| > |X|$, it is enough to prove for a suitable subset $Y_0 \subsetneq Y \setminus f(X_0)$ with $|Y_0| > |X \setminus X_0|$. Throughout this chapter, we follow the notation $x \mapsto y$ instead of writing $f(x) = y$ when the map f is understood from the context.

The rest of the chapter is organized as follows: in Section 10.2 we give a combinatorial proof of the result of Kim, Kim and Lovejoy [85], in Section 10.3 we give a proof of the conjecture of Kim, Kim and Lovejoy [85], in Section 10.4 we prove reverse parity bias as stated in Theorem 10.1.5, Section 10.5 presents the proofs of Theorems 10.1.6 and 10.1.7, and finally in Section 10.6 we present a very short discussion drawing on Section 10.5, by proposing further problems. The proofs of two preliminary lemmas (cf. Lemmas 10.2.1 and 10.2.2) are given in Section 10.7.

10.2 Proof of $p_o(n) > p_e(n)$

We begin by presenting the following two lemmas, used later in the proof of Theorem 10.1.3. For proofs, we refer to Appendix 10.7.

Lemma 10.2.1. *For all even positive integer n with $n \geq 14$, we have*

$$\sum_{k=1}^{\frac{n-6}{2}} \left\lfloor \frac{n-2k-2}{4} \right\rfloor > 1 + \sum_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} \left\lfloor \frac{n-6k+2}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} \left\lfloor \frac{n-6k-2}{4} \right\rfloor.$$

Lemma 10.2.2. *For all odd positive integer n with $n \geq 11$, we have*

$$\sum_{k=1}^{\frac{n-5}{2}} \left\lfloor \frac{n-2k-1}{4} \right\rfloor > 1 + \left\lfloor \frac{n-4}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} \left\lfloor \frac{n-6k-1}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} \left\lfloor \frac{n-6k-5}{4} \right\rfloor.$$

Let

$$\begin{aligned}
G_e^0(n) &:= \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 0 \pmod 2\},\\
\overline{G_e^0}(n) &:= \{\lambda \in G_e^0(n) : \lambda_3 \geq 3\},\\
G_e^1(n) &:= \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 1 \pmod 2\},\\
G_e^2(n) &:= \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) \geq 2\},\\
\text{and } \ G_e(n) &:= G_e^1(n) \cup G_e^2(n).
\end{aligned}$$

We split the set $G_e(n)$ into the parity of length of partition as $G_e(n) = G_{e,0}(n) \cup G_{e,1}(n)$ with $G_{e,0}(n) = \{\lambda \in G_e(n) : \ell(\lambda) \equiv 0 \pmod 2\}$, $G_{e,1}(n) = \{\lambda \in G_e(n) : \ell(\lambda) \equiv 1 \pmod 2\}$ and let $\overline{G_e}(n) := G_{e,0}(n) \cup G_{e,1}(n) \cup \overline{G_e^0}(n)$. Therefore,

$$P_e(n) \setminus \overline{G_e}(n) = \{\lambda \in G_e^0(n) : 0 \leq \lambda_3 \leq 2\}. \tag{10.1}$$

We construct a map $f : \overline{G_e}(n) \to P_o(n)$ by defining maps $f|_{G_{e,0}(n)} = f_1$, $f|_{G_{e,1}(n)} = f_2$ and $f|_{\overline{G_e^0}(n)} = f_3$ such that $\{f_i\}_{1\leq i\leq 3}$ are injective with the following properties

- $f_1(G_{e,0}(n)) \cap f_2(G_{e,1}(n)) = \emptyset$,
- $f_1(G_{e,0}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$, and
- $f_2(G_{e,1}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$,

so as to conclude the map f is injective. Then we will choose a subset $\overline{P_o}(n) \subsetneq P_o(n) \setminus f(\overline{G_e}(n))$ with $|\overline{P_o}(n)| > |P_e(n) \setminus \overline{G_e}(n)|$.

Let $\lambda \in G_{e,0}(n)$ with $\lambda_e = (\lambda_{e_1}, \ldots, \lambda_{e_k})$ and $\lambda_o = (\lambda_{o_1}, \ldots, \lambda_{o_m})$ where $k + m = \ell(\lambda)$. Since $\lambda \in G_{e,0}(n)$, $\ell(\lambda) = 2r$ for some $r \in \mathbb{Z}_{>0}$ and $k > r$ because, $k - m \geq 1$ implies $2k \geq k + m + 1 = 2r + 1$.

We define $f_1 : G_{e,0}(n) \to P_o(n)$ by $f_1(\lambda) := \mu$ with

$$\mu_e = ((\lambda_{o_1} + 1), \ldots, (\lambda_{o_m} + 1))$$

and

$$\mu_o = ((\lambda_{e_1} + 1), \ldots, (\lambda_{e_{k-r}} + 1), (\lambda_{e_{k-r+1}} - 1), \ldots, (\lambda_{e_k} - 1)).$$

Here we note that $\mu \in P(n)$ and f_1 reverses the parity of parts; i.e., for λ with k even and m odd parts, we get $f_1(\lambda) = \mu$ with k odd and m even parts and $\mu \in P_o(n)$. Suppose for $\lambda' \neq \lambda''(\in G_{e,0}(n))$ with $\ell(\lambda') = \ell(\lambda'')$, we have $\mu' = f_1(\lambda') = f_1(\lambda'') = \mu''$. Then $\ell_e(\lambda) = \ell_e(\lambda'')$ and so, $\ell_o(\lambda) = \ell_o(\lambda'')$. Now, since λ' and λ'' being distinct, by the definition of f_1 we have at least a tuple $(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $\mu_i' \neq \mu_j''$. Next, we consider the case when $\lambda' \neq \lambda''(\in G_{e,0}(n))$ with $\ell(\lambda') \neq \ell(\lambda'')$ and it is immediate that $\ell(\mu') \neq \ell(\mu'')$ and therefore, $\mu' \neq \mu''$. So, f_1 is an injective map.

For $\lambda \in G_{e,1}(n)$ with $\lambda_e = (\lambda_{e_1}, \ldots, \lambda_{e_k})$ and $\lambda_o = (\lambda_{o_1}, \ldots, \lambda_{o_m})$ where $k + m = \ell(\lambda) = 2r + 1$ for some $r \in \mathbb{Z}_{>0}$. Here we note that, $k > r$ and $k - m \geq 3$ but $k - m = 1$ holds only when $a(\lambda)$ is odd, because $k = m + 1$ and $a(\lambda)$ is even implies that $\lambda \in G_e^0(n)$. We exclude the condition $k = m + 2$ as it contradicts that $k + m = 2r + 1$.

We define $f_2 : G_{e,1}(n) \to P_o(n)$ with $f_2(\lambda) := \mu$, where

$$\mu_e = \begin{cases} ((\lambda_{o_1} + 1), \ldots, (\lambda_{o_m} + 1)) \cup (\lambda_{e_1} + 2) & \text{if } a(\lambda) \text{ is even,} \\ ((\lambda_{o_2} + 1), \ldots, (\lambda_{o_m} + 1)) & \text{if } a(\lambda) \text{ is odd,} \end{cases} \tag{10.2}$$

and

$$\mu_o = \begin{cases} ((\lambda_{e_2}+1),\ldots,(\lambda_{e_{k-r-1}}+1),(\lambda_{e_{k-r}}-1),\ldots,(\lambda_{e_k}-1)) & \text{if } a(\lambda) \text{ even,} \\ ((\lambda_{e_1}+1),\ldots,(\lambda_{e_{k-r-1}}+1),(\lambda_{e_{k-r}}-1),\ldots,(\lambda_{e_k}-1)) \cup (\lambda_{o_1}+2) & \text{otherwise.} \end{cases} \tag{10.3}$$

For $a(\lambda)$ even,

$$\ell_o(\mu) - \ell_e(\mu) = k - 1 - (m+1) = k - m - 2 \geq 1$$

and for $a(\lambda)$ odd,

$$\ell_o(\mu) - \ell_e(\mu) = k + 1 - (m-1) = k - m + 2 \geq 3.$$

Hence, $\mu \in P_o(n)$ and by similar argument as give before, one can show that f_2 is injective.

Next, for $\lambda \in \overline{G_e^0}(n)$ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we define $f_3 : \overline{G_e^0}(n) \to P_o(n)$ by

$$f_3(\lambda) = \mu = ((\lambda_1+1), \lambda_4, \ldots, \lambda_\ell) \cup ((\lambda_2 - 2), (\lambda_3 - 2)) \cup (2,1).$$

Independent of whether λ_2 and λ_3 are odd or even, we can observe that $\ell_o(\mu)-\ell_e(\mu) = 1$ and $a(\mu) = \lambda_1 + 1$ is odd. By definition, f_3 is an injective map. Next, we show that images of $\{f_i\}_{1\leq i\leq 3}$ are mutually disjoint by considering the following cases

1. By definition of the maps given before, $f_1(G_{e,0}(n)) \subsetneq P_o^0(n)$ where

   $$P_o^0(n) := \{\mu \in P_o(n) : \ell(\mu) \equiv 0 \pmod 2\}$$

   and $f_2(G_{e,1}(n)) \subsetneq P_o^1(n)$ with

   $$P_o^1(n) := \{\mu \in P_o(n) : \ell(\mu) \equiv 1 \pmod 2\}.$$

   So, $f_1(G_{e,0}(n)) \cap f_2(G_{e,1}(n)) = \emptyset$.

2. For $\lambda \in G_{e,0}(n)$ with $\ell_e(\lambda) - \ell_o(\lambda) \geq 2$, we have $f_1(\lambda) = \mu \in P_o(n)$ with $\ell_o(\mu) - \ell_e(\mu) \geq 2$ and for $\lambda \in G_{e,0}(n)$ with $\ell_e(\lambda) - \ell_o(\lambda) = 1$, $f_1(\lambda) = \mu \in P_o(n)$ with $\ell_o(\mu) - \ell_e(\mu) = 1$ but then $a(\mu) = \lambda_1 + 1$ is even. Considering $\lambda \in \overline{G_e^0}(n)$, $f_3(\lambda) = \mu \in P_o(n)$ with $a(\mu) = \mu_1$ is odd and $\ell_o(\mu) - \ell_e(\mu) = 1$. Therefore, $f_1(G_{e,0}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$.

3. Let us consider $\lambda \in G_{e,1}(n)$ with $a(\lambda)$ is even and $\ell_e(\lambda) - \ell_o(\lambda) = 3$. Then $f_2(\lambda) = \mu \in P_o(n)$ with $a(\mu)$ is even and $\ell_o(\mu)-\ell_e(\mu) = 1$. For $\ell_e(\lambda)-\ell_o(\lambda) \geq 4$, we have $f_2(\lambda) = \mu$ with $\ell_o(\mu) - \ell_e(\mu) \geq 2$. Moreover, if $a(\lambda)$ is odd then it is immediate that $\ell_o(\mu) - \ell_e(\mu) \geq 3$ and consequently, $f_2(G_{e,1}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$. So, the map $f : \overline{G_e}(n) \to P_o(n)$ is injective.

For $\mu \in P_o(n)$ with its odd component $\mu_o = (\mu_{o_1}, \ldots, \mu_{o_s})$, we define

$$\overline{P_o}(n) := \{\mu \in P_o(n) : \ell_e(\mu) = 2 \text{ and } \mu_{o_i} = 1 \text{ for all } 1 \leq i \leq s\}.$$

By the definition of f, it is clear that $\overline{P_o}(n) \subsetneq P_o(n) \setminus f(\overline{G_e}(n))$. Now, it remains to show that $|\overline{P_o}(n)| > |P_e(n) \setminus \overline{G_e}(n)|$.

For n even and for $\lambda \in \overline{P_o}(n)$, we have $\ell_o(\lambda) = 2k + 2$ for some $k \in \mathbb{Z}_{>0}$. Here we observe that

$$|\{\lambda \in P(n) : \lambda_1 + \lambda_2 = n; \lambda_1, \lambda_2 \text{ both even}\}| = \left\lfloor \frac{n}{4} \right\rfloor \tag{10.4}$$

and

$$|\{\lambda \in P(n) : \lambda_1 + \lambda_2 = n; \lambda_1 \text{ even}, \lambda_2 \text{ odd and } \lambda_2 \in \mathbb{Z}_{\geq 3}\}| = \left\lfloor \frac{n-3}{4} \right\rfloor. \tag{10.5}$$

Since, $\lambda \in \overline{P_o}(n)$ with n even positive integer and $\ell_o(\lambda) = 2k + 2$, for each $k \in \mathbb{Z}_{>0}$, then by (10.4),

$$|\{\lambda \in \overline{P_o}(n) : \lambda_1 + \lambda_2 + (2k+2) \times 1 = n; \lambda_1, \lambda_2 \text{ both even}\}| = \left\lfloor \frac{n-2k-2}{4} \right\rfloor \tag{10.6}$$

and $1 \leq k \leq \dfrac{n-6}{2}$ because k maximizes only when both λ_1 and λ_2 minimum; i.e., only the instance $2 + 2 + (2k+2) \times 1 = n$ which implies $k = \dfrac{n-6}{2}$. Therefore we have,

$$|\overline{P_o}(n)| = \sum_{k=1}^{\frac{n-6}{2}} \left\lfloor \frac{n-2k-2}{4} \right\rfloor. \tag{10.7}$$

Similarly, for n odd, we have $\ell_o(\lambda) = 2k + 1$ with $1 \leq k \leq \dfrac{n-5}{2}$ and

$$|\{\lambda \in \overline{P_o}(n) : \lambda_1 + \lambda_2 + (2k+1) \times 1 = n; \lambda_1, \lambda_2 \text{ even}\}| = \left\lfloor \frac{n-2k-1}{4} \right\rfloor. \tag{10.8}$$

Consequently,

$$|\overline{P_o}(n)| = \sum_{k=1}^{\frac{n-5}{2}} \left\lfloor \frac{n-2k-1}{4} \right\rfloor. \tag{10.9}$$

Now for n even we will show that

$$|P_e(n) \setminus \overline{G_e}(n)| = 1 + \sum_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} \left\lfloor \frac{n-6k+2}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} \left\lfloor \frac{n-6k-2}{4} \right\rfloor. \tag{10.10}$$

We interpret the set $P_e(n) \setminus \overline{G_e}(n)$ as a disjoint union of its three proper subsets given by $P_e(n) \setminus \overline{G_e}(n) = A_1 \cup A_2 \cup A_3$ where,

$$A_1 = \{\lambda \in P_e(n) \setminus \overline{G_e}(n) : 0 \le \lambda_3 \le 1\},\ A_2 = \bigcup_{k\ge 1} A_{2,k},\ \text{and}\ \ A_3 = \bigcup_{k\ge 1} A_{3,k};$$

with

$$\begin{aligned} A_{2,k} &= \{\lambda = (\lambda_1\lambda_2 2^{2k-1}1^{2k}) \vdash n : \lambda_1 \text{ and } \lambda_2 \text{ even}\},\\ A_{3,k} &= \{\lambda = (\lambda_1\lambda_2 2^{2k}1^{2k-1}) \vdash n : \lambda_1 \text{ even and } \lambda_2 \text{ odd}\}. \end{aligned} \tag{10.11}$$

Next, we explicitly describe the sets and will derive their cardinality by separating into three cases.

