Ramanujan J (2017) 44:281–299
DOI 10.1007/s11139-016-9842-9

The log-behavior of $\sqrt[n]{p(n)}$ and $\sqrt[n]{p(n)/n}$

William Y. C. Chen[1] · Ken Y. Zheng[2]

Received: 8 November 2015 / Accepted: 11 August 2016 / Published online: 1 November 2016

Abstract Let $p(n)$ denote the partition function and let Δ be the difference operator with respect to n. In this paper, we obtain a lower bound for $\Delta^2 \log \sqrt[n-1]{p(n-1)/(n-1)}$, leading to a proof of a conjecture of Sun on the log-convexity of $\{\sqrt[n]{p(n)/n}\}_{n\geq 60}$. Using the same argument, it can be shown that for any real number α, there exists an integer $n(\alpha)$ such that the sequence $\{\sqrt[n]{p(n)/n^{\alpha}}\}_{n\geq n(\alpha)}$ is log-convex. Moreover, we show that $\lim_{n\to+\infty} n^{\frac{5}{2}} \Delta^2 \log \sqrt[n]{p(n)} = 3\pi/\sqrt{24}$. Finally, by finding an upper bound for $\Delta^2 \log \sqrt[n-1]{p(n-1)}$, we establish an inequality on the ratio $\frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}$.

Keywords Partition function · Log-convex sequence · Hardy–Ramanujan–Rademacher formula · Lehmer's error bound

Mathematics Subject Classification 05A20

This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

✉ William Y. C. Chen
chenyc@tju.edu.cn

Ken Y. Zheng
kenzheng@aliyun.com

1 Center for Applied Mathematics, Tianjin University, Tianjin 300072, People's Republic of China

2 Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, People's Republic of China

1 Introduction

The objective of this paper is to study the log-behavior of the sequences $\sqrt[n]{p(n)}$ and $\sqrt[n]{p(n)/n}$, where $p(n)$ denotes the number of partitions of a nonnegative integer n. A positive sequence $\{a_n\}_{n\geq 0}$ is called log-convex if for $n \geq 1$,

$$a_n^2 - a_{n-1}a_{n+1} \leq 0,$$

and it is called log-concave if for $n \geq 1$,

$$a_n^2 - a_{n-1}a_{n+1} \geq 0.$$

Let $r(n) = \sqrt[n]{p(n)/n}$ and let Δ be the difference operator with respect to n. Sun [11] conjectured that the sequence $\{r(n)\}_{n\geq 60}$ is log-convex. Desalvo and Pak [5] noticed that the log-convexity of $\{r(n)\}_{n\geq 60}$ can be derived from an estimate for $\Delta^2 \log r(n-1)$, see [5, Final Remark 7.7]. They also remarked that their approach to bounding $-\Delta^2 \log p(n-1)$ does not seem to apply to $\Delta^2 \log r(n-1)$. In this paper, we obtain a lower bound for $\Delta^2 \log r(n-1)$, leading to a proof of the log-convexity of $\{r(n)\}_{n\geq 60}$.

Theorem 1.1 *The sequence $\{r(n)\}_{n\geq 60}$ is log-convex.*

The log-convexity of $\{r(n)\}_{n\geq 60}$ implies the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 26}$, because the sequence $\{\sqrt[n]{n}\}_{n\geq 4}$ is log-convex [11]. It is known that $\lim\limits_{n\to+\infty} \sqrt[n]{p(n)} = 1$. For a combinatorial proof of this fact, see Andrews [1]. Sun [11] proposed the conjecture that $\{\sqrt[n]{p(n)}\}_{n\geq 6}$ is strictly decreasing, which has been proved by Wang and Zhu [12]. The log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 26}$ was also conjectured by Sun [11]. It is easy to see that the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 26}$ implies the decreasing property.

It should be noted that there is an alternative way to prove the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 26}$. Chen, Guo, and Wang [3] introduced the notion of a ratio log-convex sequence and showed that the ratio log-convexity implies the log-convexity under a certain initial condition. A sequence $\{a_n\}_{n\geq k}$ is called ratio log-convex if $\{a_{n+1}/a_n\}_{n\geq k}$ is log-convex, or, equivalently, for $n \geq k+1$,

$$\log a_{n+2} - 3\log a_{n+1} + 3\log a_n - \log a_{n-1} \geq 0.$$

Chen et al. [4] showed that for any $r \geq 1$, one can determine a number $n(r)$ such that for $n > n(r)$, $(-1)^{r-1}\Delta^r \log p(n)$ is positive. For $r = 3$, it can be shown that for $n \geq 116$,

$$\Delta^3 \log p(n-1) > 0.$$

Since

$$\Delta^3 \log p(n-1) = \log p(n+2) - 3\log p(n+1) + 3\log p(n) - \log p(n-1),$$

we see $\{p(n)\}_{n\geq 115}$ is ratio log-convex. So we are led to the following assertion.

Theorem 1.2 *The sequence* $\{\sqrt[n]{p(n)}\}_{n\geq 26}$ *is log-convex.*

Moreover, as pointed out by the referee, we may consider the log-behavior of $\sqrt[n]{p(n)/n^{\alpha}}$ for any real number α. To this end, we obtain the following generalization of Theorems 1.1 and 1.2.

Theorem 1.3 *Let* α *be a real number. There exists a positive integer* $n(\alpha)$ *such that the sequence* $\{\sqrt[n]{p(n)/n^{\alpha}}\}_{n\geq n(\alpha)}$ *is log-convex.*

We also establish the following inequality on the ratio $\frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}$.

Theorem 1.4 *For* $n \geq 2$*, we have*

$$\frac{\sqrt[n]{p(n)}}{\sqrt[n+1]{p(n+1)}}\left(1+\frac{3\pi}{\sqrt{24}n^{5/2}}\right) > \frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}. \tag{1.1}$$

Desalvo and Pak [5] have shown that the limit of $-n^{\frac{3}{2}}\Delta^2 \log p(n)$ is $\pi/\sqrt{24}$. By bounding $\Delta^2 \log \sqrt[n]{p(n)}$, we derive the following limit of $n^{\frac{5}{2}}\Delta^2 \log \sqrt[n]{p(n)}$:

$$\lim_{n\to+\infty} n^{\frac{5}{2}}\Delta^2 \log \sqrt[n]{p(n)} = \frac{3\pi}{\sqrt{24}}. \tag{1.2}$$

From the above relation (1.2), it can be seen that the coefficient $3\pi/\sqrt{24}$ in (1.1) is the best possible.

