The spt-Function of Andrews

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Abstract

The spt-function $\mathrm{spt}(n)$ was introduced by Andrews as the weighted counting of partitions of n with respect to the number of occurrences of the smallest part. Andrews showed that $\mathrm{spt}(5n+4) \equiv 0 \pmod 5$, $\mathrm{spt}(7n+5) \equiv 0 \pmod 7$ and $\mathrm{spt}(13n+6) \equiv 0 \pmod{13}$. Since then, congruences of $\mathrm{spt}(n)$ have been extensively studied. Folsom and Ono obtained congruences of $\mathrm{spt}(n)$ mod 2 and 3. They also showed that the generating function of $\mathrm{spt}(n)$ mod 3 is related to a weight 3/2 Hecke eigenform with Nebentypus. Combinatorial interpretations of congruences of $\mathrm{spt}(n)$ mod 5 and 7 have been found by Andrews, Garvan and Liang by introducing the spt-crank of a vector partition. Chen, Ji and Zang showed that the set of partitions counted by $\mathrm{spt}(5n+4)$ (or $\mathrm{spt}(7n+5)$) can be divided into five (or seven) equinumerous classes according to the spt-crank of a doubly marked partition. Let $N_S(m,n)$ denote the net number of S-partitions of n with spt-crank m. Andrews, Dyson and Rhoades conjectured that $\{N_S(m,n)\}_m$ is unimodal for any n. Chen, Ji and Zang gave a constructive proof of this conjecture. In this survey, we summarize developments on congruence properties of $\mathrm{spt}(n)$ established by Andrews, Bringmann, Folsom, Garvan, Lovejoy and Ono et al., as well as their combinatorial interpretations. Generalizations and variations of the spt-function are also discussed. We also give an overview of asymptotic formulas of $\mathrm{spt}(n)$ obtained by Ahlgren, Andersen and Rhoades et al. We conclude with some conjectures on inequalities on $\mathrm{spt}(n)$, which are reminiscent of inequalities on $p(n)$ due to DeSalvo and Pak, and Bessenrodt and Ono. Furthermore, we observe that, beyond the log-concavity, $p(n)$ and $\mathrm{spt}(n)$ satisfy higher order inequalities based on polynomials arising in the invariant theory of binary forms. In particular, we conjecture that the higher order Turán inequality $4(a_n^2-a_{n-1}a_{n+1})(a_{n+1}^2-a_na_{n+2})-(a_na_{n+1}-a_{n-1}a_{n+2})^2>0$ holds for $p(n)$ when $n \geq 95$ and for $\mathrm{spt}(n)$ when $n \geq 108$.

1 Introduction

Andrews [12] introduced the spt-function $\mathrm{spt}(n)$ as the weighted counting of partitions with respect to the number of occurrences of the smallest part and he discovered that the spt-function bears striking resemblance to the classical partition function $p(n)$. Since then, the spt-function has drawn much attention and has been extensively studied. In this survey, we shall summarize developments on the spt-function including congruence properties derived from q-identities and modular forms, along with their combinatorial interpretations, as well as generalizations, variations and asymptotic properties. For the background on partitions, we refer to [8, 10, 20], and for the background on modular forms, we refer to [26, 59, 98, 111].

The spt-function spt(n), called the smallest part function, is defined to be the total number of smallest parts in all partitions of n. More precisely, for a partition λ of n, we use $n_s(\lambda)$ to denote the number of occurrences of the smallest part in λ. Let $P(n)$ denote the set of partitions of n, then

$$\mathrm{spt}(n) = \sum_{\lambda \in P(n)} n_s(\lambda). \tag{1.1}$$

For example, for $n = 4$, we have spt$(4) = 10$. Partitions in $P(4)$ and the values of $n_s(\lambda)$ are listed below:

$\lambda \in P(4)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$n_s(\lambda)$	1	1	2	2	4

The spt-function spt(n) can also be interpreted by marked partitions, see Andrews, Dyson and Rhoades [19]. A marked partition of n is meant to be a pair (λ, k), where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is an ordinary partition of n and k is an integer identifying one of its smallest parts. If λ_k is the identified smallest part of λ, we then use (λ, k) to denote this marked partition. For example, there are ten marked partitions of 4.

$$((4),1), \quad ((3,1),2), \quad ((2,2),1), \quad ((2,2),2), \quad ((2,1,1),2),$$
$$((2,1,1),3), \quad ((1,1,1,1),1), \quad ((1,1,1,1),2), \quad ((1,1,1,1),3), \quad ((1,1,1,1),4).$$

Using the definition (1.1), it is easy to derive the following generating function, see Andrews [12],

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty}. \tag{1.2}$$

Here we have adopted the common notation [10]:

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

The spt-function is closely related to the rank and the crank of a partition. Recall that the rank of a partition was introduced by Dyson [63] as the largest part of the partition minus the number of parts. The crank of a partition was defined by Andrews and Garvan [21] as the largest part if the partition contains no ones, otherwise as the number of parts larger than the number of ones minus the number of ones. For $n \geq 1$, let $N(m,n)$ denote the number of partitions of n with rank m, and for $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m. For $n = 1$, set

$$M(0,1) = -1,\ M(1,1) = M(-1,1) = 1,$$

and for $n = 1$ and $m \neq -1, 0, 1$, set

$$M(m,1) = 0.$$

Atkin and Garvan [28] defined the k-th moment $N_k(n)$ of ranks as

$$N_k(n) \quad = \quad \sum_{m=-\infty}^{\infty} m^k N(m,n), \tag{1.3}$$

and the k-th moment $M_k(n)$ of cranks as

$$M_k(n) = \sum_{m=-\infty}^{\infty} m^k M(m,n).$$

It is worth mentioning that Atkin and Garvan [28] showed that the generating functions of the moments of cranks are related to quasimodular forms. Bringmann, Garvan and Mahlburg [41] showed that the generating functions of the moments of ranks are related to quasimock theta functions. Asymptotic formulas for the moments of ranks and cranks were derived by Bringmann, Mahlburg and Rhoades [47].

Based on the generating function (1.2) and Watson's q-analog of Whipple's theorem [80, p. 43, eq. (2.5.1)], Andrews [12] showed that the spt-function can be expressed in terms of the second moment $N_2(n)$ of ranks introduced by Atkin and Garvan [28],

$$\mathrm{spt}(n) = np(n) - \frac{1}{2}N_2(n). \tag{1.4}$$

Ji [93] found a combinatorial proof of (1.4) using rooted partitions.

By means of a relation due to Dyson [64], namely,

$$M_2(n) = 2np(n), \tag{1.5}$$

Garvan [72] observed that the expression

$$\mathrm{spt}(n) = \frac{1}{2}M_2(n) - \frac{1}{2}N_2(n) \tag{1.6}$$

implies that $M_2(n) > N_2(n)$ for $n \geq 1$. In general, he conjectured and later proved that $M_{2k}(n) > N_{2k}(n)$ for $k \geq 1$ and $n \geq 1$, see [72, 73].

In view of the relation (1.4) and identities on refinements of $N(m,n)$ established by Atkin and Swinnerton-Dyer [30] and O'Brien [108], Andrews proved that $\mathrm{spt}(n)$ satisfies congruences mod 5, 7 and 13 which are reminiscent of Ramanujan's congruences for $p(n)$. Let ℓ be a prime. A Ramanujan congruence modulo ℓ for the sequence $\{a(n)\}_{n\geq 0}$ means a congruence of the form

$$a(\ell n + \beta) \equiv 0 \pmod{\ell}$$

for all nonnegative integers n and a fixed integer β.

Ramanujan [122] discovered the following congruences for $p(n)$,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.7}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.8}$$

$$p(11n+6) \equiv 0 \pmod{11}, \tag{1.9}$$

and proclaimed that "it appears that there are no equally simple properties for any moduli involving primes other than these three (i.e. $\ell = 5, 7, 11$)." See also Berndt [34, p. 27].

Elementary proofs of the congruences (1.7) and (1.8) were given by Ramanujan [122] and an elementary proof of the congruence (1.9) was given by Winquist [133]. Alternative proofs of (1.9) were found by Berndt, Chan, Liu and Yesilyurt [36] and Hirschhorn [84]. Recently, Paule and Radu [116] found a recurrence relation of the generating function of $p(11n+6)$, from which (1.9) is an immediate consequence. Berndt [35] provided simple proofs of (1.7)–(1.9) by using Ramanujan's differential equations for the Eisenstein series. Uniform proofs of (1.7)–(1.9) were found by Hirschhorn [83].

Concerning Ramanujan's conjecture, Kiming and Olsson [97] showed that if there exists a Ramanujan's congruence $p(\ell n+\beta) \equiv 0 \pmod{\ell}$, then $24\beta \equiv 1 \pmod{\ell}$. According to this condition, Ahlgren and Boylan [4] confirmed Ramanujan's conjecture. More precisely, they showed that for a prime ℓ, if there is a Ramanujan's congruence modulo ℓ for $p(n)$, then it must be one of the congruences (1.7), (1.8) and (1.9).

Combinatorial studies of Ramanujan's congruences of $p(n)$ go back to Dyson [63]. He conjectured that the rank of a partition can be used to divide the set of partitions of $5n+4$ (or $7n+5$) into five (or seven) equinumerous classes. More precisely, let $N(i,t,n)$ denote the number of partitions of n with rank congruent to i modulo t. Dyson [63] conjectured that

$$N(i,5,5n+4) = \frac{p(5n+4)}{5} \quad \text{for} \quad 0 \le i \le 4, \tag{1.10}$$

$$N(i,7,7n+5) = \frac{p(7n+5)}{7} \quad \text{for} \quad 0 \le i \le 6. \tag{1.11}$$

These relations were proved by Atkin and Swinnerton-Dyer [30], which imply (1.7) and (1.8). Dyson also pointed out that the rank of a partition cannot be used to interpret (1.9). To give a combinatorial explanation of this congruence modulo 11, Garvan [70] introduced the crank of a vector partition and showed that this statistic leads to interpretations of the above congruences of $p(n)$ mod 5, 7 and 11. Andrews and Garvan [21] found an equivalent definition of the crank in terms of an ordinary partition. For the history of the rank and the crank, see, for example, Andrews and Berndt [14] and Andrews and Ono [24].

Although Dyson's rank fails to explain Ramanujan's congruence (1.9) combinatorially, the generating functions for the rank differences have been extensively studied. For example, the generating functions for the rank differences $N(s,\ell,\ell n+d) - N(t,\ell,\ell n+d)$ for $\ell = 2, 9, 11, 12, 13$ have been determined by Atkin and Hussain [29], O'Brien [108], Lewis [102,103] and Santa-Gadea [126].

By the relations (1.4), (1.10) and (1.11), Andrews [12] showed that

$$\mathrm{spt}(5n+4) \equiv 0 \pmod{5}, \tag{1.12}$$

$$\mathrm{spt}(7n+5) \equiv 0 \pmod{7}. \tag{1.13}$$

He also obtained that

$$\mathrm{spt}(13n+6) \equiv 0 \pmod{13}, \tag{1.14}$$

by considering the properties of $N(i,13,13n+6)$ due to O'Brien [108]. Let

$$r_{a,b}(d) = \sum_{n=0}^{\infty} (N(a,13,13n+d) - N(b,13,13n+d))q^{13n},$$

and for $1 \le i \le 5$, and let

$$S_i(d) = r_{(i-1),i}(d) - (7-i)r_{5,6}(d).$$

O'Brien [108] deduced that

$$S_1(6) + 2S_2(6) - 5S_5(6) \equiv 0 \pmod{13} \tag{1.15}$$

and

$$S_2(6) + 5S_3(6) + 3S_4(6) + 3S_5(6) \equiv 0 \pmod{13}. \tag{1.16}$$

Employing (1.4), Andrews derived an expression for $\mathrm{spt}(13n+6)$ in terms of $N(i, 13, 13n+6)$ modulo 13. Then the congruence (1.14) follows from (1.15) and (1.16).

This paper is organized as follows. In Section 2, we recall the spt-crank of an S-partition defined by Andrews, Garvan and Liang, which leads to combinatorial interpretations of the congruences of the spt-function mod 5 and 7. Motivated by a problem of Andrews, Garvan and Liang on constructive proofs of the congruences mod 5 and 7, Chen, Ji and Zang introduced the notion of a doubly marked partition and its spt-crank. Such an spt-crank can be used to divide the set counted by $\mathrm{spt}(5n+4)$ (resp. $\mathrm{spt}(7n+5)$) into five (resp. seven) equinumerous classes. The unimodality of the spt-crank and related topics are also discussed. In Section 3, we begin with Ramanujan-type congruences of $\mathrm{spt}(n)$ mod 11,17, 19, 29, 31 and 37 obtained by Garvan. We then consider Ramanujan-type congruences of $\mathrm{spt}(n)$ modulo any prime $\ell \geq 5$ due to Ono and the ℓ-adic generalization due to Ahlgren, Bringmann and Lovejoy. The congruences of $\mathrm{spt}(n)$ mod powers of 5, 7 and 13 established by Garvan will also be discussed. We finish this section with congruences of $\mathrm{spt}(n)$ mod $2, 3$ and powers of 2 due to Folsom and Ono, and Garvan and Jennings-Shaffer. Section 4 is devoted to generalizations and variations of the spt-function. We first recall the higher order spt-function defined by Garvan, as a generalization of the spt-function. We then concentrate on two generalizations of the spt-function based on the j-rank, given by Dixit and Yee. The first variation of the spt-function was defined by Andrews, Chan and Kim as the difference between the first rank and crank moments. At the end of this section, we present three variations of the spt-function, which are restrictions of the spt-function to three classes of partitions. The generating functions, combinatorial interpretations and congruences of these generalizations and variations of the spt-function will also be discussed. In Section 5, we summarize asymptotic formulas of the spt-function and its variations. Section 6 contains some conjectures on inequalities on $\mathrm{spt}(n)$, which are analogous to those on $p(n)$, due to DeSalvo and Pak, and Bessenrodt and Ono. Beyond the log-concavity, we conjecture that $p(n)$ and $\mathrm{spt}(n)$ satisfy higher order inequalities induced from invariants of binary forms. In particular, we conjecture that the higher order Turán inequality holds for both $p(n)$ and $\mathrm{spt}(n)$ when n is large enough.

2 The spt-crank

To give combinatorial interpretations of congruences on spt(n), Andrews, Garvan and Liang [22] introduced the spt-crank of an S-partition, which is analogous to Garvan's crank of a vector partition [70]. They showed that the spt-crank of an S-partition can be used to divide the set of S-partitions with signs counted by spt($5n+4$) (or spt($7n+5$)) into five (or seven) equinumerous classes which leads to the congruences (1.12) and (1.13).

Andrews, Dyson and Rhoades [19] proposed the problem of finding an equivalent definition of the spt-crank for a marked partition. Chen, Ji and Zang [53] introduced the structure of a doubly marked partition and established a bijection between marked partitions and doubly marked partitions. Then they defined the spt-crank of a doubly marked partition in order to divide the set of marked partitions counted by spt($5n+4$) (or spt($7n+5$)) into five (or seven) equinumerous classes. Hence, in principle, the spt-crank of a doubly marked partition can be considered as a solution to the problem of Andrews, Dyson and Rhoades. It would be interesting to find an spt-crank directly defined on marked partitions.

Let $N_S(m,n)$ denote the net number, or the sum of signs, of S-partitions of n with spt-crank m. Andrews, Dyson and Rhoades [19] conjectured that $\{N_S(m,n)\}_m$ is unimodal for any given n and showed that this conjecture is equivalent to an inequality between the rank and the crank of a partition. Using the notion of the rank-set of a partition introduced by Dyson [64], Chen, Ji and Zang [52] gave a proof of this conjecture by constructing an injection from the set of partitions of n such that m appears in the rank-set to the set of partitions of n with rank not less than $-m$.