Case 1(E): We observe that $|A_1| = 1$ because we have only one possibility $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 0)$. We reject the other three possibilities; i.e., $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 1)$ as $\lambda_2 \ge \lambda_3$, $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 0)$ as n even and $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 1)$ as $\lambda \in P_e(n) \setminus \overline{G_e}(n)$. Next, we look at the subset of A_1, say $A_{1,\ge 2} := \{\lambda \in A_1 : \lambda_2 \ge 2\}$ and note that $A_{1,\ge 2} = 0\emptyset$. This is because for $\lambda \in A_{1,\ge 2}$, there are altogether four possibilities for $\lambda_3 \in \{0, 1\}$.

For $\lambda_3 = 0$, the choice $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, \lambda_2, 0)$ and λ_2 is even is impossible as $\lambda \in P_e(n) \setminus \overline{G_e}(n)$ and if λ_2 is odd, again an impossible option since n is even. Whereas for $\lambda_3 = 1$, the choice $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, \lambda_2, 1)$ and λ_2 is even is impossible as n is even and if λ_2 is odd, again an impossible option since $\lambda \in P_e(n) \setminus \overline{G_e}(n)$.

Case 2(E): By (10.4),

$$|A_{2,k}| = \left\lfloor \frac{n - 6k + 2}{4} \right\rfloor \tag{10.12}$$

and $1 \le k \le \lfloor \frac{n-2}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimum; i.e., the instance $2 + 2 + (2k-1) \times 2 + (2k) \times 1 = n$ which implies $k \le \lfloor \frac{n-2}{6} \rfloor$. By (10.12),

$$A_2 = \bigcup_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} A_{2,k} \text{ and } |A_2| = \sum_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} \left\lfloor \frac{n - 6k + 2}{4} \right\rfloor. \tag{10.13}$$

Case 3(E): From (10.5), it follows that

$$|A_{3,k}| = \left\lfloor \frac{(n - 6k + 1) - 3}{4} \right\rfloor = \left\lfloor \frac{n - 6k - 2}{4} \right\rfloor \tag{10.14}$$

and $1 \le k \le \lfloor \frac{n-6}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimum; i.e., the instance $4+3+(2k)\times 2+(2k-1)\times 1 = n$ which implies $k \le \lfloor \frac{n-6}{6} \rfloor$. By (10.14),

$$A_3 = \bigcup_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} A_{3,k} \text{ and } |A_3| = \sum_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} \left\lfloor \frac{n-6k-2}{4} \right\rfloor. \tag{10.15}$$

By *Case 1(E)*, (10.13) and (10.15) we have (10.10).

For all n odd integers greater equal 9, we will show that

$$|P_e(n) \setminus \overline{G_e}(n)| = 1 + \left\lfloor \frac{n-4}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} \left\lfloor \frac{n-6k-1}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} \left\lfloor \frac{n-6k-5}{4} \right\rfloor. \tag{10.16}$$

Similarly as before, we write $P_e(n) \setminus \overline{G_e}(n)$ as a disjoint union of its four proper subsets given by $P_e(n) \setminus \overline{G_e}(n) = B_0 \cup B_1 \cup B_2 \cup B_3$ where,

$$\begin{aligned} B_0 &= \{\lambda = (\lambda_1, \lambda_2, 1) \in P_e(n) \setminus \overline{G_e}(n) : \lambda_2 \ge 4\}, \\ B_1 &= \{\lambda \in P_e(n) \setminus \overline{G_e}(n) : 0 \le \lambda_2 \le 2 \text{ and } 0 \le \lambda_3 \le 1\}, \\ B_2 &= \bigcup_{k\ge 1} B_{2,k}, \text{and } B_3 = \bigcup_{k \ge 1} B_{3,k}; \end{aligned} \tag{10.17}$$

with

$$\begin{aligned} B_{2,k} &= \{\lambda = (\lambda_1 \lambda_2 2^{2k}, 1^{2k+1}) \vdash n : \lambda_1 \text{ and } \lambda_2 \text{ even}\}, \\ B_{3,k} &= \{\lambda = (\lambda_1 \lambda_2 2^{2k+1} 1^{2k}) \vdash n : \lambda_1 \text{ even and } \lambda_2 \text{ odd}\}. \end{aligned}$$

Case 1(O): For $\lambda = (\lambda_1, \lambda_2, 1) \in B_0$ and n is odd, it follows that both λ_1 and λ_2 are even. Therefore minimal choice for n is 9 because otherwise $\lambda_1 \ge \lambda_2 \ge 4$ with the constraint that both λ_1 and λ_2 even would be an impossibility in such context. Moreover, we can observe that

$$|B_0| = \left\lfloor \frac{n-4}{4} \right\rfloor. \tag{10.18}$$

Case 2(O): We observe that $|B_1| = 1$ because we have only one possibility $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 2, 1)$. We reject the other three possibilities; i.e., $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 1)$ as $\lambda_2 \ge \lambda_3$, $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 0)$ and $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 2, 0)$ as n odd, $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 0)$ and $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 1)$ as $\lambda \in P_e(n) \setminus \overline{G_e}(n)$.

Case 3(O): By (10.4),

$$|B_{2,k}| = \left\lfloor \frac{n-6k-1}{4} \right\rfloor \tag{10.19}$$

and $1 \leq k \leq \lfloor \frac{n-5}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimum; i.e., the instance $2+2+(2k)\times 2+(2k+1)\times 1 = n$ which implies $k \leq \lfloor \frac{n-5}{6} \rfloor$. By (10.19),

$$B_2 = \bigcup_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} B_{2,k} \text{ and } |B_2| = \sum_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} \left\lfloor \frac{n-6k-1}{4} \right\rfloor. \tag{10.20}$$

Case 4(O): From (10.5), it follows that

$$|B_{3,k}| = \left\lfloor \frac{(n-6k-2)-3}{4} \right\rfloor = \left\lfloor \frac{n-6k-5}{4} \right\rfloor \tag{10.21}$$

and $1 \leq k \leq \lfloor \frac{n-9}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimum; i.e., the instance $4+3+(2k+1)\times 2+(2k)\times 1 = n$ which implies $k \leq \lfloor \frac{n-9}{6} \rfloor$. By (10.21),

$$B_3 = \bigcup_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} B_{3,k} \text{ and } |B_3| = \sum_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} \left\lfloor \frac{n-6k-5}{4} \right\rfloor. \tag{10.22}$$

By *Case 2(O)*, (10.18), (10.20) and (10.22) we have (10.16).

Therefore, by Lemmas 10.2.1 and 10.2.2, $|\overline{P_o}(n)| > |P_e(n) \setminus \overline{G_e}(n)|$ for all $n \geq \mathbb{Z}_{\geq 14} \cup \{11, 13\}$. To conclude the proof, it remains to check for $n \in \{1, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$ which we did by numerically checking in Mathematica.

10.3 Proof of $d_o(n) > d_e(n)$

Following the definitions in Section 10.2, set

$$\begin{aligned} H_e^0(n) &:= G_e^0(n) \cap D(n), \\ \overline{H_e^0}(n) &:= \{\lambda \in H_e^0(n) : \ell_o(\lambda) > 1\}, \\ \text{and} \quad H_e(n) &:= G_e(n) \cap D(n). \end{aligned}$$

We split $H_e(n)$ into $H_{e,0}(n) = G_{e,0}(n) \cap D(n)$ and $H_{e,1}(n) = G_{e,1}(n) \cap D(n)$. Similarly, define the map $f : H_e(n) \to D_o(n)$ by $f|_{H_{e,0}(n)} = f_1$ and $f|_{H_{e,1}(n)} = f_2$. Since $H_e(n) \subsetneq G_e(n)$, we conclude that the map f is injective by (10.2) and (10.3).

Now we are to show that $d_o(n) - |f(H_e(n))| > d_e(n) - |H_e(n)|$ for all $n > 31$. The subset $D_o(n) \setminus f(H_e(n))$ contains different classes of partitions. One of which is

$$\overline{D_o}(n) := \{\lambda \in D_o(n) \setminus f(H_e(n)) : \ell_o(\lambda) - \ell_e(\lambda) = 1 \text{ and } a(\lambda) \equiv 1 (\text{mod } 2)\}.$$

We note that $\overline{D_o}(n)$ may contain other classes of partitions depending on n is even or odd.

For a partition $\lambda \in \overline{H_e^0}(n)$, we split λ into its even component $\lambda_e = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_{m+1}})$ and $\lambda_o = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_m})$ for some $m \in \mathbb{Z}_{\geq 2}$. Now, me make a transformation of λ into λ^* with

$$\lambda^* = (\lambda_{e_1} + \lambda_{o_1}, \lambda_{e_2} + \lambda_{o_2}) \cup (\lambda_{e_3}, \lambda_{e_4}, \ldots, \lambda_{e_{m+1}}) \cup (\lambda_{o_3}, \lambda_{o_4}, \ldots, \lambda_{o_m}) \in \overline{D_o}(n).$$

We observe that two partitions, say $\lambda, \overline{\lambda} \in \overline{H_e^0}(n)$, where

$$\begin{aligned} \lambda &= (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_{m+1}}) \cup (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_m}) \\ \text{and } \overline{\lambda} &= (\overline{\lambda}_{e_1}, \overline{\lambda}_{e_2}, \ldots, \overline{\lambda}_{e_{m+1}}) \cup (\overline{\lambda}_{o_1}, \overline{\lambda}_{o_2}, \ldots, \overline{\lambda}_{o_m}), \end{aligned}$$

transform to a same partition, say $\mu \in \overline{D_o}(n)$ if and only if

$$\lambda_{e_1} - \overline{\lambda}_{e_1} = \overline{\lambda}_{o_1} - \lambda_{o_1} \equiv 0 (\text{mod } 2) \text{ or/and } \lambda_{e_2} - \overline{\lambda}_{e_2} = \overline{\lambda}_{o_2} - \lambda_{o_2} \equiv 0 (\text{mod } 2).$$

If those cases arise we subtract some multiple of 2 from the greatest part of the resultant partition and add the multiple of 2 to the other even parts which are present in the partition, and continue this process till we have a repetition among the parts of the partition. This process is injective by its definition, and we denote it by g. For example, consider partitions $\lambda = (12, 10, 6, 2) \cup (7, 3, 1)$ and $\overline{\lambda} = (10, 8, 6, 2) \cup (9, 5, 1)$ in $\overline{H_e^0}(41)$, then both λ and $\overline{\lambda}$ maps to the same partition $\mu = (19, 13, 6, 2, 1) \in \overline{D_o}(41)$. Consequently, by the process g, finally $\lambda \mapsto (19, 13, 6, 2, 1)$ whereas $\overline{\lambda} \mapsto (17, 13, 8, 2, 1)$. As a trivial remark, $\overline{H_e^0}(n) = \emptyset$ for all positive even integers $n \leq 14$, since $6 + 4 + 2 + 3 + 1 = 16$ is the least possible option.

Depending on the parity of n, it remains to analyze the left over set

$$\widetilde{H_e^0}(n) := \{\lambda \in H_e^0(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1, \ell_o(\lambda) \leq 1 \text{ and } a(\lambda) \equiv 0 (\text{mod } 2)\}, \quad (10.23)$$

which is unmapped yet (after applying the map f and g).

For n to be an even positive integer, we observe that $\widetilde{H_e^0}(n)$ consists of only one partition (n). An even integer n can be expressed as a sum of two consecutive odd integers if and only if n is divisible by 4. If n is divisible by 4, then for some definite odd integer λ_{o_1} we get $(\lambda_{o_1}, \lambda_{o_1} - 2) \in D_o(n)$, which is not mapped yet. So we map (n) to $(\lambda_{o_1}, \lambda_{o_1} - 2)$. If n is not divisible by 4, then for some definite odd integer λ_{o_1} we get $(\lambda_{o_1}, \lambda_{o_1} - 2, 2) \in D_o(n)$, which is not mapped yet. So in this case we map (n) to $(\lambda_{o_1}, \lambda_{o_1} - 2, 2)$. Therefore, by some elementary observations we get that the theorem is true for all even integer $n > 6$.

Let n be odd. We rewrite (10.23) as

$$\widetilde{H_e^0}(n) := \{\lambda \in H_e^0(n) : \ell_e(\lambda) = 2, \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 0 (\text{mod } 2)\}.$$

Write a partition $\lambda \in \widetilde{H_e^0}(n)$ into its even component $\lambda_e = (\lambda_{e_1}, \lambda_{e_2})$ and odd component $\lambda_o = (\lambda_{o_1})$. We split $\widetilde{H_e^0}(n)$ into following three classes:

1. $\widetilde{H}_{e,1}^0(n) := \{\lambda \in \widetilde{H_e^0}(n) : \lambda_{e_2} = 2\}$,
2. $\widetilde{H}_{e,2}^0(n) := \{\lambda \in \widetilde{H_e^0}(n) : \lambda_{e_2} \geq 6\}$, and
3. $\widetilde{H}_{e,3}^0(n) := \{\lambda \in \widetilde{H_e^0}(n) : \lambda_{e_2} = 4\}$.