2 The log-convexity of $r(n)$

In this section, we obtain a lower bound for $\Delta^2 \log r(n-1)$ and prove the log-convexity of $\{r(n)\}_{n\geq 60}$. First, we follow the approach of Desalvo and Pak to give an expression of $\Delta^2 \log r(n-1)$ as a sum of $\Delta^2 \widetilde{B}(n-1)$ and $\Delta^2 \widetilde{E}(n-1)$, where $\Delta^2 \widetilde{B}(n-1)$ makes a major contribution to $\Delta^2 \log r(n-1)$ with $\Delta^2 \widetilde{E}(n-1)$ being the error term, that is, $\Delta^2 \widetilde{B}(n-1)$ converges to $\Delta^2 \log r(n-1)$. The expressions for $B(n)$ and $E(n)$ will be given later. In this setting, we derive a lower bound for $\Delta^2 \widetilde{B}(n-1)$. By Lehmer's error bound, we give an upper bound for $|\Delta^2 \widetilde{E}(n-1)|$. Combining the lower bound for $\Delta^2 \widetilde{B}(n-1)$ and the upper bound for $\Delta^2 \widetilde{E}(n-1)$, we are led to a lower bound for $\Delta^2 \log r(n-1)$. By proving the positivity of this lower bound for $\Delta^2 \log r(n-1)$, we reach the log-convexity of $\{r(n)\}_{n\geq 60}$.

The strict log-convexity of $\{r(n)\}_{n\geq 60}$ can be restated as the following relation for $n \geq 61$:

$$\log r(n+1) + \log r(n-1) - 2\log r(n) > 0,$$

that is, for $n \geq 61$,

$$\Delta^2 \log r(n-1) > 0.$$

For $n \geq 1$ and any positive integer N, the Hardy–Ramanujan–Rademacher formula (see [2,6,7,10]) reads

$$p(n) = \frac{d}{\mu^2} \sum_{k=1}^{N} A_k^\star(n) \left[\left(1 - \frac{k}{\mu}\right) e^{\frac{\mu}{k}} + \left(1 + \frac{k}{\mu}\right) e^{-\frac{\mu}{k}} \right] + R_2(n, N), \tag{2.1}$$

where $d = \frac{\pi^2}{6\sqrt{3}}$, $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$, $A_k^\star(n) = k^{-\frac{1}{2}} A_k(n)$, $A_k(n)$ is a sum of the 24th roots of unity with initial values $A_1(n) = 1$ and $A_2(n) = (-1)^n$, $R_2(n, N)$ is the remainder. Lehmer's error bound (see [8,9]) for $R_2(n, N)$ is given by

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu}\right)^3 \sinh \frac{\mu}{N} + \frac{1}{6} - \left(\frac{N}{\mu}\right)^2 \right]. \tag{2.2}$$

Let us give an outline of Desalvo and Pak's approach to proving the log-concavity of $\{p(n)\}_{n>25}$. Setting $N = 2$ in (2.1), they expressed $p(n)$ as

$$p(n) = T(n) + R(n), \tag{2.3}$$

where

$$T(n) = \frac{d}{\mu(n)^2} \left[\left(1 - \frac{1}{\mu(n)}\right) e^{\mu(n)} + \frac{(-1)^n}{\sqrt{2}} e^{\frac{\mu(n)}{2}} \right], \tag{2.4}$$

$$R(n) = \frac{d}{\mu(n)^2} \left[\left(1 + \frac{1}{\mu(n)}\right) e^{-\mu(n)} - \frac{(-1)^n}{\sqrt{2}} \frac{2}{\mu(n)} + \frac{(-1)^n}{\sqrt{2}} \left(1 + \frac{2}{\mu(n)}\right) e^{-\frac{\mu(n)}{2}} \right] + R_2(n, 2). \tag{2.5}$$

They have shown that

$$\left| \Delta^2 \log p(n-1) - \Delta^2 \log T(n-1) \right| = \left| \Delta^2 \log \left(1 + \frac{R(n-1)}{T(n-1)}\right) \right| < e^{-\frac{\pi\sqrt{2n}}{10\sqrt{3}}} \tag{2.6}$$

and

$$\left| \Delta^2 \log T(n-1) - \Delta^2 \log \frac{d}{\mu(n-1)^2} \left(1 - \frac{1}{\mu(n-1)}\right) e^{\mu(n-1)} \right| < e^{-\frac{\pi\sqrt{2n}}{10\sqrt{3}}}. \tag{2.7}$$

It follows that $\Delta^2 \log \frac{d}{\mu(n-1)^2} \left(1 - \frac{1}{\mu(n-1)}\right) e^{\mu(n-1)}$ converges to $\Delta^2 \log p(n-1)$. Finally, they use $-\Delta^2 \log \frac{d}{\mu(n-1)^2} \left(1 - \frac{1}{\mu(n-1)}\right) e^{\mu(n-1)}$ to estimate $-\Delta^2 \log p(n-1)$, leading to the log-concavity of $\{p(n)\}_{n>25}$.