2.1 The spt-crank of an S-partition

Based on (1.2), Andrews, Garvan and Liang [22] noticed that the generating function of spt(n) can be expressed as

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_\infty}{(q^n;q)_\infty^2}, \tag{2.1}$$

and they introduced the structure of S-partitions to interpret the right-hand side of (2.1) as the generating function of the net number of S-partitions of n, that is, the sum of signs of S-partitions of n. More precisely, let $\mathcal{D}$ denote the set of partitions into distinct parts and $\mathcal{P}$ denote the set of partitions. For $\lambda \in \mathcal{P}$, we use $s(\lambda)$ to denote the smallest part of λ with the convention that $s(\emptyset) = +\infty$. The set of S-partitions is defined by

$$S = \{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P} \mid \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min\{s(\pi_2), s(\pi_3)\}\}. \tag{2.2}$$

For $\pi = (\pi_1, \pi_2, \pi_3) \in S$, Andrews, Garvan and Liang [22] defined the weight of π to be $|\pi_1| + |\pi_2| + |\pi_3|$ and defined the sign of π to be

$$\omega(\pi) = (-1)^{l(\pi_1)-1},$$

where $|\pi|$ denotes the sum of parts of π and $l(\pi)$ denotes the number of parts of π.

They showed that

$$\mathrm{spt}(n) = \sum_{\pi} \omega(\pi),$$

where π ranges over S-partitions of n. To give combinatorial interpretations of the congruences (1.12) and (1.13), Andrews, Garvan and Liang [22] defined the spt-crank of an S-partition, which takes the same form as the crank of a vector partition.

Let π be an S-partition, the spt-crank of π, denoted $r(\pi)$, is defined to be the number of parts of π_2 minus the number of parts of π_3, i.e.,

$$r(\pi) = l(\pi_2) - l(\pi_3).$$

Let $N_S(m, n)$ denote the net number of S-partitions of n with spt-crank m, that is,

$$N_S(m, n) = \sum_{\substack{|\pi|=n \\ r(\pi)=m}} \omega(\pi), \tag{2.3}$$

and let $N_S(k, t, n)$ denote the net number of S-partitions of n with spt-crank congruent to $k \pmod t$, namely,

$$N_S(k, t, n) = \sum_{m \equiv k \pmod t} N_S(m, n).$$

Andrews, Garvan and Liang [22] obtained the following relations.

Theorem 2.1 (Andrews, Garvan and Liang) *For* $0 \leq k \leq 4$,

$$N_S(k, 5, 5n+4) = \frac{\mathrm{spt}(5n+4)}{5},$$

and for $0 \leq k \leq 6$,

$$N_S(k, 7, 7n+5) = \frac{\mathrm{spt}(7n+5)}{7}.$$

Andrews, Garvan and Liang [22] defined an involution on the set of S-partitions:

$$\iota(\vec{\pi}) = \iota(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_3, \pi_2),$$

which leads to the symmetry property of $N_S(m, n)$:

$$N_S(m, n) = N_S(-m, n). \tag{2.4}$$

Using the generating function of $N_S(m, n)$, Andrews, Garvan and Liang [22] proved its positivity.

Theorem 2.2 (Andrews, Garvan and Liang) *For all integers m and positive integers n,*

$$N_S(m, n) \geq 0. \tag{2.5}$$

Dyson [65] gave an alternative proof of this property by establishing the relation:

$$N_S(m, n) = \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{j=0}^{k-1} p(n - k(m + j) - (k(k + 1)/2)).$$

Andrews, Garvan and Liang [22] posed the problem of finding a combinatorial interpretation of $N_S(m, n)$. Chen, Ji and Zang [53] introduced the structure of a doubly marked partition which leads to a combinatorial interpretation of $N_S(m, n)$.

2.2 The spt-crank of a doubly marked partition

In this section, we first give a definition of a doubly marked partition and then define its spt-crank. To this end, we assume that a partition λ of n is represented by its Ferrers diagram, and we use $D(\lambda)$ to denote size of the Durfee square of λ, see [10, p. 28]. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of n, the associated Ferrers diagram is the arrangement of n dots in l rows with the dots being left-justified and the i-th row having λ_i dots for $1 \leq i \leq l$. The Durfee square of λ is the largest-size square contained within the Ferrers diagram of λ.

For a partition λ, let λ' denote its conjugate. A doubly marked partition of n is a partition λ of n along with two distinguished columns indexed by s and t, denoted (λ, s, t), where

(1) $1 \leq s \leq D(\lambda)$;

(2) $s \leq t \leq \lambda_1$;

(3) $\lambda'_s = \lambda'_t$.

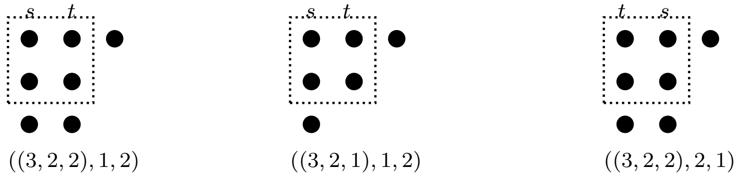

Figure 1: An illustration of the conditions for a doubly marked partition

For example, $((3, 2, 2), 1, 2)$ is a doubly marked partition, whereas $((3, 2, 1), 1, 2)$ and $((3, 2, 2), 2, 1)$ are not doubly marked partitions, see Figure 1.

To define the spt-crank of a doubly marked partition (λ, s, t), let

$$g(\lambda, s, t) = \lambda'_s - s + 1, \tag{2.6}$$

As $s \leq D(\lambda)$, we see that $\lambda'_s \geq s$, which implies that $g(\lambda, s, t) \geq 1$.

Let (λ, s, t) be a doubly marked partition, and let $g = g(\lambda, s, t)$. The spt-crank of (λ, s, t) is defined by

$$c(\lambda, s, t) = g - \lambda_g + t - s. \tag{2.7}$$

For example, for the doubly marked partition $((4, 4, 1, 1), 2, 3)$, we have $g = 2 - 1 = 1$ and the spt-crank equals $1 - \lambda_1 + 3 - 2 = -2$.

The following theorem in [53] gives a combinatorial interpretation of $N_S(m, n)$.

Theorem 2.3 (Chen, Ji and Zang) *For any integer m and any positive integer n, $N_S(m, n)$ equals the number of doubly marked partitions of n with spt-crank m.*

For example, for $n = 4$, the sixteen S-partitions of 4, their spt-cranks and the ten doubly marked partitions of 4 and their spt-cranks are listed in Table 1.

The proof of Theorem 2.3 relies on the generating function of $N_S(m, n)$ given by Andrews, Garvan and Liang [22].

Andrews, Dyson and Rhoades [19] proposed the problem of finding a definition of the spt-crank for a marked partition so that the set of marked partitions of $5n + 4$ (or $7n + 5$) can be divided into five (or seven) equinumerous classes. Chen, Ji and Zang [53] established a bijection Δ between the set of marked partitions of n and the set of doubly marked partitions of n.

S-partition	sign	spt-crank	doubly marked partition	spt-crank
$((1),(1,1,1),\emptyset)$	$+1$	3	$((1,1,1,1),1,1)$	3
$((1),(2,1),\emptyset)$	$+1$	2	$((2,1,1),1,1)$	2
$((1),(1,1),(1))$	$+1$	1	$((3,1),1,1)$	1
$((1),(3),\emptyset)$	$+1$	1	$((2,2),1,2)$	1
$((2,1),(1),\emptyset)$	-1	1		
$((2),(2),\emptyset)$	$+1$	1		
$((1),(2),(1))$	$+1$	0	$((2,2),1,1)$	0
$((1),(1),(2))$	$+1$	0	$((4),1,4)$	0
$((3,1),\emptyset,\emptyset)$	-1	0		
$((4),\emptyset,\emptyset)$	$+1$	0		
$((1),(1),(1,1))$	$+1$	-1	$((2,2),2,2)$	-1
$((1),\emptyset,(3))$	$+1$	-1	$((4),1,3)$	-1
$((2,1),\emptyset,(1))$	-1	-1		
$((2),\emptyset,(2))$	$+1$	-1		
$((1),\emptyset,(2,1))$	$+1$	-2	$((4),1,2)$	-2
$((1),\emptyset,(1,1,1))$	$+1$	-3	$((4),1,1)$	-3

Table 1: S-partitions and doubly marked partitions

Theorem 2.4 (Chen, Ji and Zang) *There is a bijection Δ between the set of marked partitions (μ,k) of n and the set of doubly marked partitions (λ,s,t) of n.*

To prove the above theorem, we adopt the notation (λ,s,t) for a partition λ with two distinguished columns λ'_s and λ'_t in the Ferrers diagram. Let Q_n denote the set of doubly marked partitions of n, and let

$$U_n=\{(\lambda,s,t)\mid|\lambda|=n,\,1\leq s\leq D(\lambda),\,1\leq t\leq\lambda_1\}.$$

Obviously, $Q_n\subseteq U_n$.

Before we give a description of the bijection Δ, we introduce a transformation τ from $U_n\setminus Q_n$ to U_n.

The transformation τ: Assume that $(\lambda,s,t)\in U_n\setminus Q_n$, that is, λ is a partition of n with two distinguished columns indexed by s and t such that $1\leq s\leq D(\lambda)$ and either $1\leq t<s$ or $\lambda'_s>\lambda'_t$. We wish to construct a partition μ with two distinguished columns indexed by a and b. Let p

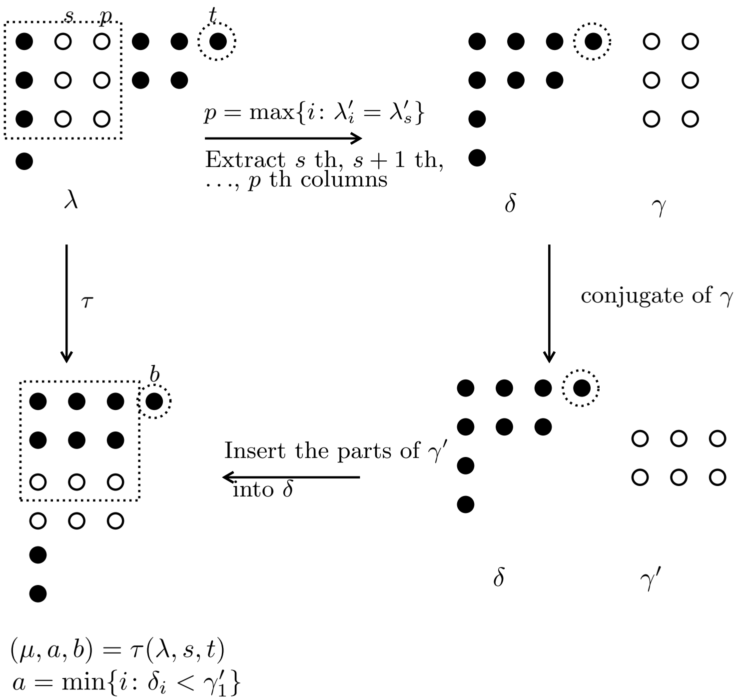

Figure 2: An illustration of the map τ

be the maximum integer such that $\lambda'_p = \lambda'_s$. Define

$$\delta = (\lambda_1 - p + s - 1, \lambda_2 - p + s - 1, \ldots, \lambda_{\lambda'_s} - p + s - 1, \lambda_{\lambda'_s+1}, \ldots, \lambda_\ell). \quad (2.8)$$

Set a to be the minimum integer such that $\delta_a < \lambda'_s$ and

$$\mu = (\delta_1, \ldots, \delta_{a-1}, \lambda'_s, \ldots, \lambda'_p, \delta_a, \ldots, \delta_\ell). \quad (2.9)$$

If $t < s$, then set $b = t$ and if $\lambda'_s > \lambda'_t$, then set $b = t - p + s - 1$. Define $\tau(\lambda, s, t) = (\mu, a, b)$. Figure 2 gives an illustration of the map $\tau\colon ((6,5,3,1),2,6) \mapsto ((4,3,3,3,1,1),3,4)$.

It was proved in [53] that the map τ is indeed an injection. Using this property, they described the bijection Δ in Theorem 2.4 based on the injection τ.

The definition of Δ: Let (μ, k) be a marked partition of n, we proceed to construct a doubly marked partition (λ, s, t) of n.

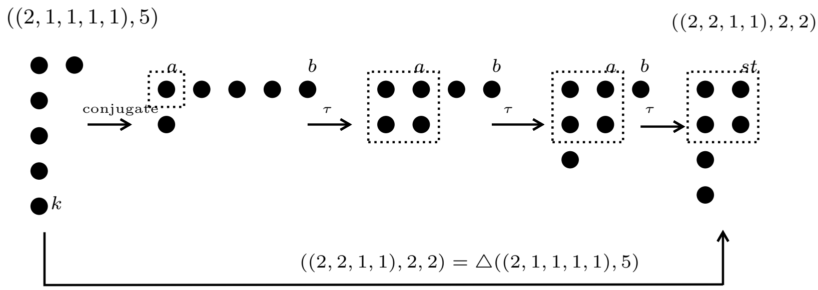

Figure 3: The bijection $\triangle$: $((2, 1, 1, 1, 1), 5) \mapsto ((2, 2, 1, 1), 2, 2)$

We first consider $(\mu', 1, k)$. If $(\mu', 1, k)$ is already a doubly marked partition, then there is nothing to be done and we just set $(\lambda, s, t) = (\mu', 1, k)$. Otherwise, we iteratively apply the map τ to $(\mu', 1, k)$ until we get a doubly marked partition (λ, s, t). We then define

$$\Delta(\mu, k) = (\lambda, s, t).$$

It can be shown that this process terminates and it is reversible. Thus Δ is well-defined and is a bijection between the set of marked partitions (μ, k) of n and the set of doubly marked partitions (λ, s, t) of n.

To give an example of the map Δ, let $n = 6$, $\mu = (2, 1, 1, 1, 1)$ and $k = 5$. We have $\mu' = (5, 1)$. Note that $(\mu', 1, k) = ((5, 1), 1, 5)$, which is not a doubly marked partition. It can be checked that $\tau(\mu', 1, k) = ((4, 2), 2, 4)$, which is not a doubly marked partition. Repeating this process, we get $\tau((4, 2), 2, 4) = ((3, 2, 1), 2, 3)$, and $\tau((3, 2, 1), 2, 3) = ((2, 2, 1, 1), 2, 2)$, which is eventually a doubly marked partition. See Figure 3. Thus, we obtain

$$\Delta((2, 1, 1, 1, 1), 5) = ((2, 2, 1, 1), 2, 2).$$

Utilizing the bijection Δ and the spt-crank for a doubly marked partition, one can divide the set of marked partitions of $5n+4$ (or $7n+5$) into five (or seven) equinumerous classes. Hence, in principle, the spt-crank of a doubly marked partition can be considered as a solution to the problem of Andrews, Dyson and Rhoades. It would be interesting to find an spt-crank directly defined on marked partitions.

For example, for $n = 4$, we have $\mathrm{spt}(4) = 10$. The ten marked partitions of 4, the corresponding doubly marked partitions, and the spt-crank modulo 5 are listed in Table 2.

For $n = 5$, we have $\mathrm{spt}(5) = 14$. The fourteen marked partitions of 5,

(μ, k)	$(\lambda, s, t) = \Delta(\mu, k)$	$c(\lambda, s, t)$	$c(\lambda, s, t) \mod 5$
$((4), 1)$	$((1,1,1,1),1,1)$	3	3
$((3,1), 2)$	$((3,1),1,1)$	1	1
$((2,2), 1)$	$((2,2),1,1)$	0	0
$((2,2), 2)$	$((2,2),1,2)$	1	1
$((2,1,1), 2)$	$((2,1,1),1,1)$	2	2
$((2,1,1), 3)$	$((2,2),2,2)$	-1	4
$((1,1,1,1), 1)$	$((4),1,1)$	-3	2
$((1,1,1,1), 2)$	$((4),1,2)$	-2	3
$((1,1,1,1), 3)$	$((4),1,3)$	-1	4
$((1,1,1,1), 4)$	$((4),1,4)$	0	0

Table 2: The case for $n = 4$

the corresponding doubly marked partitions, and the spt-crank modulo 7 are listed in Table 3.