Now we consider the following three classes of partitions from the set of partitions, say $\widetilde{D_o}(n) \subsetneq D_o(n)$ which have no preimage yet:

1. $\widetilde{D}_{o,1}(n) := \{\pi \in \widetilde{D_o}(n) : \ell(\pi) = 4 \text{ and } \pi_{o_1} - \pi_{o_2} = 2\}$,
2. $\widetilde{D}_{o,2}(n) := \{\pi \in \widetilde{D_o}(n) : \ell_o(\pi) = 3 \text{ and } \pi_{o_1} - \pi_{o_2} = 2\}$, and
3. $\widetilde{D}_{o,3}(n) := \{\pi \in \widetilde{D_o}(n) : \ell_o(\pi) - \ell_e(\pi) = 1, a(\pi) \equiv 0 (\text{mod } 2) \text{ and } \pi_{e_1} - \pi_{o_1} = 1 \text{ or } 3\}$.

Now we construct an injective map from $\widetilde{H}_{e,1}^0(n)$ to $\widetilde{D}_{o,1}(n)$. Let $\lambda = (\lambda_{e_1}, 2) \cup (\lambda_{o_1}) \in \widetilde{H}_{e,1}^0(n)$. Define a transformation S such that $S(\lambda) = (\lambda_{e_1}) \cup (\lambda_{o_1} + 1, 1)$. Now define S^* such that

$$S^*(S(\lambda)) = \begin{cases} S(\lambda) & \text{if } \lambda_{e_1} \equiv 0 (\text{mod } 4), \\ (\lambda_{e_1} - 2) \cup (\lambda_{o_1} + 1, 3) & \text{if } \lambda_{e_1} \equiv 2 (\text{mod } 4). \end{cases}$$

Now define S^{**} such that $S^{**}(S^*(S(\lambda))) = (\lambda_{o_2}, \lambda_{o_2} - 2) \cup (\lambda_{o_3}) \cup (\lambda_{o_1} + 1)$, where $\lambda_{o_2} + \lambda_{o_2} - 2 = \lambda_{e_1}$ or $\lambda_{e_1} - 2$, and $\lambda_{o_3} = 1$ or 3 accordingly. For example: $(24, 5, 2)$ maps to $(13, 11, 6, 1)$ and $(22, 7, 2)$ maps to $(11, 9, 8, 3)$. This process is injective.

Our next objective is to embed the set $\widetilde{H}_{e,2}^0(n)$ into a subset of $\widetilde{D_o}(n)$ which is not mapped till now. Define a transformation U such that $U(\lambda) = ((\lambda_{e_1 - 3}, 3) \cup (\lambda_{o_1})) \cup$

$(\lambda_{e_2}, 2)$ for $\lambda \in \widetilde{H}^0_{e,2}(n)$. Associated with U, let us define U^* in such a way that

$$U^*(U(\lambda)) = \begin{cases} U(\lambda) & \text{if } \lambda_{o_1} \neq 3 \text{ and } \lambda_{e_1} - 3 \neq \lambda_{o_1}, \\ (\lambda_{e_1} - 3, 5, 1) \cup (\lambda_{e_2} - 2, 2) & \text{if } \lambda_{o_1} = 3, \\ \big(\text{this transformation is impossible for } n \leq 17\big) & \\ \big(\text{e.g. } (10, 6, 3) \mapsto (7, 5, 4, 2, 1)\big) & \\ ((\lambda_{e_1} - 3, \lambda_{o_1} - 2) \cup (5)) \cup (\lambda_{e_2} - 2, 2) & \text{if } \lambda_{e_1} - 3 = \lambda_{o_1}, \\ \big(\text{this transformation is impossible for } n \leq 25\big) & \\ \big(\text{e.g. } (12, 9, 6) \mapsto (9, 7, 5, 4, 2)\big) & \\ ((\lambda_{e_1} - 3, \lambda_{o_1} - 4) \cup (5)) \cup (\lambda_{e_2} - 2, 4) & \text{if } \lambda_{e_1} - 1 = \lambda_{o_1}, \lambda_{e_2} \neq 6, \\ \big(\text{this transformation is impossible for } n \leq 29\big) & \\ \big(\text{e.g. } (12, 11, 8) \mapsto (9, 7, 6, 5, 4)\big) & \\ ((\lambda_{e_1} - 3, \lambda_{o_1} - 4) \cup (3)) \cup (6, 4) & \text{if } \lambda_{e_1} - 1 = \lambda_{o_1}, \text{ and } \lambda_{e_2} = 6. \\ \big(\text{this transformation is impossible for } n \leq 23\big) & \\ \big(\text{e.g. } (10, 9, 6) \mapsto (7, 6, 5, 4, 3)\big) & \end{cases}$$

Denote the resulting transformation U^*U by $\widetilde{U}$. Note that $\ell(\widetilde{U}(\lambda)) = 5$, where the resulting partitions; i.e., images under $\widetilde{U}$, contains parts from $\{3, 5\}$ and its smallest even part from $\{2, 4\}$. For a partition $\lambda \in H^0_e(n)$ with $\ell(\lambda) = 7$, $\ell(g(\lambda)) = 5$. Now, assume $\widetilde{U}(\lambda) = g(\mu)$ for some partition μ with $\ell(\mu) = 7$. In the map g, after applying the first transformation, the other transformations are applied on the even parts only (if it necessary). If $\widetilde{U}(\lambda) = g(\mu)$, then we remove one of 2, 3, 4, or 5 (which one exists in $g(\mu)$) from $g(\mu)$ by applying similar transformation.

Now we compare the number of partitions in $\widetilde{H}^0_{e,3}(n)$ with $\widetilde{D}_{o,2}(n)$ and $\widetilde{D}_{o,3}(n)$.

$$\left|\widetilde{H}^0_{e,3}(n)\right| = \left\lfloor \frac{n-3}{4} \right\rfloor.$$

Let $\pi = (\pi_{o_1}, \pi_{o_1} - 2, \pi_{o_3}) \in \widetilde{D}_{o,2}(n)$, and $\pi_{o_1} = 2k + 1$. Then the least possible value of k is given by

$$(2k + 1) + (2k - 1) \geq 2\{n - (2k + 1) - (2k - 1)\} + 6,$$

which implies $k = \lceil \frac{n+3}{6} \rceil$. So the total number of partitions in $\widetilde{D}_{o,2}(n)$ is at least $\lceil \frac{n-4k}{4} \rceil$, which is equal to $\lceil \frac{n}{4} - \lceil \frac{n+3}{6} \rceil \rceil$. Any odd integer n is of the form $12\ell + r$, where $r = 1, 3, 5, 7, 9,$ or 11. Calculating for all the six cases, we get the total number of partitions in this class is

$$= \begin{cases} \left\lfloor \frac{n}{12} \right\rfloor, \text{ if } n \neq 12\ell + 9, \\ \left\lfloor \frac{n}{12} \right\rfloor + 1, \text{ if } n = 12\ell + 9. \end{cases}$$

Now, $\left\lfloor \frac{n-3}{4} \right\rfloor - \begin{cases} \left\lfloor \frac{n}{12} \right\rfloor, \text{ if } n \neq 12\ell + 9, \\ \left\lfloor \frac{n}{12} \right\rfloor + 1, \text{ if } n = 12\ell + 9. \end{cases} = \left\lfloor \frac{n}{6} \right\rfloor - 1, \left\lfloor \frac{n}{6} \right\rfloor, \text{ or } \left\lfloor \frac{n}{6} \right\rfloor + 1.$

It remains to estimate a lower bound of $\left|\widetilde{D}_{o,3}(n)\right|$. Let $\pi = (\pi_{o_1}, \pi_{o_2}, \pi_{o_3}) \cup (\pi_{e_1}, \pi_{e_2}) \in \widetilde{D}_{o,3}(n)$. The greatest odd part π_{o_1} is the largest possible value if π contains 3, 1 and 2. So the largest possible value of π_{o_1} is $\frac{n-9}{2}$ or $\frac{n-7}{2}$ if $n \equiv 3 \pmod 4$ or $n \equiv 1 \pmod 4$ respectively. The smallest possible value of π_{o_1} is greater than $\left\lfloor \frac{n}{5} \right\rfloor$. When π_{o_1} is not the largest or the smallest possible value, then for each possible value of π_{o_1}, we get at least 6 partitions. So total number of partitions is greater than

$$6 \times \frac{1}{2} \times \left\{ \frac{n-9}{2} - 1 - \left(\left\lfloor \frac{n}{5} \right\rfloor + 1 + 1 \right) \right\} \geq 3 \times \left\{ \frac{n-9}{2} - \frac{n}{5} - 3 \right\} = \frac{9(n-25)}{10}.$$

Now, $\left\lfloor \frac{9(n-25)}{10} \right\rfloor > \left\lfloor \frac{n}{6} \right\rfloor + 1$ for all odd positive integers $n > 31$.

This proves the theorem for all $n > 31$. We can verify the result numerically for $19 < n \leq 31$.

10.4 Proof of $q_o(n) < q_e(n)$: the reverse case

Define

$$g_o(\lambda) = \ell_o(\lambda) - \ell_e(\lambda) \text{ for } \lambda \in P_o(n)$$

and

$$g_e(\pi) = \ell_e(\pi) - \ell_e(\pi) \text{ for } \pi \in P_e(n).$$

Let us split $Q_o(n)$ into the following two disjoint subsets, defined by

$$I_o^1(n) := \{\lambda \in Q_o(n) : g_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 1 (\text{mod } 2)\}$$

and

$$I_o(n) := Q_o(n) \setminus I_o^1(n).$$

Then the map $f : I_o(n) \to Q_e(n)$ defined by the restriction maps f_1 and f_2 (cf. (10.2)-(10.3)) is injective and consequently, $f(I_o(n)) := \widetilde{I}_e(n) \subsetneq Q_e(n)$.

For $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m) \in I_o^1(n)$, we split $I_o^1(n)$ into

1. $I^1_{o,1}(n) := \{\lambda \in I^1_o(n) : \lambda_3 \geq 6\}$.

2. $I^1_{o,2}(n) := \bigcup_{t=2}^{5} I^1_{o,2_t}(n)$, where $I^1_{o,2_t}(n) := \{\lambda \in I^1_o(n) : \lambda_3 = t\}$.

Define $\widetilde{I}^c_e(n) := Q_e(n) \setminus \widetilde{I}_e(n)$, and for $\pi = (\pi_1, \pi_2, \ldots, \pi_s) \in \widetilde{I}^c_e(n)$ with its even component $\pi_e = (\pi_{e_1}, \pi_{e_2}, \ldots, \pi_{e_k})$, we consider the following disjoint classes, where $n \geq 21$:

1. $\widetilde{I}^c_{e,1}(n) := \{\pi \in \widetilde{I}^c_e(n) : g_e(\pi) = 1, a(\pi) \equiv 0 (\text{mod } 2), \text{and } \pi_1 \neq \pi_2\}$.

2. $\widetilde{I}^c_{e,2}(n) := \{\pi \in \widetilde{I}^c_e(n) : g_e(\pi) = 1, a(\pi) \equiv 0 (\text{mod } 2), \text{and } \pi_1 = \pi_2\}$.
   Example: $(4, 4, 3, 3, 3, 2, 2)$.

3. $\widetilde{I}^c_{e,3}(n) := \{\pi \in \widetilde{I}^c_e(n) : g_e(\pi) = 1, a(\pi) \equiv 1 (\text{mod } 2), \text{and } \pi_1 - \pi_2 \in \{0, 1\}\}$.
   Example: $(5, 5, 3, 2, 2, 2, 2)$ and $(5, 4, 3, 3, 2, 2, 2)$.

4. $\widetilde{I}^c_{e,4}(n) := \{\pi \in \widetilde{I}^c_e(n) : \ell(\pi) = 2r \text{ and } \text{mult}_\pi(2) \geq r + 1\}$.
   Example: $(4, 4, 3, 2, 2, 2, 2, 2)$.

5. $\widetilde{I}^c_{e,5}(n) := \{\pi \in \widetilde{I}^c_e(n) : \ell(\pi) = 2r + 1 \text{ and } \text{mult}_\pi(2) \geq r + 2\}$.
   Example: $(7, 4, 2, 2, 2, 2, 2)$.

6. $\widetilde{I}^c_{e,6}(n) := \{\pi \in \widetilde{I}^c_e(n) : g_e(\pi) = 2, \ell(\pi) \equiv 0 (\text{mod } 2), \text{and } \pi_{e_1} \neq \pi_{e_2}\}$.
   Example: $(13, 8, 8, 7, 4, 2)$.

7. $\widetilde{I}^c_{e,7}(n) := \{\pi \in \widetilde{I}^c_e(n) : g_e(\pi) = 3, \ell(\pi) \equiv 1 (\text{mod } 2), \text{and } \pi_{e_1} = \pi_{e_2}\}$.
   Example: $(13, 8, 8, 7, 4, 2, 2)$.