We shall use an alternative decomposition of $p(n)$. Setting $N = 2$ in (2.1), we can express $p(n)$ as

$$p(n) = \widetilde{T}(n) + \widetilde{R}(n), \tag{2.8}$$

where

$$\widetilde{T}(n) = \frac{d}{\mu(n)^2}\left(1 - \frac{1}{\mu(n)}\right)e^{\mu(n)}, \tag{2.9}$$

$$\widetilde{R}(n) = \frac{d}{\mu(n)^2}\Bigg[\left(1 + \frac{1}{\mu(n)}\right)e^{-\mu(n)} + \frac{(-1)^n}{\sqrt{2}}\left(1 - \frac{2}{\mu(n)}\right)e^{\frac{\mu(n)}{2}} + \frac{(-1)^n}{\sqrt{2}}\left(1 + \frac{2}{\mu(n)}\right)e^{-\frac{\mu(n)}{2}}\Bigg] + R_2(n, 2). \tag{2.10}$$

Based on the decomposition (2.8) for $p(n)$, one can express $\Delta^2 \log r(n-1)$ as follows:

$$\Delta^2 \log r(n-1) = \Delta^2 \widetilde{B}(n-1) + \Delta^2 \widetilde{E}(n-1), \tag{2.11}$$

where

$$\widetilde{B}(n) = \frac{1}{n}\log \widetilde{T}(n) - \frac{1}{n}\log n, \tag{2.12}$$

$$\widetilde{y}_n = \widetilde{R}(n)/\widetilde{T}(n), \tag{2.13}$$

$$\widetilde{E}(n) = \frac{1}{n}\log(1 + \widetilde{y}_n). \tag{2.14}$$

The following lemma will be used to derive a lower bound and an upper bound for $\Delta^2 \widetilde{B}(n-1)$.

Lemma 2.1 *Suppose $f(x)$ has a continuous second derivative for $x \in [n-1, n+1]$. Then there exists $c \in (n-1, n+1)$ such that*

$$\Delta^2 f(n-1) = f(n+1) + f(n-1) - 2f(n) = f''(c). \tag{2.15}$$

If $f(x)$ has an increasing second derivative, then

$$f''(n-1) < \Delta^2 f(n-1) < f''(n+1). \tag{2.16}$$

Conversely, if $f(x)$ has a decreasing second derivative, then

$$f''(n+1) < \Delta^2 f(n-1) < f''(n-1). \tag{2.17}$$

Proof Set $\varphi(x) = f(x+1) - f(x)$. By the mean value theorem, there exists a number $\xi \in (n-1, n)$ such that

$$f(n+1) + f(n-1) - 2f(n) = \varphi(n) - \varphi(n-1) = \varphi'(\xi).$$

Again, applying the mean value theorem to $\varphi'(\xi)$, there exists a number $\theta \in (0, 1)$ such that

$$\varphi'(\xi) = f'(\xi + 1) - f'(\xi) = f''(\xi + \theta).$$

Let $c = \xi + \theta$. Then we get (2.15), which yields (2.16) and (2.17). □

In order to find a lower bound for $\Delta^2 \log r(n-1)$ and obtain the limit of $n^{\frac{5}{2}} \Delta^2 \log \sqrt[n]{p(n)}$, we need the following lower and upper bounds for $\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1)$.

Lemma 2.2 *Let*

$$B_1(n) = \frac{72\pi}{(n+1)(24n+23)^{3/2}} - \frac{4\log(\mu(n-1))}{(n-1)^3}, \tag{2.18}$$

$$B_2(n) = \frac{72\pi}{(n-1)(24n-25)^{3/2}} - \frac{4\log(\mu(n+1))}{(n+1)^3} + \frac{5}{(n-1)^3}. \tag{2.19}$$

For $n \geq 40$, we have

$$B_1(n) < \Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1) < B_2(n). \tag{2.20}$$

Proof By the definition (2.9), we may write

$$\frac{\log \widetilde{T}(n)}{n} = \sum_{i=1}^{4} f_i,$$

where

$$f_1(n) = \frac{\mu(n)}{n},$$

$$f_2(n) = -\frac{3\log \mu(n)}{n},$$

$$f_3(n) = \frac{\log(\mu(n)-1)}{n},$$

$$f_4(n) = \frac{\log d}{n}.$$

Thus

$$\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1) = \sum_{i=1}^{4} \Delta^2 f_i(n-1). \tag{2.21}$$

Since

$$f_1'''(n) = \frac{\pi}{n(24n-1)^{3/2}} \left(-\frac{216}{n} + \frac{864}{24n-1} + \frac{36}{n^2} - \frac{1}{n^3} \right),$$

we see that for $n \geq 1$, $f_1'''(n) < 0$. Similarly, it can be checked that for $n \geq 4$, $f_2'''(n) > 0$, $f_3'''(n) < 0$, and $f_4'''(n) > 0$. Consequently, for $n \geq 4$, $f_1''(n)$ and $f_3''(n)$ are decreasing, whereas $f_2''(n)$ and $f_4''(n)$ are increasing. Using Lemma 2.1, for each

i, we can get a lower bound and an upper bound for $\Delta^2 f_i(n-1)$ in terms of $f_i''(n-1)$ and $f_i''(n+1)$. For example,

$$f_1''(n+1) < \Delta^2 f_1(n-1) < f_1''(n-1).$$

So, by (2.21) we find that

$$\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1) > f_1''(n+1) + f_2''(n-1) + f_3''(n+1) + f_4''(n-1) \tag{2.22}$$

and

$$\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1) < f_1''(n-1) + f_2''(n+1) + f_3''(n-1) + f_4''(n+1), \tag{2.23}$$

where

$$f_1''(n) = \frac{72\pi}{n(24n-1)^{3/2}} - \frac{12\pi}{n^2(24n-1)^{3/2}} + \frac{\pi}{3n^3(24n-1)^{3/2}}, \tag{2.24}$$

$$f_2''(n) = -\frac{6\log\mu(n)}{n^3} + \frac{72}{(24n-1)n^2} + \frac{864}{n(24n-1)^2}, \tag{2.25}$$

$$\begin{aligned} f_3''(n) = & -\frac{4\pi^2}{(\mu(n)-1)^2(24n-1)n} + \frac{2\log(\mu(n)-1)}{n^3} \\ & - \frac{4\pi}{(\mu(n)-1)\sqrt{24n-1}n^2} - \frac{24\pi}{(\mu(n)-1)(24n-1)^{3/2}n}, \end{aligned} \tag{2.26}$$

$$f_4''(n) = \frac{2\log d}{n^3}. \tag{2.27}$$

According to (2.24), one can check that for $n \geq 2$,

$$f_1''(n+1) > \frac{72\pi}{(n+1)(24n+23)^{3/2}} - \frac{12\pi}{(n+1)^2(24n+23)^{3/2}}. \tag{2.28}$$