(μ, k)	$(\lambda, s, t) = \Delta(\mu, k)$	$c(\lambda, s, t)$	$c(\lambda, s, t) \mod 7$
$((5), 1)$	$((1,1,1,1,1),1,1)$	4	4
$((4,1), 2)$	$((4,1),1,1)$	1	1
$((3,2), 2)$	$((3,1,1),1,1)$	2	2
$((3,1,1), 2)$	$((3,2),1,1)$	0	0
$((3,1,1), 3)$	$((3,2),1,2)$	1	1
$((2,2,1), 3)$	$((2,2,1),1,1)$	2	2
$((2,1,1,1), 2)$	$((2,1,1,1),1,1)$	3	3
$((2,1,1,1), 3)$	$((3,2),2,2)$	-2	5
$((2,1,1,1), 4)$	$((2,2,1),2,2)$	-1	6
$((1,1,1,1,1), 1)$	$((5),1,1)$	-4	3
$((1,1,1,1,1), 2)$	$((5),1,2)$	-3	4
$((1,1,1,1,1), 3)$	$((5),1,3)$	-2	5
$((1,1,1,1,1), 4)$	$((5),1,4)$	-1	6
$((1,1,1,1,1), 5)$	$((5),1,5)$	0	0

Table 3: The case for $n = 5$

2.3 The unimodality of the spt-crank

The unimodality of the spt-crank was first studied by Andrews, Dyson and Rhoades [19]. They showed that the unimodality of the spt-crank is equivalent to an inequality between the rank and the crank of a partition. Define

$$N_{\leq m}(n) = \sum_{|r|\leq m} N(r,n), \tag{2.10}$$

$$M_{\leq m}(n) = \sum_{|r|\leq m} M(r,n). \tag{2.11}$$

Andrews, Dyson and Rhoades [19] established the following relation.

Theorem 2.5 (Andrews, Dyson and Rhoades) *For $m \geq 0$ and $n > 1$,*

$$N_S(m,n) - N_S(m+1,n) = \frac{1}{2}\left(N_{\leq m}(n) - M_{\leq m}(n)\right). \tag{2.12}$$

They also posed a conjecture on the spt-crank.

Conjecture 2.6 (Andrews, Dyson and Rhoades) *For $m, n \geq 0$,*

$$N_S(m,n) \geq N_S(m+1,n). \tag{2.13}$$

By the symmetry (2.4) of $N_S(m,n)$ and the relation (2.13), we see that

$$N_S(-n,n) \leq \cdots \leq N_S(-1,n) \leq N_S(0,n) \geq N_S(1,n) \geq \cdots \geq N_S(n,n).$$

In view of (2.12), Andrews, Dyson and Rhoades pointed out that Conjecture 2.6 is equivalent to the assertion

$$N_{\leq m}(n) \geq M_{\leq m}(n), \tag{2.14}$$

where $m, n \geq 0$. It was remarked in [19] that (2.14) was conjectured by Bringmann and Mahlburg [44]. When $m = 0$, (2.14) was conjectured by Kaavya [95].

Andrews, Dyson and Rhoades [19] obtained an asymptotic formula for $N_{\leq m}(n) - M_{\leq m}(n)$, which implies that Conjecture 2.6 holds for fixed m and sufficiently large n.

Theorem 2.7 (Andrews, Dyson and Rhoades) *For each $m \geq 0$,*

$$\left(N_{\leq m}(n) - M_{\leq m}(n)\right) \sim \frac{(2m+1)\pi^2}{192\sqrt{3}n^2} e^{\pi\sqrt{\frac{2n}{3}}} \quad \text{as} \quad n \to \infty. \tag{2.15}$$

$n \setminus m$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
0							1						
1							1						
2						1	1	1					
3					1	1	1	1	1				
4				1	1	2	2	2	1	1			
5			1	1	2	2	2	2	2	1	1		
6		1	1	2	3	4	4	4	3	2	1	1	
7	1	1	2	3	4	4	5	4	4	3	2	1	1

Table 4: An illustration of the unimodality of $N_S(m, n)$

Using the rank-set of a partition, Chen, Ji and Zang [52] constructed an injection from the set of partitions of n such that m appears in the rank-set to the set of partitions of n with rank not less than $-m$. This proves the inequality (2.14) for all $m \geq 0$ and $n \geq 1$, and hence Conjecture 2.6 is confirmed.

In fact, the relation (2.14) was stated by Bringmann and Mahlburg [44] in a different notation. For an integer m and a positive integer n, let

$$\overline{\mathcal{M}}(m, n) = \sum_{r \leq m} M(r, n),$$

and

$$\overline{\mathcal{N}}(m, n) = \sum_{r \leq m} N(r, n).$$

By the symmetry properties of the rank and the crank, that is,

$$N(m, n) = N(-m, n) \quad \text{and} \quad M(m, n) = M(-m, n),$$

see [63] and [70], it is not difficult to verify that (2.14) is equivalent to the following inequality for $m < 0$ and $n \geq 1$:

$$\overline{\mathcal{N}}(m, n) \leq \overline{\mathcal{M}}(m, n). \tag{2.16}$$

It turns out that the constructive approach in [52] can be used to prove the other part of the conjecture (2.16) of Bringmann and Mahlburg, that is,

$$\overline{\mathcal{M}}(m, n) \leq \overline{\mathcal{N}}(m + 1, n), \tag{2.17}$$

for $m < 0$ and $n \geq 1$. A proof of (2.17) was given in [54].

In the notation $N_{\leq m-1}(n)$ and $M_{\leq m}(n)$, the inequality (2.17) can be expressed as

$$M_{\leq m}(n) \geq N_{\leq m-1}(n), \tag{2.18}$$

for $m \geq 1$ and $n \geq 1$.

Bringmann and Mahlburg [44] also pointed out that the inequalities (2.16) and (2.17) can be restated as the existence of a re-ordering τ_n on the set of partitions of n such that $|\text{crank}(\lambda)|-|\text{rank}(\tau_n(\lambda))| = 0$ or 1 for all partitions λ of n. Chen, Ji and Zang [54] defined a re-ordering τ_n on the set of partitions of n and showed that this re-ordering τ_n satisfies the relation $|\text{crank}(\lambda)| - |\text{rank}(\tau_n(\lambda))| = 0$ or 1 for any partition λ of n. Appealing to this re-ordering τ_n, they gave a new combinatorial interpretation of the function $\text{ospt}(n)$ defined by Andrews, Chan and Kim [15], which leads to an upper bound for $\text{ospt}(n)$ due to Chan and Mao [49].

Bringmann and Mahlburg [44] also remarked that using the Cauchy-Schwartz inequality, the bijection τ_n leads to an upper bound for $\text{spt}(n)$, namely, for $n \geq 1$,

$$\text{spt}(n) \leq \sqrt{2n}\,p(n). \tag{2.19}$$

Chan and Mao [49] posed a conjecture on a sharper upper bound and a lower bound for $\text{spt}(n)$.

Conjecture 2.1 (Chan and Mao) *For $n \geq 3$,*

$$\frac{\sqrt{6n}}{\pi}p(n) \leq \text{spt}(n) \leq \sqrt{n}\,p(n). \tag{2.20}$$

The following upper bound and lower bound for $\text{spt}(n)$ were conjectured by Hirschhorn and later proved by Eichhorn and Hirschhorn [66].

Theorem 2.8 (Eichhorn and Hirschhorn) *For $n \geq 2$,*

$$p(0) + p(1) + \cdots + p(n-1) < \text{spt}(n) < p(0) + p(1) + \cdots + p(n). \tag{2.21}$$

3 More congruences

Garvan [72] obtained Ramanujan-type congruences of $\text{spt}(n)$ mod 11, 17, 19, 29, 31 and 37.

Theorem 3.1 (Garvan) *For $n \geq 0$,*

$$\text{spt}(11 \cdot 19^4 \cdot n + 22006) \quad \equiv \quad 0 \pmod{11}, \tag{3.1}$$

$$\begin{aligned}
\mathrm{spt}(17 \cdot 7^4 \cdot n + 243) &\equiv 0 \pmod{17}, && (3.2)\\
\mathrm{spt}(19 \cdot 5^4 \cdot n + 99) &\equiv 0 \pmod{19}, && (3.3)\\
\mathrm{spt}(29 \cdot 13^4 \cdot n + 18583) &\equiv 0 \pmod{29}, && (3.4)\\
\mathrm{spt}(31 \cdot 29^4 \cdot n + 409532) &\equiv 0 \pmod{31}, && (3.5)\\
\mathrm{spt}(37 \cdot 5^4 \cdot n + 1349) &\equiv 0 \pmod{37}. && (3.6)
\end{aligned}$$

Bringmann [39] showed that $\mathrm{spt}(n)$ possesses a congruence property analogous to the following theorem for $p(n)$, due to Ono [110].

Theorem 3.2 (Ono) *For any prime $\ell \geq 5$, there are infinitely many arithmetic progressions $an + b$ such that*

$$p(an + b) \equiv 0 \pmod{\ell}. \tag{3.7}$$

As for $\mathrm{spt}(n)$, Bringmann [39] proved the following assertion.

Theorem 3.3 (Bringmann) *For any prime $\ell \geq 5$, there are infinitely many arithmetic progressions $an + b$ such that*

$$\mathrm{spt}(an + b) \equiv 0 \pmod{\ell}.$$

The above theorem is a consequence of (1.4), Theorem 3.2 and the following theorem of Bringmann [39].

Theorem 3.4 (Bringmann) *For any prime $\ell \geq 5$, there are infinitely many arithmetic progressions $an + b$ such that*

$$N_2(an + b) \equiv 0 \pmod{\ell}. \tag{3.8}$$

Bringmann [39] constructed a weight 3/2 harmonic weak Maass form $\mathcal{M}(z)$ on $\Gamma_0(576)$ with Nebentypus $\chi_{12}(\bullet) = \left(\frac{12}{\bullet}\right)$, which is related to the generating function of $\mathrm{spt}(n)$. This implies that the generating function of $\mathrm{spt}(n)$ is essentially a mock theta function with Dedekind eta-function $\eta(q)$ as its shadow just as pointed out by Rhoades [124]. Ono [112] found a weight $(\ell^2+3)/2$ holomorphic modular form on $SL_2(\mathbb{Z})$ which contains the holomorphic part of $\mathcal{M}(z)$. Using this modular form, Ono [112] derived Ramanujan-type congruences of $\mathrm{spt}(n)$ modulo ℓ for any prime $\ell \geq 5$.

Theorem 3.5 (Ono) *Let $\ell \geq 5$ be a prime and let $\left(\frac{\bullet}{\circ}\right)$ denote the Legendre symbol.*

(i) *For* $n \geq 1$, *if* $\left(\frac{-n}{\ell}\right) = 1$,

$$\operatorname{spt}\left((\ell^2 n+1)/24\right) \equiv 0 \pmod{\ell}.$$

(ii) *For* $n \geq 0$,

$$\operatorname{spt}\left((\ell^3 n+1)/24\right) \equiv \left(\frac{3}{\ell}\right) \operatorname{spt}\left((\ell n+1)/24\right) \pmod{\ell}.$$

Ahlgren, Bringmann and Lovejoy [5] extended Theorem 3.5 to any prime power. An analogous congruence for $p(n)$ was found by Ahlgren [1].

Theorem 3.6 (Ahlgren, Bringmann and Lovejoy) *Let* $\ell \geq 5$ *be a prime and let* $m \geq 1$.

(i) *For* $n \geq 1$, *if* $\left(\frac{-n}{\ell}\right) = 1$,

$$\operatorname{spt}\left((\ell^{2m} n+1)/24\right) \equiv 0 \pmod{\ell^m}.$$

(ii) *For* $n \geq 0$,

$$\operatorname{spt}\left((\ell^{2m+1} n+1)/24\right) \equiv \left(\frac{3}{\ell}\right) \operatorname{spt}\left((\ell^{2m-1} n+1)/24\right) \pmod{\ell^m}.$$

Recall the following congruences of $p(n)$:

$$p(5^a n+\delta_a) \equiv 0 \pmod{5^a}, \tag{3.9}$$

$$p(7^b n+\lambda_b) \equiv 0 \pmod{7^{\lfloor \frac{b+2}{2} \rfloor}}, \tag{3.10}$$

$$p(11^c n+\varphi_c) \equiv 0 \pmod{11^c}, \tag{3.11}$$

where a, b, c are positive integers and δ_a, λ_b and φ_c are the least nonnegative residues of the reciprocals of 24 mod $5^a, 7^b$ and 11^c, respectively. The congruences (3.9) and (3.10) were proved by Watson [132] and the congruence (3.11) was proved by Atkin [27]. Folsom, Kent and Ono [67] provided alternative proofs of the congruences (3.9)–(3.11) with the aid of the theory of ℓ-adic modular forms. Recently, Paule and Radu [117] found a unified algorithmic approach to (3.9)–(3.11) resorting to elementary modular function tools only.

In the case of the spt-function, although Theorem 3.6 gives congruences for all primes $\ell \geq 5$, the congruences (1.12)–(1.14) do not follow from Theorem 3.6. Congruences for these missing cases have been obtained by Garvan [74], which are analogous to (3.9)–(3.11).

Theorem 3.7 (Garvan) *For $n \geq 0$,*

$$\mathrm{spt}(5^a n + \delta_a) \equiv 0 \pmod{5^{\lfloor \frac{a+1}{2} \rfloor}},$$

$$\mathrm{spt}(7^b n + \lambda_b) \equiv 0 \pmod{7^{\lfloor \frac{b+1}{2} \rfloor}},$$

$$\mathrm{spt}(13^c n + \gamma_c) \equiv 0 \pmod{13^{\lfloor \frac{c+1}{2} \rfloor}},$$

where a, b, c are positive integers, and δ_a, λ_b and γ_c are the least nonnegative residues of the reciprocals of 24 mod 5^a, 7^b and 13^c respectively.

Setting $a = b = c = 1$, Theorem 3.7 reduces to (1.12)–(1.14). Belmont, Lee, Musat and Trebat-Leder [32] provided another proof of the above theorem by generalizing techniques of Folsom, Kent and Ono [67] and by utilizing refinements due to Boylan and Webb [38].

Before we get into the discussions about the parity of $\mathrm{spt}(n)$, let us look back at the parity of $p(n)$. Subbarao [129] conjectured that in every arithmetic progression $r \pmod t$, there are infinitely many integers $N \equiv r \pmod t$ for which $p(N)$ is even, and infinitely many integers $M \equiv r \pmod t$ for which $p(M)$ is odd. This conjecture has been confirmed for $t = 1, 2, 3, 4, 5, 10, 12, 16$ and 40 by Garvan and Stanton [79], Hirschhorn [82], Hirschhorn and Subbarao [85], Kolberg [99] and Subbarao [129]. The even case of Subbarao's conjecture was settled by Ono [109] and the odd case was solved by Radu [121]. Radu [121] also showed that for every arithmetic progression $r \pmod t$, there are infinitely many integers $N \equiv r \pmod t$ such that $p(N) \not\equiv 0 \pmod 3$. This confirms a conjecture posed by Ahlgren and Ono [6].

For $n \geq 1$, the parity of $\mathrm{spt}(n)$ is determined by Folsom and Ono [68]. They constructed a pair of harmonic weak Maass forms with equal nonholomorphic parts, whose difference contains the generating function of $\mathrm{spt}(n)$ as a component. Based on the results in [40], Folsom and Ono showed that the difference of such pair of harmonic weak Maass forms can be expressed as the sum of the generating function for $\mathrm{spt}(n)$ and a modular form. This enables us to completely determine the parity of $\mathrm{spt}(n)$.

To be more specific, Folsom and Ono [68] first defined the mock theta functions:

$$D(z) = \frac{q^{-\frac{1}{24}}}{(q;q)_\infty} \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right) = \frac{q^{-\frac{1}{24}}}{(q;q)_\infty} E_2(z)$$

and

$$L(z) = \frac{(q^6;q)_\infty^2 (q^{24};q)_\infty^2}{(q^{12};q)_\infty^5} \left(\sum_{n=-\infty}^{\infty} \frac{(12n-1)q^{6n^2-\frac{1}{24}}}{1-q^{12n-1}} - \sum_{n=-\infty}^{\infty} \frac{(12n-5)q^{6n^2-\frac{25}{24}}}{1-q^{12n-5}} \right).$$

Then they obtained the following modular form.

Theorem 3.8 (Folsom and Ono) *The function*

$$D(24z) - 12L(24z) - 12q^{-1}S(24z)$$

is a weight $3/2$ *weakly holomorphic modular form on* $\Gamma_0(576)$ *with Nebentypus* $\left(\frac{12}{\bullet}\right)$, *where*

$$S(z) = \sum_{n=0}^{\infty} \mathrm{spt}(n)q^n.$$

By Theorem 3.8, Folsom and Ono [68] obtained a characterization of the parity of $\mathrm{spt}(n)$.