Now we define a map $\psi_1 : I^1_{o,1}(n) \to \widetilde{I}^c_{e,1}(n)$. For $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m) \in I^1_{o,1}(n)$,

$$\psi_1(\lambda) := (\lambda_1 + 1, \lambda_2 - 4, \lambda_3 - 4, \lambda_4, \ldots, \lambda_m) \cup (4, 3),$$

and hence ψ_1 is a well defined injective map.
For the rest of the proof, in a partition $\lambda \vdash n$, as a part x (resp. y) denotes 3 or 5 (resp. 2 or 4).
Now we construct injective maps on $I^1_{o,2}(n)$ by considering the following five cases.
Case 1: $\lambda \in I^1_{o,2_2}(n)$.
This case is satisfied only when n is even. For $\lambda = (\lambda_1, \lambda_2, 2) \in I^1_{o,2_2}(n)$, define

$$\psi_2(\lambda) := \pi = (\lambda_1 - \lambda_2 + 2, 2, 2, \ldots, 2) \text{ with } \text{mult}_\pi(2) = \lambda_2.$$

For example, $(23,5,2) \mapsto (20,2,2,2,2,2)$. One can observe that the map ψ_2 is injective and

$$\psi_2(I^1_{o,2_2}(n)) := \widetilde{I}^1_{e,2_2}(n) \subsetneq \widetilde{I}^c_{e,4}(n),$$

where

$$\widetilde{I}^1_{e,2_2}(n) = \begin{cases} \{\pi \in \widetilde{I}^c_{e,4}(n) : \pi_1 \geq 2 \text{ and } \pi_j = 2 \text{ for all } j > 1\} & \text{if } n \equiv 0 (\text{mod } 4), \\ \{\pi \in \widetilde{I}^c_{e,4}(n) : \pi_1 \geq 4 \text{ and } \pi_j = 2 \text{ for all } j > 1\} & \text{if } n \equiv 2 (\text{mod } 4). \end{cases}$$

Case 2: $\lambda \in I^1_{o,2_3}(n)$.
Subdivide $I^1_{o,2_3}(n) := \bigcup_{t=1}^3 I^1_{o,2_{3,t}}(n)$ with

$$\begin{aligned} & I^1_{o,2_{3,1}}(n) := \{\lambda \in I^1_{o,2_3}(n) : \lambda_2 \equiv 1 (\text{mod } 2)\}, \\ & I^1_{o,2_{3,2}}(n) := \{\lambda \in I^1_{o,2_3}(n) : \lambda_2 \equiv 0 (\text{mod } 2) \text{ and } \lambda_1 \neq 5\}, \\ \text{and} \quad & I^1_{o,2_{3,3}}(n) := \{\lambda \in I^1_{o,2_3}(n) : \lambda_2 \equiv 0 (\text{mod } 2) \text{ and } \lambda_1 = 5\}. \end{aligned}$$

We define the map $\psi_3 : I^1_{o,2_3}(n) \to \widetilde{I}^c_e(n)$ by $\psi_3|_{I^1_{o,2_{3,t}}(n)} := \psi_{3,t}$ for $1 \leq t \leq 3$.
We take $\psi_{3,1} := \psi_2$, given above and so,

$$\begin{aligned} \widetilde{I}^1_{e,2_{3,1}}(n) &:= \psi_{3,1}(I^1_{o,2_{3,1}}(n)) \\ &= \Big\{\pi \in \widetilde{I}^c_{e,4}(n) : a(\pi) = \pi_{e_1} \geq 2 \text{ or } 4, \text{mult}_\pi(3) > 1, \text{and } \pi_{e_j} = 2 \text{ for } j > 1\Big\}. \end{aligned}$$

For example, $(23,3,3,2,2) \mapsto (22,3,2,2,2,2,2)$.
We note that for any $\lambda \in I^1_{o,2_{3,2}}(n)$, $\lambda = (\lambda_1, \lambda_2, 3, \ldots, 3, 2, \ldots, 2)$ with $\text{mult}_\lambda(2) = r$ and $\text{mult}_\lambda(3) = r+1$ for some $r \in \mathbb{Z}_{\geq 0}$. Define $\psi_{3,2}(\lambda) := \pi = (\lambda_1 - 4, 3, \ldots, 3, 2, \ldots, 2)$ with $\text{mult}_\pi(3) \geq r+1$ and $\text{mult}_\pi(2) = \frac{1}{2}\lambda_2 + 2$. Consequently,

$$\begin{aligned} \widetilde{I}^1_{e,2_{3,2}}(n) &:= \psi_{3,2}(I^1_{o,2_{3,2}}(n)) \\ &= \Big\{\pi \in \cup_{k=4}^5 \widetilde{I}^c_{e,k}(n) : a(\pi) = \pi_{o_1} \geq 3, \text{mult}_\pi(3) > 1, \text{and } \pi_{e_i} = 2 \text{ for } i \geq 1\Big\}. \end{aligned}$$

For example, $(21,8,3) \mapsto (17,3,2,2,2,2,2,2)$.
Similar as before, observe that for any $\lambda \in I^1_{o,2_{3,3}}(n)$, $\lambda = (5,4,3,3,\ldots,3,2,2\ldots,2)$ with $\text{mult}_\lambda(2) = r$ and $\text{mult}_\lambda(3) = r+1$ for some $r \in \mathbb{Z}_{\geq 0}$. Define $\psi_{3,3}(\lambda) := \pi = (3,3,\ldots,3,2,2,\ldots,2)$ with $\text{mult}_\pi(3) = r+2$ and $\text{mult}_\pi(2) = r+3$. Consequently,

$$\begin{aligned} \widetilde{I}^1_{e,2_{3,3}}(n) &:= \psi_{3,3}(I^1_{o,2_{3,3}}(n)) \\ &= \Big\{\pi \in \widetilde{I}^c_{e,3}(n) : \text{mult}_\pi(3) = \text{mult}_\lambda(3) + 1 \text{ and } \text{mult}_\pi(2) = \text{mult}_\lambda(2) + 3\Big\}. \end{aligned}$$

For example, $(5,4,3,3,2) \mapsto (3,3,3,2,2,2,2)$.
Moreover, $\bigcap_{1\le i\le 3} \widetilde{I}^1_{e,2_{3,i}}(n) = \emptyset$, for each i, $\{\widetilde{I}^1_{e,2_{3,i}}(n)\} \cap \widetilde{I}^1_{e,2_2}(n) = \emptyset$; and also $\{\widetilde{I}^1_{e,2_{3,i}}(n)\}_{1\le i\le 3}$ are mutually disjoint with the images under the maps f and ψ_1 considered before.
Case 3: $\lambda \in I^1_{o,2_4}(n)$.
For each pair of (λ_1,λ_2), which exist, we get a unique partition. This is also satisfied in the above two cases, but we have not considered this in the above two cases to find different images. Now analyze the disjoint subsets of $I^1_{o,2_4}(n)$, depending on λ_2 is even or odd, defined as follows

$$\begin{aligned} I^1_{o,2_{4,1}}(n) &:= \{\lambda \in I^1_{o,2_4}(n) : \lambda_2 \equiv 1 (\text{mod } 2) \text{ and } \ell(\lambda) \le 5\}, \\ I^1_{o,2_{4,2}}(n) &:= \{\lambda \in I^1_{o,2_4}(n) : \lambda_2 \equiv 1 (\text{mod } 2) \text{ and } \ell(\lambda) > 5\}, \\ \text{and} \quad I^1_{o,2_{4,3}}(n) &:= \{\lambda \in I^1_{o,2_4}(n) : \lambda_2 \equiv 0 (\text{mod } 2)\}. \end{aligned}$$

When $\lambda \in I^1_{o,2_{4,1}}(n)$, then the partitions are of the two forms: $\lambda = (\lambda_1,\lambda_2,4)$ or $\lambda = (\lambda_1,\lambda_2,3) \cup (4,y)$, where $\lambda_1 \ge \lambda_2 \ge 5$ are odds.
a) $\lambda_1 > 2\lambda_2 + 3$:

$$(\lambda_1,\lambda_2,3)\cup(4,y) \xrightarrow{\psi_{4,1}} \begin{cases} (6^2 4) \cup \left(\dfrac{\lambda_1+\lambda_2-12}{2}, \dfrac{\lambda_1+\lambda_2-12}{2}, 3\right) \text{ if } \lambda_1-\lambda_2 \equiv 0 (\text{mod } 4), \\ (6^3 y) \cup \left(\dfrac{\lambda_1+\lambda_2-14}{2}, \dfrac{\lambda_1+\lambda_2-14}{2}, 3\right) \text{ if } \lambda_1-\lambda_2 \equiv 2 (\text{mod } 4), \end{cases}$$

and

$$(\lambda_1,\lambda_2,4) \xrightarrow{\psi_{4,1}} \begin{cases} (6,6,4) \cup \left(\dfrac{\lambda_1+\lambda_2-12}{2}, \dfrac{\lambda_1+\lambda_2-12}{2}\right) \text{ if } \lambda_1-\lambda_2 \equiv 0 (\text{mod } 4), \\ (6,6,6) \cup \left(\dfrac{\lambda_1+\lambda_2-14}{2}, \dfrac{\lambda_1+\lambda_2-14}{2}\right) \text{ if } \lambda_1-\lambda_2 \equiv 2 (\text{mod } 4). \end{cases}$$

b) $2\lambda_2 + 3 \ge \lambda_1 \ge \lambda_2 + 8$:

$$\begin{aligned} (\lambda_1,\lambda_2,3)\cup(4,y) &\xrightarrow{\psi_{4,1}} (\lambda_1-\lambda_2-4,4,4)\cup(\lambda_2,\lambda_2,3) \\ \text{and} \quad (\lambda_1,\lambda_2,4) &\xrightarrow{\psi_{4,1}} (\lambda_1-\lambda_2-4,4,4)\cup(\lambda_2,\lambda_2). \end{aligned}$$

c) $\lambda_2 + 8 > \lambda_1$: For $\lambda_2 > 5$,

$$\begin{aligned}(\lambda_1, \lambda_2, 4) &\xrightarrow{\psi_{4,1}} (\lambda_1 - 3, \lambda_2 - 3, 2, 2, 2, 2, 2),\\ (\lambda_1, \lambda_2, 4, 4, 3) &\xrightarrow{\psi_{4,1}} (\lambda_1 - 1, \lambda_2 - 1, 3, 2, 2, 2, 2, 2),\\ \text{and} \quad (\lambda_1, \lambda_2, 4, 3, 2) &\xrightarrow{\psi_{4,1}} (a - 3, b - 3, 3, 2, 2, 2, 2, 2, 2).\end{aligned}$$

Whereas for the left over partitions are mapped as follows

$$\begin{aligned}(7, 5, 4, 3, 2) &\xrightarrow{\psi_{4,1}} (5, 4, 4, 4, 4),\\ (9, 5, 4, 3, 2) &\xrightarrow{\psi_{4,1}} (5, 4, 4, 4, 4, 2),\\ \text{and} \quad (11, 5, 4, 3, 2) &\xrightarrow{\psi_{4,1}} (5, 4, 4, 4, 4, 4).\end{aligned}$$

For $\lambda \in I^1_{o,2_{4,2}}(n)$, can be written explicitly in the form

$$\lambda = (\lambda_1, \lambda_2, 4, 4, \ldots, 4, 3, 3, \ldots, 3, 2, \ldots, 2),$$

with λ_1, λ_2 both odd, $\mathrm{mult}_\lambda(4) = s + 1, \mathrm{mult}_\lambda(2) = r$, and $\mathrm{mult}_\lambda(3) = r + s$ for some $r, s \in \mathbb{Z}_{\geq 0}$ subject to the condition that $r + s \geq 2$. Then

$$\lambda \xrightarrow{\psi_{4,2}} \begin{cases}(\lambda_1, \lambda_2, 3, 2, \ldots, 2) & \text{if } r + s \equiv 1 \pmod 2,\\ (\lambda_1, \lambda_2, 2, \ldots, 2) & \text{if } r + s \equiv 0 \pmod 2.\end{cases}$$

with

$$\mathrm{mult}_{\psi_{4,2}(\lambda)}(2) = \begin{cases}2s + 2 + 3\frac{r+s-1}{2} + r & \text{if } r + s \equiv 1 \pmod 2,\\ 2s + 2 + 3\frac{r+s}{2} + r & \text{if } r + s \equiv 0 \pmod 2.\end{cases}$$

For example $(15, 7, 4, 3, 3, 3, 2, 2, 2) \mapsto (15, 7, 3, 2, 2, 2, 2, 2, 2, 2, 2)$. By definition of the map $\psi_{4,2}$, these images are different from all the above images.
For $\lambda \in I^1_{o,2_{4,3}}(n)$, can be expressed as $\lambda = (\lambda_1, \lambda_2, 4, 4, \ldots, 4, 3, 3, \ldots, 3, 2, \ldots, 2)$ with λ_1 odd, λ_2 even, $\mathrm{mult}_\lambda(4) = s + 1, \mathrm{mult}_\lambda(2) = r$, and $\mathrm{mult}_\lambda(3) = r + s + 2$ for some $r, s \in \mathbb{Z}_{\geq 0}$. Then

$$\lambda = \xrightarrow{\psi_{4,3}} \begin{cases}(\lambda_1, \lambda_2, 3, 2, \ldots, 2) & \text{if } r + s \equiv 1 \pmod 2,\\ (\lambda_1, \lambda_2, 2, \ldots, 2) & \text{if } r + s \equiv 0 \pmod 2.\end{cases}$$

with

$$\mathrm{mult}_{\psi_{4,3}(\lambda)}(2) = \begin{cases}2s + 2 + 3\frac{r+s+1}{2} + r & \text{if } r + s \equiv 1 \pmod 2,\\ 2s + 2 + 3\frac{r+s+2}{2} + r & \text{if } r + s \equiv 0 \pmod 2.\end{cases}$$

For example $(13,6,4,3,3,3,2) \mapsto (13,6,3,2,2,2,2,2,2)$. Moreover, the images under the map $\psi_{4,3}$ are different from all the above images.

Case 4: $\lambda \in I^1_{o,2_5}(n)$.

$\lambda = \lambda_e \cup \lambda_o \in D_e(n)$ with $\ell(\lambda) = 2r = k + m$ ($m = \ell_o(\lambda)$), then $\lambda_{e_i} \neq \lambda_{e_j}$ for all $1 \leq i \neq j \leq r - m + 1$ if λ has a preimage under the map f (defined by f_1 and f_2). More precisely, $\lambda_{e_{r-m}} \neq \lambda_{e_{r-m+1}}$. For example, $(15,12,12,12,8,8,3,3,2,2,2,2,2,2) \in D_e(n)$, but it has no preimage since $\lambda_{e_4} = \lambda_{e_5} = 8$. So, when $\ell(\lambda) = (m+2)+m$, then $\lambda_{e_1} \neq \lambda_{e_2}$ in order to have a preimage under f. As a conclusion, we see that $\widetilde{I}^c_{e,6}(n)$ has no preimage yet.