An easy computation shows that for $n \geq 3$,

$$\mu(n) - 1 > \frac{2}{3}\mu(n-2). \tag{2.29}$$

Substituting (2.29) into (2.26) yields that

$$f_3''(n+1) > \frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{540}{(24n-25)^2(n-1)} - \frac{36}{(24n-25)(n-1)^2}. \tag{2.30}$$

Using (2.25) and (2.30), we find that

$$\begin{aligned} & f_2''(n-1) + f_3''(n+1) \\ & \quad > \frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{6\log(\mu(n-1))}{(n-1)^3} \\ & \qquad + \frac{324}{(n-1)(24n-25)^2} + \frac{36}{(n-1)^2(24n-25)}. \end{aligned} \tag{2.31}$$

Apparently, for $n \geq 2$,

$$\frac{2}{(n+1)^3} - \frac{2}{(n-1)^3} > -\frac{12}{(n-1)^4}$$

so that

$$\begin{aligned} & \frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{6\log(\mu(n-1))}{(n-1)^3} \\ & \quad > \frac{2\log(\mu(n+1)-1)}{(n+1)^3} - \frac{2\log(\mu(n+1)-1)}{(n-1)^3} - \frac{4\log(\mu(n-1))}{(n-1)^3} \\ & \quad > -\frac{12\log(\mu(n+1)-1)}{(n-1)^4} - \frac{4\log(\mu(n-1))}{(n-1)^3}. \end{aligned} \tag{2.32}$$

Since, for $n \geq 2$,

$$\frac{324}{(n-1)(24n-25)^2} + \frac{36}{(n-1)^2(24n-25)} > \frac{2}{(n-1)^3}, \tag{2.33}$$

utilizing (2.31) and (2.32) yields, for $n \geq 3$,

$$f_2''(n-1) + f_3''(n+1) > -\frac{4\log(\mu(n-1))}{(n-1)^3} + \frac{2}{(n-1)^3} - \frac{12\log(\mu(n+1)-1)}{(n-1)^4}. \tag{2.34}$$

Using (2.27), (2.28), and (2.34), we deduce that

$$\begin{aligned} & f_1''(n+1) + f_2''(n-1) + f_3''(n+1) + f_4''(n-1) - B_1(n) \\ & \quad > \frac{2(1+\log d)}{(n-1)^3} - \frac{12\pi}{(n+1)^2(24n+23)^{3/2}} - \frac{12\log(\mu(n+1)-1)}{(n-1)^4}. \end{aligned} \tag{2.35}$$

Let $C(n)$ be the right-hand side of (2.35). By (2.22), to prove $B_1(n) < \Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1)$, it is enough to show that $C(n) > 0$ when $n \geq 40$. Since $\log x < x$ for $x > 0$ and, for $n \geq 3$,

$$\mu(n+1) - 1 < \frac{\pi}{4}\sqrt{24n-24}, \tag{2.36}$$

we get

$$-\frac{12\log(\mu(n+1)-1)}{(n-1)^4} > -\frac{12(\mu(n+1)-1)}{(n-1)^4} > -\frac{3\sqrt{24}\pi}{(n-1)^{7/2}}. \tag{2.37}$$

Note that for $n \geq 2$,

$$-\frac{12\pi}{(n+1)^2(24n+23)^{3/2}} > -\frac{\sqrt{24}\pi}{48(n-1)^{7/2}}. \tag{2.38}$$

Combining (2.37) and (2.38), we see that for $n \geq 2$,

$$C(n) > \frac{2(1+\log d)}{(n-1)^3} - \frac{(3+1/48)\sqrt{24}\pi}{(n-1)^{7/2}}. \tag{2.39}$$

It is straightforward to show that the right-hand side of (2.39) is positive if $n \geq 490$. For $40 \leq n \leq 489$, it is routine to check that $C(n) > 0$, and so $C(n) > 0$ for $n \geq 40$. It follows from (2.35) that for $n \geq 40$,

$$\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1) > B_1(n).$$

To derive the upper bound for $\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1)$, we obtain the following upper bounds which can be verified directly. The proofs are omitted. For $n \geq 2$,

$$\begin{aligned}
f_1''(n-1) &< \frac{72\pi}{(n-1)[24n-25]^{3/2}},\\
f_2''(n+1) &< -\frac{6\log\mu(n+1)}{(n+1)^3} + \frac{9}{2(n-1)^3},\\
f_3''(n-1) &< -\frac{4\pi^2}{(\mu(n-1))^2(24n-25)(n-1)} + \frac{2\log(\mu(n-1))}{(n-1)^3}\\
&\quad -\frac{4\pi}{\mu(n-1)\sqrt{24n-25}(n-1)^2} - \frac{24\pi}{\mu(n-1)(24n-25)^{3/2}(n-1)},\\
f_2''(n+1) + f_3''(n-1) &< \frac{3}{(n-1)^3} + \frac{12\log(\mu(n+1))}{(n-1)^4} - \frac{4\log(\mu(n+1))}{(n+1)^3},\\
f_4''(n+1) &< 0.
\end{aligned}$$

Combining the above upper bounds, we conclude that for $n \geq 40$,

$$f_1''(n-1) + f_2''(n+1) + f_3''(n-1) + f_4''(n+1) < B_2(n).$$

This completes the proof. □

The following lemma gives an upper bound for $|\Delta^2 \widetilde{E}(n-1)|$.