Theorem 3.9 (Folsom and Ono) *The function* $\mathrm{spt}(n)$ *is odd if and only if* $24n - 1 = pm^2$, *where* m *is an integer and* $p \equiv 23 \pmod{24}$ *is prime.*

As pointed out by Andrews, Garvan and Liang [23], Theorem 3.9 contains an error. For example, for $n = 507$, it is clear that $507 \times 24 - 1 = 12167 = 23 \times 23^2 = pm^2$, where $p = m = 23$. Obviously, 507 satisfies the condition of Theorem 3.9. But $\mathrm{spt}(507) = 6047032773755628522506 4$ is even. This error has been corrected by Andrews, Garvan and Liang [23]. By using the notion of S-partitions as defined in (2.2), they noticed that the number of S-partitions of n has the same parity as $\mathrm{spt}(n)$. Then they built an involution ι on the set of S-partitions of n as follows:

$$\iota(\vec{\pi}) = \iota(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_3, \pi_2).$$

Clearly, an S-partition (π_1, π_2, π_3) is a fixed point of ι if and only if $\pi_2 = \pi_3$. Denote the number of such S-partitions of n by $N_{SC}(n)$. It is not difficult to see that

$$\mathrm{spt}(n) \equiv N_{SC}(n) \pmod{2}.$$

By computing the generating function of $N_{SC}(n)$, Andrews, Garvan and Liang [23] established a corrected version of Theorem 3.9.

Theorem 3.10 (Andrews, Garvan and Liang) *The function* $\mathrm{spt}(n)$ *is odd if and only if* $24n - 1 = p^{4a+1}m^2$ *for some prime* $p \equiv 23 \pmod{24}$ *and some integers* a, m *with* $(p, m) = 1$.

The spt-function is also related to some combinatorial sequences, see, for example, Andrews, Rhoades and Zwegers [25] and Bryson, Ono, Pitman and Rhoades [48]. Bryson, Ono, Pitman and Rhoades [48] showed that the number of strongly unimodal sequences of size n has the same parity as $\mathrm{spt}(n)$. More specifically, a sequence of integers $\{a_i\}_{i=1}^{s}$ is said to be a strongly unimodal sequence of size n if $a_1 + \cdots + a_s = n$ and for some k,

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > a_{k+2} > \cdots > a_s > 0.$$

Let $u(n)$ be the number of strongly unimodal sequences of size n. By [13, Theorem 1], Bryson, Ono, Pitman and Rhoades [48] observed that

$$u(n) \equiv \mathrm{spt}(n) \pmod 2.$$

As for congruences of $\mathrm{spt}(n)$ modulo powers of 2, Garvan and Jennings-Shaffer [77] obtained congruences mod $2^3, 2^4$ and 2^5. Let

$$s_\ell = \frac{\ell^2 - 1}{24}.$$

Theorem 3.11 (Garvan and Jennings-Shaffer) *Let $\ell \geq 5$ be a prime, and define*

$$\beta = \begin{cases} 3, & \text{if } \ell \equiv 7, 9 \pmod{24}, \\ 4, & \text{if } \ell \equiv 13, 23 \pmod{24}, \\ 5, & \text{if } \ell \equiv 1, 11, 17, 19 \pmod{24}. \end{cases}$$

Then for $n \geq 1$,

$$\begin{aligned} \mathrm{spt}(\ell^2 n - s_\ell) &+ \left(\frac{3 - 72n}{\ell}\right) \mathrm{spt}(n) + \ell\, \mathrm{spt}\left((n + s_\ell)/\ell^2\right) \\ &\equiv \left(\frac{3}{\ell}\right) (1 + \ell)\, \mathrm{spt}(n) \pmod{2^\beta}. \end{aligned}$$

By using the Hecke algebra of a Maass form, Folsom and Ono [68] derived a congruence of $\mathrm{spt}(n)$ modulo 3.

Theorem 3.12 (Folsom and Ono) *Let $\ell \geq 5$ be a prime, then for $n \geq 1$,*

$$\begin{aligned} \mathrm{spt}(\ell^2 n - s_\ell) &+ \left(\frac{3 - 72n}{\ell}\right) \mathrm{spt}(n) + \ell\, \mathrm{spt}\left((n + s_\ell)/\ell^2\right) \\ &\equiv \left(\frac{3}{\ell}\right) (1 + \ell)\, \mathrm{spt}(n) \pmod 3. \end{aligned}$$

Corollary 3.13 (Folsom and Ono) *Let $\ell \geq 5$ be a prime such that $\ell \equiv 2 \pmod 3$. If $0 < k < \ell - 1$, then for $n \geq 1$,*

$$\operatorname{spt}(\ell^4 n + \ell^3 k - (\ell^4 - 1)/24) \equiv 0 \pmod 3.$$

For example, for $\ell = 5$, we have

$$\begin{aligned} \operatorname{spt}(625n + 99) &\equiv \operatorname{spt}(625n + 224) \\ &\equiv \operatorname{spt}(625n + 349) \\ &\equiv \operatorname{spt}(625n + 474) \\ &\equiv 0 \pmod 3. \end{aligned}$$

Garvan [75] derived congruences mod $5, 7, 13$ and 72.

Theorem 3.14 (Garvan) (i) *If $\ell \geq 5$ is prime, then for $n \geq 1$*

$$\begin{aligned} &\operatorname{spt}(\ell^2 n - s_\ell) + \left(\frac{3 - 72n}{\ell}\right) \operatorname{spt}(n) + \ell \operatorname{spt}\left((n + s_\ell)/\ell^2\right) \\ &\qquad \equiv \left(\frac{3}{\ell}\right) (1 + \ell) \operatorname{spt}(n) \pmod{72}. \end{aligned} \tag{3.12}$$

(ii) *If $\ell \geq 5$ is prime, $t = 5, 7$ or 13 and $\ell \neq t$, then for $n \geq 1$*

$$\begin{aligned} &\operatorname{spt}(\ell^2 n - s_\ell) + \left(\frac{3 - 72n}{\ell}\right) \operatorname{spt}(n) + \ell \operatorname{spt}\left((n + s_\ell)/\ell^2\right) \\ &\qquad \equiv \left(\frac{3}{\ell}\right) (1 + \ell) \operatorname{spt}(n) \pmod{t}. \end{aligned} \tag{3.13}$$

Note that Theorem 3.12 can be deduced from (3.12). Moreover, writing $32760 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, from (3.12) and (3.13), it is easy to deduce a congruence of $\operatorname{spt}(n)$ modulo 32760.

Corollary 3.15 (Garvan) *If ℓ is prime and $\ell \notin \{2, 3, 5, 7, 13\}$, then for $n \geq 1$*

$$\begin{aligned} &\operatorname{spt}(\ell^2 n - s_\ell) + \left(\frac{3 - 72n}{\ell}\right) \operatorname{spt}(n) + \ell \operatorname{spt}\left((n + s_\ell)/\ell^2\right) \\ &\qquad \equiv \left(\frac{3}{\ell}\right) (1 + \ell) \operatorname{spt}(n) \pmod{32760}. \end{aligned}$$

Garrett, McEachern, Frederick and Hall-Holt [69] obtained a recurrence relation for $\mathrm{spt}(n)$. To compute $\mathrm{spt}(n)$, they introduced two integer arrays $A(n,j)$ and $B(n,j)$, where $A(n,j)$ denotes the number of partitions of n with the smallest part at least j and $B(n,j)$ denotes the number of times that j occurs as the smallest part of partitions of n. From the definitions of $A(n,j)$ and $B(n,j)$, it is not difficult to deduce the following recurrence relations:

$$\begin{aligned} A(n,j) &= A(n-j,j) + A(n,j+1), \\ B(n,j) &= A(n-j,j) + B(n-j,j), \end{aligned}$$

where $A(n,j) = B(n,j) = 0$ whenever $n < j$ and $A(n,n) = B(n,n) = 1$.

Thus we have

$$\mathrm{spt}(n) = \sum_{j=1}^{n} B(n,j).$$

By the above relation, Garrett, McEachern, Frederick and Hall-Holt computed the first million values of $\mathrm{spt}(n)$, and found many conjectures on congruences of $\mathrm{spt}(n)$.

$$\mathrm{spt}(1331n+479) \equiv 0 \pmod{11}, \tag{3.14}$$

$$\mathrm{spt}(1331n+842) \equiv 0 \pmod{11}, \tag{3.15}$$

$$\mathrm{spt}(1331n+1084) \equiv 0 \pmod{11}, \tag{3.16}$$

$$\mathrm{spt}(1331n+1205) \equiv 0 \pmod{11}, \tag{3.17}$$

$$\mathrm{spt}(1331n+1326) \equiv 0 \pmod{11}, \tag{3.18}$$

$$\mathrm{spt}(4913n+566) \equiv 0 \pmod{17}, \tag{3.19}$$

$$\mathrm{spt}(4913n+2300) \equiv 0 \pmod{17}, \tag{3.20}$$

$$\mathrm{spt}(4913n+2878) \equiv 0 \pmod{17}, \tag{3.21}$$

$$\mathrm{spt}(4913n+3167) \equiv 0 \pmod{17}, \tag{3.22}$$

$$\mathrm{spt}(4913n+3456) \equiv 0 \pmod{17}, \tag{3.23}$$

$$\mathrm{spt}(4913n+4323) \equiv 0 \pmod{17}, \tag{3.24}$$

$$\mathrm{spt}(4913n+4612) \equiv 0 \pmod{17}, \tag{3.25}$$

$$\mathrm{spt}(4913n+4901) \equiv 0 \pmod{17}, \tag{3.26}$$

$$\mathrm{spt}(11875n+99) \equiv 0 \pmod{19}, \tag{3.27}$$

$$\mathrm{spt}(12167n+9500) \equiv 0 \pmod{23}, \tag{3.28}$$

$$\operatorname{spt}(24389n+806) \equiv 0 \pmod{29}. \tag{3.29}$$

All the above conjectures have been confirmed. The congruence (3.27) has been proved by Garvan [72], and the rest are consequences of Theorem 3.5. Indeed, Theorem 3.5 (i) implies that if $\left(\frac{-\delta}{\ell}\right)=1$, then

$$\operatorname{spt}\left(\frac{\ell^2(\ell n+\delta)+1}{24}\right) \equiv 0 \pmod{\ell}. \tag{3.30}$$

When $\ell = 11, 17, 23, 29$, (3.30) becomes (3.14)–(3.26), (3.28) and (3.29), respectively.

4 Generalizations and variations

In this section, we discuss three generalizations and one variation of the spt-function based on the relation (1.6) and three variations based on the combinatorial definition.

4.1 The higher order spt-function of Garvan

The first generalization of the spt-function was due to Garvan [73]. He defined a higher order spt-function in terms of the k-th symmetrized rank function and the k-th symmetrized crank function.

The k-th symmetrized rank function $\eta_k(n)$ was introduced by Andrews [11], and it is defined by

$$\eta_k(n) = \sum_{m=-n}^{n} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} N(m,n). \tag{4.1}$$

By using q-identities, Andrews [11] found a combinatorial interpretation of $\eta_k(n)$ in terms of k-marked Durfee symbol. Ji [94] and Kursungoz [101] found combinatorial derivations of this combinatorial interpretation of $\eta_k(n)$ directly from the definition (4.1). When $k=2$, it is easy to check that

$$\eta_2(n) = \frac{1}{2} N_2(n),$$

where the second rank moment $N_2(n)$ is defined as in (1.3).

Garvan [73] introduced the k-th symmetrized crank function $\mu_k(n)$ as follows:

$$\mu_k(n) = \sum_{m=-n}^{n} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} M(m,n). \tag{4.2}$$

A combinatorial interpretation of $\mu_k(n)$ was given by Chen, Ji and Shen [51]. When $k=2$, it is not difficult to derive that

$$\mu_2(n) = \frac{1}{2} M_2(n).$$

Garvan [73] introduced the higher order spt-function $\mathrm{spt}_k(n)$.

Definition 4.1 *For $k \geq 1$, define*

$$\mathrm{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n). \tag{4.3}$$

In view of (1.6), it is easy to see that $\mathrm{spt}_k(n)$ reduces to $\mathrm{spt}(n)$ when $k=1$. Making use of Bailey pairs [9], Garvan obtained the generating function of $\mathrm{spt}_k(n)$.

Theorem 4.2 (Garvan) *For $k \geq 1$,*

$$\sum_{n=1}^{\infty} \mathrm{spt}_k(n) q^n = \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2 (q^{n_1+1};q)_\infty}. \tag{4.4}$$

Setting $k=1$ in (4.4), we get the generating function (1.2) of $\mathrm{spt}(n)$. Furthermore, it can be seen from (4.4) that $\mathrm{spt}_k(n) \geq 0$ for $n, k \geq 1$. Together with (4.3), we find that

$$\mu_{2k}(n) \geq \eta_{2k}(n). \tag{4.5}$$

The inequality (4.5) plays a key role in the proof of an inequality between the rank moments and the crank moments, as conjectured by Garvan [72].

Conjecture 4.3 (Garvan) *For $n, k \geq 1$,*

$$M_{2k}(n) \geq N_{2k}(n). \tag{4.6}$$

Bringmann and Mahlburg [44] showed that the above conjecture is true for $k=1,2$ and sufficiently large n. For each fixed k, Garvan's conjecture was proved for sufficiently large n by Bringmann, Mahlburg and Rhoades [46]. Garvan [73] confirmed his conjecture for all k and n. He introduced an analogue of the Stirling numbers of the second kind, denoted by $S^*(k,j)$. It is defined recursively as follows:

(1) $S^*(1,1) = 1$;

(2) $S^*(k,j) = 0$ if $j \leq 0$ or $j > k$;

(3) $S^*(k+1,j) = S^*(k,j-1) + j^2 S^*(k,j)$ for $1 \leq j \leq k+1$.

It is clear from the above recurrence relation that $S^*(k,j) \geq 0$. Garvan established the following relations between the ordinary moments and symmetrized moments in terms of $S^*(k,j)$:

$$M_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k,j) \mu_{2j}(n) \tag{4.7}$$

and

$$N_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k,j) \eta_{2j}(n). \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$M_{2k}(n) - N_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k,j) \left(\mu_{2j}(n) - \eta_{2j}(n)\right). \tag{4.9}$$

Invoking (4.5) we deduce that $M_{2k}(n) - N_{2k}(n) \geq 0$ for $n, k \geq 1$, and hence Conjecture 4.3 is proved.

Garvan [73] gave a combinatorial explanation of the right-hand side of (4.4). Thus Theorem 4.2 leads to a combinatorial interpretation of $\mathrm{spt}_k(n)$.

Theorem 4.4 (Garvan) *Let λ be a partition with m different parts*

$$n_1 < n_2 < \cdots < n_m.$$

Let $k \geq 1$, define the weight $\omega_k(\lambda)$ of λ as follows:

$$\omega_k(\lambda) = \sum_{\substack{m_1+\cdots+m_r=k \\ 1\leq r\leq k}} \binom{f_1+m_1-1}{2m_1-1} \times \sum_{2\leq j_2<j_3<\cdots<j_r} \binom{f_{j_2}+m_2}{2m_2}\binom{f_{j_3}+m_3}{2m_3}\cdots\binom{f_{j_r}+m_r}{2m_r},$$

where $f_j = f_j(\lambda)$ denotes the multiplicity of the part n_j in λ. Then

$$\mathrm{spt}_k(n) = \sum_{\lambda\in P(n)} \omega_k(\lambda).$$

Garvan [73] also obtained congruences of $\mathrm{spt}_2(n)$, $\mathrm{spt}_3(n)$ and $\mathrm{spt}_4(n)$.

Theorem 4.5 (Garvan) *For* $n \geq 1$,

$$\begin{aligned}
\mathrm{spt}_2(n) &\equiv 0 \pmod 5, && \text{if } n \equiv 0,1,4 \pmod 5,\\
\mathrm{spt}_2(n) &\equiv 0 \pmod 7, && \text{if } n \equiv 0,1,5 \pmod 7,\\
\mathrm{spt}_2(n) &\equiv 0 \pmod{11}, && \text{if } n \equiv 0 \pmod{11},\\
\mathrm{spt}_3(n) &\equiv 0 \pmod 7, && \text{if } n \not\equiv 3,6 \pmod 7,\\
\mathrm{spt}_3(n) &\equiv 0 \pmod 2, && \text{if } n \equiv 1 \pmod 4,\\
\mathrm{spt}_4(n) &\equiv 0 \pmod 3, && \text{if } n \equiv 0 \pmod 3.
\end{aligned}$$

4.2 Generalized higher order spt-functions of Dixit and Yee

Other generalizations of the spt-function have been given by Dixit and Yee [62], which are based on the j-rank introduced by Garvan [71]. The j-rank is a generalization of Dyson's rank. For a partition λ and $j \geq 2$, let $n_j(\lambda)$ denote the size of the j-th successive Durfee square of λ, let $c_j(\lambda)$ denote the number of columns in the Ferrers diagram of λ with length not exceeding $n_j(\lambda)$ and let $r_j(\lambda)$ denote the number of parts of λ that lie below the j-th Durfee square. Then the j-rank of λ is defined to be $c_{j-1}(\lambda) - r_{j-1}(\lambda)$. It should be noted that the 2-rank coincides with Dyson's rank.