Similarly as before, we further split $I^1_{o,2_5}(n)$ depending on λ_2 is even or odd, given by

$$I^1_{o,2_{5,1}}(n) := \{\lambda \in I^1_{o,2_5}(n) : \lambda_2 \equiv 1 \pmod 2 \text{ and } \ell(\lambda) > 5\},$$
$$I^1_{o,2_{5,2}}(n) := \{\lambda \in I^1_{o,2_5}(n) : \lambda_2 \equiv 1 \pmod 2 \text{ and } \ell(\lambda) = 5\},$$

and

$$I^1_{o,2_{5,3}}(n) := \{\lambda \in I^1_{o,2_5}(n) : \lambda_2 \equiv 0 \pmod 2\}.$$

For $\lambda \in I^1_{o,2_{5,1}}(n)$, we want to define a map, say $\psi_{5,1} : I^1_{o,2_{5,1}}(n) \to \widetilde{I}^c_{e,6}(n)$. First, we explicitly write such λ as

$$\lambda = \lambda_e \cup \lambda_o := (\lambda_1, \lambda_2, 5, x, \dots, x) \cup (\lambda_{e_1}, \lambda_{e_2}, \dots, \lambda_{e_r}),$$

where $\lambda_{e_i} \in \{4, 2\}$ for $1 \leq i \leq r$ and $r \in \mathbb{Z}_{\geq 3}$ along with the property that $\lambda_{e_i} = \lambda_{e_j}$ for some $1 \leq i \neq j \leq r$. Now define

$$\lambda \xrightarrow{\psi_{5,1}} \begin{cases} (\lambda_1, x, \dots, x) \cup \left(\hat{\lambda}_e \cup \left(\lambda_{e_i} + \frac{\lambda_2+1}{2}, \lambda_{e_j} + \frac{\lambda_2+1}{2}, 4\right)\right) & \text{if } \lambda_2 \equiv 3 \pmod 4 \\ (\lambda_1, x, \dots, x) \cup \left(\hat{\lambda}_e \cup \left(\lambda_{e_i} + \frac{\lambda_2+3}{2}, \lambda_{e_j} + \frac{\lambda_2+3}{2}, 2\right)\right) & \text{if } \lambda_2 \equiv 1 \pmod 4, \end{cases}$$

where $\hat{\lambda}_e := (\lambda_{e_1}, \dots, \lambda_{e_{i-1}}, \lambda_{e_{i+1}}, \lambda_{e_{j-1}}, \lambda_{e_{j+1}}, \dots, \lambda_{e_r})$. For example

$$(13,9,5,4,3,2,2) \mapsto (13,8,8,4,3,2).$$

The map $\psi_{5,1}$ may not be necessarily one to one. Then by adding and subtracting some multiples of 2 from the parts of the $\psi_{5,1}(\lambda)$ except the part $a(\psi_{5,1}(\lambda))$, we can map them to different partitions. For example, $(9,9,5,3,2,2,2)$ and $(9,5,5,4,4,3,2)$ both transform into $(9,8,8,3,2,2)$. Then for one preimage we can change the image to $(9,7,6,6,2,2)$.

When $\lambda \in I^1_{o,2_{5,2}}(n)$, then n must be odd. Here partition λ is of three form: $(\lambda_1, \lambda_2, 5, 4, 4)$, $(\lambda_1, \lambda_2, 5, 4, 2)$, and $(\lambda_1, \lambda_2, 5, 2, 2)$. For each of these partitions, define

$$(\lambda_1, \lambda_2, 5, 4, 4) \xrightarrow{\psi_{5,2}} \begin{cases} (\lambda_1 - 5, \lambda_2 - 1, 2, 2, 2, 2, 2) \cup (9) & \text{if } \lambda_1 \geq \lambda_2 + 4, \\ (\lambda_1 - 5, \lambda_2 - 5, 2, 2, 2, 2, 2, 2, 2) \cup (9) & \text{if } \lambda_1 < \lambda_2 + 4, \end{cases}$$

$$(\lambda_1,\lambda_2,5,4,2) \xrightarrow{\psi_{5,2}} \begin{cases} (\lambda_1-5,\lambda_2-1,2,2,2,2,2)\cup(7) & \text{if } \lambda_1\ge\lambda_2+4,\\ (\lambda_1-5,\lambda_2-5,2,2,2,2,2,2,2)\cup(7) & \text{if } \lambda_1<\lambda_2+4,\end{cases}$$

and

$$(\lambda_1,\lambda_2,5,2,2) \xrightarrow{\psi_{5,2}} \begin{cases} (\lambda_1-5,\lambda_2-1,2,2,2,2,2)\cup(5) & \text{if } \lambda_1\ge\lambda_2+4,\\ (\lambda_1-5,\lambda_2-5,2,2,2,2,2,2,2)\cup(5) & \text{if } \lambda_1<\lambda_2+4.\end{cases}$$

For example $(13,9,5,4,2)\mapsto(8^2 72^5)$ and $(11,9,5,4,4)\mapsto(9,6,4,2,2,2,2,2,2,2,2)$. Here, $\lambda_1-5,\lambda_2-5\ge 4$ for all $n\ge 31$. If a partition λ has 3 parts which are greater then equal to 3, then $\mathrm{mult}_{\psi_{5,2}(\lambda)}(2)\ge 5$ so as to belong to the $\widetilde{I}^c_{e,4}(n)$; i.e., $\psi_{5,2}$ is an injective map to $\widetilde{I}^c_{e,4}(n)$, for all $n\ge 31$.
Consider $\lambda\in I^1_{o,2_{5,3}}(n)$. Write $\lambda=(\lambda_1,5,x,\ldots,x)\cup(\lambda_2,y,\ldots,y)$ with $\lambda_1\ge 7$ odd, $\lambda_2\ge 6$ even, and total no. of x = total no. of $y=r$ for some non-negative integer r.
a) If $\lambda_1>2\lambda_2+3$, then

$$\lambda \xrightarrow{\psi_{5,3}} \begin{cases} (7,5,x,\ldots,x)\cup\left(\frac{\lambda_1-7}{2},\frac{\lambda_1-7}{2},\lambda_2,y,\ldots,y\right) & \text{if } \lambda_1\equiv 3 \pmod 4,\\ (5,5,x,\ldots,x)\cup\left(\frac{\lambda_1-5}{2},\frac{\lambda_1-5}{2},\lambda_2,y,\ldots,y\right) & \text{if } \lambda_1\equiv 1 \pmod 4.\end{cases}$$

b) If $2\lambda_2+3\ge\lambda_1>\lambda_2+5$, then $\lambda\xrightarrow{\psi_{5,3}}(\lambda_1-\lambda_2-5,5,y,\ldots,y)\cup(\lambda_2,\lambda_2,x,\ldots,x,2)$.
c) If $\lambda_1=\lambda_2+5$, or λ_2+3, then define

$$\lambda \xrightarrow{\psi_{5,3}} \begin{cases} (\lambda_2-1,5,x,\ldots,x)\cup\Big((\lambda_2-2)\cup(6,y,\ldots,y,2)\Big) & \text{if } \lambda_1=\lambda_2+5,\\ (\lambda_2-1,5,x,\ldots,x)\cup(\lambda_2-2,4,y,\ldots,y,2) & \text{if } \lambda_1=\lambda_2+3.\end{cases}$$

Here we exclude the partitions λ of the form $(13,5,x,\ldots,x)\cup(8,y,\ldots,y)$, since $(13,8,5)$ maps to $(7,6,6,5,2)$, and from (b) we see that $(15,6,5)$ also maps to $(7,6,6,5,2)$.
d) It remains to consider the two left-over classes of partitions, given by

$$\lambda=(13,5,x,\ldots,x)\cup(8,y,\ldots,y)$$
$$\text{and}\quad \lambda=(\lambda_1,5,x,\ldots,x)\cup(\lambda_2,y\ldots,y) \text{ with } \lambda_1=\lambda_2+1.$$

For these two classes of λ, define

$$(13,5,x,\ldots,x)\cup(8,y,\ldots,y)\xrightarrow{\psi_{5,3}}(x,\ldots,x)\cup(12,12,y,\ldots,y,2)$$
$$\text{and}\quad (\lambda_2+1,5,x,\ldots,x)\cup(\lambda_2,y\ldots,y)\xrightarrow{\psi_{5,3}}(x,\ldots,x)\cup(\lambda_2,\lambda_2,6,y,\ldots,y).$$

If in two partitions of the form $(\lambda_2+1,5,x,\ldots,x)\cup(\lambda_2,y,\ldots,y)$ the part λ_2 is different, then the partitions are not equal. Again, when λ_2 is equal in two such partitions, then the partitions are different if and only if the combination of x's and y's are different. The same is true for the partitions of the form $(13,5,x,\ldots,x)\cup(8,y,\ldots,y)$. So the map gives us different images, which are in the subset $\widetilde{I}^c_{e,7}(n)$. By similar argument to the partitions of the $\widetilde{I}^c_{e,6}(n)$, we observe that $\widetilde{I}^c_{e,7}(n)$ have also no preimage under f.
Case 5: The partition $\lambda=(n)$.
If n is odd, then the partition $\lambda\in Q_o(n)$, which has not mapped yet. Let

$$\pi=(4,4,\ldots,4,3) \text{ or } (4,4,\ldots,4,3,2).$$

Then $\pi\in Q_e(n)$, and it has no preimage yet. So we map (n) to π.
Now listing the partitions of n for $n\leq 20$ and $n=21,23,25,27,29$, we get the inequality is true for all even numbers, and for all odd numbers greater than 7.

10.5 Proofs of $p_o^S(n)>p_e^S(n)$ with $S=\{2\}$ and $S=\{1,2\}$

Proof of Theorem 10.1.6: Throughout this proof, $S:=\{2\}$. Define

$$P^S_{e,e}(n):=\{\lambda\in P^S_e(n):\ell(\lambda)\equiv 0 \pmod 2\}$$

and

$$P^S_{e,o}(n):=\{\lambda\in P^S_e(n):\ell(\lambda)\equiv 1 \pmod 2\}.$$

We split $P^S_e(n)$ into the following five disjoint classes.

1. $P^S_{e,1}(n):=\{\lambda\in P^S_{e,e}(n):\lambda_i\neq 1 \text{ for all } i\}$,
2. $P^S_{e,2}(n):=\{\lambda\in P^S_{e,o}(n):a(\lambda) \text{ even}, \ell_e(\lambda)-\ell_o(\lambda)\geq 2, \text{and } \lambda_i\neq 1 \text{ for all } i\}$,
3. $P^S_{e,3}(n):=\{\lambda\in P^S_{e,e}(n):\lambda_j=1 \text{ for some } j\}$,
4. $P^S_{e,4}(n):=\{\lambda\in P^S_{e,o}(n):\lambda_j=1 \text{ for some } j\}$, and
5. $P^S_{e,5}(n):=\{\lambda\in P^S_{e,o}(n):a(\lambda) \text{ even}, \ell_e(\lambda)=\ell_o(\lambda)+1, \text{and } \lambda_i\neq 1 \text{ for all } i\}$.

We apply the injective map f on $P^S_{e,1}(n)$ and $P^S_{e,2}(n)$. For the sets $P^S_{e,3}(n)$ and $P^S_{e,4}(n)$, we apply f in a slightly different way. For a partition $\lambda\in P^S_{e,3}(n)\cup P^S_{e,4}(n)$, we first

split it as $\lambda = \hat{\lambda} \cup (1, 1, \ldots, 1)$, where $\hat{\lambda}$ has no parts equal to 1. Then we define $\hat{f}(\lambda) := f(\hat{\lambda}) \cup (1, \ldots, 1)$. For example, $(8, 8, 7, 6, 4, 4, 1, 1) \mapsto (9, 9, 8, 5, 3, 3, 1, 1)$, and $(8, 8, 7, 6, 4, 1, 1) \mapsto (10, 8, 7, 5, 3, 1, 1)$.
Now, we dissect the set $P_{e,5}^S(n)$ into two disjoint classes. For $\lambda \in P_{e,5}^S(n)$ with its even component (resp. odd component) $\lambda_e = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_{r+1}})$ (resp. $\lambda_o = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r})$), define

(5a) $\overline{P}_{e,5}^S(n) := \{\lambda \in P_{e,5}^S(n) : \lambda_{e_1} \neq \lambda_{e_2}$ and $\lambda_{e_{r+1}} \geq 6\}$, and

(5b) $\overline{P}_{e,5}^{S,c}(n) := P_{e,5}^S(n) \setminus \overline{P}_{e,5}^S(n)$.

Define a map, say $\phi : \overline{P}_{e,5}^S(n) \to P_o^S(n)$ by $\phi(\lambda) := \pi_o \cup \pi_e$, where

$$\pi_o = \lambda_o \cup (1, 1, \ldots, 1, 1) \quad \text{and} \quad \pi_e = (\lambda_{e_1} - 4, \lambda_{e_2} - 2, \ldots, \lambda_{e_{r+1}} - 2).$$

By definition of ϕ, for $\lambda \in \overline{P}_{e,5}^S(n)$ with $\ell_o(\lambda) = r$ it follows that

$$\phi(\overline{P}_{e,5}^S(n)) := \overline{P}_{o,5}^S(n) := \{\pi \in P_o^S(n) : \ell_e(\pi) = r + 1, \ell_{o,1}(\pi) = r \text{ and } \mathrm{mult}_\pi(1) = 2r + 4\},$$

where $\ell_{o,1}(\lambda)$ denote the number of odd parts of $\lambda \vdash n$ which are greater than 1.
Therefore, $\overline{P}_{o,5}^S(n)$ is disjoint with an image of any partition belongs to $\bigcup_{i=1}^4 P_{e,i}^S(n)$ under the map f or $\hat{f}$. Since the odd parts of λ are not altered, so the map is injective.
Next, we further split $\overline{P}_{e,5}^{S,c}(n)$ into four disjoint subsets.