Lemma 2.3 *For* $n \geq 40$,

$$|\Delta^2 \widetilde{E}(n-1)| < \frac{5}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}}. \tag{2.40}$$

Proof By (2.14), we find that for $n \geq 2$,

$$\Delta^2 \widetilde{E}(n-1) = \frac{1}{n-1}\log(1+\widetilde{y}_{n-1}) + \frac{1}{n+1}\log(1+\widetilde{y}_{n+1}) - \frac{2}{n}\log(1+\widetilde{y}_n), \tag{2.41}$$

where

$$\widetilde{y}_n = \widetilde{R}(n)/\widetilde{T}(n).$$

To bound $|\Delta^2 \widetilde{E}(n-1)|$, it is necessary to bound $\widetilde{y}_n$. For this purpose, we first consider $\widetilde{R}(n)$, as defined by (2.10). Since $d < 1$ and $\mu(n) > 2$, for $n \geq 1$ we have

$$\frac{d}{\mu(n)^2}\left[\left(1+\frac{1}{\mu(n)}\right)e^{-\mu(n)} + \frac{(-1)^n}{\sqrt{2}}\left(1-\frac{2}{\mu(n)}\right)e^{\frac{\mu(n)}{2}} + \frac{(-1)^n}{\sqrt{2}}\left(1+\frac{2}{\mu(n)}\right)e^{-\frac{\mu(n)}{2}}\right] < \frac{1}{\mu(n)^2}\left(1+e^{\frac{\mu(n)}{2}}+1\right).$$

For $N = 2$ and $n \geq 1$, Lehmer's bound (2.2) reduces to

$$|R_2(n,2)| < 4\left(1+\frac{4}{\mu(n)^3}e^{\frac{\mu(n)}{2}}\right).$$

By the definition of $\widetilde{R}(n)$,

$$|\widetilde{R}(n)| < \frac{1}{\mu(n)^2}\left(1+e^{\frac{\mu(n)}{2}}+1\right) + 4\left(1+\frac{4}{\mu(n)^3}e^{\frac{\mu(n)}{2}}\right) < 5 + \frac{9}{\mu(n)^2}e^{\frac{\mu(n)}{2}}. \tag{2.42}$$

Recalling the definition (2.9) of $\widetilde{T}(n)$, it follows from (2.42) that for $n \geq 1$,

$$|\widetilde{y}_n| < \frac{\mu(n)}{d(\mu(n)-1)}\left(5\mu(n)^2 e^{-\frac{2\mu(n)}{3}} + 9e^{-\frac{\mu(n)}{6}}\right)e^{-\frac{\mu(n)}{3}}. \tag{2.43}$$

Observe that for $n \geq 2$,

$$\left(5\mu(n)^2 e^{-\frac{2\mu(n)}{3}} + 9e^{-\frac{\mu(n)}{6}}\right)' < 0, \tag{2.44}$$

and

$$\left(\frac{d(\mu(n)-1)}{\mu(n)}\right)' > 0. \tag{2.45}$$

Since

$$5\mu^2(40)e^{-\frac{2\mu(40)}{3}} + 9e^{-\frac{\mu(40)}{6}} < \frac{d(\mu(40)-1)}{\mu(40)},$$

using (2.44) and (2.45), we deduce that for $n \geq 40$,

$$5\mu^2(n)e^{-\frac{2\mu(n)}{3}} + 9e^{-\frac{\mu(n)}{6}} < \frac{d(\mu(n)-1)}{\mu(n)}. \tag{2.46}$$

Now, it is clear from (2.43) and (2.46) that for $n \geq 40$,

$$|\widetilde{y}_n| < e^{-\frac{\mu(n)}{3}}. \tag{2.47}$$

In view of (2.47), for $n \geq 40$,

$$|\widetilde{y}_n| < e^{-\frac{\mu(40)}{3}} < \frac{1}{5}. \tag{2.48}$$

It is known that $\log(1+x) < x$ for $0 < x < 1$ and $-\log(1+x) < -x/(1+x)$ for $-1 < x < 0$. Thus, for $|x| < 1$,

$$|\log(1+x)| \leq \frac{|x|}{1-|x|}, \tag{2.49}$$

see also [5], and so it follows from (2.48) and (2.49) that for $n \geq 40$,

$$|\log(1+\widetilde{y}_n)| \leq \frac{|\widetilde{y}_n|}{1-|\widetilde{y}_n|} \leq \frac{5}{4}|\widetilde{y}_n|. \tag{2.50}$$

Because of (2.41), we see that for $n \geq 2$,

$$\left|\Delta^2\widetilde{E}(n-1)\right| \leq \frac{1}{n-1}|\log(1+\widetilde{y}_{n-1})| + \frac{1}{n+1}|\log(1+\widetilde{y}_{n+1})| + \frac{2}{n}|\log(1+\widetilde{y}_n)|. \tag{2.51}$$

Applying (2.50) to (2.51), we obtain that for $n \geq 40$,

$$\left|\Delta^2\widetilde{E}(n-1)\right| \leq \frac{5}{4}\left(\frac{|\widetilde{y}_{n-1}|}{n-1} + \frac{|\widetilde{y}_{n+1}|}{n+1} + \frac{2|\widetilde{y}_n|}{n}\right). \tag{2.52}$$

Plugging (2.47) into (2.52), we infer that for $n \geq 40$,

$$\left|\Delta^2 \widetilde{E}(n-1)\right| < \frac{5}{4}\left(\frac{e^{-\frac{\mu(n-1)}{3}}}{n-1} + \frac{e^{-\frac{\mu(n+1)}{3}}}{n+1} + \frac{2e^{-\frac{\mu(n)}{3}}}{n}\right). \tag{2.53}$$

But $\frac{1}{n}e^{-\frac{\mu(n)}{3}}$ is decreasing for $n \geq 1$. It follows from (2.53) that for $n \geq 40$,

$$\left|\Delta^2 \widetilde{E}(n-1)\right| < \frac{5}{n-1}e^{-\frac{\mu(n-1)}{3}}.$$

This proves (2.40). □

With the aid of Lemmas 2.2 and 2.3, we are ready to prove the log-convexity of $\{r(n)\}_{n\geq 60}$.