For example, the 3-rank of $\lambda = (9,9,7,7,7,5,3,3,2,2,1)$ is equal to -1, since $n_2(\lambda) = 3$, $c_2(\lambda) = 2$ and $r_2(\lambda) = 3$, see Figure 4.

Let $N_j(m,n)$ denote the number of partitions of n with j-rank m. Garvan [71] showed that for $j \geq 2$,

$$\sum_{n=0}^{\infty} N_j(m,n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n((2j-1)n-1)}{2}+|m|n}(1-q^n). \tag{4.10}$$

Dixit and Yee [62] defined the j-rank moment ${}_jN_k(n)$ by

$${}_jN_k(n) = \sum_{m=-\infty}^{\infty} m^k N_j(m,n). \tag{4.11}$$

In the notation ${}_jN_k(n)$, they defined $\mathrm{Spt}_j(n)$ as follows.

Definition 4.6 *For* $n, j \geq 1$,

$$\mathrm{Spt}_j(n) = np(n) - \frac{1}{2}\ {}_{j+1}N_2(n). \tag{4.12}$$

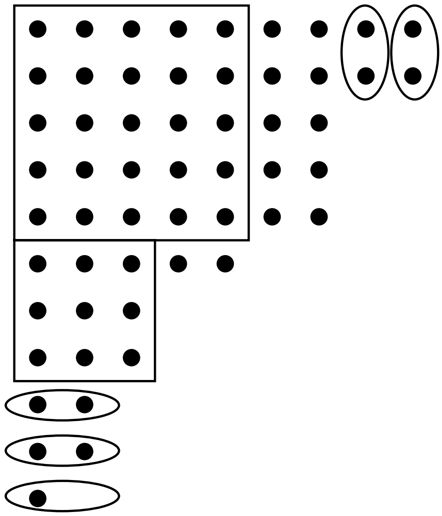

Figure 4: An illustration of 3-rank of $(9,9,7,7,7,5,3,3,2,2,1)$

In light of (1.4), it is easy to see that $\mathrm{Spt}_j(n)$ reduces to $\mathrm{spt}(n)$ when $j=1$.

Dixit and Yee [62] derived the generating function of $\mathrm{Spt}_j(n)$.

Theorem 4.7 (Dixit and Yee) *For $j \geq 1$,*

$$\sum_{n=1}^{\infty} \mathrm{Spt}_j(n) q^n$$
$$= \sum_{n_j \geq 1} \sum_{n_{j-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_j}}{(1-q^{n_j})(q^{n_j};q)_\infty} \begin{bmatrix} n_j \\ n_{j-1} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix} q^{n_1^2+\cdots+n_{j-1}^2}, \tag{4.13}$$

where the q-binomial coefficients or the Gaussian coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{4.14}$$

Dixit and Yee also found a combinatorial interpretation of $\mathrm{Spt}_j(n)$. To give a combinatorial explanation of the right-hand side of (4.13), they introduced the k-th lower-Durfee square of a partition λ. For a partition

λ, take the largest square that fits inside the Ferrers diagram of λ starting from the lower left corner. This square is called the lower-Durfee square. If there are remaining parts above the lower-Durfee square, then take the second lower-Durfee square in the diagram above the lower-Durfee square. Repeating this process, we are led to the third lower-Durfee square, if it exists, and so on.

The combinatorial explanation of the right-hand side of (4.13) also requires a labeling of a partition, as given by Dixit and Yee. For a partition λ, let f_i denote the multiplicity of i in λ. For the f_i occurrences of i, we label these f_i parts from left to right by $1, 2, \ldots, f_i$. The labels are represented by subscripts. For instance, $(9, 8, 8, 8, 8, 6, 6, 5, 4, 4, 3)$ can be labeled as $(9_1, 8_1, 8_2, 8_3, 8_4, 6_1, 6_2, 5_1, 4_1, 4_2, 3_1)$.

Using the lower-Durfee squares and the above labeling of a partition, for a partition λ and $j \geq 1$, Dixit and Yee defined the weight of λ , denoted $W_j(\lambda)$. There are two cases:

Case 1: λ does not contain the $(j-1)$-th lower-Durfee square. Then $W_j(\lambda)$ is defined to be the sum of the labels of λ.

Case 2: λ contains the $(j-1)$-th lower-Durfee square. Then $W_j(\lambda)$ is defined to be the sum of labels of all the parts that are contained in and below the $(j-1)$-th lower-Durfee square and the label of the part just right above the $(j-1)$-th lower-Durfee square.

For example, for $\lambda = (9, 8, 8, 8, 8, 6, 6, 5, 4, 4, 3)$ and $j = 3$, we have $W_3(\lambda) = 2 + 3 + 4 + 1 + 2 + 1 + 1 + 2 + 1 = 17$, see Figure 5.

We are now ready to state the combinatorial interpretation of $\mathrm{Spt}_j(n)$.

Theorem 4.8 (Dixit and Yee) *For $j \geq 1$,*

$$\mathrm{Spt}_j(n) = \sum_{\lambda \in P(n)} W_j(\lambda).$$

Analogous to the k-th symmetrized rank moments $\eta_k(n)$ and the k-th symmetrized crank moments $\mu_k(n)$, Dixit and Yee [62] defined the k-th symmetrized j-rank function ${}_j\mu_k(n)$ by

$${}_j\mu_k(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N_j(m, n).$$

It can be checked that ${}_1\mu_k(n) = \mu_k(n)$ and ${}_2\mu_k(n) = \eta_k(n)$. By the definition (4.3) of the higher order spt-function $\mathrm{spt}_k(n)$, we see that

$$\mathrm{spt}_k(n) = {}_1\mu_{2k}(n) - {}_2\mu_{2k}(n). \tag{4.15}$$

The generalized higher order spt-function ${}_j\,\mathrm{spt}_k$ is defined as follows.

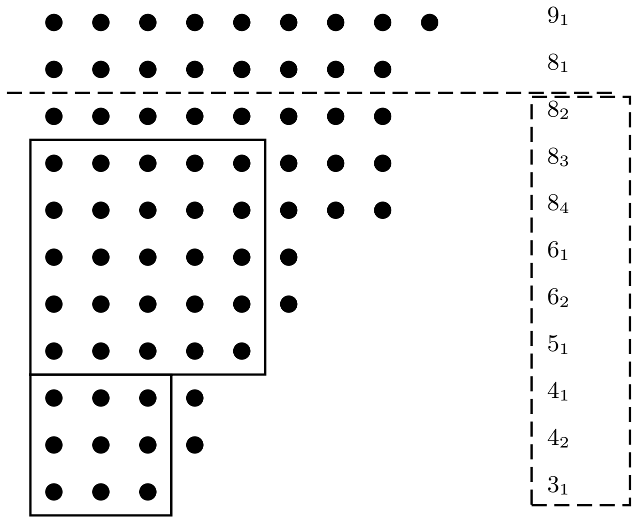

Figure 5: An illustration of weight $W_3(\lambda)$

Definition 4.9 *For $j, k \geq 1$,*

$$_j\,\mathrm{spt}_k(n) = {}_j\mu_{2k}(n) - {}_{j+1}\mu_{2k}(n).$$

Dixit and Yee [62] derived the generating function of $_j\,\mathrm{spt}_k(n)$:

Theorem 4.10 (Dixit and Yee) *For $j, k \geq 1$,*

$$\sum_{n=1}^{\infty} {}_j\,\mathrm{spt}_k(n) q^n = \sum_{\substack{n_k \geq \cdots \geq n_1 \geq \\ m_1 \geq \cdots \geq m_{j-1} \geq 1}} \left(\frac{q^{n_k + \cdots + n_1}(q;q)_{n_1}}{(1-q^{n_k})^2 \cdots (1-q^{n_1})^2 (q^{n_1+1};q)_\infty} \right.$$
$$\left. \times \frac{q^{m_1^2 + \cdots + m_{j-1}^2}}{(q;q)_{n_1 - m_1}(q;q)_{m_1 - m_2} \cdots (q;q)_{m_{j-1}}} \right). \quad (4.16)$$

They also gave a combinatorial explanation of the right-hand side of (4.16). Let λ be a partition, and let f_t denote the number of occurrences of t in λ. We shall use the same labeling of λ as given before. For a positive integer k and a part t in λ with label a, define

$$g_k(\lambda, t_a) = \binom{a+k-1}{2k-1}$$

$$+\sum_{r=2}^{k} \sum_{\substack{m_1,m_2,\ldots,m_r\geq 1\\ m_1+\cdots+m_r=k\\ t<t_2<\cdots<t_r\leq\lambda_1}} \binom{a+m_1-1}{2m_1-1}\binom{f_{t_2}+m_2}{2m_2}\cdots\binom{f_{t_r}+m_r}{2m_r}.$$

Definition 4.11 *For $j,k\geq 1$, define*

$$_j\omega_k(\lambda)=\sum_{t_a} g_k(\lambda,t_a), \tag{4.17}$$

where the sum ranges over the parts that are contained in the $(j-1)$-th lower-Durfee square except for the last part, but also contains the part immediately above the $(j-1)$-th lower-Durfee square.

For example, let $\lambda=(5,5,5,3,3,2,2,2)$, $j=3$ and $k=2$. Label λ as $(5_1,5_2,5_3,3_1,3_2,2_1,2_2,2_3)$. Then

$$g_2(\lambda,3_1)=0+1\cdot\binom{3+1}{2}=6$$

and

$$g_2(\lambda,3_2)=1+2\cdot\binom{3+1}{2}=13.$$

Moreover, from (4.17) we find that

$$_3\omega_2(\lambda)=g_2(\lambda,3_1)+g_2(\lambda,3_2)=6+13=19.$$

Figure 6 gives an illustration of this example.

Dixit and Yee [62] proved that $_j\,\mathrm{spt}_k(n)$ can be expressed in terms of $_j\omega_k(\lambda)$.

Theorem 4.12 (Dixit and Yee) *We have*

$$_j\,\mathrm{spt}_k(n)=\sum_{\lambda\in P(n)} {_j\omega_k(\lambda)}.$$

4.3 The ospt-function of Andrews, Chan and Kim

A variation of the spt-function based on relation (1.6) was given by Andrews, Chan and Kim [15]. In view of the symmetry properties $N(-m,n)=N(m,n)$ and $M(-m,n)=M(m,n)$, it is known that

$$N_{2k+1}(n)=M_{2k+1}(n)=0.$$

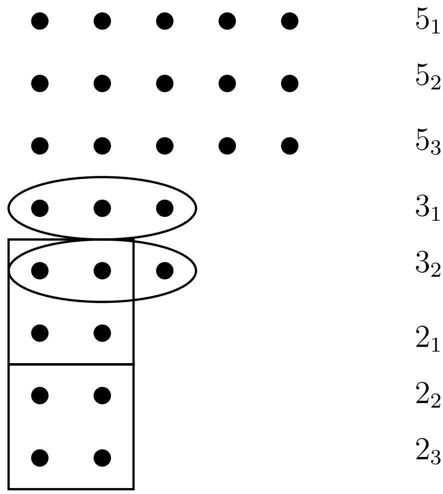

Figure 6: An illustration of ${}_j\omega_k(\pi)$

To avoid the trivial odd moments, Andrews, Chan and Kim [15] introduced the modified rank and crank moments $N_j^+(n)$ and $M_j^+(n)$ by considering the unilateral sums:

$$N_j^+(n) = \sum_{m\geq 0} m^j N(m,n)$$

and

$$M_j^+(n) = \sum_{m\geq 0} m^j M(m,n).$$

They proved the following inequality.

Theorem 4.13 (Andrews, Chan and Kim) *For $n, k \geq 1$,*

$$M_k^+(n) > N_k^+(n). \tag{4.18}$$

Bringmann and Mahlburg [45] proved that the above inequality (4.18) holds for any fixed positive integer k and sufficiently large n by deriving an asymptotic formula for $M_k^+(n) - N_k^+(n)$ stated in Theorem 5.2. When k is even, this inequality (4.18) is equivalent to the inequality (4.6) of Garvan between the rank moments and the crank moments. Chen, Ji and Zang [52] showed that the Andrews-Dyson-Rhoades conjecture (2.13) implies the inequality (4.18).

Andrews, Chan and Kim [15] defined the ospt-function $\mathrm{ospt}(n)$ as given below:

Definition 4.14 *For $n \geq 1$,*

$$\mathrm{ospt}(n) = M_1^+(n) - N_1^+(n). \tag{4.19}$$

They obtained the generating function of $\mathrm{ospt}(n)$.

Theorem 4.15 (Andrews, Chan and Kim) *We have*

$$\sum_{n=0}^{\infty} \mathrm{ospt}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2}(1-q^{4i+2})(1-q^{4i+2j+3}) \right.$$
$$\left. + \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+1}(1-q^{2i+1})(1-q^{4i+2j+2}) \right). \tag{4.20}$$

Andrews, Chan and Kim found a combinatorial interpretation of the right-hand side of (4.20), which leads to a combinatorial interpretation of $\mathrm{ospt}(n)$. In doing so, they defined even strings and odd strings of a partition.

Definition 4.16 *Let λ be a partition. A maximal consecutive sequence $(r, r-1, \ldots, s)$ in λ is called an even string of λ if it satisfies the following restrictions:*

(1) $r \geq 2s - 2$;

(2) *r and s are even.*

Similarly, a consecutive sequence $(r, r-1, \ldots, s)$ in λ, not necessarily maximal, is called an odd string of λ if it satisfies the following restrictions:

(1) *$r + 1$ is not a part of λ;*

(2) *s is odd and it appears only once in λ;*

(3) $r \geq 2s - 1$.

For example, the partition $\lambda = (5, 4, 4, 3, 2, 2)$ contains only one odd string $(5, 4, 3)$, and it does not contain any even string. For $\lambda = (6, 4, 4, 3, 2)$, it contains an even string $(4, 3, 2)$, but it does not contain any odd string.

Andrews, Chan and Kim [15] found a combinatorial interpretation of $\mathrm{ospt}(n)$.

Theorem 4.17 (Andrews, Chan and Kim) *For a partition λ, let $ST(\lambda)$ denote the total number of even strings and odd strings in λ. For $n \geq 1$,*

$$\mathrm{ospt}(n) = \sum_{\lambda \in P(n)} ST(\lambda).$$

In light of Theorem 4.17, Bringmann and Mahlburg [45] proved a monotone property of $\mathrm{ospt}(n)$ by a combinatorial argument.

Theorem 4.18 (Bringmann and Mahlburg) *For $n \geq 1$,*

$$\mathrm{ospt}(n+1) \geq \mathrm{ospt}(n).$$

They also noticed that $\mathrm{ospt}(n)$ and $\mathrm{spt}(n)$ have the same parity. This fact can be justified as follows: Since

$$M_1^+(n) = \sum_{m \geq 0} mM(m,n) \equiv \sum_{m \geq 0} m^2 M(m,n) = M_2^+(n) \pmod 2$$

and

$$N_1^+(n) = \sum_{m \geq 0} mN(m,n) \equiv \sum_{m \geq 0} m^2 N(m,n) = N_2^+(n) \pmod 2,$$

we see that

$$\mathrm{ospt}(n) = M_1^+(n) - N_1^+(n) \equiv M_2^+(n) - N_2^+(n) = \mathrm{spt}(n) \pmod 2.$$

With the aid of the characterization of the parity of $\mathrm{spt}(n)$, Bringmann and Mahlburg [45] determined the parity of $\mathrm{ospt}(n)$.

Theorem 4.19 (Bringmann and Mahlburg) *The ospt-function $\mathrm{ospt}(n)$ is odd if and only if $24n - 1 = p^{4a+1}m^2$ for some prime $p \equiv 23 \pmod{24}$ and some integers a, m, where $(p, m) = 1$.*

Chan and Mao [49] established an upper bound and a lower bound for $\mathrm{ospt}(n)$, leading to an asymptotic estimate of $\mathrm{ospt}(n)$.