(5b1) $\overline{P}_{e,5_1}^{S,c}(n) := \{\lambda \in \overline{P}_{e,5}^{S,c}(n) : \lambda_i \neq 3\}$,

(5b2) $\overline{P}_{e,5_2}^{S,c}(n) := \{\lambda \in \overline{P}_{e,5}^{S,c}(n) : \ell(\lambda) \neq 3$ and $\mathrm{mult}_\lambda(3) = 1\}$,

(5b3) $\overline{P}_{e,5_3}^{S,c}(n) := \{\lambda \in \overline{P}_{e,5}^{S,c}(n) : \ell(\lambda) \neq 5$ and $\mathrm{mult}_\lambda(3) \geq 2\}$, and

(5b4) $\overline{P}_{e,5_4}^{S,c}(n) := \bigcup_{k=1}^2 \overline{P}_{e,5_{4,k}}^{S,c}(n)$, with

$$\overline{P}_{e,5_{4,k}}^{S,c}(n) = \{\lambda \in \overline{P}_{e,5}^{S,c}(n) : \ell(\lambda) = 2k + 1 \text{ and } \mathrm{mult}_\lambda(3) = k\}.$$

We construct a map $\rho : \overline{P}_{e,5}^{S,c}(n) \to P_o^S(n)$ by defining

$$\text{for } 1 \leq t \leq 3, \quad \psi|_{\overline{P}_{e,5_t}^{S,c}(n)} := \rho_t.$$

Case 1: Let $\lambda = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_{r+1}}) \cup (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r}) \in \overline{P}^{S,c}_{e,5}(n)$. For $1 \leq t \leq 3$,

$$\rho_t(\lambda) := \lambda_e \cup (\lambda_{o_1} - 2, \lambda_{o_2} - 2, \ldots, \lambda_{o_{r-t}} - 2, 1, 1, \ldots, 1)$$

with $\text{mult}_\pi(1) = 2(r-t) + \sum_{k=0}^{t-1} \lambda_{o_{r-k}}$ and consequently for $1 \leq t \leq 2$,

$$\begin{aligned}\rho_t(\overline{P}^{S,c}_{e,5_t}(n)) &:= \overline{P}^{S,c}_{o,5_t}(n)\\ &:= \Big\{\pi \in P^S_o(n) : \text{mult}_\pi(1) = 2(r-t) + \sum_{k=0}^{t-1} \lambda_{o_{r-k}}, \ell_e(\pi) = r+1, \ell_{o,1}(\pi) = r-t\Big\};\end{aligned}$$

whereas for $t = 3$,

$$\begin{aligned}\rho_3(\overline{P}^{S,c}_{e,5_3}(n)) &:= \overline{P}^{S,c}_{o,5_3}(n)\\ &:= \Big\{\pi \in P^S_o(n) : \text{mult}_\pi(1) = 2(r-t) + \sum_{k=0}^{2} \lambda_{o_{r-k}}, \ell_e(\pi) = r+1, \ell_{o,1}(\pi) \leq r-3\Big\}.\end{aligned}$$

So for each t we have,

$$2r - 2t + \sum_{k=0}^{t-1} \lambda_{o_{r-k}} > \frac{\ell(\pi)}{2} + 1.$$

For example,

$$\begin{aligned}(10,9,7,7,6,4,4) &\xrightarrow{\rho_1} (10,7,6,5,4,4,1,\ldots,1),\\ (10,9,7,6,4,4,3) &\xrightarrow{\rho_2} (10,7,6,4,4,1,\ldots,1),\\ (10,9,6,4,4,3,3) &\xrightarrow{\rho_3} (10,6,4,4,1,\ldots,1),\\ (10,9,8,7,6,6,4,4,4,3,3,3,3) &\xrightarrow{\rho_3} (10,8,7,6,6,5,4,4,4,1,\ldots,1).\end{aligned}$$

Case 2: For n even, we have to consider $\overline{P}^{S,c}_{e,5_{4,2}}(n) \subset \overline{P}^{S,c}_{e,5_4}(n)$ as any partition $\lambda \in \overline{P}^{S,c}_{e,5_{4,2}}(n)$ is of the form $(\lambda_{e_1}, \lambda_{e_2}, \lambda_{e_3}) \cup (3,3)$. Define $\rho_4(\lambda) := \pi = (\lambda_{e_1}+3, \lambda_{e_2}+3, \lambda_{e_3})$ so that $\ell_o(\pi) - \ell_e(\pi) = 1$ and $a(\pi)$ is odd. One can observe that $\rho_4(\overline{P}^{S,c}_{e,5_{4,2}}(n))$ is disjoint from the sets which have already a preimage either by the map $f, \hat{f}, \phi$ or by $\{\rho_t\}_{1 \leq t \leq 3}$.
Similarly, if n is odd, then we consider $\overline{P}^{S,c}_{e,5_{4,1}}(n) \subset \overline{P}^{S,c}_{e,5_4}(n)$. We compare the partitions $\lambda = (\lambda_{e_1}, \lambda_{e_2}) \cup (3) \in \overline{P}^{S,c}_{e,5_{4,1}}(n)$ with the partitions of the form $((\pi_{o_1}, \pi_{o_2}) \cup$

$(3)) \cup (4,4) \in P_o^S(n)$ where $\pi_{o_1} \geq 5$, and $((\pi_{o_1}, \pi_{o_2}) \cup (3)) \cup (6,4) \in P_o^S(n)$ where $\pi_{o_1} \geq 7$, for all $n > 23$. The total number of such partitions is

$$\left(\left\lfloor \frac{n-9}{4} \right\rfloor - 1\right) + \left(\left\lfloor \frac{n-11}{4} \right\rfloor - 1\right),$$

and

$$\left|\overline{P}^{S,c}_{e,5_{4,1}}(n)\right| = \left\lfloor \frac{n-3}{4} \right\rfloor - 1.$$

Since n is odd, it is of the form $4s+1$ or $4s+3$. For any form of n we get,

$$\left(\left\lfloor \frac{n-9}{4} \right\rfloor - 1\right) + \left(\left\lfloor \frac{n-11}{4} \right\rfloor - 1\right) > \left\lfloor \frac{n-3}{4} \right\rfloor - 1.$$

We note that the set $\overline{P}^{S,c}_{e,5_{4,1}}(n) \neq \emptyset$ for $n = 11, 13, \ldots, 23$. In these cases $\lambda = (\lambda_{e_1}, \lambda_{e_2}) \cup (3)$ maps to $\rho_5(\lambda) := \pi = (\lambda_{e_2}-1, \lambda_{e_2}-1, 1) \cup (\lambda_{e_1}-\lambda_{e_2}+4)$. $\ell_o(\pi)-\ell_e(\pi) = 2$ and the two greatest odd parts are equal. So it was not an image in the above cases. For example, $(14,6,3)$ maps to $(12,5,5,1)$.
Now, we observe that $P_o^S(n)$ has more classes of partitions for all n, which are not mapped yet. For example, the partition where all parts are 1. Hence $p_o^{\{2\}}(n) > p_e^{\{2\}}(n)$ for all $n \geq 1$. This completes the proof. □

Proof of Theorem 10.1.7: For $\lambda \in p_o^S(n)$, it is immediate that $\lambda_{e_i} \geq 4$, where $\lambda_e = (\lambda_{e_1}^{k_1} \lambda_{e_2}^{k_2} \ldots \lambda_{e_s}^{k_s})$ is the even component of λ. So we can apply the injective map f (cf. Section 10.2) on $G_e(n) \subseteq P_e(n)$. Consequently for $\lambda = (\lambda_1^{m_1} \lambda_2^{m_2} \ldots \lambda_k^{m_k}) \in G_e^0(n)$, divide the subset $G_e^0(n)$ into two disjoint classes given as follows

$$G_e^0(n) := G_{e,1}^0(n) \cup G_{e,2}^0(n),$$

with

$$G_{e,1}^0(n) = \{\lambda \in G_e^0(n) : \lambda_3 > 6\}$$

and

$$G_{e,2}^0(n) = \{\lambda \in G_e^0(n) : \lambda_3 \in T\}, \text{ where } T = \{3,4,5,6\}.$$

By a similar method to the proof of Theorem 10.1.5, this result can be proved. We leave this as an exercise to the readers. □

10.6 Concluding remarks

In continuation with the study on parity of parts, as discussed in the previous section (cf. Section 10.5), we conclude this chapter by presenting a somewhat more general

discussion on the prospect towards further study.
First, by allowing only distinct partitions in $Q_e(n)$ and $Q_o(n)$ into Theorem 10.1.5, we propose the following problem.

Problem 10.6.1. *For all $m > 6$ we have*

$$dq_o(2m) > dq_e(2m),$$

and

$$dq_o(2m+1) < dq_e(2m+1).$$

Remark 10.6.2. *In fact, $dq_o(2m+1) < dq_e(2m+1)$ for $m = 4, 5$ as well.*

Whereas, Theorems 10.1.6 and 10.1.7 suggest a more general family of inequalities in the following sense.

Problem 10.6.3. *For all $k > 2$ we have $p_o^{\{k\}}(n) > p_e^{\{k\}}(n)$ and $p_e^{\{1,k\}}(n) > p_o^{\{1,k\}}(n)$, for all $n > N(k)$, for some constant $N(k)$, depending on k.*
Moreover, it would be worthwhile to understand the threshold $N(k)$ asymptotically.

10.7 Appendix: Proofs of Lemmas 10.2.1 and 10.2.2

Proof of Lemma 10.2.1: Take $n = 2m$ with $m \in \mathbb{Z}_{\geq 7}$. So, we have to show

$$\sum_{k=1}^{m-3} \Big\lfloor \frac{m-k-1}{2} \Big\rfloor > 1 + \sum_{k=1}^{\lfloor \frac{m-1}{3} \rfloor} \Big\lfloor \frac{m-3k+1}{2} \Big\rfloor + \sum_{k=1}^{\lfloor \frac{m-3}{3} \rfloor} \Big\lfloor \frac{m-3k-1}{2} \Big\rfloor. \tag{10.24}$$

Note that

$$\begin{aligned}\sum_{k=1}^{m-3} \Big\lfloor \frac{m-k-1}{2} \Big\rfloor = \sum_{k=0}^{m-4} \Big\lfloor \frac{k+2}{2} \Big\rfloor > \sum_{k=0}^{m-4} \Big(\frac{k+2}{2} - 1 \Big) \ \Big(\text{since, } \lfloor x \rfloor > x - 1\Big) \\ = \frac{(m-4)(m-3)}{4} = \frac{m^2 - 7m + 12}{4},\end{aligned}$$

and

$$1 + \sum_{k=1}^{\lfloor \frac{m-1}{3} \rfloor} \Big\lfloor \frac{m-3k+1}{2} \Big\rfloor + \sum_{k=1}^{\lfloor \frac{m-3}{3} \rfloor} \Big\lfloor \frac{m-3k-1}{2} \Big\rfloor < \frac{m^2 + 3m - 3}{6}. \tag{10.25}$$

Now, $\dfrac{m^2-7m+12}{4} > \dfrac{m^2+3m-3}{6}$ for $m \in \mathbb{Z}_{\geq 26}$. We finish the proof by checking the inequality (10.24) for $7 \leq m \leq 25$ numerically in Mathematica. $\square$

Proof of Lemma 10.2.2: The proof is exactly similar to the proof of Lemma 10.2.1. $\square$

Part IV

Epilogue

In this final part of the thesis, we address a few problems that arose in Chapters 3-8 which so far remained unanswered.

In Chapter 3, we proposed the Conjecture 3.1.2 on the asymptotic growth of the cubic partition function $a(n)$. Using Sussman's theorem [138, Theorem 1.1], we obtain a Hardy-Ramanujan-Rademacher type expansion for $a(n)$. The next task would be to estimate the absolute value of the exponential sum $A_k(n)$ and apply Theorem 8.3.9 to get inequalities for $I_2(x)$ so that one can finally arrive at the inequalities for $a(n)$ similar to Theorem 3.4.4. Then we can follow the proof strategy as done for Theorem 3.6.6 so as to verify the conjecture for $a(n)$. This is work in progress.

In Chapter 4, the full asymptotic expansion for $(-1)^{r-1}\Delta^r \log p(n)$ is given in Theorem 4.4.7. As we have already mentioned, a full asymptotic expansion merely depicts the growth of a sequence whereas computation of error bounds is essential in order to get an infinite family of inequalities for the sequence. Therefore, after Theorem 4.4.7, our next goal would be to obtain a lower and upper bound for the error term

$$E(r,n) := (-1)^{r-1}\Delta^r \log p(n) - \sum_{u=2r-1}^{w} G_u\Big(\frac{1}{\sqrt{n}}\Big)^u,$$

where $w \geq 2r-1$, $r \geq 2$, for all $u \geq 1$,

$$G_{2u} = \left[\frac{(-1)^u}{u}\left\{{u \atop r}\right\} + \sum_{k=1}^{u-r} g_{2k}\binom{-k}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r! \quad \text{for all } u \geq r,$$

and for all $u \geq r-1$,

$$G_{2u+1} = \left[\pi\sqrt{\frac{2}{3}}\binom{1/2}{u+1}\left\{{u+1 \atop r}\right\} + \sum_{k=0}^{u-r} g_{2k+1}\binom{-k-1/2}{u-k}\left\{{u-k \atop r}\right\}\right](-1)^{r+1}r!.$$

The main result of Chapter 6 is the infinite family of inequalities for $p(n-\ell)$ with $\ell \in \mathbb{Z}_{\geq 0}$ stated in Theorem 6.4.5. A similar result for $p(n+\ell)$ would be highly desirable because then we can directly apply both inequalities for $p(n+\ell)$ and $p(n-\ell)$ without making any shifts to a given problem on inequalities for the partition function. From Section 6.7, we propose the following problems.

- Using Theorem 6.4.5 and by choosing the cut-off $w(m)$ approximately $5m$ for all $m \geq 12$, then in order to confirm that $(p(n))_{n \geq N_B(m)}$ satisfies the higher order Laguerre inequalities of order m, what would be the growth of $N_B(m)$ as m getting larger? Moreover, how far $N_B(m)$ would be away from the real cut-off $N_L(m)$? Is it possible to choose the sequence of truncation points $(w(m))_{m \geq 12}$, so that $N_B(m)/N_L(m) \to C$ as $m \to \infty$ for some $C > 0$? This would be a nice extension of Theorem 6.7.1.

- Similar to the problem of computing bounds of $E(r,n)$ for $(-1)^{r-1}\Delta^r \log p(n)$, we ask for an explicit error bound for $\Delta_j^r(p(n))$ followed by Theorem 6.7.8.