Proof of Theorem 1.1. To prove the strict log-convexity of $\{r(n)\}_{n\geq 60}$, we proceed to show that for $n \geq 61$,

$$\Delta^2 \log r(n-1) > 0.$$

Evidently, for $n \geq 40$,

$$\left(-\frac{\log n}{n}\right)''' > 0.$$

By Lemma 2.1,

$$-\Delta^2 \frac{\log(n-1)}{n-1} > \left(-\frac{\log(n-1)}{n-1}\right)'',$$

that is,

$$-\Delta^2 \frac{\log(n-1)}{n-1} > -\frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3}. \tag{2.54}$$

It follows from (2.12) that

$$\Delta^2 \widetilde{B}(n-1) = \Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1) - \Delta^2 \frac{\log(n-1)}{n-1}.$$

Applying Lemma 2.2 and (2.54) to the above relation, we deduce that for $n \geq 40$,

$$\Delta^2 \widetilde{B}(n-1) > \widetilde{B}_1(n) - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3},$$

that is,

$$\Delta^2 \widetilde{B}(n-1) > \frac{72\pi}{(n+1)(24n+23)^{3/2}} - \frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3}. \tag{2.55}$$

By (2.11) and Lemma 2.3, we find that for $n \geq 40$,

$$\Delta^2 \log r(n-1) > \Delta^2 \widetilde{B}(n-1) - \frac{5}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}}. \tag{2.56}$$

It follows from (2.55) and (2.56) that for $n \geq 40$,

$$\Delta^2 \log r(n-1) > \frac{72\pi}{(n+1)(24n+23)^{3/2}} - \frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3} - \frac{5}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}}.$$

Let $D(n)$ denote the right-hand side of the above relation. Clearly, for $n \geq 5505$,

$$\frac{72\pi}{(n+1)(24n+23)^{3/2}} > \frac{3\pi}{\sqrt{24}(n+1)^{5/2}} > \frac{1}{(n-1)^{5/2}}. \tag{2.57}$$

To prove that $D(n) > 0$ for $n \geq 5505$, we wish to show that for $n \geq 5505$,

$$-\frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3} - \frac{5}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}} > -\frac{1}{(n-1)^{5/2}}. \tag{2.58}$$

Using the fact that for $x > 5504$, $\log x < x^{1/4}$, we deduce that for $n \geq 5505$,

$$\frac{4\log[\mu(n-1)]}{(n-1)^3} < \frac{4\sqrt[4]{\mu(n-1)}}{(n-1)^3} < \frac{4\sqrt[4]{\frac{\pi}{4}\sqrt{24n-24}}}{(n-1)^3} < \frac{6}{(n-1)^{23/8}}, \tag{2.59}$$

and

$$\frac{2\log(n-1)}{(n-1)^3} < \frac{2(n-1)^{1/4}}{(n-1)^3} < \frac{2}{(n-1)^{11/4}}. \tag{2.60}$$

Since $e^x > x^6/720$ for $x > 0$, we see that for $n \geq 2$,

$$\frac{1}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}} < \frac{1}{n-1} e^{-\frac{\pi\sqrt{23n}}{18}} < \frac{2094}{n^3(n-1)} < \frac{2094}{(n-1)^4}. \tag{2.61}$$

Combining (2.59), (2.60), and (2.61), we find that for $n \geq 5505$,

$$
\begin{aligned}
&-\frac{4\log[\mu(n-1)]}{(n-1)^3}-\frac{2\log(n-1)}{(n-1)^3}+\frac{3}{(n-1)^3}-\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}\\
&\quad>-\frac{6}{(n-1)^{23/8}}-\frac{2}{(n-1)^{11/4}}+\frac{3}{(n-1)^3}-\frac{10470}{(n-1)^4}\\
&\quad>-\frac{6}{(n-1)^{23/8}}-\frac{2}{(n-1)^{11/4}}\\
&\quad>-\frac{1}{(n-1)^{5/2}}.
\end{aligned}
$$

This proves the inequality (2.58). By (2.58) and (2.57), we obtain that $D(n) > 0$ for $n \geq 5505$. Verifying that $\Delta^2 \log r(n-1) > 0$ for $61 \leq n \leq 5504$ completes the proof. □

Clearly, Theorem 1.3 is a generalization as well as a unification of Theorems 1.1 and 1.2. In fact, it can be proved in the same manner as the proof of Theorem 1.1.

Proof of Theorem 1.3. Let α be a real number. When $\alpha \leq 0$, it is clear that $\frac{1}{\sqrt[n]{n^\alpha}}$ is log-convex. It follows from Theorem 1.2 that $\sqrt[n]{p(n)/n^\alpha}$ is log-convex for $n \geq 26$.

We now consider the case $\alpha > 0$. A similar argument to the proof of Theorem 1.1 shows that for $n \geq 40$,

$$
\begin{aligned}
&\Delta^2 \log \sqrt[n-1]{p(n-1)/(n-1)^\alpha}\\
&\quad=\Delta^2\frac{1}{n-1}\log T(n)+\Delta^2\frac{1}{n-1}\log(1+y_{n-1})-\alpha\Delta^2\frac{\log(n-1)}{n-1}\\
&\quad>\frac{72\pi}{(n+1)(24n+23)^{3/2}}-\frac{4\log[\mu(n-1)]}{(n-1)^3}-\frac{2\alpha\log(n-1)}{(n-1)^3}\\
&\qquad+\frac{3\alpha}{(n-1)^3}-\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}.
\end{aligned}
\tag{2.62}
$$

It is easy to check that for $n \geq \max\left\{\left[\frac{3490}{\alpha}\right]+2, 5505\right\}$,

$$
\frac{3\alpha}{(n-1)^3}-\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}>\frac{3\alpha}{(n-1)^3}-\frac{10470}{(n-1)^4}>0
$$

and that for $n \geq \max\{[(2\alpha+3)^4]+2, 5505\}$,

$$
-\frac{4\log[\mu(n-1)]}{(n-1)^3}-\frac{2\alpha\log(n-1)}{(n-1)^3}>-\frac{6}{(n-1)^{23/8}}-\frac{2\alpha}{(n-1)^{11/4}}>-\frac{1}{(n-1)^{5/2}}.
$$

Let

$$
n(\alpha)=\max\left\{\left[\frac{3490}{\alpha}\right]+2, [(2\alpha+3)^4]+2, 5505\right\}.
$$

It can be seen that for $n > n(\alpha)$,

$$-\frac{4\log[\mu(n-1)]}{(n-1)^3} - \frac{2\alpha\log(n-1)}{(n-1)^3} + \frac{3\alpha}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}} > -\frac{1}{(n-1)^{5/2}}. \tag{2.63}$$

Combining (2.57) and (2.63), we deduce that the right-hand side of (2.62) is positive for $n > n(\alpha)$. So we are led to the log-convexity of the sequence $\{\sqrt[n]{p(n)/n^\alpha}\}_{n\geq n(\alpha)}$.□

3 An inequality on the ratio $\frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}$

In this section, we employ Lemmas 2.2 and 2.3 to find the limit of $n^{\frac{5}{2}}\Delta^2\log\sqrt[n]{p(n)}$. Then we give an upper bound for $\Delta^2\log\sqrt[n-1]{p(n-1)}$. This leads to the inequality (1.1).