Theorem 4.20 (Chan and Mao) *We have*

$$\mathrm{ospt}(n) > \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} \quad \text{for } n \geq 8, \tag{4.21}$$

$$\mathrm{ospt}(n) < \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} + \frac{N(1,n)}{2} \quad \text{for } n \geq 7, \tag{4.22}$$

$$\mathrm{ospt}(n) < \frac{p(n)}{2} \quad \textit{for } n \geq 3. \tag{4.23}$$

An asymptotic estimate of $\mathrm{ospt}(n)$ can be deduced from the bounds (4.21) and (4.22), along with an asymptotic property of $M(m,n)$ and $N(m,n)$ due to Mao [106].

Theorem 4.21 (Mao) *For any integer m, as $n \to \infty$*

$$M(m,n) \sim N(m,n) \sim \frac{\pi}{4\sqrt{6n}} p(n). \tag{4.24}$$

By (4.24), we see that as $n \to \infty$,

$$\frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} \sim \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} + \frac{N(1,n)}{2} \sim \frac{1}{4} p(n).$$

Combining (4.21) and (4.22), we arrive at the asymptotic estimate (5.2) due to Bringmann and Mahlburg [45] as given in Section 5.

4.4 The first variation of Ahlgren, Bringmann and Lovejoy

We now turn to three variations of the spt-function based on the combinatorial definition. The first variation of the spt-function was given by Ahlgren, Bringmann and Lovejoy [5]. They defined the M2spt-function as follows.

Definition 4.22 *The function* $\mathrm{M2spt}(n)$ *is defined to be the total number of smallest parts in all partitions of n without repeated odd parts and the smallest part is even.*

For example, there are two partitions of 7 without repeated odd parts and the smallest part is even, namely,

$$(5, \mathbf{2}), (3, \mathbf{2}, \mathbf{2}).$$

So we have $\mathrm{M2spt}(7) = 3$.

By [43, Section 7], Ahlgren, Bringmann and Lovejoy [5] derived the generating function of $\mathrm{M2spt}(n)$.

Theorem 4.23 (Ahlgren, Bringmann and Lovejoy) *We have*

$$\sum_{n=1}^{\infty} \mathrm{M2spt}(n) q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \times \left(\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{2n^2+n}}{(1-q^{2n})^2} \right). \tag{4.25}$$

Jennings-Shaffer [88] showed that the function M2spt(n) can be expressed as the difference between the symmetrized M_2-rank moments and the symmetrized residue crank moments of partitions without repeated odd parts. Let us first recall the definitions of the M_2-rank of a partition without repeated odd parts and the residue crank of a partition without repeated odd parts.

Let λ be a partition without repeated odd parts, the M_2-rank of λ was defined by Berkovich and Garvan [33] as stated below:

$$M_2\text{-rank}(\lambda) = \left\lceil \frac{\lambda_1}{2} \right\rceil - l(\lambda). \tag{4.26}$$

The residue crank of λ was defined by Garvan and Jennings-Shaffer [76] which is related to the crank of an ordinary partition. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition without repeated odd parts, define λ^e to be the ordinary partition obtained from λ by omitting odd parts of λ and dividing each even part by 2. The residue crank of λ is defined to be the crank of λ^e.

For example, let $\lambda = (11, 7, 6, 5, 4, 4, 3, 2, 2)$, then $\lambda_1 = 11$, $l(\lambda) = 9$ and $\lambda^e = (3, 2, 2, 1, 1)$. Hence the M_2-rank of λ is equal to -3 and the residue crank of λ is equal to the crank of λ^e, which equals -1.

Let $N2(m, n)$ denote the number of partitions of n without repeated odd parts such that M_2-rank is equal to m. Let $M2(m, n)$ denote the number of partitions of n without repeated odd parts such that the residue crank is equal to m. The k-th symmetrized M_2-rank moments $\eta 2_k(n)$ and the k-th symmetrized residue crank moments $\mu 2_k(n)$ of partitions without repeated odd parts were defined by Jennings-Shaffer [88] as follows:

$$\eta 2_k(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N2(m, n),$$

$$\mu 2_k(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M2(m, n).$$

Analogue to the relation (1.6) for spt(n), Jennings-Shaffer [88] established the following connection.

Theorem 4.24 (Jennings-Shaffer) *For $n \geq 1$,*

$$\text{M2spt}(n) = \mu 2_2(n) - \eta 2_2(n). \tag{4.27}$$

The following congruences of M2spt(n) mod 3 and 5 were given by Garvan and Jennings-Shaffer [76].

Theorem 4.25 (Garvan and Jennings-Shaffer) *For* $n \geq 0$,

$$\begin{aligned} \mathrm{M2spt}(3n+1) &\equiv 0 \pmod 3, \\ \mathrm{M2spt}(5n+1) &\equiv 0 \pmod 5, \\ \mathrm{M2spt}(5n+3) &\equiv 0 \pmod 5. \end{aligned}$$

Jennings-Shaffer [89] provided alternative proofs of the above congruences. Furthermore, he showed that

Theorem 4.26 (Jennings-Shaffer) *For* $n \geq 0$,

$$\begin{aligned} \mathrm{M2spt}(27n+26) &\equiv 0 \pmod 5, \\ \mathrm{M2spt}(125n+97) &\equiv 0 \pmod 5, \\ \mathrm{M2spt}(125n+122) &\equiv 0 \pmod 5. \end{aligned}$$

Ahlgren, Bringmann and Lovejoy [5] established Ramanujan-type congruences of $\mathrm{M2spt}(n)$ modulo powers of ℓ for any prime $\ell \geq 3$.

Theorem 4.27 (Ahlgren, Bringmann and Lovejoy) *Let* $\ell \geq 3$ *be a prime, and let* $m, n \geq 1$.

(i) *If* $\left(\frac{-n}{\ell}\right) = 1$, *then*

$$\mathrm{M2spt}\left((\ell^{2m}n+1)/8\right) \equiv 0 \pmod{\ell^m}.$$

(ii)

$$\mathrm{M2spt}\left((\ell^{2m+1}n+1)/8\right) \equiv \left(\frac{2}{\ell}\right) \mathrm{M2spt}\left((\ell^{2m-1}n+1)/8\right) \pmod{\ell^m}.$$

Hecke-type congruences of $\mathrm{M2spt}(n)$ mod $2, 2^2, 2^3, 3$ and 5 have been found by Andersen [7].

Theorem 4.28 (Andersen) *Let* $\ell \geq 3$ *be a prime. Define* $s_\ell = (\ell^2-1)/8$ *and*

$$\beta = \begin{cases} 1, & \text{if } \ell \equiv 3 \pmod 8, \\ 2, & \text{if } \ell \equiv 5 \pmod 8, \\ 3, & \text{if } \ell \equiv 1, 7 \pmod 8. \end{cases}$$

For $t \in \{2^\beta, 3, 5\}, \ell \neq t$ *and* $n \geq 1$,

$$\mathrm{M2spt}(\ell^2 n - s_\ell) + \left(\frac{2}{\ell}\right)\left(\frac{1-8n}{\ell}\right) \mathrm{M2spt}(n) + \ell\, \mathrm{M2spt}\left((n+s_\ell)/\ell^2\right)$$

$$\equiv \left(\frac{2}{\ell}\right)(1+\ell)\,\mathrm{M2spt}(n) \pmod{t}.$$

In analogy with the higher order spt-function $\mathrm{spt}_k(n)$, Jennings-Shaffer [89] defined the higher order function $\mathrm{M2spt}_k(n)$ in terms of the k-th symmetrized M_2-rank moments $\eta 2_k(n)$ and the k-th symmetrized residue crank moments $\mu 2_k(n)$ for partitions without repeated odd parts.

Definition 4.29 *For $k \geq 1$, define*

$$\mathrm{M2spt}_k(n) = \mu 2_{2k}(n) - \eta 2_{2k}(n).$$

Using (4.27), it is clear to see that $\mathrm{M2spt}_k(n)$ reduces to $\mathrm{M2spt}(n)$ when $k=1$. Jennings-Shaffer [88] also obtained the generating function of $\mathrm{M2spt}_k(n)$.

Theorem 4.30 (Jennings-Shaffer) *We have*

$$\sum_{n=1}^{\infty} \mathrm{M2spt}_k(n) q^n = \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-q^{2n_1+1};q^2)_\infty q^{2n_1+2n_2+\cdots+2n_k}}{(1-q^{2n_k})^2(1-q^{2n_{k-1}})^2\cdots(1-q^{2n_1})^2(q^{2n_1+2};q^2)_\infty}. \tag{4.28}$$

By interpreting the right-hand side of (4.28) combinatorially, Jennings-Shaffer [88] found a combinatorial interpretation of $\mathrm{M2spt}_k(n)$. Let $P_o(n)$ denote the set of partitions of n without repeated odd parts and the smallest part is even. For a partition $\lambda \in P_o(n)$, assume that there are r different even parts in λ, namely,

$$2t_1 < 2t_2 < \cdots < 2t_r.$$

Let $f_j = f_j(\lambda)$ denote the frequency of the part $2t_j$ in λ. For a fixed integer $k \geq 1$, Jennings-Shaffer [88] defined $\omega_k(\lambda)$ as follows:

$$\omega_k(\lambda) = \sum_{\substack{m_1+m_2+\cdots+m_s=k \\ 1\leq s\leq k}} \binom{f_1+m_1-1}{2m_1-1} \times \sum_{2\leq j_2<j_3<\cdots<j_s} \prod_{i=2}^{s} \binom{f_{j_i}+m_i}{2m_i}. \tag{4.29}$$

For example, let $k=2$ and $\lambda = (10,10,9,5,4,3,2,2,2)$ be a partition in $P_o(47)$, there are three distinct even parts in λ. Thus $r=3$, $f_1=3$, $f_2=1$ and $f_3=2$. By the definition (4.29) of $\omega_k(\lambda)$, we have $\omega_2(\lambda)=16$.

With the above notation, Jennings-Shaffer [88] found a combinatorial interpretation of $\mathrm{M2spt}_k(n)$.

Theorem 4.31 (Jennings-Shaffer) *For $n \geq 1$,*

$$\mathrm{M2spt}_k(n) = \sum_{\lambda \in P_o(n)} \omega_k(\lambda). \tag{4.30}$$

Jennings-Shaffer [89] also obtained the following congruences of $\mathrm{M2spt}_2(n)$.

Theorem 4.32 (Jennings-Shaffer) *For $n \geq 1$,*

$$\begin{aligned}
\mathrm{M2spt}_2(n) &\equiv 0 \pmod 3, \ \textit{if } n \equiv 0 \pmod 9,\\
\mathrm{M2spt}_2(n) &\equiv 0 \pmod 5, \ \textit{if } n \equiv 0 \pmod 5,\\
\mathrm{M2spt}_2(n) &\equiv 0 \pmod 5, \ \textit{if } n \equiv 1 \pmod 5,\\
\mathrm{M2spt}_2(n) &\equiv 0 \pmod 5, \ \textit{if } n \equiv 3 \pmod 5.
\end{aligned}$$

4.5 The second variation of Bringmann, Lovejoy and Osburn

The second variation of the spt-function was due to Bringmann, Lovejoy and Osburn [42], which is defined on overpartitions. Recall that Corteel and Lovejoy [56] defined an overpartition of n as a partition of n in which the first occurrence of a part may be overlined. Bringmann, Lovejoy and Osburn [42] introduced three spt-type functions.

Definition 4.33 (Bringmann, Lovejoy and Osburn)

(1) *The function $\overline{\mathrm{spt}}(n)$ is defined to be the total number of smallest parts in all overpartitions of n.*

(2) *The function $\overline{\mathrm{spt}\,1}(n)$ is defined to be the total number of smallest parts in all overpartitions of n with the smallest part being odd.*

(3) *The function $\overline{\mathrm{spt}\,2}(n)$ is defined to be the total number of smallest parts in all overpartitions of n with the smallest part being even.*

For example, there are 14 overpartitions of 4:

$$\begin{array}{lllllll}
(4) & (\bar{4}) & (3,1) & (\bar{3},1) & (3,\bar{1}) & (\bar{3},\bar{1}) & (2,2),\\
(\bar{2},2) & (2,1,1) & (\bar{2},1,1) & (2,\bar{1},1) & (\bar{2},\bar{1},1) & (1,1,1,1) & (\bar{1},1,1,1).
\end{array}$$

We have $\overline{\mathrm{spt}}(4) = 26$, $\overline{\mathrm{spt}\,1}(4) = 20$ and $\overline{\mathrm{spt}\,2}(4) = 6$.

Analogous to the relation (1.6) for the spt-function, the functions $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}\,2}(n)$ can also be expressed as the differences of the rank and the crank moments of overpartitions. The definitions of the rank and the crank moments of overpartitions are based on the two definitions of the rank of an overpartition and the two definitions of the crank of an overpartition. Although there are four possibilities, only two of them have been studied.

For an overpartition λ, there are two kinds of ranks. One is called the D-rank introduced by Lovejoy [104] and the other is called the M_2-rank introduced by Lovejoy [105]. The D-rank of λ is defined as the largest part minus the number of parts. To define the M_2-rank, let λ_o denote the partition consisting of non-overlined odd parts of λ. Then M_2-rank(λ) can be defined as follows:

$$M_2\text{-rank}(\lambda) = \left\lceil \frac{\lambda_1}{2} \right\rceil - l(\lambda) + l(\lambda_o) - \chi(\lambda),$$

where $\chi(\lambda) = 1$ if the largest part of λ is odd and non-overlined and $\chi(\lambda) = 0$ otherwise.

For example, for an overpartition $\lambda = (\bar{9}, 9, 7, \bar{6}, 5, 5, \bar{4}, 3, 2, \bar{1})$, we see that D-rank$(\lambda) = 9 - 10 = -1$. Moreover, since $\lambda_o = (9, 7, 5, 5, 3)$ and $\chi(\lambda) = 0$, we have M_2-rank$(\lambda) = 0$.

Bringmann, Lovejoy and Osburn [42] defined the first and second residue crank of an overpartition. The first residue crank of an overpartition is defined as the crank of the partition consisting of non-overlined parts. The second residue crank is defined as the crank of the subpartition consisting of all the even non-overlined parts divided by two.

For example, for $\lambda = (\bar{9}, 9, 7, \bar{6}, 5, 5, \bar{4}, 4, 3, 2, \bar{1})$, the partition consisting of non-overlined parts of λ is $(9, 7, 5, 5, 4, 3, 2)$. The first residue crank of λ is 9. The partition formed by even non-overlined parts of λ is $(4, 2)$. So the second residue crank of λ is equal to the crank of $(2, 1)$, which is equal to 0.

We are now in a position to present the definitions of the rank and the crank moments of overpartitions. Let $\overline{N}(m, n)$ denote the number of overpartitions of n with the D-rank m, and let $\overline{N2}(m, n)$ denote the number of overpartitions of n with the M_2-rank m. Notice that there are two kinds of ranks of overpartitions. Consequently, there are two possibilities to define the rank moments of overpartitions. The two rank moments are defined as follows:

$$\overline{N}_k(n) = \sum_{m=-\infty}^{\infty} m^k \overline{N}(m, n), \tag{4.31}$$

$$\overline{N2}_k(n) = \sum_{m=-\infty}^{\infty} m^k \overline{N2}(m, n). \tag{4.32}$$

Similarly, let $\overline{M}(m,n)$ denote the number of overpartitions of n with the first residue crank m and let $\overline{M2}(m,n)$ denote the number of overpartitions of n with the second residue crank m. The two crank moments are defined by

$$\overline{M}_k(n) = \sum_{m=-\infty}^{\infty} m^k \overline{M}(m,n), \tag{4.33}$$

$$\overline{M2}_k(n) = \sum_{m=-\infty}^{\infty} m^k \overline{M2}(m,n). \tag{4.34}$$

Bringmann, Lovejoy and Osburn [42] deduced the following relations on $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}\,2}(n)$.