Concerning Chapter 8, we now focus only on Lemma 8.3.8. The reason behind this is the bound on the absolute value of the error term $E(\nu, N, x)$ with $\nu \leq N$. Roughly speaking, $|E(\nu, N, x)|$ is bounded by $|a_{N+1}(\nu)|$ (the absolute value of the next term of the asymptotic expansion of $I_\nu(x)$, see equation (8.20)) multiplied by a "bad" term of the form $\sqrt{N}\log N$. Immediately, a question appears: (i) does $\sqrt{N}\log N$ actually appear or is there some overestimation done in the proofs? (ii) if $\sqrt{N}\log N$ appears at all, then for which value of x, $|E(\nu, N, x)|$ attains $|a_{N+1}(\nu)|\sqrt{N}\log N$? We[1] proved that for $x = O(N/2)$, $E(\nu, N, x) \sim a_{N+1}(\nu)\sqrt{N}\log N$ as $N \to \infty$ for arbitrary but fixed $x \in \mathbb{R}_{\geq 1}$. Therefore, the next question which springs up is: for which interval of x, is $|E(\nu, N, x)|$ bounded by $|a_{N+1}(\nu)|$ and what is the range of x when the bad factor $\sqrt{N}\log N$ appears? In short, what is the asymptotic growth of $|E(\nu, N, x)|$?

Finally, we trace back to the real rootedness of the Jensen polynomial $J_p^{d,n}(x)$ of degree d and shift n associated with $(p(n))_{n\geq 0}$. Griffin, Ono, Rolen, and Zagier proved that $J_p^{d,n}(x)$ has all real and distinct roots for sufficiently large n from the following limit formula

$$\lim_{n\to\infty} \widehat{J}_p^{d,n}(x) := \lim_{n\to\infty}\left(\frac{\delta_n^{-d}}{p(n)} J_p^{d,n}\left(\frac{\delta_n x - 1}{e^{A_n}}\right)\right) = H_d(x), \tag{10.26}$$

with

$$A_n = \frac{2\pi}{\sqrt{24n-1}} - \frac{24}{24n-1} \quad \text{and} \quad \delta_n = \sqrt{\frac{12\pi}{(24n-1)^{3/2}} - \frac{288}{(24n-1)^2}}.$$

In order to verify their conjecture for the cut-off $N(d)$ is approximately $10d^2\log d$ or at least to get close to it; i.e., approximating $N(d)$ by a polynomial in d, we need to demystify (10.26). To do so, first we need to compute an effective error bound for the absolute value of the difference between the normalized Jensen polynomial $\widehat{J}_p^{d,n}(x)$ and the Hermite polynomial $H_d(x)$ for each $d \geq 2$. Larson and Wagner [95] took the approach following Hermite's criteria to show $J_p^{d,n}(x)$ has all real and distinct roots for $n \geq N(d)$ but their estimation of $N(d)$ is roughly of the form $(3d)^{24d}(50d)^{3d^2}$ which is approximately the exponential of the conjectured real cut-off; i.e., $e^{d^2\log d}$. This motivates us to study (10.26) in the future through the lens of inequalities.

[1] This work is in progress (jointly with Silviu Radu)

Bibliography

[1] J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, C. Schneider, and K. Schönwald. The two-mass contribution to the three-loop polarized operator matrix element $A_{gg,Q}^{(3)}$. *Nuclear Physics B*, **955** (2020), pp. 1–70.

[2] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. United States Department of Commerce, National Bureau of Standards (NBS), 1964.

[3] A. Akande, T. Genao, S. Haag, M. Hendon, N. Pulagam, R. Schneider, and A. V. Sills. Computational study of non-unitary partitions. (2021). https://arxiv.org/abs/2112.03264.

[4] H. L. Alder. The nonexistence of certain identities in the theory of partitions and compositions. *Bulletin of American Mathematical Society*, **54** (1948), pp. 712–722.

[5] G. Almkvist. On the differences of the partition function. *Acta Arithmetica*, **61** (1992), pp. 173–181.

[6] G. E. Andrews. Differences of partition functions: the anti-telescoping method. *From Fourier analysis and number theory to Radon transforms and geometry, Springer*, (2013), pp. 1–20.

[7] G. E. Andrews. Euler's pentagonal number theorem. *Mathematics Magazine*, **56** (1983), pp. 279–284.

[8] G. E. Andrews. *The Theory of Partitions*. Addison-Wesley, Reading, Mass, 1976; reprinted, Cambridge University Press., 1998.

[9] G. E. Andrews and M. Merca. The truncated pentagonal number theorem. *Journal of Combinatorial Theory, Series A*, **119** (2012), pp. 1639–1643.

[10] G. E. Andrews and M. Merca. Truncated theta series and a problem of Guo and Zeng. *Journal of Combinatorial Theory, Series A*, **154** (2018), pp. 610–619.

[11] G. E. Andrews, P. Paule, and C. Schneider. Plane partitions VI: Stembridge's TSPP theorem. *Advances in Applied Mathematics*, **34** (2005), pp. 709–739.

[12] C. Ballantine, M. Merca, D. Passary, and A. J. Yee. Combinatorial proofs of two truncated theta series theorems. *Journal of Combinatorial Theory, Series A*, **160** (2018), pp. 168–185.

[13] K. Banerjee. A unified framework to prove multiplicative inequalities for the partition function. *Submitted for publication*, (2022). https://www3.risc.jku.at/publications/download/risc_6614/inequalityproductform.pdf.

[14] K. Banerjee. Higher order Laguerre inequalities for the partition function and its consequences. *In preparation*, (2022).

[15] K. Banerjee. Inequalities for the modified Bessel function of first kind of non-negative order. *Journal of Mathematical Analysis and Applications*, **524** (2023), p. 127082.

[16] K. Banerjee. Invariants of the quartic binary form and proofs of Chen's conjectures for partition function inequalities. *Submitted for publication*, (2022). https://www3.risc.jku.at/publications/download/risc_6615/Chen.pdf.

[17] K. Banerjee. Positivity of second shifted difference of partitions and overpartitions: a combinatorial approach. *Enumerative Combinatorics and Applications*, **3:2** (2023).

[18] K. Banerjee, S. Bhattacharjee, M. G. Dastidar, P. J. Mahanta, and M. P. Saikia. Parity biases in partitions and restricted partitions. *European Journal of Combinatorics*, **103** (2022), p. 103522.

[19] K. Banerjee and M. G. Dastidar. Inequalities for the partition function arising from truncated theta series. *Submitted for publication*, (2022). https://www3.risc.jku.at/publications/download/risc_6622/AndMer.pdf.

[20] K. Banerjee and M. G. Dastidar. Ramanujan's theta functions and parity of parts and cranks of partitions. *Annals of Combinatorics*, (2022). https://doi.org/10.1007/s00026-022-00615-1.

[21] K. Banerjee, P. Paule, C. S. Radu, and C. Schneider. Error bounds for the asymptotic expansion of the partition function. *Submitted for publication*, (2022). https://arxiv.org/abs/2209.07887.

[22] K. Banerjee, P. Paule, C. S. Radu, and W. H. Zeng. New inequalities for $p(n)$ and $\log p(n)$. *Ramanujan Journal*, (2022). https://doi.org/10.1007/s11139-022-00653-6.

[23] A. Berkovich and A. K. Uncu. Some elementary partition inequalities and their implications. *Annals of Combinatorics*, **23** (2019), pp. 263–284.

[24] F. W. Bessel. Analytische Auflösung der Keplerschen Aufgabe. *Abhandlungen der Berliner Akademie*, (1819), pp. 49–55.

[25] F. W. Bessel. Untersuchung des Theils der planetarischen Störungen, welcher aus der Bewegung der Sonne entsteht. *Abhandlungen der Berliner Akademie*, (1826), pp. 1–52.

[26] C. Bessenrodt and K. Ono. Maximal multiplicative properties of partitions. *Annals of Combinatorics*, **20** (2016), pp. 59–64.

[27] R. F. Boisvert, C. W. Clark, H. Cohl, A. B. O. Dalhuis, D. W. Lozier, M. McClain, B. Miller, F. W. J. Olver, B. Saunders, and B. Schneider. Digital library of mathematical functions. http://dlmf.nist.gov/, Release 1.0.26 of 2020-03-15, 2010.

[28] G. Boros and V. H. Moll. *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals*. Cambridge University Press, (2004).

[29] P. Brändén. Iterated sequences and the geometry of zeros. *Journal für die Reine und Angewandte Mathematik*, **658** (2011), pp. 115–131.

[30] S. Brassesco and A. Meyroneinc. An expansion for the number of partitions of an integer. *Ramanujan Journal*, **51** (2020), pp. 563–592.

[31] F. Brenti. Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update. In *Jerusalem combinatorics '93, volume 178 of Contemp. Math.*, American Mathematical Society, Providence, RI, (1994), pp. 71–89.

[32] K. Bringmann, B. Kane, L. Rolen, and Z. Tripp. Fractional partitions and conjectures of Chern–Fu–Tang and Heim–Neuhauser. *Transactions of the American Mathematical Society, Series B*, **8** (2021), pp. 615–634.

[33] E. R. Canfield. From recursions to asymptotics: On Szekeres' formula for the number of partitions. *The Electronic Journal of Combinatorics*, **4** (1997), pp. 1–16.

[34] A. Cayley. Researches on the partition of numbers. *Philosophical Transactions of the Royal Society of London*, **146** (1856), pp. 127–140.

[35] W. Y. C. Chen. Recent developments on log-concavity and q-log-concavity of combinatorial polynomials. *DMTCS Proceeding of 22nd International Conference on Formal Power Series and Algebraic Combinatorics*, 2010.

[36] W. Y. C. Chen. The spt-function of Andrews. *Surveys in combinatorics*, Cambridge Univ. Press, Cambridge, 2017, pp. 141–203.

[37] W. Y. C. Chen, D. Jia, and L. Wang. Higher order Turán inequalities for the partition function. *Transactions of the American Mathematical Society*, **372** (2019), pp. 2143–2165.

[38] W. Y. C. Chen and B. L. S. Lin. Congruences for the number of cubic partitions derived from modular forms. (2009). https://arxiv.org/abs/0910.1263.

[39] W. Y. C. Chen, L. Wang, and G. Xie. Finite differences of the logarithm of the partition function. *Mathematics of Computation*, **85** (2016), pp. 825–847.

[40] S. Chern. A further look at the truncated pentagonal number theorem. *Acta Arithmetica*, **189** (2019), pp. 397–403.

[41] S. Chern. Asymptotics for the Fourier coefficients of eta-quotients. *Journal of Number Theory*, **199** (2019), pp. 168–191.

[42] S. Chern. Further results on biases in integer partitions. *Bulletin of the Korean Mathematical Society*, **59** (2022), pp. 111–117.

[43] S. Chern, S. Fu, and D. Tang. Some inequalities for k-colored partition functions. *Ramanujan Journal*, **46** (2018), pp. 713–725.

[44] G. E. Collins. Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition. *Lecture Notes in Computer Science*, **33** (1975), 134–183, pp. 134–183.

[45] R. M. Corless and L. R. Sevyeri. Stirling's original asymptotic series from a formula like one of Binet's and its evaluation by sequence acceleration. *Experimental Mathematics*, **31** (2022), pp. 184–191.

[46] S. Corteel and J. Lovejoy. Overpartitions. *Transactions of the American Mathematical Society*, **356** (2004), pp. 1623–1635.

[47] T. Craven and G. Csordas. Iterated Turán and Laguerre inequalities. *J. Inequal. Pure Appl. Math.*, **3** (2002) p. 39.

[48] T. Craven and G. Csordas. Jensen polynomials and the Turán and Laguerre inequalities. *Pacific Journal of Mathematics*, **136** (1989), pp. 241–260.

[49] G. Csordas and R. S. Varga. Necessary and sufficient conditions and the Riemann hypothesis. *Advances in Applied Mathematics*, **11** (1990), pp. 328–357.

[50] M. Dawsey and R. Masri. Effective bounds for the Andrews spt-function. *Forum Mathematicum*, **31** (2019), pp. 743–767.

[51] N. G. De Bruijn. *Asymptotic Methods in Analysis*. North Holland Publishing Co., Amsterdam, (1958).

[52] C. J. De la Vallée Poussin. Recherches analytiques sur la théorie des nombres premiers. *Annales de la Société scientifique de Bruxelles*, Imprimeur de l'Académie royale de Belgique, (1897).

[53] S. DeSalvo and I. Pak. Log-concavity of the partition function. *Ramanujan Journal*, **38** (2015), pp. 61–73.

[54] F. Diamond and J. Shurman. *A First Course in Modular Forms*. Springer, (2005).

[55] L. E. Dickson. *History of the Theory of Numbers*. Vol. 2, Carnegie Institution, Washington 1920; reprinted: Chelsea, New York, 1952.

[56] D. Dimitrov. Higher order Turán inequalities. *Proceedings of the American Mathematical Society*, **126** (1998), pp. 2033–2037.

[57] J. J. W. Dong, K. Q. Ji, and D. X. Q. Jia. Turán inequalities for the broken k-diamond partition function. (2022). https://arxiv.org/abs/2206.09512.

[58] L. M. Dou and L. X. W. Wang. Higher order Laguerre inequalities. *Submitted for publication*, 2022.

[59] M. E. Bachraoui. Positive alternating sums of integer partitions. *Ramanujan Journal*, **55** (2021), pp. 697–711.

[60] P. Erdős. On an elementary proof of some asymptotic formulas in the theory of partitions. *Annals of Mathematics*, **43** (1942), pp. 437–450.

[61] L. Euler. Methodus universalis series summandi ulterius promota. *Commentarii academiae scientiarum imperialis Petropolitanae*; reprinted in Opera Omnia, Vol. XIV, **8** (1736), pp. 147–158.

[62] L. Euler. Observationes analyticae variae de combinationibus. *Comm. Acad. Petrop.*, **13** (1741-1743, 1751), pp. 64–93.

[63] S. Fisk. Questions about determinants and polynomials. (2008). https://arxiv.org/abs/0808.1850.

[64] F. G. Garvan. Generalizations of Dyson's rank and non-Rogers-Ramanujan partitions. *Manuscripta Mathematica*, **84** (1994), pp. 343–359.