Theorem 3.1 *Let $\beta = 3\pi/\sqrt{24}$. We have*

$$\lim_{n\to+\infty} n^{\frac{5}{2}}\Delta^2\log\sqrt[n]{p(n)} = \beta. \tag{3.1}$$

Proof Using (2.8), that is, the $N = 2$ case of the Hardy–Ramanujan–Rademacher formula for $p(n)$, we find that

$$\log\sqrt[n]{p(n)} = \frac{1}{n}\log\widetilde{T}(n) + \frac{1}{n}\log(1+\widetilde{y}_n),$$

where $\widetilde{T}(n)$ and $\widetilde{y}_n$ are given by (2.9) and (2.13). By the definition (2.14) of $\widetilde{E}(n)$, we get

$$\Delta^2\log\sqrt[n-1]{p(n-1)} = \Delta^2\frac{1}{n-1}\log\widetilde{T}(n-1) + \Delta^2\widetilde{E}(n-1). \tag{3.2}$$

Applying Lemma 2.2, we see that

$$\lim_{n\to+\infty}(n-1)^{\frac{5}{2}}\Delta^2\frac{1}{n-1}\log\widetilde{T}(n-1) = \beta. \tag{3.3}$$

From Lemma 2.3, we get

$$\lim_{n\to+\infty}(n-1)^{\frac{5}{2}}\Delta^2\widetilde{E}(n-1) = 0. \tag{3.4}$$

Using (3.2), (3.3), and (3.4), we deduce that

$$\lim_{n\to+\infty} n^{\frac{5}{2}}\Delta^2\log\sqrt[n]{p(n)} = \beta,$$

as required. □

To prove Theorem 1.4, we need the following upper bound for $\Delta^2 \log \sqrt[n-1]{p(n-1)}$.

Theorem 3.2 *For $n \geq 2$,*

$$\Delta^2 \log \sqrt[n-1]{p(n-1)} < \frac{3\pi}{\sqrt{24}n^{5/2} + 3\pi}. \tag{3.5}$$

Proof By the upper bound for $\Delta^2 \frac{1}{n-1} \log \widetilde{T}(n-1)$ given in Lemma 2.2, the upper bound for $\Delta^2 \widetilde{E}(n-1)$ given in Lemma 2.3, and the relation (3.2), we obtain the following upper bound of $\Delta^2 \log \sqrt[n-1]{p(n-1)}$ for $n \geq 40$:

$$\Delta^2 \log \sqrt[n-1]{p(n-1)} < \frac{72\pi}{(n-1)(24n-25)^{3/2}} + \frac{5}{(n-1)^3} - \frac{4\log[\mu(n+1)]}{(n+1)^3} + \frac{5}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}}.$$

To prove (3.5), we claim that for $n \geq 2095$,

$$\frac{72\pi}{(n-1)(24n-25)^{3/2}} + \frac{5}{(n-1)^3} - \frac{4\log[\mu(n+1)]}{(n+1)^3} + \frac{5}{n-1} e^{-\frac{\pi\sqrt{24n-25}}{18}} < \frac{3\pi}{\sqrt{24}n^{5/2} + 3\pi}. \tag{3.6}$$

First, we show that for $n \geq 60$,

$$\frac{72\pi}{(n-1)(24n-25)^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2} + 3\pi} < \frac{1}{(n-1)^3}. \tag{3.7}$$

For $0 < x \leq \frac{1}{48}$, it can be checked that

$$\frac{1}{(1-x)^{3/2}} < 1 + \frac{3}{2}x + \frac{3}{8}x^{\frac{3}{2}}. \tag{3.8}$$

In the notation $\beta = 3\pi/\sqrt{24}$, we have

$$\frac{72\pi}{(n-1)(24n-25)^{3/2}} = \frac{\beta}{(n-1)n^{3/2}(1-\frac{25}{24n})^{3/2}}. \tag{3.9}$$

Setting $x = \frac{25}{24n}$, we have $x \leq \frac{1}{48}$ for $n \geq 60$. Applying (3.8) to the right-hand side of (3.9), we find that for $n \geq 60$,

$$\frac{\beta}{(n-1)n^{3/2}(1-\frac{25}{24n})^{3/2}} < \frac{\beta}{(n-1)n^{3/2}} \left[1 + \frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}}\right], \tag{3.10}$$

so that for $n \geq 60$,

$$\frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} < \frac{\beta}{(n-1)n^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} + \frac{\beta}{(n-1)n^{3/2}}\left[\frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}}\right]. \tag{3.11}$$

To prove (3.7), we proceed to show that the right-hand side of (3.11) is bounded by $\frac{1}{(n-1)^3}$. Noting that for $n \geq 2$,

$$\frac{\beta}{(n-1)n^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} = \frac{\beta}{(n^{5/2}+\beta)(n-1)} + \frac{\beta^2}{(n^{5/2}+\beta)(n-1)n^{3/2}},$$

and using the fact $n^{5/2}+\beta > (n-1)^{5/2}$, together with $n^{3/2} > (n-1)^{3/2}$, we deduce that

$$\frac{\beta}{(n-1)n^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} < \frac{\beta}{(n-1)^{7/2}} + \frac{\beta^2}{(n-1)^5}. \tag{3.12}$$