Theorem 4.34 (Bringmann, Lovejoy and Osburn) *For $n \geq 1$,*

$$\overline{\mathrm{spt}}(n) = \overline{M}_2(n) - \overline{N}_2(n), \tag{4.35}$$

$$\overline{\mathrm{spt}\,2}(n) = \overline{M2}_2(n) - \overline{N2}_2(n). \tag{4.36}$$

In light of Theorem 4.34, Bringmann, Lovejoy and Osburn [42] proved the following congruences:

Theorem 4.35 (Bringmann, Lovejoy and Osburn) *For $n \geq 1$,*

$$\overline{\mathrm{spt}\,2}(n) \equiv \overline{\mathrm{spt}\,2}(n) \equiv 0 \pmod 3, \text{ if } n \equiv 0,1 \pmod 3, \tag{4.37}$$

$$\overline{\mathrm{spt}}(n) \equiv 0 \pmod 3, \text{ if } n \equiv 0 \pmod 3, \tag{4.38}$$

$$\overline{\mathrm{spt}\,2}(n) \equiv 0 \pmod 5, \text{ if } n \equiv 3 \pmod 5, \tag{4.39}$$

$$\overline{\mathrm{spt}\,1}(n) \equiv 0 \pmod 5, \text{ if } n \equiv 0 \pmod 5. \tag{4.40}$$

Moreover, if $\ell \geq 5$ is a prime, then the following congruence holds:

$$\overline{\mathrm{spt}\,1}(\ell^2 n) + \left(\frac{-n}{\ell}\right)\overline{\mathrm{spt}\,1}(n) + \ell\overline{\mathrm{spt}\,1}\left(n/\ell^2\right) \equiv (\ell+1)\overline{\mathrm{spt}\,1}(n) \pmod 3.$$

An alternative proof of the congruence (4.37) was given by Jennings-Shaffer [87]. The combinatorial interpretations of the congruences (4.37)–(4.40) were given by Garvan and Jennings-Shaffer [76]. Ahlgren, Bringmann and Lovejoy [5] derived Ramanujan-type congruences of $\overline{\mathrm{spt}\,1}(n)$ modulo powers of a prime ℓ, which are similar to the Ramanujan-type congruences of $\mathrm{spt}(n)$ modulo powers of a prime ℓ.

Theorem 4.36 (Ahlgren, Bringmann and Lovejoy) *Let $\ell \geq 3$ be a prime, and let $m, n \geq 1$.*

(1) *If* $(\frac{-n}{\ell}) = 1$, *then*

$$\overline{\mathrm{spt}\,1}(\ell^{2m} n) \equiv 0 \pmod{\ell^m}.$$

(2)
$$\overline{\mathrm{spt}\,1}(\ell^{2m+1} n) \equiv \overline{\mathrm{spt}\,1}(\ell^{2m-1} n) \pmod{\ell^m}.$$

Andersen [7] obtained Hecke-type congruences of $\overline{\mathrm{spt}\,1}(n)$ mod 2^6, 2^7, 2^8, 3 and 5.

Theorem 4.37 (Andersen) *Let $\ell \geq 3$ be a prime, and define*

$$\beta = \begin{cases} 6, & \text{if } \ell \equiv 3 \pmod{8}, \\ 7, & \text{if } \ell \equiv 5, 7 \pmod{8}, \\ 8, & \text{if } \ell \equiv 1 \pmod{8}. \end{cases}$$

For $t \in \{2^\beta, 3, 5\}$, $\ell \neq t$ and $n \geq 1$,

$$\overline{\mathrm{spt}\,1}(\ell^2 n) + \left(\frac{-n}{\ell}\right) \overline{\mathrm{spt}\,1}(n) + \ell \overline{\mathrm{spt}\,1}(n/\ell^2) \equiv (1+\ell) \overline{\mathrm{spt}\,1}(n) \pmod{t}.$$

It is readily seen that $\overline{\mathrm{spt}\,1}(n)$, $\overline{\mathrm{spt}\,2}(n)$ and $\overline{\mathrm{spt}}(n)$ are all even. Congruences of these functions modulo 4 were investigated by Garvan and Jennings-Shaffer [76].

Theorem 4.38 (Garvan and Jennings-Shaffer) *For $n \geq 1$,*

(1) $\overline{\mathrm{spt}}(n) \equiv 2 \pmod{4}$ *if and only if n is a square or twice a square*;

(2) $\overline{\mathrm{spt}\,1}(n) \equiv 2 \pmod{4}$ *if and only if n is an odd square*;

(3) $\overline{\mathrm{spt}\,2}(n) \equiv 2 \pmod{4}$ *if and only if n is an even square or twice a square.*

Moreover, they introduced a statistic $\overline{\mathrm{sptcrank}}$ defined on a marked overpartition, which leads to combinatorial interpretations of the above congruences.

The following recurrence relation of $\overline{\mathrm{spt}\,1}(n)$ was given by Ahlgren and Andersen [2].

Theorem 4.39 (Ahlgren and Andersen) *Let*

$$s(n) = \sum_{d|n} \min\left(d, \frac{n}{d}\right).$$

For $n > 0$,

$$\sum_{k} (-1)^k \overline{\mathrm{spt}\,1}(n - k^2) = b(n),$$

where

$$b(n) = \begin{cases} 2s(n), & \text{if } n \text{ is odd}, \\ -4s(n/4), & \text{if } n \equiv 0 \pmod 4, \\ 0, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In view of the symmetry properties $\overline{N}(-m,n) = \overline{N}(m,n)$ and $\overline{M}(-m,n) = \overline{M}(m,n)$, we see that

$$\overline{N}_{2k+1}(n) = \overline{M}_{2k+1}(n) = 0.$$

Similarly, to avoid the trivial odd moments, Andrews, Chan, Kim and Osburn [16] introduced the modified rank and crank moments $\overline{N}_k^+(n)$ and $\overline{M}_k^+(n)$ for overpartitions:

$$\overline{N}_k^+(n) = \sum_{m \geq 1} m^k \overline{N}(m,n) \tag{4.41}$$

and

$$\overline{M}_k^+(n) = \sum_{m \geq 1} m^k \overline{M}(m,n). \tag{4.42}$$

They defined the ospt-function $\overline{\mathrm{ospt}}(n)$ for overpartitions which is in the spirit of the ospt-function $\mathrm{ospt}(n)$ for ordinary partitions.

Definition 4.40 *For* $n \geq 1$,

$$\overline{\mathrm{ospt}}(n) = \overline{M}_1^+(n) - \overline{N}_1^+(n). \tag{4.43}$$

Andrews, Chan, Kim and Osburn [16] defined even strings and odd strings of an overpartition, and provided a combinatorial interpretation of $\overline{\mathrm{ospt}}(n)$.

Jennings-Shaffer [88] defined the higher order spt-functions for overpartitions by using the k-th symmetrized rank and crank moments for overpartitions. There are two symmetrized rank moments for overpartitions:

$$\overline{\eta}_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{N}(m,n) \tag{4.44}$$

and

$$\overline{\eta 2}_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{N2}(m,n). \tag{4.45}$$

There are also two symmetrized crank moments for overpartitions:

$$\overline{\mu}_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{M}(m,n) \tag{4.46}$$

and

$$\overline{\mu 2}_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{M2}(m,n). \tag{4.47}$$

The two higher order spt-functions for overpartitions are defined as follows.

Definition 4.41 *For $k \geq 1$,*

$$\overline{\mathrm{spt}}_k(n) = \overline{\mu}_{2k}(n) - \overline{\eta}_{2k}(n), \tag{4.48}$$

$$\overline{\mathrm{spt}\,2}_k(n) = \overline{\mu 2}_{2k}(n) - \overline{\eta 2}_{2k}(n). \tag{4.49}$$

Using Bailey pairs, Jennings-Shaffer [88] obtained the generating functions of $\overline{\mathrm{spt}}_k(n)$ and $\overline{\mathrm{spt}\,2}_k(n)$.

Theorem 4.42 (Jennings-Shaffer) *For $k \geq 1$,*

$$\sum_{n=1}^{\infty} \overline{\mathrm{spt}}_k(n) q^n = \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}(-q^{n_1+1};q)_\infty}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2(q^{n_1+1};q)_\infty}, \tag{4.50}$$

$$\sum_{n=1}^{\infty} \overline{\mathrm{spt}\,2}_k(n) q^n = \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\cdots+2n_k}(-q^{2n_1+1};q)_\infty}{(1-q^{2n_k})^2(1-q^{2n_{k-1}})^2\cdots(1-q^{2n_1})^2(q^{2n_1+1};q)_\infty}. \tag{4.51}$$

By interpreting the right-hand sides of (4.50) and (4.51) based on vector partitions, Jennings-Shaffer found combinatorial explanations of $\overline{\mathrm{spt}}_k(n)$ and $\overline{\mathrm{spt}\,2}_k(n)$.

4.6 The third variation of Andrews, Dixit and Yee

The third variation of the spt-function was introduced by Andrews, Dixit and Yee [18]. Let $p_\omega(n)$ denote the number of partitions of n in which each odd part is less than twice the smallest part. They defined $\mathrm{spt}_\omega(n)$ as follows.

Definition 4.43 *The function* $\mathrm{spt}_\omega(n)$ *is defined to be the number of smallest parts in the partitions enumerated by* $p_\omega(n)$.

For example, for $n = 4$, there are four partitions counted by $p_\omega(4)$, namely,

$$(4) \quad (2,2) \quad (2,1,1) \quad (1,1,1,1).$$

We have $p_\omega(4) = 4$ and $\mathrm{spt}_\omega(4) = 9$.

They derived the generating function of $\mathrm{spt}_\omega(n)$.

Theorem 4.44 (Andrews, Dixit and Yee) *We have*

$$\sum_{n=1}^{\infty} \mathrm{spt}_\omega(n) q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^{2n}) q^{n(3n+1)}}{(1-q^{2n})^2}. \quad (4.52)$$

Using the above generating function, Andrews, Dixit and Yee [18] proved the following congruences of $\mathrm{spt}_\omega(n)$.

Theorem 4.45 (Andrews, Dixit and Yee) *For* $n \geq 0$,

$$\mathrm{spt}_\omega(5n+3) \equiv 0 \pmod 5, \quad (4.53)$$
$$\mathrm{spt}_\omega(10n+7) \equiv 0 \pmod 5, \quad (4.54)$$
$$\mathrm{spt}_\omega(10n+9) \equiv 0 \pmod 5. \quad (4.55)$$

Employing the generating function (4.52), Wang [131] derived the generating function of $\mathrm{spt}_\omega(2n+1)$.

Theorem 4.46 (Wang) *We have*

$$\sum_{n=0}^{\infty} \mathrm{spt}_{\omega}(2n+1)q^n = \frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^5}. \tag{4.56}$$

Wang [131] also posed two conjectures on congruences of $\mathrm{spt}_{\omega}(n)$ modulo arbitrary powers of 5.

Conjecture 4.47 (Wang) *For $k \geq 1$ and $n \geq 0$,*

$$\mathrm{spt}_{\omega}\left(2 \cdot 5^{2k-1}n + \frac{7 \cdot 5^{2k-1}+1}{12}\right) \equiv 0 \pmod{5^{2k-1}}.$$

Conjecture 4.48 (Wang) *For $k \geq 1$ and $n \geq 0$,*

$$\mathrm{spt}_{\omega}\left(2 \cdot 5^{2k}n + \frac{11 \cdot 5^{2k}+1}{12}\right) \equiv 0 \pmod{5^{2k}}.$$

Jang and Kim [86] obtained a congruence of $\mathrm{spt}_{\omega}(n)$ via the mock modularity of its generating function.

Theorem 4.49 (Jang and Kim) *Let $\ell \geq 5$ be a prime, and let j, m and n be positive integers with $\left(\frac{n}{\ell}\right) = -1$. If m is sufficiently large, then there are infinitely many primes $Q \equiv -1 \pmod{576\ell^j}$ satisfying*

$$\mathrm{spt}_{\omega}\left(\frac{Q^3\ell^m n+1}{12}\right) \equiv 0 \pmod{\ell^j}. \tag{4.57}$$

An overpartition analogue of the function $\mathrm{spt}_{\omega}(n)$ was defined by Andrews, Dixit, Schultz and Yee [17].

Definition 4.50 *The function $\overline{\mathrm{spt}}_{\omega}(n)$ is defined to be the number of smallest parts in the overpartitions of n in which the smallest part is always overlined and all odd parts are less than twice the smallest part.*

They obtained the generating function of $\overline{\mathrm{spt}}_{\omega}(n)$.

Theorem 4.51 (Andrews, Dixit, Schultz and Yee) *We have*

$$\sum_{n=1}^{\infty} \overline{\mathrm{spt}}_{\omega}(n)q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)} + 2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n(n+1)}}{(1-q^{2n})^2}. \tag{4.58}$$

Based on the generating function (4.58), they derived the following congruences of $\overline{\mathrm{spt}}_\omega(n)$ mod 3, 5 and 6.

Theorem 4.52 (Andrews, Dixit, Schultz and Yee) *For $n \geq 0$,*

$$\begin{aligned}
\overline{\mathrm{spt}}_\omega(3n) &\equiv 0 \pmod{3},\\
\overline{\mathrm{spt}}_\omega(3n+2) &\equiv 0 \pmod{3},\\
\overline{\mathrm{spt}}_\omega(10n+6) &\equiv 0 \pmod{5},\\
\overline{\mathrm{spt}}_\omega(6n+5) &\equiv 0 \pmod{6}.
\end{aligned}$$

They also characterized the parity of $\overline{\mathrm{spt}}_\omega(n)$.

Theorem 4.53 (Andrews, Dixit, Schultz and Yee) *For $n \geq 1$, $\overline{\mathrm{spt}}_\omega(n)$ is odd if and only if $n = k^2$ or $2k^2$ for some $k \geq 1$.*

Moreover, they found a congruence of $\overline{\mathrm{spt}}_\omega(n)$ modulo 4.

Theorem 4.54 (Andrews, Dixit, Schultz and Yee) *For $n \geq 1$,*

$$\overline{\mathrm{spt}}_\omega(7n) \equiv \overline{\mathrm{spt}}_\omega(n/7) \pmod{4},$$

where we adopt the convention that $\overline{\mathrm{spt}}_\omega(x) = 0$ if x is not a positive integer.

By (4.52), Wang [131] obtained the generating function of $\overline{\mathrm{spt}}_\omega(2n+1)$.

Theorem 4.55 (Wang) *We have*

$$\sum_{n=0}^{\infty} \overline{\mathrm{spt}}_\omega(2n+1)q^n = \frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^6}. \tag{4.59}$$

In light of (4.59), Wang derived the following congruences of $\overline{\mathrm{spt}}_\omega(n)$.

Theorem 4.56 (Wang) *For $n \geq 0$,*

$$\begin{aligned}
\overline{\mathrm{spt}}_\omega(8n+7) &\equiv 0 \pmod{4},\\
\overline{\mathrm{spt}}_\omega(6n+5) &\equiv 0 \pmod{9},\\
\overline{\mathrm{spt}}_\omega(18n+r) &\equiv 0 \pmod{9}, \quad \text{for } r = 9 \text{ or } 15,\\
\overline{\mathrm{spt}}_\omega(22n+r) &\equiv 0 \pmod{11}, \quad \text{for } r = 7, 11, 13, 17, 19, \text{ or } 21,\\
\overline{\mathrm{spt}}_\omega(162n+r) &\equiv 0 \pmod{27}, \quad \text{for } r = 81 \text{ or } 135.
\end{aligned}$$

There are other variations of the spt-function, and we just mention the main ideas of these variations. Jennings-Shaffer [90–92] introduced several spt-type functions arising from Bailey pairs and derived several Ramanujan-type congruences. Garvan and Jennings-Shaffer [78] discovered more spt-type functions and found some congruences of these spt-type functions. Patkowski [113–115] also defined several spt-type functions based on Bailey pairs. Furthermore, Patkowski obtained generating functions and congruences of these functions. Sarma, Reddy, Gunakala and Comissiong [127] defined a more general function, in the notation $\mathrm{spt}_i(n)$, as the total number of the i-th smallest part in all partitions of n.

5 Asymptotic properties

In this section, we present asymptotic formulas for the spt-function and its variations. By applying the circle method to the second symmetrized rank moment $\eta_2(n)$, Bringmann [39] obtained an asymptotic expression of the spt-function $\mathrm{spt}(n)$.

Theorem 5.1 (Bringmann) *As $n \to \infty$,*

$$\mathrm{spt}(n) \sim \frac{\sqrt{6}}{\pi}\sqrt{n}p(n) \sim \frac{1}{2\sqrt{2}\pi\sqrt{n}} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{5.1}$$

The above formula also follows from an asymptotic estimate of the difference of the positive rank moments and the positive crank moments, due to Bringmann and Mahlburg [45].