[65] C. Goldbach. Lettre CXXXI. http://eulerarchive.maa.org/correspondence/letters/OO0859.pdf, (18 July, 1750).

[66] K. Gomez, J. Males, and L. Rolen. The second shifted difference of partitions and its applications. *Bulletin of Australian Mathematical Society*, (2022). https://doi.org/10.1017/S0004972722000764.

[67] I. J. Good. The differences of the partition function, Problem 6137. *American Mathematical Monthly*, **84** (1977), p. 141.

[68] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison Wesley, Reading MA, second edition, 1994.

[69] M. Griffin, K. Ono, L. Rolen, and D. Zagier. Jensen polynomials for the Riemann zeta function and other sequences. *Proceedings of the National Academy of Sciences*, **116** (2019), pp. 11103–11110.

[70] M. J. Griffin, K. Ono, L. Rolen, J. Thorner, Z. Tripp, and I. Wagner. Jensen polynomials for the Riemann xi-function. *Advances in Mathematics*, **397** (2022), p. 108186.

[71] H. Gupta. Finite differences of the partition function. *Mathematics of Computation*, **32** (1978), pp. 1241–1243.

[72] H. Gupta. Partitions: a survey. *J. Res. Nat. Bur. Standards*, **74B** (1970), pp. 1–29.

[73] R. K. Guy. Letter to Morris Newman concerning A2513 [E]. https://oeis.org/A002513/a002513.pdf, 21 August, 1986.

[74] J. Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bulletin de la Societé mathematique de France*, **24** (1896), pp. 199–220.

[75] G. H. Hardy and J. E. Littlewood. *Inequalities*. Cambridge university press, (1952).

[76] G. H. Hardy and S. Ramanujan. Asymptotic formulæ in combinatory analysis. *Proceedings of the London Mathematical Society*, **2** (1918), pp. 75–115.

[77] J. F. W. Herschel. On circulating functions, and on the integration of a class of equations of finite differences into which they enter as coefficients. *Philosophical Transactions of the Royal Society of London*, **108** (1818), pp. 144–168.

[78] D. Hilbert. *Theory of Algebraic Invariants*. Cambridge University Press, 1993.

[79] Q. H. Hou and Z. R. Zhang. r-log-concavity of partition functions. *Ramanujan Journal*, **48** (2019), pp. 117–129.

[80] J. L. W. V. Jensen. Recherches sur la théorie des équations. *Acta Mathematica*, **36** (1913), pp. 181–195.

[81] D. X. Q. Jia. Inequalities for the broken k-diamond partition function. (2022). https://arxiv.org/abs/2209.07056.

[82] D. X. Q. Jia and L. X. W. Wang. Determinantal inequalities for the partition function. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, **150** (2020), pp. 1451–1466.

[83] J. Katriel. Asymptotically trivial linear homogeneous partition inequalities. *Journal of Number Theory*, **184** (2018), pp. 107–121.

[84] B. Kim and E. Kim. Biases in integer partitions. *Bulletin of the Australian Mathematical Society*, **104** (2021), pp. 177–186.

[85] B. Kim, E. Kim, and J. Lovejoy. Parity bias in partitions. *European Journal of Combinatorics*, **89** (2020), p. 103159.

[86] G. Kirchhoff. Ueber den inducirten Magnetismus eines unbegrenzten Cylinders von weichem Eisen. *J. Reine Angew. Math.*, **48** (1854), pp. 348–376.

[87] C. Knessl and J. B. Keller. Asymptotic behavior of high-order differences of the partition function. *Communications on pure and applied mathematics*, **44** (1991), pp. 1033–1045.

[88] N. Koblitz. *Introduction to Elliptic Curves and Modular Forms*. Graduate Texts in Mathematics, Springer-Verlag, New York, (1993).

[89] L. W. Kolitsch. Another approach to the truncated pentagonal number theorem. *International Journal of Number Theory*, **11** (2015), pp. 1563–1569.

[90] L. W. Kolitsch and M. Burnette. Interpreting the truncated pentagonal number theorem using partition pairs. *Electronic Journal of Combinatorics*, **22** (2015) 2.55.

[91] V. Kotesovec. A method of finding the asymptotics of q-series based on the convolution of generating functions. (2015). https://arxiv.org/abs/1509.08708.

[92] J. P. S. Kung and G. C. Rota. The invariant theory of binary forms. *Bulletin of the American Mathematical Society*, **10** (1984), pp. 27–85.

[93] E. Laguerre. *Oeuvres*. vol.1, Gaauthier-Villars, Paris, (1989).

[94] E. Landau. Vorlesungen über Zahlentheorie. *American Mathematical Society*, **26** (2004).

[95] H. Larson and I. Wagner. Hyperbolicity of the partition Jensen polynomials. *Research in Number Theory*, **5** (2019), pp. 1–12.

[96] D. H. Lehmer. On the Hardy-Ramanujan series for the partition function. *Journal of the London Mathematical Society*, **1** (1937), pp. 171–176.

[97] D. H. Lehmer. On the maxima and minima of Bernoulli polynomials. *American Mathematical Monthly*, **47** (1940), pp. 533–538.

[98] D. H. Lehmer. On the remainders and convergence of the series for the partition function. *Transactions of the American Mathematical Society*, **46** (1939), pp. 362–373.

[99] D. H. Lehmer. On the series for the partition function. *Transactions of the American Mathematical Society*, **43** (1938), pp. 271–295.

[100] G. W. Leibniz. Math. Schriften, Vol IV 2, Specimen de divulsionibus aequationum. (see D. Mahnke, Leibniz auf der Suche nach einer allgemeinen Primzahlgleichung, Bibliotheca Math., **13** (1912-1913)), Letter 3 dated Sept. 2, 1674, pp. 29–61.

[101] J. E. Littlewood. *Littlewood's Miscellany*. Cambridge University Press, 1986.

[102] C. Maclaurin. *A treatise of fluxions*. Edinburgh, **1** (1742).

[103] P. A. MacMahon. *Combinatory Analysis, Volumes I and II*. Cambridge University Press, Cambridge, 1915-1916.

[104] C. Mallinger. Algorithmic Manipulations and Transformations of Univariate Holonomic Functions and Sequences. *Master's thesis, RISC, J. Kepler University, Linz*, (1996).

[105] J. McLaughlin. Refinements of some partition inequalities. A66. *INTEGERS: The Electronic Journal of Combinatorial Number Theory*, **16** (2016), p. 1.

[106] P. R. W. McNamara and B. E. Sagan. Infinite log-concavity: developments and conjectures. *Advances in Applied Mathematics*, **44** (2010), pp. 1–15.

[107] C. S. Meijer. Asymptotische Entwicklungen von Besselschen, Hankelschen und verwandten funktionen I-IV. *Proc. Kon. Akad. Wet. Amsterdam*, **35** (1932), pp. 656–667, 852–866, 948–958, 1079–1090.

[108] M. Merca and J. Katriel. A general method for proving the non-trivial linear homogeneous partition inequalities. *Ramanujan Journal*, **51** (2020), 245-266, pp. 245–266.

[109] G. Nemes. Error bounds and exponential improvements for the asymptotic expansions of the gamma function and its reciprocal. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, **145** (2015), pp. 571–596.

[110] G. Nemes. Error bounds for the large-argument asymptotic expansions of the Hankel and Bessel functions. *Acta Applicandae Mathematicae*, **150** (2017), pp. 141–177.

[111] J. L. Nicolas. Sur les entiers N pour lesquels il y a beaucoup de groupes abéliens d'ordre N. *Annales de l'institut Fourier*, **28** (1978), pp. 1–16.

[112] C. O'Sullivan. Detailed asymptotic expansions for partitions into powers. (2022). https://arxiv.org/abs/2205.13468.

[113] N. Obrechkoff. *Zeros of Polynomials*. Publ. Bulg. Acad. Sci., Sofia, (1963).

[114] A. M. Odlyzko. Differences of the partition function. *Acta Arithmetica*, **49** (1988), pp. 237–254.

[115] F. W. J. Olver. *Asymptotics and Special Functions*. AK Peters/CRC Press, 1997.

[116] F. W. J. Olver. Error bounds for asymptotic expansions, with an application to cylinder functions of large argument. *Asymptotic solutions of differential equations and their applications*, C. H. Wilcox, ed., John Wiley, New York, (1964), pp. 163–183.

[117] K. Ono. *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*. CBMS Regional Conference Series in Mathematics, 102, (2004).

[118] P. Paule and M. Schorn. A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. *Journal of Symbolic Computation*, **20** (1995), pp. 673–698.

[119] R. Pérez-Marco. On the definition of the Euler Gamma function. *L'Enseignement Mathématique*, **68** (2022), pp. 135–160.

[120] M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.

[121] G. Pólya. Über die algebraisch-funktionentheoretische Untersuchungen von J. L. W. V. Jensen. Kgl Danske Vid Sel Math-Fys Medd, **7** (1927), pp. 3–33.

[122] H. Rademacher. A convergent series for the partition function $p(n)$. *Proceedings of the National Academy of Sciences of the United States of America*, **23** (1937), p. 78.

[123] H. Rademacher. On the expansion of the partition function in a series. *Annals of Mathematics*, **44** (1943), pp. 416–422.

[124] H. Rademacher. On the partition function $p(n)$. *Proceedings of the London Mathematical Society*, **2** (1938), pp. 241–254.

[125] Q. I. Rahman and G. Schmeisser. *Analytic Theory of Polynomials*. Oxford University Press, 2002.

[126] L. Schläfli. Sull'uso delle linee lungo le quali il valore assoluto di una funzione è costante. *Annali di Matematica Pura ed Applicata*, **6** (1873), pp. 1–20.

[127] C. Schneider. Simplifying multiple sums in difference fields. In: *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions* (Eds: Schneider, C., Blümlein, J.), Texts, and Monographs in Symbolic Computation, Springer, (2013), pp. 325–360.

[128] C. Schneider. Symbolic summation assists combinatorics. *Séminaire Lotharingien de Combinatoire*, **56** (2007), 1–36.

[129] C. Schneider. Term algebras, canonical representations and difference ring theory for symbolic summation. In: *Anti-Differentiation and the Calculation of Feynman Amplitudes* (Eds: Schneider, C., Blümlein, J.), Texts, and Monographs in Symbolic Computuation, Springer, (2021), pp. 423–485.

[130] J. Schur and G. Pólya. Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. *J. Reine Angew. Math.*, **144** (1914), pp. 89–113.

[131] A. Selberg. An elementary proof of the prime-number theorem. *Annals of Mathematics*, **50** (1949), pp. 305–313.

[132] A. Selberg. Reflections around the Ramanujan centenary. *Resonance*, **1** (1996), pp. 81–91.

[133] M. Shen, Z. Yang, and Zeng. Z. Some applications of sparse interpolation via compressive sensing. *MICA2016, Milestones in computer algebra*, July 16-18, Waterloo, 2016. https://cs.uwaterloo.ca/conferences/mica2016/MICA_abstract.pdf.

[134] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at https://oeis.org, (2014).

[135] R. P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. *Ann. New York Acad. Sci.*, **576** (1989), pp. 500–535.

[136] E. M. Stein and R. Shakarchi. *Complex Analysis*. Princeton University Press, (2010).

[137] B. Sturmfels. *Algorithms in Invariant Theory*. Springer Science and Business Media, 2008.

[138] E. Sussman. Rademacher Series for η-quotients. (2017). https://arxiv.org/abs/1710.03415.

[139] J. J. Sylvester. *On subvariants, that is, semi-invariants to binary quantics of an unlimited order. American Journal of Mathematics*, **5** (1882), pp. 79–136.

[140] G Szegö. On an inequality of P. Turán concerning Legendre polynomials. *Bulletin of the American Mathematical Society*, **54** (1948), pp. 401–405.

[141] G. Szegö. *Orthogonal Polynomials*. American Mathematical Society, 1939.

[142] G. Szekeres. Some asymptotic formulae in the theory of partitions (II). *The Quarterly Journal of Mathematics*, **4** (1953), pp. 96–111.

[143] P. Turán. On the zeros of the polynomials of Legendre. *Časopis Pěst. Mat. Fys.*, **75** (1950), pp. 113–122.

[144] J. V. Uspensky. Asymptotic formulae for numerical functions which occur in the theory of partitions. *Bulletin of the Russian Academy of Sciences*, **14** (1920), pp. 199–218.

[145] I. Wagner. On a new class of Laguerre-Pólya type functions with applications in number theory. (2021). https://arxiv.org/abs/2108.01827.

[146] C. Wang and A. J. Yee. Truncated Hecke-Rogers type series. *Advances in Mathematics*, **365** (2020), p. 107051.

[147] L. Wang. Truncated series with nonnegative coefficients from the Jacobi triple product. *Hardy-Ramanujan Journal*, **44** (2022).

[148] L. X. W. Wang, G. Y. B. Xie, and A. Q. Zhang. Finite difference of the overpartition function. *Advances in Applied Mathematics*, **92** (2018), pp. 51–72.

[149] L. X. W. Wang and E. Y. Y. Yang. Laguerre inequalities for discrete sequences. *Advances in Applied Mathematics*, **139** (2022), p. 102357.

[150] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 1995.

[151] H. Weber. Zur Theorie der Bessel'schen Functionen. *Mathematische Annalen*, **37** (1890), pp. 404–416.

[152] E. M. Wright. Asymptotic partition formulae. III. Partitions into k-th powers. *Acta Mathematica*, **63** (1934), pp. 143–191.

[153] E. M. Wright. The asymptotic expansion of the generalized Bessel function. *Proceedings of the London mathematical society*, **2** (1935), pp. 257–270.

[154] A. J. Yee. A truncated Jacobi triple product theorem. *Journal of Combinatorial Theory, Series A*, **130** (2015), pp. 1–14.

[155] W. H. Zeng. Some Properties of Morris Newman Sequence. Thesis for Bachelor degree in mathematics. East China Normal University, Shanghai, China, 2016.

[156] H. S. Zuckerman. On the coefficients of certain modular forms belonging to subgroups of the modular group. *Transactions of the American Mathematical Society*, **45** (1939), pp. 298–321.