Applying (3.12) to (3.11), we obtain that for $n \geq 60$,

$$\frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} < \frac{\beta}{(n-1)^{7/2}} + \frac{\beta^2}{(n-1)^5} + \frac{\beta}{(n-1)n^{3/2}}\left[\frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}}\right]. \tag{3.13}$$

Since $\frac{75}{48n} < \frac{2}{n-1}$ and $\frac{3}{8}\left(\frac{25}{24n}\right)^{\frac{3}{2}} < \frac{1}{(n-1)^{3/2}}$ for $n \geq 2$, it follows from (3.13) that for $n \geq 60$,

$$\frac{72\pi}{(n-1)[24n-25]^{3/2}} - \frac{3\pi}{\sqrt{24}n^{5/2}+3\pi} < \frac{\beta}{(n-1)^{7/2}} + \frac{\beta^2}{(n-1)^5} + \frac{2\beta}{(n-1)^{7/2}} + \frac{\beta}{(n-1)^4}.$$

Using the fact that $\beta < 2$, we see that

$$\frac{3\beta}{(n-1)^{7/2}} + \frac{\beta^2}{(n-1)^5} + \frac{\beta}{(n-1)^4} < \frac{6}{(n-1)^{7/2}} + \frac{4}{(n-1)^5} + \frac{2}{(n-1)^4}. \tag{3.14}$$

For $n \geq 60$, it is easily checked that the right-hand side of (3.14) is bounded by $\frac{1}{(n-1)^3}$. This confirms (3.7).

To prove the claim (3.6), it is enough to show that for $n \geq 2095$,

$$\frac{1}{(n-1)^3} < \frac{4\log[\mu(n+1)]}{(n+1)^3} - \frac{5}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}}. \tag{3.15}$$

From (2.61), it can be seen that for $n \geq 2095$,

$$\frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}} < \frac{5}{(n-1)^3}. \tag{3.16}$$

Since $4\log[\mu(n+1)] > 18$ for $n \geq 2095$, it follows from (3.16) that for $n \geq 2095$,

$$\begin{aligned} &\frac{4\log[\mu(n+1)]}{(n+1)^3} - \frac{5}{(n-1)^3} - \frac{5}{n-1}e^{-\frac{\pi\sqrt{24n-25}}{18}} \\ &> \frac{18}{(n+1)^3} - \frac{10}{(n-1)^3} > \frac{1}{(n-1)^3}. \end{aligned}$$

So we obtain (3.15). Combining (3.15) and (3.7), we arrive at (3.6). For $2 \leq n \leq 2094$, the inequality (3.5) can be easily checked. This completes the proof. □

We are now in a position to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. It is known that for $x > 0$,

$$\frac{x}{1+x} < \log(1+x),$$

so that for $n \geq 1$,

$$\frac{3\pi}{\sqrt{24}n^{5/2} + 3\pi} < \log\left(1 + \frac{3\pi}{\sqrt{24}n^{5/2}}\right).$$

In light of the above relation, Theorem 3.2 implies that for $n \geq 2$,

$$\Delta^2 \log \sqrt[n-1]{p(n-1)} < \log\left(1 + \frac{3\pi}{\sqrt{24}n^{5/2}}\right),$$

that is,

$$\sqrt[n+1]{p(n+1)}\ \sqrt[n-1]{p(n-1)} < \left(1 + \frac{3\pi}{\sqrt{24}n^{5/2}}\right)\left(\sqrt[n]{p(n)}\right)^2,$$

as required. □

We remark that $\beta = 3\pi/\sqrt{24}$ is the smallest possible number for the inequality in Theorem 1.4. Suppose that $0 < \gamma < \beta$. By Theorem 3.1, there exists an integer N such that for $n > N$,

$$n^{5/2}\Delta^2 \log \sqrt[n-1]{p(n-1)} > \gamma.$$

It follows that

$$\Delta^2 \log \sqrt[n-1]{p(n-1)} > \frac{\gamma}{n^{5/2}} > \log\left(1 + \frac{\gamma}{n^{5/2}}\right),$$

which implies that for $n > N$,

$$\frac{\sqrt[n]{p(n)}}{\sqrt[n+1]{p(n+1)}}\left(1 + \frac{\gamma}{n^{5/2}}\right) < \frac{\sqrt[n-1]{p(n-1)}}{\sqrt[n]{p(n)}}.$$

Acknowledgements We wish to thank the referee for helpful comments.

References

1. Andrews, G.E.: Combinatorial proof of a partition function limit. Am. Math. Mon. **78**, 276–278 (1971)
2. Andrews, G.E.: The Theory of Partitions. Cambridge University Press, Cambridge (1998)
3. Chen, W.Y.C., Guo, J.J.F., Wang, L.X.W.: Infinitely log-monotonic combinatorial sequences. Adv. Appl. Math. **52**, 99–120 (2014)
4. Chen, W.Y.C., Wang, L.X.W., Xie, G.Y.B.: Finite differences of the logarithm of the partition function. Math. Comput. **85**, 825–847 (2016)
5. Desalvo, S., Pak, I.: Log-concavity of the partition function. Ramanujan J. **38**, 61–73 (2016)
6. Hardy, G.H.: Twelve Lectures on Subjects Suggested by His Life and Work. Cambridge University Press, Cambridge (1940)
7. Hardy, G.H., Ramanujan, S.: Asymptotic formulae in combinatory analysis. Proc. Lond. Math. Soc. **17**, 75–175 (1918)
8. Lehmer, D.H.: On the series for the partition function. Trans. Am. Math. Soc. **43**, 271–292 (1938)
9. Lehmer, D.H.: On the remainders and convergence of the series for the partition function. Trans. Am. Math. Soc. **46**, 362–373 (1939)
10. Rademacher, H.: A convergent series for the partition function $p(n)$. Proc. Natl. Acad. Sci. USA **23**, 78–84 (1937)
11. Sun, Z.W.: On a sequence involving sums of primes. Bull. Aust. Math. Soc. **88**, 197–205 (2013)
12. Wang, Y., Zhu, B.X.: Proofs of some conjectures on monotonicity of number-theoretic and combinatorial sequences. Sci. China Math. **57**, 2429–2435 (2014)