Theorem 5.2 (Bringmann and Mahlburg) *For $r \geq 1$, as $n \to \infty$,*

$$M_r^+(n) - N_r^+(n) \sim \delta_r n^{\frac{r}{2}-\frac{3}{2}} e^{\pi\sqrt{\frac{2n}{3}}},$$

where

$$\delta_r = r!\zeta(r-2)\left(1 - 2^{3-r}\right) \frac{6^{\frac{r-1}{2}}}{4\sqrt{3}\pi^{r-1}}.$$

Using Hardy and Ramanujan's asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}, \quad \text{as} \quad n \to \infty,$$

the $r = 2$ case of Theorem 5.2 implies Theorem 5.1, since

$$\mathrm{spt}(n) = M_2^+(n) - N_2^+(n).$$

Bringmann and Mahlburg [45] pointed out that for $r = 1$, Theorem 5.2 leads to an asymptotic formula for $\mathrm{ospt}(n)$, as defined in (4.19).

Theorem 5.3 (Bringmann and Mahlburg) *As $n \to \infty$,*

$$\mathrm{ospt}(n) \sim \frac{p(n)}{4} \sim \frac{1}{16\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{5.2}$$

Eichhorn and Hirschhorn [66] provided an alternative proof of Theorem 5.1. In fact, they showed that

$$\frac{\mathrm{spt}(n)}{p(n)} \sim \frac{\sqrt{6}}{\pi}\sqrt{n}, \quad \text{as} \quad n \to \infty. \tag{5.3}$$

Let λ be a partition of n, define $n_s(\lambda)$ to be the number of smallest parts of λ. It is clear that the left-hand side of (5.3) can be viewed as the mean of the statistic $n_s(\lambda)$ over all partitions of n. Eichhorn and Hirschhorn [66] obtained formulas for the mean and the standard deviation of $n_s(\lambda)$.

Theorem 5.4 (Eichhorn and Hirschhorn) *As $n \to \infty$, the statistic $n_s(\lambda)$ is distributed roughly as a negative exponential, with mean*

$$\mu = \frac{\sqrt{6}}{\pi}\sqrt{n} + \frac{3}{\pi^2} + O\left(\frac{1}{\sqrt{n}}\right) \tag{5.4}$$

and standard derivation

$$\sigma = \frac{\sqrt{6}}{\pi}\sqrt{n} - \frac{1}{4} + O\left(\frac{1}{\sqrt{n}}\right). \tag{5.5}$$

An asymptotic formula with a power saving error term for $\mathrm{spt}(n)$ has been obtained by Banks, Barquero-Sanchez, Masri, Sheng [31] based on an asymptotic formula for $p(n)$ due to Masri [107].

In analogy with the explicit formula for $p(n)$ due to Rademacher [118–120], Ahlgren and Andersen [3] obtained an exact expression for the spt-function.

Theorem 5.5 (Ahlgren and Andersen) *For $n \geq 1$,*

$$\mathrm{spt}(n) = \frac{\pi}{6}(24n-1)^{\frac{1}{4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} (I_{1/2} - I_{3/2}) \left(\frac{\pi\sqrt{24n-1}}{6c}\right),$$

where I_ν is the I-Bessel function, $A_c(n)$ is the Kloosterman sum

$$A_c(n) = \sum_{\substack{d \mod c \\ (d,c)=1}} e^{\pi i s(d,c) - 2i\pi \frac{dn}{c}},$$

and $s(d,c)$ is the Dedekind sum

$$s(d,c) = \sum_{r=1}^{c-1} \frac{r}{c}\left(\frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2}\right).$$

Asymptotic properties of generalizations and variations of the spt-function have also been well-studied. Recall that the higher order spt-function $\mathrm{spt}_k(n)$ introduced by Garvan is defined in (4.3). Its asymptotic property was first conjectured by Bringmann and Mahlburg [44], and then confirmed by Bringmann, Mahlburg and Rhoades [46].

Theorem 5.6 (Bringmann, Mahlburg and Rhoades) *As $n \to \infty$,*

$$\mathrm{spt}_k(n) \sim \beta_{2k} n^{k-\frac{1}{2}} p(n),$$

where $\beta_{2k} \in \frac{\sqrt{6}}{\pi}\mathbb{Q}$ is positive.

The following asymptotic formula for $\mathrm{Spt}_j(n)$, as defined in (4.12), is due to Rhoades [123].

Theorem 5.7 (Rhoades) *As $n \to \infty$,*

$$\mathrm{Spt}_j(n) = \frac{j}{2\pi\sqrt{2n}} e^{\pi\sqrt{\frac{2n}{3}}}(1 + o_j(1)).$$

Waldherr [130] obtained an asymptotic property of the j-rank moment ${}_jN_k(n)$ defined in (4.11).

Theorem 5.8 (Waldherr) *For $1 \le j \le 12$, as $n \to \infty$,*

$$ {}_jN_{2k}(n) \sim 2\sqrt{3}(-1)^k B_{2k}\left(\frac{1}{2}\right)(24n)^{k-1} e^{\pi\sqrt{\frac{2n}{3}}}, \tag{5.6}$$

where $B_r(\cdot)$ is a Bernoulli polynomial. Furthermore,

$$ {}_{j-1}N_{2k}(n) - {}_jN_{2k}(n) \sim \sqrt{3}\frac{(2k)!}{(2k-2)!}(-1)^{k+1} B_{2k-2}(24n)^{k-\frac{3}{2}} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{5.7}$$

In particular, ${}_{j-1}N_{2k}(n) > {}_jN_{2k}(n)$ for all sufficiently large n.

Kim, Kim and Seo [96] derived an asymptotic expression for $\overline{\mathrm{ospt}}(n)$, as defined in (4.43).

Theorem 5.9 (Kim, Kim and Seo) *As $n \to \infty$,*

$$\overline{\mathrm{ospt}}(n) \sim \frac{1}{64n} e^{\pi\sqrt{n}} \sim \frac{\bar{p}(n)}{8}, \tag{5.8}$$

where $\bar{p}(n)$ denotes the number of overpartitions of n.

The above theorem is a consequence of an asymptotic formula for the difference of the modified rank and crank moments for overpartitions due to Rolon [125].

Theorem 5.10 (Rolon) *As $n \to \infty$,*

$$\overline{M}_r^+(n) - \overline{N}_r^+(n) \sim \delta_r n^{\frac{r}{2}-\frac{3}{2}} e^{\pi\sqrt{n}}, \tag{5.9}$$

where

$$\delta_r = r!\pi^{-r+1} 2^{r-5} \zeta(r-2) \left(1 - 2^{3-r}\right).$$

Combining (4.43) and (5.9) with $r = 1$, we arrive at (5.8).

6 Conjectures on inequalities

In this section, we pose some conjectures on inequalities on the spt-function, which are reminiscent of inequalities on $p(n)$. We first state some results and conjectures on $p(n)$. Then we present corresponding conjectures on $\mathrm{spt}(n)$.

Recall that a sequence $\{a_n\}_{n\geq 0}$ is called log-concave if for $n \geq 1$,

$$a_n^2 - a_{n-1}a_{n+1} \geq 0. \tag{6.1}$$

It was conjectured in [50] that the partition function $p(n)$ is log-concave for $n \geq 26$, that is, (6.1) is true for $p(n)$ when $n \geq 26$. DeSalvo and Pak [58] confirmed this conjecture by using the Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound.

Theorem 6.1 (DeSalvo and Pak) *For $n \geq 26$,*

$$p(n)^2 > p(n-1)p(n+1). \tag{6.2}$$

They also proved the following inequalities conjectured in [50].

Theorem 6.2 (DeSalvo and Pak) *For $n \geq 2$,*

$$\frac{p(n-1)}{p(n)}\left(1 + \frac{1}{n}\right) > \frac{p(n)}{p(n+1)}. \tag{6.3}$$

Theorem 6.3 (DeSalvo and Pak) *For $n > m > 1$,*

$$p(n)^2 \geq p(n-m)p(n+m). \tag{6.4}$$

DeSalvo and Pak further proved that the term $(1+1/n)$ in (6.3) can be improved to $(1+O(n^{-3/2}))$.

Theorem 6.4 (DeSalvo and Pak) *For $n \geq 7$,*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{6.5}$$

DeSalvo and Pak [58] conjectured that the coefficient of $1/n^{3/2}$ in the inequality (6.5) can be improved to $\pi/\sqrt{24}$, which was proved by Chen, Wang and Xie [55].

Theorem 6.5 (Chen, Wang and Xie) *For $n \geq 45$,*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{6.6}$$

Bessenrodt and Ono [37] obtained an inequality on $p(n)$.

Theorem 6.6 (Bessenrodt and Ono) *If a, b are integers with $a, b > 1$ and $a + b > 8$, then*

$$p(a)p(b) \geq p(a+b), \tag{6.7}$$

where the equality can occur only if $\{a, b\} = \{2, 7\}$.

We now turn to conjectures on $\mathrm{spt}(n)$.

Conjecture 6.7 *For $n \geq 36$,*

$$\mathrm{spt}(n)^2 > \mathrm{spt}(n-1)\,\mathrm{spt}(n+1). \tag{6.8}$$

Conjecture 6.8 *For $n \geq 13$,*

$$\frac{\mathrm{spt}(n-1)}{\mathrm{spt}(n)}\left(1+\frac{1}{n}\right) > \frac{\mathrm{spt}(n)}{\mathrm{spt}(n+1)}. \tag{6.9}$$

Like the case for $p(n)$, the term $(1+1/n)$ in Conjecture 6.8 can be sharpened to $(1+O(n^{-3/2}))$.

Conjecture 6.9 *For $n \geq 73$,*

$$\frac{\operatorname{spt}(n-1)}{\operatorname{spt}(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{\operatorname{spt}(n)}{\operatorname{spt}(n+1)}. \tag{6.10}$$

The following conjectures are analogous to (6.4) and (6.7).

Conjecture 6.10 *For $n > m > 1$,*

$$\operatorname{spt}(n)^2 > \operatorname{spt}(n-m)\operatorname{spt}(n+m). \tag{6.11}$$

Conjecture 6.11 *If a, b are integers with $a, b > 1$ and $(a, b) \neq (2, 2)$ or $(3, 3)$, then*

$$\operatorname{spt}(a)\operatorname{spt}(b) > \operatorname{spt}(a+b). \tag{6.12}$$

Beyond quadratic inequalities, we observe that many combinatorial sequences including $\{p(n)\}_{n\geq 1}$ and $\{\operatorname{spt}(n)\}_{n\geq 1}$ seem to satisfy higher order inequalities except for a few terms at the beginning. Notice that $I(a_0, a_1, a_2) = a_1^2 - a_0a_2$ is an invariant of the quadratic binary form

$$a_2x^2 + 2a_1xy + a_0y^2.$$

For a sequence $a_0, a_1, a_2, \ldots$ of indeterminates, let

$$I_{n-1}(a_0, a_1, a_2) = I(a_{n-1}, a_n, a_{n+1}) = a_n^2 - a_{n-1}a_{n+1}.$$

Then Conjecture 6.7 says that for $a_n = \operatorname{spt}(n)$, $I_{n-1}(a_0, a_1, a_2) > 0$ holds when $n \geq 36$.

This phenomenon occurs for other invariants as well. For the background on the theory of invariants, see, for example, Hilbert [81], Kung and Rota [100] and Sturmfels [128]. A binary form $f(x, y)$ of degree n is a homogeneous polynomial of degree n in two variables x and y:

$$f(x, y) = \sum_{i=0}^{n} \binom{n}{i} a_i x^i y^{n-i},$$

where the coefficients a_i are complex numbers.

Let

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

be an invertible complex matrix. Under the linear transformation

$$x = c_{11}\overline{x} + c_{12}\overline{y},$$

$$y = c_{21}\overline{x} + c_{22}\overline{y},$$

the binary form $f(x, y)$ is transformed into another binary form

$$\overline{f}(\overline{x}, \overline{y}) = \sum_{i=0}^{n} \binom{n}{i} \overline{a}_i\, \overline{x}^i\, \overline{y}^{n-i},$$

where the coefficients $\overline{a}_i$ are polynomials in a_i and c_{ij}. Let g be a nonnegative integer. A polynomial $I(a_0, a_1, \ldots, a_n)$ in the coefficients $a_0, a_1, \ldots, a_n$ is an invariant of index g of the binary form $f(x, y)$ if for any invertible matrix C,

$$I(\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n) = (c_{11}c_{22} - c_{12}c_{21})^g I(a_0, a_1, \ldots, a_n).$$

For example,

$$I(a_0, a_1, a_2, a_3) = 3a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2 + 6a_0a_1a_2a_3 \tag{6.13}$$

is an invariant of the cubic binary form

$$f(x, y) = a_3x^3 + 3a_2x^2y + 3a_1xy^2 + a_0y^3. \tag{6.14}$$

Note that $27I(a_0, a_1, a_2, a_3)$ is called the discriminant of (6.14). The polynomial $I(a_{n-1}, a_n, a_{n+1}, a_{n+2})$ is related to the higher order Turán inequality. Recall that a sequence $\{a_n\}_{n\geq 0}$ satisfies the higher order Turán inequality if for $n \geq 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_na_{n+2}) - (a_na_{n+1} - a_{n-1}a_{n+2})^2 > 0, \tag{6.15}$$

and we say that $\{a_n\}_{n\geq 0}$ satisfies the Turán inequality if it is log-concave.

A simple calculation shows that for $n = 1$, the polynomial in (6.15) reduces to the invariant $I(a_0, a_1, a_2, a_3)$ in (6.13), namely,

$$\begin{aligned} &3a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2 + 6a_0a_1a_2a_3 \\ &\qquad = 4(a_1^2 - a_0a_2)(a_2^2 - a_1a_3) - (a_1a_2 - a_0a_3)^2. \end{aligned}$$

Csordas, Norfolk and Varga [57] proved that the coefficients of the Riemann ξ-function satisfy the Turán inequality. This settles a conjecture of Pólya. Dimitrov [60] showed under the Riemann hypothesis, the coefficients of the Riemann ξ-function satisfy the higher order Turán inequality. Dimitrov and Lucas [61] proved this assertion without the Riemann hypothesis.

Numerical evidence indicates that both $p(n)$ and $\mathrm{spt}(n)$ satisfy the high order Turán inequality.

Conjecture 6.12 *For $n \geq 95$, $p(n)$ satisfies the higher order Turán inequality* (6.15)*, whereas* $\mathrm{spt}(n)$ *satisfies* (6.15) *for $n \geq 108$.*

We next consider the invariant of the quartic binary form

$$f(x,y) = a_4x^4 + 4a_3x^3y + 6a_2x^2y^2 + 4a_1xy^3 + a_0y^4. \tag{6.16}$$

It appears that for large n, both $p(n)$ and $\mathrm{spt}(n)$ satisfy the inequalities derived from the following invariants of (6.16):

$$\begin{aligned} A(a_0,a_1,a_2,a_3,a_4) &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\ B(a_0,a_1,a_2,a_3,a_4) &= -a_0a_2a_4 + a_2^3 + a_0a_3^2 + a_1^2a_4 - 2a_1a_2a_3, \\ I(a_0,a_1,a_2,a_3,a_4) &= A(a_0,a_1,a_2,a_3,a_4)^3 - 27B(a_0,a_1,a_2,a_3,a_4)^2. \end{aligned}$$

Notice that $256I(a_0,a_1,a_2,a_3,a_4)$ is the discriminant of $f(x,y)$ in (6.16). To be more specific, we have the following conjectures: Setting $a_n = p(n)$,

$$\begin{aligned} A(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) &> 0, \qquad \text{for } n \geq 185, \\ B(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) &> 0, \qquad \text{for } n \geq 221, \\ I(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) &> 0, \qquad \text{for } n \geq 207. \end{aligned}$$

Setting $a_n = \mathrm{spt}(n)$,

$$\begin{aligned} A(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) &> 0, \qquad \text{for } n \geq 205, \\ B(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) &> 0, \qquad \text{for } n \geq 241, \\ I(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) &> 0, \qquad \text{for } n \geq 227. \end{aligned}$$

In general, it would be interesting to further study higher order inequalities on $p(n)$ and $\mathrm{spt}(n)$ based on polynomials arising in the invariant theory of binary forms.

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